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Weighted Mixed Weak-Type Inequalities for Littlewood–Paley Square Functions Related to Schrödinger Operators

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Abstract

Let $L = -\Delta + V$ be a Schrödinger operator, where Δ is the Laplacian defined on $\mathbb{R}^n$ and the nonnegative potential V satisfies the reverse Hölder inequality. In this paper, we mainly establish weighted mixed weak-type endpoint estimates for Littlewood–Paley square functions associated with the heat semigroup e^{-tL} and their commutators. Such inequalities not only provide a refined characterization of the endpoint behavior of these operators in weighted spaces—extending the classical weak $(1,1)$ bounds, but also have potential applications to the regularity theory of Schrödinger equations with non-smooth potentials and to the boundedness of spectral multipliers. Our approach is based on the Calderón–Zygmund decomposition adapted to Schrödinger setting, which allows for a direct proof without invoking the associated maximal operator. These results are new even in the unweighted setting, and their weighted formulations offer a robust foundation for further developments in Fourier analysis and elliptic equations.

Keywords: weighted mixed inequalities; Littlewood–Paley functions; Schrödinger operators; critical radius functions; commutators

MSC: 42B20; 42B25; 35J10

1. Introduction

It is well known that mixed weak-type estimates for operators are critical in studying the behavior of singular integrals, maximal operators and their commutators, particularly in fractional and non-convolution settings. Historically, the foundational breakthrough in the theory of weighted mixed inequalities dates back to 1976 and the seminal work of Muckenhoupt and Wheeden [1], where they established fundamental weighted weak-type inequalities for operators associated with A_q weights. Specifically, they obtained that for all $q \geq 1$ and $v \in A_q$,

$$\left|\left\{x \in \mathbb{R} : Mf(x)v^{1/q}(x) > \lambda\right\}\right| \leq \frac{C}{\lambda^q} \int_{\mathbb{R}} |f(x)|^q v(x) dx,$$

where M denotes the Hardy–Littlewood maximal operator. Similar results also hold for the maximal Hilbert transform. Their proof was built on the celebrated corona decomposition, a technique centered on constructing intervals on the real line with favorable properties, known as "principal intervals".

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Building on this work, Sawyer [2] later proved the following weighted mixed inequality for M via adapting the corona method,

$$uv\left(\left\{x \in \mathbb{R} : \frac{M(fv)(x)}{v(x)} > \lambda\right\}\right) \leq \frac{C}{\lambda}\int_{\mathbb{R}} |f(x)|u(x)v(x)dx, \tag{1}$$

for $u, v \in A_1$ and every positive λ. The search for an alternative proof of Muckenhoupt's classical L^p-boundedness result for the Hardy–Littlewood maximal operator with A_p weights given in [3] was a primary motivation for investigating the present inequality. Compared to the well-known weak $(1,1)$ result for M, the estimate above provides a more general statement, since the operator appearing in (1) is a perturbation of M by the A_1 weight v. Numerous extensions and refinements of Sawyer's inequality (1) have been established over the last decades. For example, Cruz-Uribe, et al. [4] further improved (1) to higher-dimensional case and extended the weight conditions to $u \in A_1$ and $uv \in A_\infty$ via an extrapolation argument. Later on, Li, Ombrosi and Pérez [5] improved the conditions to $u \in A_1$ and $v \in A_\infty$ via sparse domination, which also solved the so-called Sawyer conjecture raised by [4]. Subsequently, Berra, Carena and Pradolini [6] established the weighted mixed endpoint estimates for fractional maximal functions and fractional integrals. Since then, weighted mixed endpoint inequalities have also been extensively studied for a wide range of other operators, including generalized maximal operators [7–9], variation of Calderón–Zygmund operators with Dini-type kernels [10], pseudo-differential operators [11] and Littlewood–Paley square functions [12,13].

Recently, Schrödinger type operators have also been extensively studied. Recall that the Schrödinger operator on $\mathbb{R}^n$ with $n \geq 3$ is defined as

$$L = -\Delta + V,$$

where Δ is the Laplacian operator on $\mathbb{R}^n$ and the potential V is nonnegative and does not vanish identically. Moreover, for some $q > \frac{n}{2}$, V satisfies the reverse Hölder inequality

$$\left(\frac{1}{|Q|}\int_Q V(y)^q\, dy\right)^{\frac{1}{q}} \leq \frac{C}{|Q|}\int_Q V(y)\, dy,$$

for every cube $Q \subset \mathbb{R}^n$. In view of V, the critical radius function $\rho\colon \mathbb{R}^n \to (0,\infty)$ is defined by

$$\rho(x) = \sup\left\{r > 0 : \frac{1}{r^{n-2}}\int_{Q(x,r)} V(y)\, dy \leq 1\right\}.$$

The critical radius function ρ satisfies a fundamental result from Lemma 1.4 of Shen [14]: if V satisfies the reverse Hölder inequality, then there exist $C_0 > 0$ and $N_0 \geq 1$ such that

$$C_0^{-1}\rho(x)\left(1 + \frac{|x-y|}{\rho(x)}\right)^{-N_0} \leq \rho(y) \leq C_0\,\rho(x)\left(1 + \frac{|x-y|}{\rho(x)}\right)^{\frac{N_0}{N_0+1}} \tag{2}$$

holds for every $x, y \in \mathbb{R}^n$. In particular, $\rho(y) \sim \rho(x)$ whenever $y \in Q(x, \rho(x))$.

The systematic study of quantitative estimates for Schrödinger operators began with Shen's foundational work [14], which established the L^p-boundedness of Riesz transforms associated with L. This was extended by Bongioanni, Harboure and Salinas [15,16], who introduced weight classes adapted to the critical radius function $\rho(x)$ and proved weighted inequalities for Riesz transforms, fractional integrals and Littlewood–Paley functions and their commutators, in the Schrödinger setting. Li and Peng [17] established L^1-type estimates for commutators in the endpoint case. Recently, the weighted theory for Schrödinger-type operators was further extended to spaces of homogeneous

type [18] and Morrey spaces [19–21]. For recent progress in the approximation theory of Schrödinger operators, we refer to [22] on spectral subword approximation and [23] on point-interaction approximations.

It was Berra, Pradolini and Quijano [24] who first studied the weighted mixed inequalities within the Schrödinger framework. They established the following weighted mixed weak-type inequalities for maximal operators $M^{\rho,\sigma}$ related to L and Schrödinger–Calderón–Zygmund operators and applied these results to a broad class of singular integrals of Schrödinger type $\mathcal{T}$.

Theorem 1 (cf. [24]). *Assume that ρ is a critical radius function, let $u \in A_1^\rho$ and $v \in A_\infty^\rho(u)$ (see Section 2 for details). Then for any $\lambda > 0$,*

$$uv\left(\left\{x \in \mathbb{R}^n : \frac{|\mathcal{T}(fv)(x)|}{v(x)} > \lambda\right\}\right) \leq \frac{C}{\lambda}\int_{\mathbb{R}^n} |f(x)|u(x)v(x)\,dx.$$

The same results also hold for maximal operator $M^{\rho,\sigma}$ with some $\sigma > 0$.

By considering weights $u \in A_1^\rho$ and $v \in A_\infty^\rho(u)$, many classical techniques, including the Calderón–Zygmund decomposition, can be extended to this setting. Motivated by Theorem 1, Wen and Wu [25] established the weighted mixed weak-type inequality for the variation operators of heat semigroup $T_s = e^{-sL}$. Similar results are extended to the fractional type integrals related to L by Wen, Hou and Zhang [26]. Nevertheless, few works address the weighted mixed estimates for Littlewood–Paley square functions associated with the heat semigroup.

In this paper, we mainly investigate the Littlewood–Paley functions generated by the heat semigroup T_s. For $(x,t) \in \mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0,\infty)$, define the convolution-type operator Q_t by

$$(Q_t f)(x) = t^2\left(\frac{dT_s}{ds}\bigg|_{s=t^2} f\right)(x).$$

In order to characterize the space BMO_L associated with L, Dziubaski, Garrigós, et al. [27] introduced the following Littlewood–Paley square functions G related to Q_t,

$$G(f)(x) = \left(\int_0^\infty |Q_t(f)(x)|^2 \frac{dt}{t}\right)^{1/2}. \tag{3}$$

Subsequently, the weighted strong type estimates of G have been considered by [15,28], that is, let $1 < p < \infty$ and $v \in A_p^\rho$, then there is a constant $C > 0$ such that

$$\|G(f)\|_{L^p(v)} \leq C\|f\|_{L^p(v)}.$$

However, when $p = 1$, the endpoint estimates of G still remain unknown. Therefore, an interesting question arises: can we establish the weighted mixed weak-type estimates (1) for Littlewood–Paley square functions G?

In this paper, we first establish the following weighted mixed weak-type inequalities for G, which exactly addresses the question above.

Theorem 2. *Suppose that $u \in A_1^\rho$ and $v \in A_\infty^\rho(u)$. If G is the operator defined as in (3), then there is a constant $C > 0$ such that*

$$uv\left(\left\{x \in \mathbb{R}^n : \frac{G(fv)(x)}{v(x)} > \lambda\right\}\right) \leq \frac{C}{\lambda}\int_{\mathbb{R}^n} |f(x)|u(x)v(x)\,dx$$

holds for any $\lambda > 0$.

A direct application of Theorem 2 yields the weighted endpoint estimates for the operator G.

Corollary 1. *Let $u \in A_1^\rho$. If G is the operator defined as in (3), then there is a constant $C > 0$ such that*

$$u(\{x \in \mathbb{R}^n : G(f)(x) > \lambda\}) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)| u(x)\, dx$$

holds for any $\lambda > 0$.

The second goal of this paper is to address the weighted mixed weak-type inequalities for the commutator of the square function G. We introduce the classical commutator formed by a locally integrable function b and an operator S by

$$S_b f(x) := [b, S] f(x) = b(x) S f(x) - S(bf)(x).$$

In 1976, in order to study Hardy space and Riesz system, Coifman, et al. [29] first studied the commutator T_b of a Calderón–Zygmund singular integral operator with smooth kernels. They proved that if $b \in \mathrm{BMO}$, then T_b enjoys the L^p-boundedness for all $1 < p < \infty$. Here, we call a locally integrable function b a BMO function when

$$\|b\|_{\mathrm{BMO}} = \sup_Q \frac{1}{|Q|} \int_Q |b - b_Q| dx < \infty,$$

where $b_Q = \frac{1}{|Q|} \int_Q b(x) dx$ is the average of b over cube Q.

Later on, Pérez [30] investigated the weighted endpoint estimates by using maximal operators and pointed out that T_b is not of weak type (1,1) but rather satisfies a weak $L \log L$-type estimates. Since then, the endpoint estimates of commutators for many different operators have attracted the attention of numerous scholars. For example, Tang [31] established the weighted endpoint inequality of $L \log L$ type for the commutators of Schrödinger type Calderón–Zygmund operators $\mathcal{T}$ and $b \in \mathrm{BMO}(\rho)$, by using the sharp maximal operators associated with nonnegative potentials and the corresponding weighted "good-λ" inequalities. Here the space $\mathrm{BMO}(\rho)$ associated with critical radius functions ρ shall be understood as the set of the $\mathrm{BMO}_\theta(\rho)$ spaces for $\theta \geq 0$, namely,

$$\mathrm{BMO}(\rho) = \bigcup_{\theta \geq 0} \mathrm{BMO}_\theta(\rho).$$

Moreover, for a given $\theta \geq 0$ and a critical radius function ρ, $\mathrm{BMO}_\theta(\rho)$ functions b satisfy that there is $C > 0$ such that

$$\frac{1}{|Q|} \int_Q |b(y) - b_Q| dy \leq C \left(1 + \frac{r}{\rho(x)}\right)^\theta,$$

for each cube $Q = Q(x, r)$. The smallest constant C satisfying the inequality above is denoted by $\|f\|_{\mathrm{BMO}_\theta(\rho)}$.

Recently, Berra, Carena and Pradolini [32] extended the endpoint estimates of T_b in [30] to the weighted mixed weak-type setting via a direct proof, avoiding the use of the associated maximal operator of $L \log L$ type, as follows.

Theorem 3 (cf. [32])**.** *Suppose that u, v be weights such that $u \in A_1$ and $v \in A_\infty(u)$, $b \in \mathrm{BMO}$. If T is a Calderón–Zygmund operator. Then, for any $\lambda > 0$,*

$$uv\left(\left\{x \in \mathbb{R}^n : \left|\frac{T_b(fv)(x)}{v(x)}\right| > \lambda\right\}\right) \leq C \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) u(x) v(x) dx,$$

where and in what follows, $\Phi(t) := t(1+\log^+ t)$.

Subsequently, Berra, Pradolini and Recchi [33] extended Theorem 3 to the commutators formed by Schrödinger type operators and BMO(ρ) functions associated with critical radius functions ρ. To be precise, they established the following weighted mixed inequalities for commutators of Schrödinger type Calderón–Zygmund operators.

Theorem 4 (cf. [33])**.** *Suppose that* ρ *is a critical radius function,* $b \in \mathrm{BMO}(\rho)$, $u \in A_1^\rho$ *and* $v \in A_\infty^\rho(u)$ *(see Section 2 for details). If* $\mathcal{T}_b$ *is the commutator of Schrödinger–Calderón–Zygmund operator* $\mathcal{T}$*, then for any* $\lambda > 0$,

$$uv\left(\left\{x \in \mathbb{R}^n : \frac{|\mathcal{T}_b(fv)(x)|}{v(x)} > \lambda\right\}\right) \leq C\|b\|_{\mathrm{BMO}_\theta(\rho)} \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) u(x)v(x)\,dx.$$

Motivated by Theorem 4, we now investigate the commutator of the Littlewood–Paley square function G. Theorem 2 provides the required weighted mixed weak-type inequality for G, which forms the basis for our commutator analysis. The commutator G_b, generated by G and a locally integrable BMO(ρ) function b, is a nontrivial extension featuring extra oscillatory terms not present in Theorem 2. We define G_b by

$$G_b(f)(x) = \left(\int_0^\infty |Q_t((b(x)-b(\cdot))f)(x)|^2 \frac{dt}{t}\right)^{1/2}, \tag{4}$$

with

$$Q_t((b(x)-b(\cdot))f)(x) = \int_{\mathbb{R}^n} Q_t(x,y)(b(x)-b(y))f(y),$$

where we denote $Q_t(x,y)$ the integral kernels of operators Q_t.

Recently, the following weighted strong type estimates of G_b have also been established by [15,28]: let $1 < p < \infty$, $v \in A_p^\rho$ and $b \in \mathrm{BMO}(\rho)$, then there exists a constant C such that

$$\|G_b(f)\|_{L^p(v)} \leq C\|b\|_{\mathrm{BMO}(\rho)}\|f\|_{L^p(v)}.$$

Later on, by using this result, He and Li [34] proved that G_b is a compact operator on $L^p(v)$ provided $b \in \mathrm{CMO}(\rho)$, where CMO(ρ) is defined as the closure of $C_c^\infty(\mathbb{R}^n)$ in the BMO(ρ) topology and $C_c^\infty(\mathbb{R}^n)$ consists of all smooth functions having compact support on $\mathbb{R}^n$.

Naturally, one would like to know what occurs in the case $p = 1$ for G_b, and this is exactly what we shall address. Building on the weighted estimates from Theorem 2, we adjust our decomposition method to control the oscillation of G_b, and the corresponding weighted bounds for G_b are given in the theorem below.

Theorem 5. *Let* $u \in A_1^\rho$, $v \in A_\infty^\rho(u)$ *and* $b \in \mathrm{BMO}(\rho)$*. If* G_b *is the commutator of* G *defined as in* (4)*, then for any* $\lambda > 0$*, there is a constant* $C > 0$ *such that*

$$uv\left(\left\{x \in \mathbb{R}^n : \frac{|G_b(fv)(x)|}{v(x)} > \lambda\right\}\right) \leq C\int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) u(x)v(x)\,dx.$$

As corollary, we obtain the weighted weak-type boundedness of $L\log L$ type for the commutator G_b.

Corollary 2. *Let* $v \in A_1^\rho$, $b \in \mathrm{BMO}(\rho)$*. If* G_b *is the commutator of* G *defined as in* (4)*, then there is a constant* $C > 0$ *such that*

$$v(\{x \in \mathbb{R}^n : G_b(f)(x) > \lambda\}) \leq C\int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) v(x)\,dx$$

holds for every $\lambda > 0$.

Remark 1. *Theorems 2 and 5 provide the first weighted mixed endpoint estimates for sublinear operators beyond the convolution setting. The main technical difficulty for square functions is their inherent quadratic structure, which complicates the analysis and prevents a direct reduction to pointwise domination by a maximal operator. Our main results are novel even in the unweighted case.*

Unlike the maximal-operator-based approaches (e.g., [5,10–13,25,26]), we adopt a direct proof that avoids invoking the corresponding maximal operator of $L\log L$ type to deal with Littlewood–Paley square functions, following from the strategy developed by Berra et al. [8,32]. This strategy provides an alternative framework for establishing weighted mixed norm inequalities for operators that lack such maximal domination. Moreover, in the Schrödinger setting, weighted mixed estimates for commutators of maximal operators are still missing, so we cannot obtain our results from known work on maximal commutators. The Calderón–Zygmund decomposition is the key tool here: it lets us split the function into a good part and a bad part. The bad part can then be controlled using the size and smoothness of the kernel, which allows us to avoid the missing $L\log L$-type maximal operator altogether.

Our proof relies on Calderón–Zygmund decomposition combined with generalized Hölder, Minkowski, Chebyshev and John–Nirenberg inequalities to establish weighted mixed norm inequalities for the square function G and its commutators G_b. Kernel size and smoothness conditions are essential to our argument and restrict straightforward extensions to operators with poorly regular kernels.

Beyond the theoretical interest, the weighted mixed weak-type inequalities established in this paper are expected to have several future applications. In harmonic analysis, they may serve as a key ingredient for proving endpoint estimates for other operators associated with Schrödinger operators, such as Riesz transforms and maximal functions, within the weighted framework. In particular, they could be used to characterize weighted Hardy spaces adapted to L. In the context of partial differential equations, these inequalities may contribute to the study of regularity and solvability of Schrödinger-type equations with nonsmooth coefficients or critical potentials, where strong-type bounds are often unavailable. Moreover, the commutator estimates obtained here could pave the way for investigating higher-order commutators and for developing function spaces of BMO-type adapted to L. From the perspective of operator theory, our results may provide a route to proving boundedness of spectral multipliers on weighted L^p spaces and to deriving improved Fourier multiplier theorems for operators with nonnegative potentials.

The outline of the paper is as follows. We will present several important definitions and preliminary lemmas in Section 2. Sections 3 and 4 are devoted to establishing weighted mixed inequalities for Littlewood–Paley square functions G and their commutators G_b related to Schrödinger operators by using Calderón–Zygmund decomposition adapted to Schrödinger setting, which enables us to provide a direct approach without involving the corresponding maximal operator of $L\log L$ type. In Section 5, we summarize the main contributions of this work, discuss its limitations, and outline potential directions for future research.

Throughout this paper, for an exponent p, we denote $p' = \frac{p}{p-1}$ by its conjugate exponent. The symbol C will be reserved for a positive constant that is independent of the essential parameters of the problem and it may vary in different cases. The notation $A \lesssim B$ indicates A is bounded by CB with a positive constant C. If $A \lesssim B$ and $B \lesssim A$ are both true, we shall simply write $A \sim B$.

2. Preliminaries and Definitions

In this paper, every cube has sides parallel to the coordinate axes. We write $Q(x_Q, r_Q)$ for the cube centred at x_Q, where $r_Q = \sqrt{n}\ell(Q)/2$ with $\ell(Q)$ meaning its side length. We frequently refer to *critical cubes* $Q(x_0, \rho(x_0))$ and *subcritical cubes* $Q(x_0, r)$ with $r \le \rho(x_0)$.

In order to deal with the operator Q_t, Dziubański, et al. [27] established key estimates for the corresponding integral kernels $Q_t(\cdot,\cdot)$, as follows.

Lemma 1 (cf. [27]). *There exist constants $c, \delta_0 > 0$ such that for every $l > 0$, there is a positive constant C_l such that*

$$|Q_t(x,y)| \le C_l t^{-n}\left(1+\frac{t}{\rho(x)}+\frac{t}{\rho(y)}\right)^{-l} e^{-\frac{c|x-y|^2}{t^2}}, \quad x,y\in\mathbb{R}^n; \tag{5}$$

$$|Q_t(x,y+h)-Q_t(x,y)| \le C_l \frac{|h|^{\delta_0}}{|t|^{n+\delta_0}}\left(1+\frac{t}{\rho(x)}+\frac{t}{\rho(y)}\right)^{-l} e^{-\frac{c|x-y|^2}{t^2}}, \quad x,y,h\in\mathbb{R}^n, \tag{6}$$

with $|h| \le t$ and $t \in (0,\infty)$.

The following lemma on the kernels will make our proof proceed more smoothly, which was proved in [34] (page 23).

Lemma 2 (cf. [34] (page 23)). *There exists constant $\delta_0 > 0$ such that for every l there is a constant $C_l > 0$ so that*

$$\left(\int_0^\infty |Q_t(x,y)|^2\frac{dt}{t}\right)^{1/2} \le C_l\left(1+\frac{|x-y|}{\rho(x)}+\frac{|x-y|}{\rho(y)}\right)^{-l}\frac{1}{|x-y|^n}, \tag{7}$$

and

$$\left(\int_0^\infty |Q_t(x,y)-Q_t(x,\xi)|^2\frac{dt}{t}\right)^{1/2} \le C_l\left(1+\frac{|x-y|}{\rho(x)}+\frac{|x-y|}{\rho(y)}\right)^{-l}\frac{|y-\xi|^{\delta_0}}{|x-y|^{n+\delta_0}}, \tag{8}$$

for $|x-y| > 2|y-\xi|$.

In this paper, we consider certain weight classes related to the critical radius function $\rho(x)$, which extend the A_p classes of Munkenhoupt and were introduced by Bongioanni, et al. in [15].

Definition 1 ($A_p^{\rho,\theta}$ weights). *Assume that u is a locally integrable function and let $\theta \ge 0$.*

(i) For $1 < p < \infty$, we call $v \in A_p^{\rho,\theta}(u)$, if there is a constant $C > 0$ such that

$$\left(\frac{1}{u(Q)}\int_Q v(y)u(y)dy\right)^{1/p}\left(\frac{1}{u(Q)}\int_Q v^{1-p'}(y)u(y)dy\right)^{1/p'} \le C\left(1+\frac{r}{\rho(x)}\right)^{\theta} \tag{9}$$

holds for every cube $Q = Q(x,r)$. Here and in what follows, $u(Q) := \int_Q u(x)dx$.

(ii) For $p = 1$, we say $v \in A_1^{\rho,\theta}(u)$ if there exists $C > 0$ such that

$$\frac{1}{u(Q)}\int_Q v(y)u(y)dy \le C\left(1+\frac{r}{\rho(x)}\right)^{\theta}\inf_Q v, \tag{10}$$

for every cube $Q = Q(x,r)$.

(iii) For $p = \infty$, set $A_\infty^{\rho,\theta}(u) = \bigcup_{p\ge1} A_p^{\rho,\theta}(u)$. We denote the minimal constant C appearing in (9) and (10) by $[v]_{A_p^{\rho,\theta}(u)}$. When $u = 1$ we shall denote it by $[v]_{A_p^{\rho,\theta}}$.

For $1 \le p \le \infty$, the class $A_p^\rho(u)$ gathers all $A_p^{\rho,\theta}(u)$ over $\theta \ge 0$, namely,

$$A_p^\rho(u) = \bigcup_{\theta \ge 0} A_p^{\rho,\theta}(u).$$

If $u = 1$, we shall directly write $A_p^{\rho,\theta} = A_p^{\rho,\theta}(u)$ and $A_p^\rho = A_p^\rho(u)$. Obviously, if $u \in A_p^\rho$, there is $\theta > 0$ such that $u \in A_p^{\rho,\theta}$.

Remark 2. *It is known that $v \in A_p^\rho$ implies $v^{1-p'} \in A_{p'}^\rho$ for $1 < p < \infty$, see [15].*

Remark 3. *We find that if $v \in A_p^\rho$ for some $1 \le p \le \infty$, then the measure μ given by $d\mu(x) = v(x)\,dx$ satisfies the doubling property, that is, there exist constants $C > 0$ and $\alpha \ge 0$ such that the inequality*

$$\mu(2Q) \le C\mu(Q)\left(1 + \frac{r}{\rho(x)}\right)^\alpha \tag{11}$$

holds for each cube $Q = Q(x, r)$.

The following lemma establishes some properties between $A_p^\rho(u)$ and A_p^ρ weights, see [24,33].

Lemma 3 (cf. [24,33])**.** *The following properties hold.*

(i) *For any $1 \le p \le \infty$, $uv \in A_p^\rho$, whenever $u \in A_1^\rho$ and $v \in A_p^\rho(u)$.*
(ii) *If $u \in A_1^\rho$ then $A_p^\rho(u) \subseteq A_p^\rho$.*

It is well known that A_p^ρ weights satisfy the corresponding reverse Hölder inequality.

Definition 2 (Reverse Hölder class $\mathrm{RH}_s^{\rho,\theta}$)**.** *Let $\theta \ge 0$ and v be a weight.*

(i) *For $1 < s < \infty$, we call $v \in \mathrm{RH}_s^{\rho,\theta}$, if the inequality*

$$\left(\frac{1}{|Q|}\int_Q v^s(y)dy\right)^{1/s} \le C\left(1 + \frac{r}{\rho(x)}\right)^\theta \left(\frac{1}{|Q|}\int_Q v(y)dy\right)$$

holds for each cube $Q = Q(x, r)$ and for a constant $C > 0$.

(ii) *For $s = \infty$, we say $v \in \mathrm{RH}_\infty^{\rho,\theta}$ if*

$$\sup_Q v \le C\left(1 + \frac{r}{\rho(x)}\right)^\theta \left(\frac{1}{|Q|}\int_Q v(y)dy\right)$$

holds for each cube $Q = Q(x, r)$ and for a constant $C > 0$.

The minimal constant C satisfying the above inequalities is denoted by $[v]_{\mathrm{RH}_s^{\rho,\theta}}$. For $1 < s \le \infty$, we define

$$\mathrm{RH}_s^\rho = \bigcup_{\theta \ge 0} \mathrm{RH}_s^{\rho,\theta},$$

meaning that RH_s^ρ consists of all classes $\mathrm{RH}_s^{\rho,\theta}$ with $\theta \ge 0$.

We recall here the notions of the Young function and the Luxemburg average, which will be needed in the proof of Theorem 5.

Definition 3 (Young function). *We recall that a Young function is a strictly increasing and convex function $\psi\colon [0,\infty) \to [0,\infty)$ satisfying $\psi(0) = 0$ and $\psi(t) \to \infty$ when $t \to \infty$. Let ψ^{-1} denote the generalized inverse of ψ, then*

$$\psi^{-1}(t) = \inf\{s \geq 0 : \psi(s) \geq t\},$$

where we understand $\inf \emptyset = \infty$. The complementary function $\widetilde{\psi}$ of such ψ is defined by

$$\widetilde{\psi}(t) = \sup\{ts - \psi(s) : s \geq 0\}.$$

Suppose ψ and $\widetilde{\psi}$ are Young functions; the following estimate is valid for each $t > 0$,

$$\psi^{-1}(t)\widetilde{\psi}^{-1}(t) \approx t. \tag{12}$$

Definition 4 (Luxemburg average). *Given a Young function ψ and a weight v, we call $\|f\|_{\psi,Q,v}$ the weighted Luxemburg average of f over Q, defined as*

$$\|f\|_{\psi,Q,v} = \inf\left\{\tau > 0 : \frac{1}{v(Q)}\int_Q \psi\left(\frac{|f(x)|}{\tau}\right)v(x)dx \leq 1\right\}.$$

An equivalent expression for the norm $\|\cdot\|_{\psi,Q,v}$ will be useful in our estimates, see [35] or [36]. Specifically, if v is a weight and function f satisfies that $\psi(|f|) \in L^1_{\mathrm{loc}}(v)$, then for every cube Q we have

$$\|f\|_{\psi,Q,v} \approx \inf_{\tau>0}\left\{\tau + \frac{\tau}{v(Q)}\int_Q \psi\left(\frac{|f(x)|}{\tau}\right)v(x)dx\right\}. \tag{13}$$

The following generalized Hölder inequality is crucial.

Lemma 4 (cf. [36]). *Let φ, ψ, η be Young functions and satisfy*

$$\eta^{-1}(t)\varphi^{-1}(t) \lesssim \psi^{-1}(t)$$

for all $t \geq t_0 > 0$. Then there is $k_0 > 0$ such that

$$\psi(st) \lesssim \eta(s) + \varphi(t)$$

whenever $s, t \geq k_0$. Then

$$\|fh\|_{\psi,Q,v} \lesssim \|f\|_{\eta,Q,v}\|h\|_{\varphi,Q,v},$$

valid for any weight v and cube Q.

The following lemmas involve the function $\widetilde{\Phi}(t) = \mathrm{e}^t - 1$, which is the complementary Young function of $\Phi(t) = t(1 + \log^+ t)$.

Lemma 5 (cf. [33]). *Assume that v is a weight satisfying $v \in \mathrm{RH}_s^{\rho,\theta}$ for some $s > 1$ and $\theta \geq 0$. Then*

$$\|f\|_{\widetilde{\Phi},Q,v} \leq s'\,2^{1/s'}[v]_{\mathrm{RH}_s^{\rho,\theta}}\left(1 + \frac{r}{\rho(x)}\right)^{\theta}\|f\|_{\widetilde{\Phi},Q},$$

for any $Q = Q(x,r)$.

Lemma 6 (cf. [33]). *Let $\theta > 0$, suppose that $b \in \mathrm{BMO}_\theta(\rho)$, $\widetilde{\Phi}(t) = e^t - 1$, and $Q = Q(x,r)$. Then there exist constants $\vartheta > 0$ and $C > 0$ such that*

$$\|b - b_Q\|_{\widetilde{\Phi},Q} \leq C\|b\|_{\mathrm{BMO}_\theta(\rho)}\left(1 + \frac{r}{\rho(x)}\right)^{\vartheta}.$$

Lemma 7 below is a useful consequence of the John–Nirenberg inequality adapted to the $\mathrm{BMO}_\theta(\rho)$ functions.

Lemma 7 (cf. [16]). *Let $\theta > 0$, $1 < s < \infty$ and $b \in \mathrm{BMO}_\theta(\rho)$. Then the inequality*

$$\left(\frac{1}{|Q|}\int_Q |b(x) - b_Q|^s dx\right)^{1/s} \le \|b\|_{\mathrm{BMO}_\theta(\rho)} \left(1 + \frac{r}{\rho(x)}\right)^{(N_0+1)\theta}$$

holds for all cubes $Q = Q(x, r)$, where N_0 is the constant that appears in (2).

For the convenience of readers, we summarize the notations of weight classes and spaces appearing in this paper in Table 1.

Table 1. Notations of weight classes and spaces.

Symbol	Definition	Brief Explanation
$A_p^{\rho,\theta}$, $A_p^{\rho,\theta}(u)$	Weight classes related to Schrödinger operator	Muckenhoupt weights adapted to critical radius factor $(1 + r/\rho(x))^\theta$
A_p^ρ, $A_p^\rho(u)$	$A_p^\rho(u) = \bigcup_{\theta\ge 0} A_p^{\rho,\theta}(u)$, $A_p^\rho = \bigcup_{\theta\ge 0} A_p^{\rho,\theta}$	Muckenhoupt weights adapted to critical radius function ρ
$\mathrm{BMO}_\theta(\rho)$	BMO space related to Schrödinger operator	Space of functions with bounded mean oscillation related to ρ
$\mathrm{BMO}(\rho)$	$\mathrm{BMO}(\rho) = \bigcup_{\theta\ge 0} \mathrm{BMO}_\theta(\rho)$	Space of functions with bounded mean oscillation with critical radius factor $(1 + r/\rho(x))^\theta$
$\mathrm{RH}_s^{\rho,\theta}$	Reverse Hölder weight class related to Schrödinger operator	Modified RH_s condition with critical radius factor $(1 + r/\rho(x))^\theta$

The following decomposition will be a fundamental ingredient of our argument.

Lemma 8 (cf. [37]). *There is a sequence $\{x_j\}_{j\in\mathbb{N}}$ such that the critical cubes $Q_j = Q(x_j, \rho(x_j))$ enjoy the following properties:*

(i) $\bigcup_{j\in\mathbb{N}} Q_j = \mathbb{R}^n$.

(ii) *There exist constants $C > 0$ and $N_1 > 0$ such that for any $\beta \ge 1$, $\sum_{j\in\mathbb{N}} \chi_{\beta Q_j} \le C\beta^{N_1}$.*

The argument of our first main result relies on Calderón–Zygmund decomposition applied to a function supported on a fixed cube, see [38].

Lemma 9 (cf. [38]). *Let $Q \subset \mathbb{R}^n$ be a cube, v be a weight satisfying doubling property, and $f \in L^1(Q, v\,dx)$. If*

$$\frac{1}{v(Q)}\int_Q f(x)v(x)\,dx < \lambda,$$

then there exist functions H, h and a family of dyadic sub-cubes $\{P_i\}_{i\in\mathbb{N}}$ of Q satisfying

(i) $f = H + h$;

(ii) $|H(x)| \le C\lambda$, *for almost every x;*

(iii) $h = \sum_{i\in\mathbb{N}} h_i$, *where* $\operatorname{supp} h_i \subset P_i$ *for each $i \in \mathbb{N}$.*

Lemma 9 presents the classical Calderón–Zygmund decomposition for doubling weights over standard dyadic cubes, a fundamental decomposition technique in weighted harmonic analysis. However, this classical decomposition does not involve the critical

radius function $\rho(x)$, so it is not directly applicable to the ρ-related weight classes A_p^ρ and $A_\infty^\rho(u)$ introduced for the Schrödinger framework.

To fix this issue, Berra, Pradolini and Recchi adjusted the stopping-time rule for picking cubes in the standard Calderón–Zygmund algorithm. Rather than partitioning the domain purely via dyadic sub-cubes, they introduced an additional scaling term $\left(1+\frac{r_j}{\rho(x_j)}\right)$ to constrain the weighted mean value on each chosen cube Q_j. The resulting improved decomposition is given in Lemma 10. This version is specially designed for weights of type A_p^ρ, and fits naturally with the multi-scale characteristics of the critical radius function $\rho(x)$.

Lemma 10 (cf. [33] (Lemma 3.1))**.** *If ρ is a critical radius function satisfying (2) and let $\Theta \geq 0$, $v \in A_p^\rho$ for $1 \leq p \leq \infty$. Then for any positive λ, there is a countable family of disjoint cubes $\{Q_j\}$ with $Q_j = Q(x_j, r_j)$ satisfying*

$$\left(1+\frac{r_j}{\rho(x_j)}\right)^{\Theta}\lambda < \frac{1}{v(Q_j)}\int_{Q_j}|f(x)|v(x)dx \leq C\left(1+\frac{r_j}{\rho(x_j)}\right)^{\sigma}\lambda,$$

for some $\sigma \geq \Theta$ and $C > 0$ depending only on the constants appearing in (2) and (11).

At the same time, for almost every $x \notin \bigcup_j Q_j$, $|f(x)| \leq \lambda$.

Throughout the proofs, several parameters are introduced to control the decay of the kernel estimates, the summability of dyadic series, and the interaction with the weight classes. For clarity, we collect here the required relationships between these parameters and the weight conditions, as summarized in Table 2.

Table 2. Constraints among parameters.

Parameter	Description	Constraint
M	Any fixed constant large enough	depending on n and desired accuracy
l	A decay exponent ensuring convergence of the time integral	can be taken large enough
Θ	The scale parameter in the Calderón–Zygmund decomposition	can be taken large enough
α	Exponent for doubling condition of weights	$\exists\, \alpha \geq 0$, such that $\mu(2Q) \leq C\mu(Q)\left(1+\frac{r}{\rho(x)}\right)^{\alpha}$
θ_1	Growth exponent from weight v in $A_1^\rho(u)$	$\exists\, \theta_1 \geq 0$, such that $v \in A_1^{\rho,\theta_1}(u)$
r, θ_2	Exponent for product weight uv in reverse Hölder class	$\exists\, r > 1$, $\theta_2 \geq 0$, $uv \in \mathrm{RH}_r^{\rho,\theta_2}$
s, θ_3	Exponent for u in reverse Hölder class $\mathrm{RH}_s^{\rho,\theta_3}$	$\exists\, s > 1$, $\theta_3 \geq 0$, $u \in \mathrm{RH}_s^{\rho,\theta_3}$
θ_4	Exponent for u in weighted A_1 class	$\exists\, \theta_4 \geq 0$, such that $u \in A_1^{\rho,\theta_4}$

3. Proof of Theorem 2

We will devote this section to proving the first main theorem. The following inequality for G is useful, which can be obtained by similar methods as [28]; thus, we omit the proof.

Lemma 11. *Set $1 < p < \infty$. If $v \in A_p^\rho$, then there is a positive constant C such that*

$$\|G(f)\|_{L^p(v)} \le C\|f\|_{L^p(v)}.$$

With all the necessary preliminary lemmas established above, we begin to prove Theorem 2.

Proof of Theorem 2. We may assume that, without loss of generality, f is nonnegative. Let $\{Q_j\}_{j\in\mathbb{N}}$ denote the collection of critical cubes from Lemma 8. Fix $\lambda > 0$, we have

$$\begin{aligned} uv\left(\left\{x \in \mathbb{R}^n : \frac{|G(fv)(x)|}{v(x)} > \lambda\right\}\right) &\le \sum_{j\in\mathbb{N}} uv\left(\left\{x \in Q_j : \frac{|G(fv)(x)|}{v(x)} > \lambda\right\}\right) \\ &\le \sum_{j\in\mathbb{N}} uv\left(\left\{x \in Q_j : \frac{|G(f_1^j v)(x)|}{v(x)} > \frac{\lambda}{2}\right\}\right) \\ &\quad + \sum_{j\in\mathbb{N}} uv\left(\left\{x \in Q_j : \frac{|G(f_2^j v)(x)|}{v(x)} > \frac{\lambda}{2}\right\}\right), \end{aligned}$$

where $f_1^j = f\chi_{(2Q_j)^c}$ and $f_2^j = f\chi_{2Q_j}$.

We bound the term for f_1^j. An application of Chebyshev's inequality yields

$$\begin{aligned} \sum_{j\in\mathbb{N}} uv\left(\left\{x \in Q_j : \frac{|G(f_1^j v)(x)|}{v(x)} > \frac{\lambda}{2}\right\}\right) &\lesssim \sum_{j\in\mathbb{N}} \frac{1}{\lambda}\int_{Q_j} |G(f_1^j v)(x)|u(x)\,dx \\ &\lesssim \frac{1}{\lambda}\sum_{j\in\mathbb{N}}\int_{Q_j}\left(\int_0^\infty \left|Q_t(f_1^j v)(x)\right|^2 \frac{dt}{t}\right)^{\frac{1}{2}} u(x)\,dx. \end{aligned}$$

By using (5), for any $M > 0$, we get

$$\begin{aligned} \left|Q_t(f_1^j v)(x)\right| &= \left|\int_{\mathbb{R}^n\setminus 2Q_j} Q_t(x,y)f(y)v(y)\,dy\right| \\ &\lesssim t^{-n}\left(1+\frac{t}{\rho(x)}\right)^{-l}\int_{\mathbb{R}^n\setminus 2Q_j} e^{-\frac{c|x-y|^2}{t^2}} f(y)v(y)\,dy \\ &\lesssim t^{M-n}\left(1+\frac{t}{\rho(x)}\right)^{-l}\int_{\mathbb{R}^n\setminus 2Q_j} \frac{f(y)v(y)}{|x-y|^M}\,dy \\ &\lesssim t^{M-n}\rho(x_j)^{-M}\left(1+\frac{t}{\rho(x)}\right)^{-l}\sum_{k=1}^\infty 2^{-kM}\int_{2^{k+1}Q_j\setminus 2^k Q_j} f(y)v(y)\,dy. \end{aligned}$$

For $x \in Q_j$, note that $\rho(x) \sim \rho(x_j)$ from (2). Therefore

$$\begin{aligned} &\left(\int_0^\infty \left|\int_{\mathbb{R}^n\setminus 2Q_j} Q_t(x,y)f(y)v(y)\,dy\right|^2 \frac{dt}{t}\right)^{\frac{1}{2}} \\ &\lesssim \rho(x_j)^{-n}\sum_{k=1}^\infty 2^{-kM}\int_{2^{k+1}Q_j\setminus 2^k Q_j} f(y)v(y)dy\left(\int_0^\infty \left(\frac{t}{\rho(x)}\right)^{2(M-n)}\left(1+\frac{t}{\rho(x)}\right)^{-2l}\frac{dt}{t}\right)^{\frac{1}{2}} \\ &\lesssim \rho(x_j)^{-n}\sum_{k=1}^\infty 2^{-kM}\int_{2^{k+1}Q_j\setminus 2^k Q_j} f(y)v(y)dy, \end{aligned}$$

choosing M satisfying $M > n$ and $l > M - n$. As $u \in A_1^\rho$, there exists some $\theta_4 > 0$ for which $u \in A_1^{\rho,\theta_4}$. Then, it yields that

$$\begin{aligned}
&\sum_{j\in\mathbb{N}} uv\left(\left\{x\in Q_j:\frac{|G(f_1^j v)(x)|}{v(x)}>\frac{\lambda}{2}\right\}\right)\\
&\lesssim \frac{1}{\lambda}\sum_{j\in\mathbb{N}}\int_{Q_j}\sum_{k=1}^{\infty}\frac{2^{-kM}}{\rho(x_j)^n}\Big(\int_{2^{k+1}Q_j} f(y)v(y)\,dy\Big)u(x)\,dx\\
&\lesssim \frac{1}{\lambda}\sum_{j\in\mathbb{N}}\sum_{k=1}^{\infty}2^{-k(M-n)}\frac{u(2^{k+1}Q_j)}{|2^{k+1}Q_j|}\int_{2^{k+1}Q_j} f(x)v(x)\,dx\\
&\lesssim \frac{1}{\lambda}\sum_{j\in\mathbb{N}}\sum_{k=1}^{\infty}2^{-k(M-n-\theta_4)}\int_{2^{k+1}Q_j} f(x)v(x)u(x)\,dx\\
&\lesssim \frac{1}{\lambda}\sum_{k=1}^{\infty}2^{-k(M-n-\theta_4)}\int_{\mathbb{R}^n}\Big(\sum_{j\in\mathbb{N}}\chi_{2^{k+1}Q_j}(x)\Big)f(x)v(x)u(x)\,dx\\
&\lesssim \frac{1}{\lambda}\sum_{k=1}^{\infty}2^{-k(M-n-\theta_4-N_1)}\int_{\mathbb{R}^n} f(x)u(x)v(x)dx\\
&\lesssim \frac{1}{\lambda}\int_{\mathbb{R}^n} f(x)u(x)v(x)dx,
\end{aligned}$$

where we have used part (ii) of Lemma 8 for some $N_1>0$ and chosen $M>n+\theta_4+N_1$.

We now focus on the term associated with f_2^j and prove that

$$uv\left(\left\{x\in Q_j:\frac{|G(f_2^j v)(x)|}{v(x)}>\frac{\lambda}{2}\right\}\right)\le\frac{C}{\lambda}\int_{2Q_j} f(x)u(x)v(x)\,dx,$$

for each $j\in\mathbb{N}$. Fix $j\in\mathbb{N}$ and assume first that

$$(f_2^j)^v_{2Q_j}=\frac{1}{v(2Q_j)}\int_{2Q_j} f_2^j(x)v(x)\,dx\ge\frac{\lambda}{2}.$$

From Theorem 4 in [24], combining $u\in A_1^\rho$ with $v\in A_\infty^\rho(u)$, it yields that there exist exponent $\theta_1\ge 0$ and a constant $C>0$ and such that

$$\frac{1}{v(Q)}\int_Q u(y)v(y)dy\le C\left(1+\frac{r}{\rho(x)}\right)^{\theta_1}\inf_{y\in Q}u(y)\tag{14}$$

holds for all cubes $Q=Q(x,r)$. Hence, by using (14), it follows that

$$\begin{aligned}
uv\left(\left\{x\in Q_j:\frac{|G(f_2^j v)(x)|}{v(x)}>\frac{\lambda}{2}\right\}\right)&\le\frac{uv(2Q_j)}{v(2Q_j)}v(2Q_j)\\
&\le\frac{2uv(2Q_j)}{v(2Q_j)\lambda}\int_{2Q_j} f_2^j(x)v(x)\,dx\\
&\lesssim\frac{1}{\lambda}\big(\inf_{2Q_j}u\big)\int_{2Q_j} f(x)v(x)\,dx\\
&\lesssim\frac{1}{\lambda}\int_{2Q_j} f(x)u(x)v(x)\,dx.
\end{aligned}$$

On the other hand, if $(f_2^j)^v_{2Q_j}<\lambda/2$, we apply Lemma 9 on $2Q_j$ at level $\lambda/2$. This produces a disjoint family $\{P_{i,j}\}_{i\in\mathbb{N}}$ of dyadic sub-cubes and two functions H_j, h_j satisfying $f_2^j=H_j+h_j$, where

$$H_j(x)=\begin{cases}\frac{1}{v(P_{i,j})}\int_{P_{i,j}} f_2^j(y)\,v(y)dy, & \text{if}\quad x\in P_{i,j},\\ f_2^j(x), & \text{if}\quad x\in 2Q_j\setminus P_{i,j},\end{cases}$$

h_j is given by

$$h_j(x) = \sum_{i\in\mathbb{N}} \left(f_2^j(x) - \frac{1}{v(P_{i,j})} \int_{P_{i,j}} f_2^j(y)\, v(y)dy \right) \chi_{P_{i,j}}(x) =: \sum_{i\in\mathbb{N}} h_{i,j}(x).$$

We use $x_{i,j}$ and $r_{i,j}$ to represent the center and radius of $P_{i,j}$, respectively, and set $\Omega_j = \bigcup_{i\in\mathbb{N}} P_{i,j}$. Define $\widetilde{P}_{i,j} = 2P_{i,j}$ and $\widetilde{\Omega}_j = \bigcup_{i\in\mathbb{N}} \widetilde{P}_{i,j}$. Then we have

$$\begin{aligned} uv\left(\left\{ x \in Q_j : \frac{|G(f_2^j v)(x)|}{v(x)} > \frac{\lambda}{2} \right\} \right) &\le uv\left(\left\{ x \in Q_j : \frac{|G(H_j v)(x)|}{v(x)} > \frac{\lambda}{4} \right\} \right) + uv(\widetilde{\Omega}_j) \\ &\quad + uv\left(\left\{ x \in Q_j \setminus \widetilde{\Omega}_j : \frac{|G(h_j v)(x)|}{v(x)} > \frac{\lambda}{4} \right\} \right) \\ &=: \mathcal{I} + \mathcal{II} + \mathcal{III}. \end{aligned}$$

For $\mathcal{I}$, we observe that $v \in A_\infty^\rho(u)$ implies $v \in A_{q'}^\rho(u)$, for some $q' > 1$. Therefore, by Remark 2 and item (i) in Lemma 3, we have $uv^{1-q} \in A_q^\rho$. By using Chebyshev's inequality and Lemma 11, we get

$$\begin{aligned} \mathcal{I} &\le \frac{C}{\lambda^q} \int_{\mathbb{R}^n} |G(H_j v)(x)|^q u(x) v^{1-q}(x) dx \\ &\le \frac{C}{\lambda^q} \int_{\mathbb{R}^n} |H_j(x)|^q u(x) v(x) dx \\ &\le \frac{C}{\lambda} \int_{\mathbb{R}^n} |H_j(x)| u(x) v(x) dx, \end{aligned}$$

where we also employ the bound $|H_j(x)| \le C\lambda$ for almost every $x \in 2Q_j$.

Then by using (14), we obtain

$$\begin{aligned} \mathcal{I} &\le \frac{C}{\lambda} \sum_{i\in\mathbb{N}} \int_{2Q_j \setminus P_{i,j}} f(x) u(x) v(x)\, dx + \frac{C}{\lambda} \sum_{i\in\mathbb{N}} \frac{uv(P_{i,j})}{v(P_{i,j})} \int_{P_{i,j}} f(x) v(x)\, dx \\ &\le \frac{C}{\lambda} \sum_{i\in\mathbb{N}} \int_{2Q_j \setminus P_{i,j}} f(x) u(x) v(x)\, dx + \frac{C}{\lambda} \sum_{i\in\mathbb{N}} \int_{P_{i,j}} f(x) u(x) v(x)\, dx \\ &\le \frac{C}{\lambda} \int_{2Q_j} f(x) u(x) v(x)\, dx. \end{aligned}$$

To handle term $\mathcal{II}$, we note that $uv \in A_\infty^\rho$, so uv is doubling on all sub-cubes $P_{i,j}$. From (11), (14) and Lemma 10, we proceed as follows

$$\begin{aligned} \mathcal{II} = uv(\widetilde{\Omega}_j) &\le \sum_{i\in\mathbb{N}} uv(\widetilde{P}_{i,j}) \le C \sum_{i\in\mathbb{N}} v(P_{i,j}) \left(1 + \frac{r_{i,j}}{\rho(x_{i,j})} \right)^\alpha \frac{uv(P_{i,j})}{v(P_{i,j})} \\ &\le \frac{C}{\lambda} \sum_{i\in\mathbb{N}} \left(1 + \frac{r_{i,j}}{\rho(x_{i,j})} \right)^{\alpha+\theta_1-\Theta} \left(\inf_{y\in P_{i,j}} u(y) \right) \int_{P_{i,j}} f(x) v(x)\, dx \\ &\le \frac{C}{\lambda} \int_{2Q_j} f(x) u(x) v(x)\, dx, \end{aligned} \tag{15}$$

where we take $\Theta > \theta_1 + \alpha$. The desired estimate follows.

The definition of $h_{i,j}$ gives that

$$\int_{P_{i,j}} h_{i,j}(x) v(x)\, dx = 0. \tag{16}$$

Then, by using the Chebyshev inequality, the Minkowski inequality, (8), (16) and (2), we have

$$\begin{aligned}
\mathcal{III} &= uv\left(\left\{x \in Q_j \setminus \widetilde{\Omega}_j : \frac{|G(h_j v)(x)|}{v(x)} > \frac{\lambda}{2}\right\}\right) \\
&\leq \frac{C}{\lambda}\int_{2Q_j\setminus\widetilde{\Omega}_j} |G(h_j v)(x)|u(x)\,dx \\
&\leq \frac{C}{\lambda}\sum_{i\in\mathbb{N}}\int_{2Q_j\setminus\widetilde{P}_{i,j}}\left(\int_0^\infty |Q_t(h_{i,j}v)(x)|^2\frac{dt}{t}\right)^{\frac{1}{2}}u(x)\,dx \\
&\leq \frac{C}{\lambda}\sum_{i\in\mathbb{N}}\int_{2Q_j\setminus\widetilde{P}_{i,j}}\left(\int_0^\infty\left(\int_{P_{i,j}}|Q_t(x,y)-Q_t(x,x_{i,j})||h_{i,j}(y)|v(y)\,dy\right)^2\frac{dt}{t}\right)^{\frac{1}{2}}u(x)\,dx \\
&\leq \frac{C}{\lambda}\sum_{i\in\mathbb{N}}\int_{2Q_j\setminus\widetilde{P}_{i,j}}\int_{P_{i,j}}\left(\int_0^\infty|Q_t(x,y)-Q_t(x,x_{i,j})|^2\frac{dt}{t}\right)^{\frac{1}{2}}|h_{i,j}(y)|v(y)\,dyu(x)\,dx \\
&\leq \frac{C}{\lambda}\sum_{i\in\mathbb{N}}\int_{2Q_j\setminus\widetilde{P}_{i,j}}\int_{P_{i,j}}\left(1+\frac{|x-y|}{\rho(x)}+\frac{|x-y|}{\rho(y)}\right)^{-l}\frac{|y-x_{i,j}|^{\delta_0}}{|x-y|^{n+\delta_0}}|h_{i,j}(y)|v(y)\,dyu(x)\,dx \\
&\leq \frac{C}{\lambda}\sum_{i\in\mathbb{N}}\int_{P_{i,j}}\int_{2Q_j\setminus\widetilde{P}_{i,j}}\left(1+\frac{|x-y|}{\rho(x)}+\frac{|x-y|}{\rho(y)}\right)^{-l}\frac{|y-x_{i,j}|^{\delta_0}}{|x-y|^{n+\delta_0}}u(x)\,dx|h_{i,j}(y)|v(y)\,dy \\
&\leq \frac{C}{\lambda}\sum_{i\in\mathbb{N}}\int_{P_{i,j}}\sum_{k\in\mathbb{N}}\int_{2^kP_{i,j}\setminus 2^{k-1}P_{i,j}}\frac{|y-x_{i,j}|^{\delta}}{|x-x_{i,j}|^{n+\delta}}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-l}u(x)\,dx|h_{i,j}(y)|v(y)\,dy \\
&\leq \frac{C}{\lambda}\sum_{i\in\mathbb{N}}\int_{P_{i,j}}|h_{i,j}(y)|v(y)\,dy\sum_{k\in\mathbb{N}}\frac{r_{i,j}^{\delta}}{(2^kr_{i,j})^{n+\delta}}\left(1+\frac{2^kr_{i,j}}{\rho(x_{i,j})}\right)^{-\frac{l}{N_0+1}}\int_{2^kP_{i,j}}u(x)\,dx \\
&\leq \frac{C}{\lambda}\sum_{i\in\mathbb{N}}\int_{P_{i,j}}u(y)\sum_{k\in\mathbb{N}}2^{-k\delta}\left(1+\frac{2^kr_{i,j}}{\rho(x_{i,j})}\right)^{-\frac{l}{N_0+1}+\theta_4}|h_{i,j}(y)|v(y)\,dy \\
&\leq \frac{C}{\lambda}\sum_{i\in\mathbb{N}}\int_{P_{i,j}}|h_{i,j}(y)|u(y)v(y)\,dy,
\end{aligned}$$

where we used the fact that $u \in A_1^\rho$, which means that there is an exponent $\theta_4 \geq 0$ such that $u \in A_1^{\rho,\theta_4}$. Consequently, using the definition of $h_{i,j}$ and (14), we obtain

$$\begin{aligned}
\mathcal{III} &\leq \frac{C}{\lambda}\sum_{i\in\mathbb{N}}\int_{P_{i,j}}|h_{i,j}(y)|\,u(y)v(y)\,dy \qquad (17)\\
&\leq \frac{C}{\lambda}\sum_{i\in\mathbb{N}}\int_{P_{i,j}}f(y)\,u(y)v(y)\,dy+\frac{C}{\lambda}\sum_{i\in\mathbb{N}}\frac{uv(P_{i,j})}{v(P_{i,j})}\int_{P_{i,j}}f(y)\,v(y)\,dy \\
&\leq \frac{C}{\lambda}\int_{2Q_j}f(y)\,u(y)v(y)\,dy+\frac{C}{\lambda}\sum_{i\in\mathbb{N}}\left(1+\frac{r_{i,j}}{\rho(x_{i,j})}\right)^{\theta_1}\left(\inf_{P_{i,j}}u\right)\int_{P_{i,j}}f(y)\,v(y)\,dy \\
&\leq \frac{C}{\lambda}\int_{2Q_j}f(y)\,u(y)v(y)\,dy.
\end{aligned}$$

Summing over j gives the claimed estimate,

$$\sum_{j\in\mathbb{N}}uv\left(\left\{x\in Q_j : \frac{|G(f_2^jv)(x)|}{v(x)} > \frac{\lambda}{2}\right\}\right) \leq \frac{C}{\lambda}\int_{\mathbb{R}^n}f(y)\,u(y)v(y)\,dy,$$

which completes the proof. □

4. Proof of Theorem 5

In the sequel, we intend to prove weighted mixed inequalities for G_b. To this end, the following result plays a crucial role.

Lemma 12 (cf. [34] (Lemma 2.15)). *Given $1 < p < \infty$ and $b \in \mathrm{BMO}(\rho)$. Suppose that $v \in A_p^\rho$, then the following inequality*

$$\|G_b(f)\|_{L^p(v)} \leq C\|b\|_{\mathrm{BMO}(\rho)}\|f\|_{L^p(v)}$$

holds for a positive constant C.

An extension to Schröding setting of the classical result of BMO functions is given in the following, see [33].

Lemma 13 (cf. [33]). *For any $\theta \geq 0$, let $b \in \mathrm{BMO}_\theta(\rho)$, $Q = Q(x,r)$ be a cube and $k \in \mathbb{N}$. Then the following estimate holds*

$$|b_Q - b_{2^kQ}| \leq Ck\left(1 + \frac{2^k r}{\rho(x)}\right)^\theta \|b\|_{\mathrm{BMO}_\theta(\rho)}.$$

With the preceding estimates at hand, we shall now establish Theorem 5.

Proof of Theorem 5. Fix $\lambda > 0$. Since $b \in \mathrm{BMO}(\rho)$, one can find a positive exponent γ such that $b \in \mathrm{BMO}_\gamma(\rho)$. We may, by homogeneity, reduce to the case $\|b\|_{\mathrm{BMO}_\gamma(\rho)} = 1$, that is,

$$\frac{1}{|Q|}\int_Q |b(y) - b_Q|dy \leq \left(1 + \frac{r}{\rho(x)}\right)^\gamma,$$

for each cube $Q = Q(x,r)$. Since our hypotheses on the weights imply that $v \in A_\infty^\rho$, by Remark 3 we can make use of Lemma 9 to obtain the Calderón–Zygmund decomposition of $|f|$ at level λ in terms of the measure $d\mu(x) = v(x)\,dx$, with Θ taken sufficiently large and specified later. Thus we obtain cubes $\{Q_j\}_{j\in N}$, $Q_j = Q(x_j, r_j)$ such that

$$\left(1 + \frac{r_j}{\rho(x_j)}\right)^\Theta \lambda \leq \frac{1}{v(Q_j)}\int_{Q_j} |f|\,v \leq C\left(1 + \frac{r_j}{\rho(x_j)}\right)^\sigma \lambda, \tag{18}$$

for some $C > 0$, $\sigma \geq \Theta$ and every j. Denoting $J_1 = \{j : r_j \leq \rho(x_j)\}$ and $J_2 = \mathbb{N} \setminus J_1$, we can define $\Omega_1 = \bigcup_{j\in J_1} Q_j$ and $\Omega_2 = \bigcup_{j\in J_2} Q_j$. We now write $f = h_1 + h_2 + h_3$, where

$$h_1(x) = \begin{cases} \frac{1}{v(Q_j)}\int_{Q_j} f(y)\,v(y)dy, & \text{if } x \in Q_j,\ j \in J_1, \\ 0, & \text{if } x \in Q_j,\ j \in J_2, \\ f(x), & \text{if } x \notin (\Omega_1 \cup \Omega_2), \end{cases}$$

$$h_2(x) = \begin{cases} f(x) - \frac{1}{v(Q_j)}\int_{Q_j} f(y)\,v(y)dy, & \text{if } x \in Q_j,\ j \in J_1, \\ 0, & \text{otherwise,} \end{cases}$$

and $h_3(x) = f(x)\chi_{\Omega_2}$. If $\Omega = \Omega_1 \cup \Omega_2$, $\widetilde{Q}_j = Q(x_j, 3\sqrt{n}\, r_j)$ and $\widetilde{\Omega} = \bigcup_j \widetilde{Q}_j$, then we can obtain that,

$$\begin{aligned} uv\left(\left\{x \in \mathbb{R}^n : \frac{|G_b(fv)(x)|}{v(x)} > \lambda\right\}\right) &\leq uv\left(\left\{x \in \mathbb{R}^n : \frac{|G_b(h_1 v)(x)|}{v(x)} > \frac{\lambda}{3}\right\}\right) + uv(\widetilde{\Omega}) \\ &\quad + uv\left(\left\{x \in \mathbb{R}^n \setminus \widetilde{\Omega} : \frac{|G_b(h_2 v)(x)|}{v(x)} > \frac{\lambda}{3}\right\}\right) \\ &\quad + uv\left(\left\{x \in \mathbb{R}^n \setminus \widetilde{\Omega} : \frac{|G_b(h_3 v)(x)|}{v(x)} > \frac{\lambda}{3}\right\}\right) \\ &=: X_1 + X_2 + X_3 + X_4. \end{aligned}$$

For X_2, by repeating the argument in step (15), we get the desired estimate,

$$X_2 \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)|\, u(x)v(x)dx.$$

For X_1, observe that $v \in A_\infty^\rho(u)$ implies $v \in A_{q'}^\rho(u)$ for some $q' > 1$. Hence, from Remark 2 and item (i) of Lemma 3, it follows that $uv^{1-q} \in A_q^\rho$. By using Chebyshev's inequality and Lemma 12, we get

$$\begin{aligned} X_1 &\lesssim \frac{1}{\lambda^q} \int_{\mathbb{R}^n} |G_b(h_1 v)(x)|^q u(x)v(x)^{1-q}dx \\ &\lesssim \frac{1}{\lambda^q} \int_{\mathbb{R}^n} |h_1(x)|^q u(x)v(x)dx \\ &\lesssim \frac{1}{\lambda} \int_{\mathbb{R}^n} |h_1(x)| u(x)v(x)dx, \end{aligned}$$

using the fact $|h_1(x)| \leq C\lambda$, since Q_j with $j \in J_1$ is subcritical and $|f(x)| \leq \lambda$ for $x \notin \Omega$. We then proceed via (14),

$$\begin{aligned} &\frac{C}{\lambda} \int_{\mathbb{R}^n} |h_1(x)| u(x)v(x)dx \\ &\leq \frac{C}{\lambda} \sum_{j \in J_1} \frac{uv(Q_j)}{v(Q_j)} \int_{Q_j} |f(x)|\, v(x)dx + \frac{C}{\lambda} \int_{\mathbb{R}^n \setminus \Omega} |f(x)|\, u(x)v(x)dx \\ &\leq \frac{C}{\lambda} \sum_{j \in J_1} \left(1 + \frac{r_j}{\rho(x_j)}\right)^{\theta_1} (\inf_{Q_j} u) \int_{Q_j} |f(x)|\, v(x)dx + \frac{C}{\lambda} \int_{\mathbb{R}^n \setminus \Omega} |f(x)|\, u(x)v(x)dx \\ &\leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)|\, u(x)v(x)dx, \end{aligned}$$

since the cubes Q_j for $j \in J_1$ are subcritical. Thus we finish the estimate of X_1.

For X_3, recall that

$$h_2(x) = \sum_{j \in J_1} \left(f(x) - \frac{1}{v(Q_j)} \int_{Q_j} f(y)v(y)dy\right)\chi_{Q_j}(x) = \sum_{j \in J_1} h_2^j(x).$$

It is not difficult to see that

$$\int_{Q_j} h_2^j(x)v(x)dx = 0. \tag{19}$$

By using the Minkowski inequality, one can write

$$\begin{aligned}
G_b(h_2v)(x) &= \left(\int_0^\infty |Q_t((b(x)-b(\cdot))h_2v)(x)|^2\frac{dt}{t}\right)^{\frac{1}{2}}\\
&= \left(\int_0^\infty \Big|\int_{\mathbb{R}^n}(b(x)-b_{Q_j}+b_{Q_j}-b(y))\,Q_t(x,y)\Big(\sum_{j\in J_1}h_2^j(y)v(y)\Big)dy\Big|^2\frac{dt}{t}\right)^{\frac{1}{2}}\\
&\le \sum_{j\in J_1}|b(x)-b_{Q_j}|\left(\int_0^\infty\Big|\int_{\mathbb{R}^n}Q_t(x,y)h_2^j(y)v(y)\,dy\Big|^2\frac{dt}{t}\right)^{\frac{1}{2}}\\
&\quad+\left(\int_0^\infty\Big|\int_{\mathbb{R}^n}\sum_{j\in J_1}(b_{Q_j}-b(y))\,Q_t(x,y)h_2^j(y)v(y)\,dy\Big|^2\frac{dt}{t}\right)^{\frac{1}{2}}\\
&\le \sum_{j\in J_1}|b(x)-b_{Q_j}|G(h_2^jv)(x)+G\Big(\sum_{j\in J_1}(b_{Q_j}-b)h_2^jv\Big)(x).
\end{aligned}$$

Therefore

$$\begin{aligned}
X_3 &\le uv\left(\left\{x\in\mathbb{R}^n\setminus\widetilde{\Omega}:\sum_{j\in J_1}\frac{|b(x)-b_{Q_j}|G(h_2^jv)(x)}{v(x)}>\frac{\lambda}{6}\right\}\right)\\
&\quad+uv\left(\left\{x\in\mathbb{R}^n\setminus\widetilde{\Omega}:\frac{G\left(\sum_{j\in J_1}(b_{Q_j}-b)h_2^jv\right)(x)}{v(x)}>\frac{\lambda}{6}\right\}\right)\\
&=: X_3^1+X_3^2.
\end{aligned}$$

To estimate X_3^1, we apply the Chebyshev inequality, the decay of the kernel estimate (8), (19) and Minkowski's inequality to get

$$\begin{aligned}
X_3^1 &\lesssim \frac{1}{\lambda}\int_{\mathbb{R}^n\setminus\widetilde{\Omega}}\sum_{j\in J_1}|G(h_2^jv)(x)||b(x)-b_{Q_j}|u(x)dx\\
&\lesssim \frac{1}{\lambda}\int_{\mathbb{R}^n\setminus\widetilde{\Omega}}\sum_{j\in J_1}\left(\int_0^\infty\left(\int_{Q_j}|Q_t(x,y)-Q_t(x,x_j)||h_2^j(y)|v(y)dy\right)^2\frac{dt}{t}\right)^{\frac{1}{2}}|b(x)-b_{Q_j}|u(x)dx\\
&\lesssim \frac{1}{\lambda}\int_{\mathbb{R}^n\setminus\widetilde{\Omega}}\sum_{j\in J_1}|b(x)-b_{Q_j}|\int_{Q_j}\left(\int_0^\infty|Q_t(x,y)-Q_t(x,x_j)|^2\frac{dt}{t}\right)^{\frac{1}{2}}|h_2^j(y)|v(y)dyu(x)dx\\
&\lesssim \frac{1}{\lambda}\int_{\mathbb{R}^n\setminus\widetilde{\Omega}}\sum_{j\in J_1}|b(x)-b_{Q_j}|\int_{Q_j}\left(1+\frac{|x-y|}{\rho(x)}+\frac{|x-y|}{\rho(y)}\right)^{-l}\frac{|y-x_j|^{\delta_0}}{|x-y|^{n+\delta_0}}|h_2^j(y)|v(y)dyu(x)dx\\
&\lesssim \frac{1}{\lambda}\sum_{j\in J_1}\int_{Q_j}\int_{\mathbb{R}^n\setminus\widetilde{\Omega}}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-l}|b(x)-b_{Q_j}|\frac{|y-x_j|^{\delta_0}}{|x-y|^{n+\delta_0}}u(x)\,dx|h_2^j(y)|v(y)\,dy\\
&\lesssim \frac{1}{\lambda}\sum_{j\in J_1}\int_{Q_j}\sum_{k=1}^\infty\int_{2^{k+1}Q_j\setminus 2^kQ_j}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-l}|b(x)-b_{Q_j}|\frac{|y-x_j|^{\delta_0}}{|x-y|^{n+\delta_0}}u(x)dx|h_2^j(y)|v(y)dy\\
&\lesssim \frac{1}{\lambda}\sum_{j\in J_1}\int_{Q_j}|h_2^j(y)|v(y)E_j(y)\,dy,
\end{aligned}$$

since $|x-y| > 2|y-x_j|$ when $x \notin \widetilde{Q}_j$ and $y \in Q_j$. Due to $2Q_j \subseteq \widetilde{Q}_j$, for every $j \in J_1$, by using (2), we get

$$\begin{aligned}
E_j(y) &\le C\sum_{k=1}^{\infty}\int_{2^{k+1}Q_j\setminus 2^kQ_j}|b(x)-b_{Q_j}|\left(1+\frac{|x-y|}{\rho(x)}\right)^{-l}\frac{|y-x_j|^{\delta_0}}{|x-y|^{n+\delta_0}}u(x)\,dx\\
&\le C\sum_{k=1}^{\infty}\int_{2^{k+1}Q_j\setminus 2^kQ_j}|b(x)-b_{Q_j}|\left(1+\frac{2^{k+1}r_j}{\rho(x_j)}\right)^{-l/(N_0+1)}\frac{|y-x_j|^{\delta_0}}{|x-y|^{n+\delta_0}}u(x)\,dx\\
&\le C\sum_{k=1}^{\infty}\frac{\ell(Q_j)^{\delta_0}}{\ell(2^kQ_j)^{n+\delta_0}}\left(1+\frac{2^{k+1}r_j}{\rho(x_j)}\right)^{-l/(N_0+1)}\int_{2^{k+1}Q_j\setminus 2^kQ_j}|b(x)-b_{Q_j}|u(x)\,dx\\
&\le C\sum_{k=1}^{\infty}\left(1+\frac{2^{k+1}r_j}{\rho(x_j)}\right)^{-l/(N_0+1)}\frac{2^{-k\delta_0}}{|2^{k+1}Q_j|}\int_{2^{k+1}Q_j}|b(x)-b_{Q_j}|u(x)\,dx.
\end{aligned}$$

To estimate the above integral, we decompose the oscillatory term as follows,

$$\begin{aligned}
\frac{1}{|2^{k+1}Q_j|}\int_{2^{k+1}Q_j}|b(x)-b_{Q_j}|\,u(x)\,dx &\le \frac{1}{|2^{k+1}Q_j|}\int_{2^{k+1}Q_j}|b(x)-b_{2^{k+1}Q_j}|\,u(x)\,dx\\
&\quad+\frac{1}{|2^{k+1}Q_j|}\int_{2^{k+1}Q_j}|b_{2^{k+1}Q_j}-b_{Q_j}|\,u(x)\,dx\\
&=: E_j^1+E_j^2.
\end{aligned}$$

Since $u \in A_1^{\rho}$, there exists $s>1$ and $\theta_3,\theta_4 \ge 0$ such that $u \in \mathrm{RH}_s^{\rho,\theta_3}\cap A_1^{\rho,\theta_4}$. Applying Hölder's inequality together with Lemma 7, we obtain

$$\begin{aligned}
E_j^1 &\le \left(\frac{1}{|2^{k+1}Q_j|}\int_{2^{k+1}Q_j}|b(x)-b_{2^{k+1}Q_j}|^{s'}dx\right)^{1/s'}\left(\frac{1}{|2^{k+1}Q_j|}\int_{2^{k+1}Q_j}u^s(x)dx\right)^{1/s}\\
&\le C\|b\|_{\mathrm{BMO}_\gamma(\rho)}[u]_{\mathrm{RH}_s^{\rho,\theta_3}}\left(1+\frac{2^{k+1}r_j}{\rho(x_j)}\right)^{(N_0+1)\gamma+\theta_3}\left(\frac{1}{|2^{k+1}Q_j|}\int_{2^{k+1}Q_j}u(x)dx\right)\\
&\le C\|b\|_{\mathrm{BMO}_\gamma(\rho)}[u]_{\mathrm{RH}_s^{\rho,\theta_3}}[u]_{A_1^{\rho,\theta_4}}\left(1+\frac{2^{k+1}r_j}{\rho(x_j)}\right)^{(N_0+1)\gamma+\theta_3+\theta_4}\inf_{2^{k+1}Q_j}u.
\end{aligned}\tag{20}$$

For E_j^2, we apply Lemma 13 to get

$$\begin{aligned}
E_j^2 &\le C\|b\|_{\mathrm{BMO}_\gamma(\rho)}(k+1)\left(1+\frac{2^{k+1}r_j}{\rho(x_j)}\right)^{\gamma}\frac{u(2^{k+1}Q_j)}{|2^{k+1}Q_j|}\\
&\le C\|b\|_{\mathrm{BMO}_\gamma(\rho)}[u]_{A_1^{\rho,\theta_4}}(k+1)\left(1+\frac{2^{k+1}r_j}{\rho(x_j)}\right)^{\gamma+\theta_4}\inf_{2^{k+1}Q_j}u.
\end{aligned}$$

Substituting the estimates for E_j^1 and E_j^2 in the inequality for X_3^1, we have

$$\begin{aligned}
X_3^1 &\le \frac{C}{\lambda}\sum_{j\in J_1}\sum_{k=1}^{\infty}2^{-k\delta_0}(k+1)\left(1+\frac{2^{k+1}r_j}{\rho(x_j)}\right)^{(N_0+1)\gamma+\theta_3+\theta_4-l/(N_0+1)}\inf_{2^{k+1}Q_j}u\int_{Q_j}|h_2^j(y)|v(y)dy\\
&\le \frac{C}{\lambda}\sum_{j\in J_1}\int_{Q_j}|h_2^j(y)|u(y)v(y)dy,
\end{aligned}$$

for each $y \in Q_j$, where we choose l sufficiently large such that $(N_0+1)\gamma+\theta_3+\theta_4-l/(N_0+1)<0$. Therefore, by repeating the argument in step (17), we deduce

$$X_3^1 \le \frac{C}{\lambda}\sum_{j\in J_1}\int_{Q_j}|h_2^j(y)|\,u(y)v(y)\,dy \le \frac{C}{\lambda}\int_{\mathbb{R}^n}|f(y)|\,u(y)v(y)\,dy.$$

For X_3^2, we write

$$X_3^2 = uv\left(\left\{x \in \mathbb{R}^n \setminus \widetilde{\Omega} : \frac{G\left(\sum_{j\in J_1}(b_{Q_j}-b)h_2^j v\right)(x)}{v(x)} > \frac{\lambda}{6}\right\}\right),$$

since the functions h_2^j are supported on the cubes Q_j, which are disjoint. By applying Theorem 2, we have

$$\begin{aligned}
X_3^2 &\le \frac{C}{\lambda}\int_{\mathbb{R}^n}\sum_{j\in J_1}|h_2^j(x)||b(x)-b_{Q_j}|\,u(x)v(x)\,dx\\
&\le \frac{C}{\lambda}\sum_{j\in J_1}\int_{Q_j}|h_2^j(x)||b(x)-b_{Q_j}|\,u(x)v(x)\,dx\\
&\le \frac{C}{\lambda}\sum_{j\in J_1}\int_{Q_j}|f(x)||b(x)-b_{Q_j}|\,u(x)v(x)\,dx\\
&\quad+\frac{C}{\lambda}\sum_{j\in J_1}\left(\frac{1}{v(Q_j)}\int_{Q_j}|f(y)|\,v(y)dy\right)\left(\int_{Q_j}|b(x)-b_{Q_j}|\,u(x)v(x)dx\right)\\
&=: X_3^{2,1}+X_3^{2,2}.
\end{aligned}$$

To deal with $X_3^{2,1}$, the generalized Hölder inequality is applied to the measure $uvdx$ and the functions Φ and $\widetilde{\Phi}(t)\approx e^t-1$. Since $uv\in A_\infty^\rho$, there exist $r>1$ and $\theta_2\ge 0$ such that $uv\in \mathrm{RH}_r^{\rho,\theta_2}$. By Lemma 5, we get

$$\|b-b_{Q_j}\|_{\widetilde{\Phi},Q_j,uv}\le C\left(1+\frac{r_j}{\rho(x_j)}\right)^{\theta_2}\|b-b_{Q_j}\|_{\widetilde{\Phi},Q_j}. \tag{21}$$

Then, combining Lemma 6, (13), (18), (14) and (21), we obtain

$$\begin{aligned}
&\frac{1}{\lambda}\sum_{j\in J_1}\int_{Q_j}|f(x)||b(x)-b_{Q_j}|u(x)v(x)dx\\
&\lesssim \frac{1}{\lambda}\sum_{j\in J_1}uv(Q_j)\,\|f\|_{\Phi,Q_j,uv}\,\|b-b_{Q_j}\|_{\widetilde{\Phi},Q_j,uv}\\
&\lesssim \frac{1}{\lambda}\sum_{j\in J_1}uv(Q_j)\left(1+\frac{r_j}{\rho(x_j)}\right)^{\vartheta+\theta_2}\inf_{\tau>0}\left\{\tau+\frac{\tau}{uv(Q_j)}\int_{Q_j}\Phi\left(\frac{|f(x)|}{\tau}\right)u(x)v(x)dx\right\}\\
&\lesssim \frac{1}{\lambda}\sum_{j\in J_1}uv(Q_j)\left(\lambda+\frac{\lambda}{uv(Q_j)}\int_{Q_j}\Phi\left(\frac{|f(x)|}{\lambda}\right)u(x)v(x)dx\right)\\
&\lesssim \sum_{j\in J_1}uv(Q_j)+\sum_{j\in J_1}\int_{Q_j}\Phi\left(\frac{|f(x)|}{\lambda}\right)u(x)v(x)dx\\
&\lesssim \sum_{j\in J_1}\left(1+\frac{r_j}{\rho(x_j)}\right)^{-\Theta}\frac{uv(Q_j)}{v(Q_j)}\int_{Q_j}\frac{|f(x)|}{\lambda}v(x)dx+\int_{\Omega_1}\Phi\left(\frac{|f(x)|}{\lambda}\right)u(x)v(x)dx\\
&\lesssim \sum_{j\in J_1}\left(1+\frac{r_j}{\rho(x_j)}\right)^{-\Theta+\theta_1}\left(\inf_{Q_j}u\right)\int_{Q_j}\frac{|f(x)|}{\lambda}v(x)dx+\int_{\Omega_1}\Phi\left(\frac{|f(x)|}{\lambda}\right)u(x)v(x)dx\\
&\lesssim \int_{\mathbb{R}^n}\Phi\left(\frac{|f(x)|}{\lambda}\right)u(x)v(x)dx,
\end{aligned}$$

here θ_1 denotes the exponent from (14), and we have used the fact that the cubes Q_j for $j\in J_1$ are subcritical.

For $X_3^{2,2}$, by Lemma 6, (18) and (21), we get

$$\begin{aligned}
&\frac{C}{\lambda}\sum_{j\in J_1}\left(\frac{1}{v(Q_j)}\int_{Q_j}|f(y)|\,v(y)dy\right)\left(\int_{Q_j}|b(x)-b_{Q_j}|\,u(x)v(x)dx\right)\\
&\lesssim \frac{1}{\lambda}\sum_{j\in J_1}\left(1+\frac{r_j}{\rho(x_j)}\right)^{\sigma}\lambda uv(Q_j)\|b-b_{Q_j}\|_{\widetilde{\Phi},Q_j,uv}\\
&\lesssim \sum_{j\in J_1}\left(1+\frac{r_j}{\rho(x_j)}\right)^{\sigma+\vartheta+\theta_2}uv(Q_j)\\
&\lesssim \sum_{j\in J_1}uv(Q_j)\\
&\lesssim \int_{\mathbb{R}^n}\Phi\left(\frac{|f(x)|}{\lambda}\right)u(x)v(x)dx,
\end{aligned}$$

here, for the last step, we repeated the process in step $X_3^{2,1}$.

For X_4, we have $h_3(x)=f(x)\chi_{\Omega_2}(x)$ where $\Omega_2=\bigcup_{j\in J_2}Q_j$. Then

$$h_3(x)=\sum_{j\in J_2}f(x)\chi_{Q_j}=\sum_{j\in J_2}h_3^j(x).$$

As in X_3, we also have

$$\begin{aligned}
X_4 &\le uv\left(\left\{x\in\mathbb{R}^n\setminus\widetilde{\Omega}:\sum_{j\in J_2}\frac{|b(x)-b_{Q_j}|G(h_3^jv)(x)}{v(x)}>\frac{\lambda}{6}\right\}\right)\\
&\quad+uv\left(\left\{x\in\mathbb{R}^n\setminus\widetilde{\Omega}:\frac{G(\sum_{j\in J_2}(b-b_{Q_j})h_3^jv)(x)}{v(x)}>\frac{\lambda}{6}\right\}\right)\\
&=: X_4^1+X_4^2.
\end{aligned}$$

Applying Chebyshev's inequality, Minkowski's inequality and (7), we have

$$\begin{aligned}
X_4^1 &\lesssim \frac{1}{\lambda}\int_{\mathbb{R}^n\setminus\widetilde{\Omega}}\sum_{j\in J_2}|G(h_3^jv)(x)||b(x)-b_{Q_j}|u(x)\,dx\\
&\lesssim \frac{1}{\lambda}\int_{\mathbb{R}^n\setminus\widetilde{\Omega}}\sum_{j\in J_2}\left(\int_0^\infty\left(\int_{Q_j}|Q_t(x,y)||h_3^j(y)|v(y)\,dy\right)^2\frac{dt}{t}\right)^{\frac{1}{2}}|b(x)-b_{Q_j}|u(x)\,dx\\
&\lesssim \frac{1}{\lambda}\int_{\mathbb{R}^n\setminus\widetilde{\Omega}}\sum_{j\in J_2}\int_{Q_j}\left(\int_0^\infty|Q_t(x,y)|^2\frac{dt}{t}\right)^{\frac{1}{2}}|b(x)-b_{Q_j}||h_3^j(y)|v(y)\,dyu(x)\,dx\\
&\lesssim \frac{1}{\lambda}\sum_{j\in J_2}\int_{\mathbb{R}^n\setminus\widetilde{Q}_j}\int_{Q_j}\left(1+\frac{|x-y|}{\rho(x)}+\frac{|x-y|}{\rho(y)}\right)^{-l}\frac{|b(x)-b_{Q_j}|}{|x-y|^n}|h_3^j(y)|v(y)dyu(x)dx\\
&\lesssim \frac{1}{\lambda}\sum_{j\in J_2}\int_{Q_j}|h_3^j(y)|v(y)\sum_{k=1}^{\infty}\int_{2^{k+1}Q_j\setminus 2^kQ_j}\frac{|b(x)-b_{Q_j}|}{|x-y|^n}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-l}u(x)\,dxdy\\
&= \frac{1}{\lambda}\sum_{j\in J_2}\int_{Q_j}|h_3^j(y)|\,v(y)F_j(y)\,dy.
\end{aligned}$$

In order to estimate $F_j(y)$, we split it as follows,

$$\begin{aligned} F_j(y) &\leq \sum_{k=1}^{\infty} \int_{2^{k+1}Q_j \setminus 2^k Q_j} \frac{|b(x) - b_{2^{k+1}Q_j}|}{|x-y|^n} \left(1 + \frac{|x-y|}{\rho(x)}\right)^{-l} u(x)\, dx \\ &\quad + \sum_{k=1}^{\infty} \int_{2^{k+1}Q_j \setminus 2^k Q_j} \frac{|b_{2^{k+1}Q_j} - b_{Q_j}|}{|x-y|^n} \left(1 + \frac{|x-y|}{\rho(x)}\right)^{-l} u(x)\, dx \\ &=: F_j^1(y) + F_j^2(y). \end{aligned}$$

For $F_j^1(y)$, we apply (2) to obtain

$$F_j^1(y) \leq C \sum_{k=1}^{\infty} \left(1 + \frac{2^{k+1} r_j}{\rho(x_j)}\right)^{-l/(N_0+1)} \frac{1}{|2^{k+1}Q_j|} \int_{2^{k+1}Q_j \setminus 2^k Q_j} |b(x) - b_{2^{k+1}Q_j}| u(x)\, dx.$$

Proceeding as in (20) for E_j^1, we arrive to

$$\begin{aligned} F_j^1(y) &\leq C \sum_{k=1}^{\infty} \left(1 + \frac{2^{k+1} r_j}{\rho(x_j)}\right)^{(N_0+1)\gamma+\theta_3+\theta_4-l/(N_0+1)} \inf_{2^{k+1}Q_j} u \\ &\leq C \inf_{Q_j} u \sum_{k=1}^{\infty} 2^{-k(l/(N_0+1)-\theta_3-\theta_4-(N_0+1)\gamma)} \\ &\leq C \inf_{Q_j} u, \end{aligned}$$

since $j \in J_2$ and we take l sufficiently large so that $l/(N_0+1) - \theta_3 - \theta_4 - (N_0+1)\gamma > 0$.

For $F_j^2(y)$, due to $u \in A_1^\rho$, there is a constant $\theta_4 \geq 0$ such that $u \in A_1^{\rho,\theta_4}$. Applying Lemma 13 and (2), we get

$$\begin{aligned} F_j^2(y) &\lesssim \sum_{k=1}^{\infty} (k+1) \left(1 + \frac{2^{k+1} r_j}{\rho(x_j)}\right)^{-l/(N_0+1)+\gamma} \frac{u(2^{k+1}Q_j)}{|2^{k+1}Q_j|} \\ &\lesssim \sum_{k=1}^{\infty} (k+1) \left(1 + \frac{2^{k+1} r_j}{\rho(x_j)}\right)^{-l/(N_0+1)+\gamma+\theta_4} \inf_{2^{k+1}Q_j} u \\ &\lesssim \inf_{Q_j} u \sum_{k=1}^{\infty} (k+1) 2^{-k(l/(N_0+1)-\gamma-\theta_4)} \\ &\lesssim \inf_{Q_j} u, \end{aligned}$$

since $j \in J_2$ and we have chosen l large enough such that $l/(N_0+1) - \gamma - \theta_4 > 0$. Therefore, we obtain

$$X_4^1 \leq \frac{C}{\lambda} \sum_{j \in J_2} \int_{Q_j} |h_3^j(y)|\, u(y) v(y)\, dy \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(y)|\, u(y) v(y)\, dy.$$

For X_4^2, by applying the Chebyshev inequality, Minkowski inequality and (7), one has

$$
\begin{aligned}
X_4^2 &\lesssim \frac{1}{\lambda}\int_{\mathbb{R}^n\setminus\tilde{\Omega}}\sum_{j\in J_2}|G((b-b_{Q_j})h_3^j v)(x)|u(x)\,dx\\
&\lesssim \frac{1}{\lambda}\int_{\mathbb{R}^n\setminus\tilde{\Omega}}\sum_{j\in J_2}\left(\int_0^\infty\left(\int_{Q_j}|Q_t(x,y)||h_3^j(y)||b(y)-b_{Q_j}|v(y)\,dy\right)^2\frac{dt}{t}\right)^{\frac{1}{2}}u(x)\,dx\\
&\lesssim \frac{1}{\lambda}\int_{\mathbb{R}^n\setminus\tilde{\Omega}}\sum_{j\in J_2}\int_{Q_j}\left(\int_0^\infty|Q_t(x,y)|^2\frac{dt}{t}\right)^{\frac{1}{2}}|h_3^j(y)||b(y)-b_{Q_j}|v(y)\,dyu(x)\,dx\\
&\lesssim \frac{1}{\lambda}\sum_{j\in J_2}\int_{\mathbb{R}^n\setminus\tilde{Q}_j}\int_{Q_j}\left(1+\frac{|x-y|}{\rho(x)}+\frac{|x-y|}{\rho(y)}\right)^{-l}\frac{1}{|x-y|^n}|h_3^j(y)||b(y)-b_{Q_j}|v(y)\,dyu(x)\,dx\\
&\lesssim \frac{1}{\lambda}\sum_{j\in J_2}\int_{Q_j}|b(y)-b_{Q_j}||h_3^j(y)|v(y)\sum_{k=1}^\infty\int_{2^{k+1}Q_j\setminus 2^kQ_j}\frac{1}{|x-y|^n}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-l}u(x)\,dxdy.
\end{aligned}
$$

By (2), this gives

$$
\begin{aligned}
&\sum_{k=1}^\infty\int_{2^{k+1}Q_j\setminus 2^kQ_j}|x-y|^{-n}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-l}u(x)\,dx\\
&\lesssim\sum_{k=1}^\infty\int_{2^{k+1}Q_j\setminus 2^kQ_j}|x-y|^{-n}\left(1+\frac{2^{k+1}r_j}{\rho(x_j)}\right)^{-l/(N_0+1)}u(x)\,dx\\
&\lesssim\sum_{k=1}^\infty\left(1+\frac{2^{k+1}r_j}{\rho(x_j)}\right)^{-l/(N_0+1)}\frac{1}{|2^{k+1}Q_j|}\int_{2^{k+1}Q_j}u(x)\,dx\\
&\lesssim\left(1+\frac{r_j}{\rho(x_j)}\right)^{-\frac{l}{2(N_0+1)}}\sum_{k=1}^\infty\left(1+\frac{2^{k+1}r_j}{\rho(x_j)}\right)^{-\frac{l}{2(N_0+1)}+\theta_4}\inf_{2^{k+1}Q_j}u\\
&\lesssim\left(1+\frac{r_j}{\rho(x_j)}\right)^{-\frac{l}{2(N_0+1)}}\sum_{k=1}^\infty 2^{-k\left(\frac{l}{2(N_0+1)}-\theta_4\right)}\inf_{Q_j}u\\
&\lesssim\left(1+\frac{r_j}{\rho(x_j)}\right)^{-\frac{l}{2(N_0+1)}}\inf_{Q_j}u,
\end{aligned}
$$

since $j\in J_2$ and l is chosen such that $l>2\theta_4(N_0+1)$. Therefore,

$$
X_4^2\lesssim\frac{1}{\lambda}\sum_{j\in J_2}\left(1+\frac{r_j}{\rho(x_j)}\right)^{-\frac{l}{2(N_0+1)}}\int_{Q_j}|f(y)||b(y)-b_{Q_j}|u(y)v(y)\,dy.
$$

We now invoke the extended Hölder inequality associated with Φ and $\widetilde{\Phi}$ under the measure $d\mu=uvdx$. Combining this with (13) and (21), we obtain

$$
\begin{aligned}
&\int_{Q_j}|b(y)-b_{Q_j}||f(y)|u(y)v(y)\,dy\\
&\le Cuv(Q_j)\|b-b_{Q_j}\|_{\tilde{\Phi},Q_j,uv}\|f\|_{\Phi,Q_j,uv}\\
&\le C\lambda\,uv(Q_j)\left(1+\frac{r_j}{\rho(x_j)}\right)^{\theta_2}\|b-b_{Q_j}\|_{\tilde{\Phi},Q_j}\\
&\quad+C\lambda\left(1+\frac{r_j}{\rho(x_j)}\right)^{\theta_2}\|b-b_{Q_j}\|_{\tilde{\Phi},Q_j}\int_{Q_j}\Phi\left(\frac{|f(y)|}{\lambda}\right)u(y)v(y)dy\\
&=: X_4^{2,1}+X_4^{2,2}.
\end{aligned}
$$

In order to estimate $X_4^{2,1}$, by Lemma 6, there are exponent $\vartheta > 0$ and constant $C > 0$ such that

$$\begin{aligned} X_4^{2,1} &\leq C\lambda\left(1+\frac{r_j}{\rho(x_j)}\right)^{\theta_2+\vartheta}\frac{uv(Q_j)}{v(Q_j)}v(Q_j) \\ &\leq C\lambda\left(1+\frac{r_j}{\rho(x_j)}\right)^{\theta_1+\theta_2+\vartheta}(\inf_{Q_j}u)v(Q_j) \\ &\leq C\lambda\left(1+\frac{r_j}{\rho(x_j)}\right)^{\theta_1+\theta_2+\vartheta-\Theta}\int_{Q_j}\frac{|f(y)|}{\lambda}u(y)v(y)dy, \end{aligned}$$

where we additionally make use of (14) and (18). Choosing $\Theta > \theta_1 + \theta_2 + \vartheta$, we have

$$X_4^{2,1} \leq C\int_{Q_j}|f(y)|u(y)v(y)dy.$$

For $X_4^{2,2}$, we apply again Lemma 6 and (21) to obtain

$$X_4^{2,2} \leq C\lambda\left(1+\frac{r_j}{\rho(x_j)}\right)^{\theta_2+\vartheta}\int_{Q_j}\Phi\left(\frac{|f(y)|}{\lambda}\right)u(y)v(y)dy.$$

According to the above estimates, one can conclude that

$$X_4^2 \lesssim \sum_{j\in J_2}\left(1+\frac{r_j}{\rho(x_j)}\right)^{\theta_2+\vartheta-\frac{l}{2(N_0+1)}}\int_{Q_j}\Phi\left(\frac{|f(y)|}{\lambda}\right)u(y)v(y)dy.$$

By choosing l large enough, we finish the proof. □

5. Conclusions

In this paper, the authors obtain the weighted mixed endpoint estimates for Littlewood–Paley square operators related to the heat semigroup and their commutators by using the Calderón–Zygmund decomposition adapted to Schrödinger setting, combined with some classical inequalities. Such inequalities not only provide a refined characterization of the endpoint behavior of these operators in weighted spaces—extending the classical weak $(1,1)$ bounds, but also have potential applications to the regularity theory of Schrödinger equations with non-smooth potentials and to the boundedness of spectral multipliers. These results offer a robust foundation for further developments in Fourier analysis and elliptic equations. Our approach relies on the size and smoothness of kernel $Q_t(x,y)$, which prevent direct generalization to operators with low-regularity kernels.

In the future, we will devote research to relaxing the weight conditions, namely, establishing analogous weighted mixed weak-type estimates under more general weight classes, with a particular attempt to weaken the assumption $v \in A_\infty^\rho(u)$. Another parallel direction is to extend the Calderón–Zygmund approach developed in this paper to other operators, such as area integrals and singular integral operators whose kernels have limited regularity.

Furthermore, the theory of BMO commutators could be further advanced by studying the L^p-boundedness and compactness property of square-function commutators on wider function spaces, such as weighted Hardy spaces and weighted Morrey spaces. Finally, it would be interesting to explore possible connections between the weighted estimates obtained here and operator-theoretic methods arising in other contexts, such as deep learning operator optimization [39] or q-analogue approximation theory [40].

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