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Exponential Decay Rate of a One-Dimensional Tree-Structured Timoshenko Beam System with Boundary Feedbacks

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Abstract

In this paper, we investigate a one-dimensional tree-structured Timoshenko beam system composed of three beams with boundary feedback controls and derive its explicit exponential decay rate. We first formulate a dynamic model incorporating end loads, boundary feedback, and multi-beam coupling relationships. Then, by applying Lyapunov's method, we construct an energy functional suitable for this coupled system. This study not only provides rigorous proof of the exponential stability of the coupled Timoshenko beam system, but also yields an explicit decay rate under prescribed conditions. Our results offer theoretical support and a quantitative foundation for vibration suppression of multi-beam-coupled flexible structures.

Keywords: partial differential equations; timoshenko beam; boundary control; tree-structured network; exponential decay; decay rate

MSC: 35Q93; 35B40; 93D15; 93D20

1. Introduction

The Timoshenko beam theory provides a fundamental framework in structural mechanics and vibration control. Prior to its proposal, the classical Euler–Bernoulli beam theory was widely used in engineering practice owing to its simplified assumptions. However, that theory neglects shear deformation and rotational inertia, which significantly limits its applicability to short or thick beams and high-frequency vibrations. In 1921, Timoshenko [1] derived a governing equation for transverse vibrations that incorporates both shear and rotary inertia effects, thereby establishing what is now known as the Timoshenko beam model. This theory not only overcomes the limitations of classical beam models but also extends the range of problems to which beam theory can be applied. It has become a cornerstone for structural design and vibration control in aerospace, civil, and mechanical engineering, and it continues to be of great practical relevance today.

The shear coefficient is a key parameter accounting for shear deformation, and its value governs the agreement between theoretical predictions and engineering practice. In 1966, Cowper [2] conducted a thorough investigation of this coefficient in Timoshenko beam theory and derived an explicit formula for its evaluation. This work transformed the Timoshenko beam theory from a purely theoretical model into a practical engineering tool and facilitated its widespread adoption in various engineering structures. Han, Benaroya, and Wei [3] examined the dynamic responses of transversely vibrating beams using four distinct

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engineering beam theories, further validating the accuracy of the Timoshenko beam theory in complex vibration scenarios and establishing a solid foundation for its application.

As engineering structures become lighter, more precise, and topologically more complex, extended modeling and numerical innovations for Timoshenko beam systems have emerged as a research focus. A great deal of research has been devoted to various structural forms and constraint conditions. Arboleda-Monsalve [4] formulated the dynamic stiffness matrix and the corresponding load vector for Timoshenko beam-columns resting on an elastic foundation with generalized boundary conditions. In a related study, Calio and Greco [5] examined the free vibration behavior of Timoshenko beam columns supported by a Pasternak foundation. Together, these two studies advanced the dynamic analysis framework for Timoshenko beams under elastic foundation constraints. In [6], Schnabl developed a locking-free two-layer Timoshenko beam element that accounts for interlayer slip, thereby addressing the numerical challenges in the dynamic analysis of laminated structures. More recently, Zougheib [7] constructed a fully discrete energy equation for thermoviscoelastic Timoshenko systems and analyzed their exponential stability, thus advancing numerical methods for multi-physics coupling problems. While these advances have provided valuable tools for the quantitative stability analysis of Timoshenko beam systems with complex topological configurations, their applicability to tree-structured multi-beam-coupled systems remains to be further validated.

Exponential stability is a fundamental metric for evaluating the dynamic performance and control quality of Timoshenko beam systems, and has long been a topic of extensive research interest. In recent years, considerable attention has been paid to the exponential stability of Timoshenko beam systems under various complex operating conditions. Researchers have conducted in-depth studies on key issues such as damping mechanisms and time-delay effects, proposing a range of stability criteria and analytical strategies and accumulating a substantial body of results. Rivera analyzed the global stability of damped Timoshenko systems and subsequently extended the analysis to systems with indefinite damping, establishing the stability behavior at the boundary under non-positive-definite damping conditions [8,9]. Raposo [10] proved the exponential stability of Timoshenko systems with two weak damping terms and provided a quantitative characterization of the decay rate. Shi and Feng [11] investigated Timoshenko beams under locally distributed feedback and derived the exponential decay rate of the system energy. Calsavara, Tavares, and Silva [12] further extended the study to thermoviscoelastic Timoshenko systems with fading memory, thereby refining the stability analysis framework for viscoelastic damping scenarios. In a separate development, Kirane [13] established stability criteria for Timoshenko systems with time-varying delays in the internal feedback and clarified the role of the delay effect. In [14], Rivera carried out the stability analysis of Timoshenko systems involving history-dependent terms. Nonato [15] further explored the stability of solutions for Timoshenko systems with distributed delays and performed corresponding numerical validations. Xu [16,17] investigated Timoshenko beams with lumped and distributed delays in boundary control, respectively, and proposed corresponding stabilization schemes. In multi-factor coupling scenarios, Laribi [18] analyzed the exponential stability of thermoelastic Timoshenko systems by incorporating diffusion effects, memory terms, and distributed delays, and presented supporting numerical results. Chen [19] explored the bending vibration behavior of Timoshenko beams subjected to locally distributed axial loads, revealing the coupling mechanism between axial forces and damping. Jiang [20] employed the reverberation-ray method to analyze the dynamic response of continuous Timoshenko beams under moving loads, providing a theoretical reference for dynamic loading scenarios.

As theoretical models and numerical methods continue to advance, the application of Timoshenko beam systems to multi-body coupling and engineering control problems has drawn increasing research interest. Wu, Endo, and Matsuno [21] established the exponential stability of a system in which two Timoshenko arms are used to grasp and manipulate a rigid object. In [22], Mattioni studied a clamped Timoshenko beam with an attached tip mass and achieved exponential stability through actuation at the free end. By applying Timoshenko beam theory to flexible robotic manipulators, Liu [23] proposed an observer-based boundary control strategy that not only effectively suppresses vibrations but also satisfies output constraints, highlighting the potential of this theory for practical engineering applications.

It is worth noting that existing studies on multi-beam-coupled Timoshenko systems have largely been confined to configurations such as two-beam series connections and clamped-free single-branch structures. In practice, however, many engineering structures involve more complex layouts, as exemplified by tree-structured flexible robotic manipulators in aerospace engineering and branched support beams in civil engineering. Such configurations are typically characterized by a three-beam tree-shaped topology with a concentrated load applied at the tip. Several studies have addressed the stabilization of tree-shaped Timoshenko beam networks. Han and Xu proved exponential stability for a forked triple Timoshenko beam system [24], and later analyzed the dynamical stability of non-uniform Timoshenko beam networks with boundary time-delay inputs, establishing the spectrum-determined-growth condition and discussing exponential stability for a special case [25]. Meanwhile, port-Hamiltonian formulations have provided a powerful framework for modeling and control of Timoshenko beams [26], emphasizing structural preservation and modular interconnection. However, current theories cannot provide a reliable quantitative basis for vibration suppression, and there is a clear need for targeted investigations.

Unlike existing studies on single-beam or two-beam configurations, our work addresses a tree-shaped three-beam network with non-trivial topological coupling. The branching geometry introduces energy redistribution at the junction, which makes standard methods inapplicable. Moreover, we provide for the first time an explicit lower bound for the decay rate, offering a quantitative tool for parameter tuning. Related topological effects have also been emphasized in recent studies on shell buckling and metamaterial wave propagation [27,28]. The key difficulty of the tree-shaped topology arises from the junction transmission conditions. These conditions include continuity of displacement and rotation, together with force and moment balance. They couple the three beams simultaneously, which prevents the direct application of standard single-beam or two-beam methods. Therefore, a new Lyapunov functional must be constructed to handle these coupling effects. The main novelty of our approach is a Lyapunov functional specifically designed for the tree-shaped topology. We assign exponential weights to each beam individually. These weights are systematically chosen to absorb the cross-coupling terms that arise from the transmission conditions. The transmission conditions include the continuity of displacement and rotation, together with force and moment balance. These conditions couple the three beams simultaneously. Consequently, single-beam methods cannot be directly applied to this system. This constructive framework yields a quantitative decay rate formula that is directly applicable to engineering design. Our work thus differs from previous qualitative stability analyses.

The rest of this paper is organized as follows. Section 2 presents the problem formulation, the system model, and the necessary fundamental definitions. Section 3 provides a detailed analysis of the one-dimensional tree-structured Timoshenko beam system with end loads. We propose an appropriate boundary feedback control strategy, analyze the exponential stability of the resulting closed-loop system, and derive an estimate of the

exponential decay rate under specific conditions. Section 4 validates the exponential stability of the system through numerical simulations. Finally, Section 5 summarizes the main contributions and key findings of this work.

2. Problem Description

In this section, we consider a one-dimensional tree-shaped Timoshenko beam network system, the governing equations for which are given as follows:

$$\begin{cases} \rho_i \omega_{i,tt}(x,t) - K_i[\omega_{i,xx}(x,t) - \varphi_{i,x}(x,t)] = 0,\ i = 1,2,3, \\ I_{\rho_i}\varphi_{i,tt}(x,t) - EI_i\varphi_{i,xx}(x,t) - K_i[\omega_{i,x}(x,t) - \varphi_i(x,t)] = 0,\ i = 1,2,3, \\ \omega_1(0,t) = \varphi_1(0,t) = 0, \\ \omega_1(1,t) = \omega_2(0,t) = \omega_3(0,t), \\ \varphi_1(1,t) = \varphi_2(0,t) = \varphi_3(0,t), \\ K_1[\omega_{1,x}(1,t) - \varphi_1(1,t)] = K_2[\omega_{2,x}(0,t) - \varphi_2(0,t)] + K_3[\omega_{3,x}(0,t) - \varphi_3(0,t)], \\ EI_1\varphi_{1,x}(1,t) = EI_2\varphi_{2,x}(0,t) + EI_3\varphi_{3,x}(0,t), \\ K_2[\omega_{2,x}(1,t) - \varphi_2(1,t)] = u_2(t),\ K_3[\omega_{3,x}(1,t) - \varphi_3(1,t)] = u_3(t), \\ EI_2\varphi_{2,x}(1,t) = v_2(t),\ EI_3\varphi_{3,x}(1,t) = v_3(t), \\ \omega_i(x,0) = \omega_i^0(x),\ \omega_{i,t}(x,0) = \omega_i^1(x), \\ \varphi_i(x,0) = \varphi_i^0(x),\ \varphi_{i,t}(x,0) = \varphi_i^1(x). \end{cases} \tag{1}$$

Here, $x \in (0,1)$ denotes the one-dimensional spatial coordinate, and $t \in (0,\infty)$ denotes time. The deflection of the beam from its equilibrium is represented by $\omega(x,t)$, while $\varphi(x,t)$ is the rotation angle of the beam at position x. Subscripts such as those in $\omega_t(x,t)$, $\omega_x(x,t)$ indicate partial derivatives with respect to time and space, respectively. The physical parameters I_ρ (mass inertia), ρ (mass density), EI (bending stiffness), and K (shear stiffness) are all positive constants. Finally, $u_i(t)$ and $v_i(t)$ denote the external force and torque applied at the controlled boundary, respectively.

A simplified schematic of the model is provided in Figure 1:

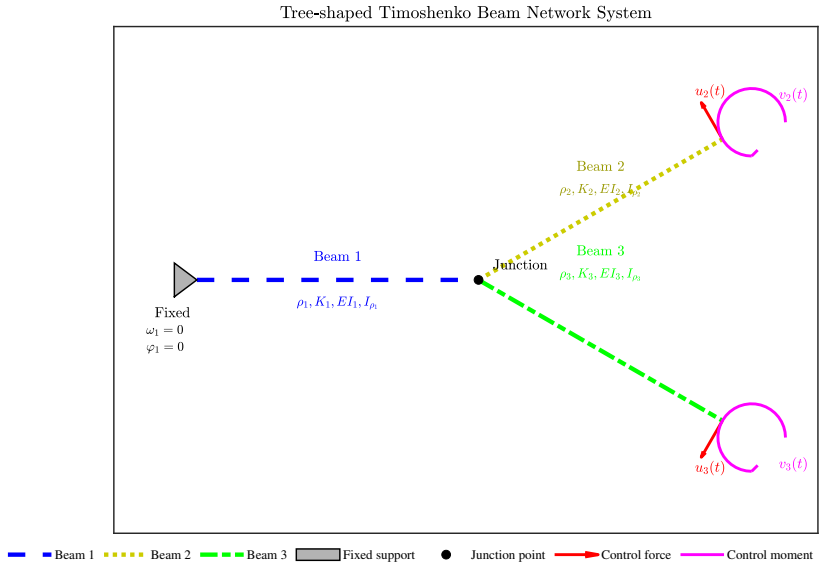


Figure 1. Tree-shaped Timoshenko Beam Network System.

3. Exponential Stability of the Timoshenko Beam System

3.1. Closed-Loop System

For Equation (1), we introduce the following auxiliary variables:

$$\begin{cases} \xi_{1,i}(x,t) = \sqrt{\rho_i}\omega_{i,t}(x,t) + \sqrt{K_i}[\omega_{i,x}(x,t) - \varphi_i(x,t)], \\ \xi_{2,i}(x,t) = \sqrt{\rho_i}\omega_{i,t}(x,t) - \sqrt{K_i}[\omega_{i,x}(x,t) - \varphi_i(x,t)], \\ \xi_{3,i}(x,t) = \sqrt{I_{\rho_i}}\varphi_{i,t}(x,t) + \sqrt{EI_i}\varphi_{i,x}(x,t), \\ \xi_{4,i}(x,t) = \sqrt{I_{\rho_i}}\varphi_{i,t}(x,t) - \sqrt{EI_i}\varphi_{i,x}(x,t), \end{cases} \tag{2}$$

where $i = 1, 2, 3$.

From Equations (1) and (2), we derive the following relations:

$$\begin{cases} \xi_{1,it}(x,t) = a\xi_{1,ix}(x,t) - b\varphi_{i,t}(x,t), \\ \xi_{2,it}(x,t) = -a\xi_{2,ix}(x,t) + b\varphi_{i,t}(x,t), \\ \xi_{3,it}(x,t) = c\xi_{3,ix}(x,t) + d[\omega_{i,x}(x,t) - \varphi_i(x,t)], \\ \xi_{4,it}(x,t) = -c\xi_{4,ix}(x,t) + d[\omega_{i,x}(x,t) - \varphi_i(x,t)], \end{cases} \tag{3}$$

where $\rho_i = \rho$, $I_{\rho_i} = I_\rho$, $K_i = K$, $EI_i = EI$, $a = \sqrt{\dfrac{K}{\rho}}$, $b = \sqrt{K}$, $c = \sqrt{\dfrac{EI}{I_\rho}}$, $d = \dfrac{K}{\sqrt{I_\rho}}$.

For each beam $i = 1, 2, 3$, we define the auxiliary variables:

$$z_i(x,t) = \omega_{i,t}(x,t), \quad \psi_i(x,t) = \varphi_{i,t}(x,t), \tag{4}$$

and define the state vector:

$$W(t) = (\omega_1, z_1, \varphi_1, \psi_1, \ \omega_2, z_2, \varphi_2, \psi_2, \ \omega_3, z_3, \varphi_3, \psi_3)^T, \tag{5}$$

Then the standard form of the system (1) is given by the following:

$$\|W(t)\|^2 = \sum_{i=1}^{3} \int_0^1 \Big[K_i|\omega_{i,x}(x,t) - \varphi_i(x,t)|^2 + \rho_i|z_i(x,t)|^2 + EI_i|\varphi_{i,x}(x,t)|^2 + I_{\rho_i}|\psi_i(x,t)|^2 \Big] dx. \tag{6}$$

The energy functional of the system (1) is given by the following:

$$E(t) = \frac{1}{2}\|W(t)\|^2. \tag{7}$$

Then, by combining the auxiliary variables, we define the energy functional of the system (1) as follows:

$$V(t) = \sum_{i=1}^{3} V_i(t), \quad i = 1, 2, 3, \tag{8}$$

and

$$V_i(t) = \int_0^1 \Big[p_{1i}(x)\xi_{1i}^2(x,t) + p_{2i}(x)\xi_{2i}^2(x,t) + p_{3i}(x)\xi_{3i}^2(x,t) + p_{4i}(x)\xi_{4i}^2(x,t) \Big] dx, \tag{9}$$

where the weight functions are given by

$$\begin{cases} p_{1i}(x) = p_{1i}e^{\frac{\gamma_{1i}x}{a}}, \\ p_{2i}(x) = p_{2i}e^{-\frac{\gamma_{2i}x}{a}}, \\ p_{3i}(x) = p_{3i}e^{\frac{\gamma_{3i}x}{c}}, \\ p_{4i}(x) = p_{4i}e^{-\frac{\gamma_{4i}x}{c}}. \end{cases} \tag{10}$$

Theorem 1. *The energy functions $E(t)$ and $V(t)$ are equivalent.*

Proof. Since $x \in [0,1]$ is a closed interval and the exponential function is continuous and bounded, constants exist:

$$\begin{cases} M_{j,i} = \max_{x\in[0,1]} p_{j,i}(x) > 0, \quad i = 1,2,3, \quad j = 1,2,3,4, \\ m_{j,i} = \min_{x\in[0,1]} p_{j,i}(x) > 0, \quad i = 1,2,3, \quad j = 1,2,3,4. \end{cases}$$

From the definitions of $\xi_i(x,t)$, we obtain the following identities:

$$\begin{cases} \xi_{i,1}^2 = \rho_i \omega_{i,t}^2 + K_i(\omega_{i,x} - \varphi_i)^2 + 2\sqrt{\rho_i K_i}\, \omega_{i,t}(\omega_{i,x} - \varphi_i), \\ \xi_{i,2}^2 = \rho_i \omega_{i,t}^2 + K_i(\omega_{i,x} - \varphi_i)^2 - 2\sqrt{\rho_i K_i}\, \omega_{i,t}(\omega_{i,x} - \varphi_i), \\ \xi_{i,3}^2 = I_{\rho_i} \varphi_{i,t}^2 + EI_i\, \varphi_{i,x}^2 + 2\sqrt{I_{\rho_i} EI_i}\, \varphi_{i,t}\varphi_{i,x}, \\ \xi_{i,4}^2 = I_{\rho_i} \varphi_{i,t}^2 + EI_i\, \varphi_{i,x}^2 - 2\sqrt{I_{\rho_i} EI_i}\, \varphi_{i,t}\varphi_{i,x}. \end{cases}$$

Summing these four equations, the cross-terms cancel, yielding

$$\xi_{i,1}^2 + \xi_{i,2}^2 + \xi_{i,3}^2 + \xi_{i,4}^2 = 2\Big[\rho_i \omega_{i,t}^2 + K_i(\omega_{i,x} - \varphi_i)^2 + I_{\rho_i}\varphi_{i,t}^2 + EI_i\varphi_{i,x}^2\Big].$$

Substituting $z_i = \omega_{i,t}$ and $\psi_i = \varphi_{i,t}$, and using the definition of $\|W(t)\|^2$, we get

$$\sum_{i=1}^{3}\sum_{j=1}^{4}\int_0^1 \xi_{i,j}^2 dx = 2\|W(t)\|^2 = 4E(t).$$

Using the uniform upper bound $p_{j,i}(x) \le M_{j,i} \le M$, we obtain

$$\begin{aligned} V(t) &= \sum_{i=1}^{3}\sum_{j=1}^{4}\int_0^1 p_{j,i}(x)\, \xi_{i,j}^2 dx \\ &\le \sum_{i=1}^{3}\sum_{j=1}^{4} M_{j,i}\int_0^1 \xi_{i,j}^2 dx \\ &\le M\sum_{i=1}^{3}\sum_{j=1}^{4}\int_0^1 \xi_{i,j}^2 dx. \end{aligned}$$

Inserting $\sum_{i=1}^{3}\sum_{j=1}^{4}\int_0^1 \xi_{i,j}^2 dx = 4E(t)$ yields

$$V(t) \le 4ME(t).$$

Let $C_2 = 4M > 0$; then $V(t) \le C_2 E(t)$.
Similarly, using $p_{j,i}(x) \ge m_{j,i} \ge m$, we obtain

$$\begin{aligned} V(t) &= \sum_{i=1}^{3}\sum_{j=1}^{4}\int_0^1 p_{j,i}(x)\, \xi_{i,j}^2 dx \\ &\ge \sum_{i=1}^{3}\sum_{j=1}^{4} m_{j,i}\int_0^1 \xi_{i,j}^2 dx \\ &\ge m\sum_{i=1}^{3}\sum_{j=1}^{4}\int_0^1 \xi_{i,j}^2 dx. \end{aligned}$$

Substituting the same integral identity gives

$$V(t) \ge 4mE(t).$$

Setting $C_1 = 4m > 0$, we have $V(t) \ge C_1 E(t)$.

Consequently, there exist positive constants $C_1, C_2 > 0$ such that $C_1 E(t) \le V(t) \le C_2 E(t)$, which proves the equivalence of $E(t)$ and $V(t)$. □

Remark 1. *The exponential form of the weight functions biases the energy functional toward the controlled boundary at ($x = 1$), where the actuators are located. The exponent γ_{ji} are determined by the wave speed a and the control gains so as to ensure that the time derivative of $V(t)$ satisfies a strict dissipative inequality. The constants M and m serve merely as uniform upper and lower bounds for the weight functions over $x \in [0,1]$; they are technical tools for establishing the equivalence between $V(t)$ and $E(t)$ and do not affect the control law or the decay rate estimate.*

Before proceeding to the stability analysis, we briefly discuss the well-posedness of the closed-loop system within the standard semigroup framework for Timoshenko beam networks [25,26]. Let the state vector be

$$W(t) = (\omega_1, z_1, \varphi_1, \psi_1, \omega_2, z_2, \varphi_2, \psi_2, \omega_3, z_3, \varphi_3, \psi_3)^T,$$

with $z_i = \omega_{i,t}$, $\psi_i = \varphi_{i,t}$, and define the energy space

$$\mathcal{H} = \prod_{i=1}^{3} \left[H^1(0,1) \times L^2(0,1)\right]^2$$

equipped with the inner product induced by the energy norm $\|W\|^2$ given in (6). The system can be rewritten as the first-order evolution equation

$$\dot{W}(t) = \mathcal{A}W(t),\ W(0) = W_0,$$

where the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is defined by

$$\mathcal{A}\begin{pmatrix} \omega_i \\ z_i \\ \varphi_i \\ \psi_i \end{pmatrix} = \begin{pmatrix} z_i \\ \frac{K}{\rho}(\omega_{i,xx} - \varphi_{i,x}) \\ \psi_i \\ \frac{EI}{I_\rho}\varphi_{i,xx} + \frac{K}{I_\rho}(\omega_{i,x} - \varphi_i) \end{pmatrix}, \quad i = 1,2,3,$$

with domain

$$D(\mathcal{A}) = \left\{ W \in \mathcal{H} : \omega_i, \varphi_i \in H^2(0,1),\ z_i, \psi_i \in H^1(0,1) \right\}.$$

The domain $D(\mathcal{A})$ is dense in $\mathcal{H}$. A direct computation using the boundary conditions yields

$$\langle \mathcal{A}W, W \rangle_{\mathcal{H}} = -\alpha_2 |z_2(1)|^2 - \alpha_3 |z_3(1)|^2 - \beta_2 |\psi_2(1)|^2 - \beta_3 |\psi_3(1)|^2 \leq 0,$$

which shows that $\mathcal{A}$ is dissipative.

To establish maximal dissipativity, it remains to prove that $0 \in \rho(\mathcal{A})$. For $\lambda > 0$, consider the resolvent equation

$$(\lambda I - \mathcal{A})W = F,\ F \in \mathcal{H}.$$

Taking the inner product with W in $\mathcal{H}$ and using the dissipative estimate, we obtain

$$\lambda \|W\|_{\mathcal{H}}^2 \leq \langle F, W \rangle_{\mathcal{H}} \leq \|F\|_{\mathcal{H}} \|W\|_{\mathcal{H}},$$

which implies the a priori bound

$$\|W\|_{\mathcal{H}} \leq \lambda^{-1} \|F\|_{\mathcal{H}}.$$

To prove that $(\lambda I - \mathcal{A})^{-1}$ exists on the whole space $\mathcal{H}$, we invoke the standard coercivity estimate for the stationary Timoshenko beam network with the prescribed boundary conditions. Indeed, the elliptic system corresponding to $(\lambda I - \mathcal{A})W = F$ can be written componentwise as

$$\lambda\omega_i - z_i = f_{1i}, \quad \lambda z_i - \frac{K}{\rho}(\omega_{i,xx} - \varphi_{i,x}) = f_{2i},$$
$$\lambda\varphi_i - \psi_i = f_{3i}, \quad \lambda\psi_i - \frac{EI}{I_\rho}\varphi_{i,xx} - \frac{K}{I_\rho}(\omega_{i,x} - \varphi_i) = f_{4i},$$

with $F = (f_{1i}, f_{2i}, f_{3i}, f_{4i})_{i=1}^3$. Eliminating z_i and ψ_i, we obtain a fourth-order elliptic system for (ω_i, φ_i). The transmission conditions at the junction and the boundary feedback conditions ensure that the associated bilinear form is coercive on the appropriate subspace of $\prod_{i=1}^3 H^2(0,1)$. Using the Lax–Milgram theorem, this system admits a unique solution $W \in D(\mathcal{A})$ for every $F \in \mathcal{H}$. Hence $(\lambda I - \mathcal{A})$ is surjective for every $\lambda > 0$, and the resolvent estimate above guarantees boundedness of $(\lambda I - \mathcal{A})^{-1}$. Therefore $0 \in \rho(\mathcal{A})$.

Using the Lumer-Phillips theorem [29], $\mathcal{A}$ generates a C_0-semigroup of contractions on $\mathcal{H}$. Consequently, for every initial datum $W_0 \in \mathcal{H}$, there exists a unique mild solution $W \in C([0,\infty); \mathcal{H})$; for $W_0 \in D(\mathcal{A})$, the solution is classical. This provides the required functional setting for the stability analysis that follows.

To stabilize the system, we adopt the control functions as follows:

$$\begin{cases} u_2(t) = -\alpha_2\omega_{2,t}(1,t), \\ v_2(t) = -\beta_2\varphi_{2,t}(1,t), \\ u_3(t) = -\alpha_3\omega_{3,t}(1,t), \\ v_3(t) = -\beta_3\varphi_{3,t}(1,t). \end{cases}$$

Accordingly, the closed-loop system corresponding to (1) is given as follows:

$$\begin{cases} \rho_i\omega_{i,tt}(x,t) - K_i[\omega_{i,xx}(x,t) - \varphi_{i,x}(x,t)] = 0, \\ I_{\rho_i}\varphi_{i,tt}(x,t) - EI_i\varphi_{i,xx}(x,t) - K_i[\omega_{i,x}(x,t) - \varphi_i(x,t)] = 0, \\ \omega_1(0,t) = \varphi_1(0,t) = 0, \\ \omega_1(1,t) = \omega_2(0,t) = \omega_3(0,t), \\ \varphi_1(1,t) = \varphi_2(0,t) = \varphi_3(0,t), \\ K_1[\omega_{1,x}(1,t) - \varphi_1(1,t)] = K_2[\omega_{2,x}(0,t) - \varphi_2(0,t)] + K_3[\omega_{3,x}(0,t) - \varphi_3(0,t)], \\ EI_1\varphi_{1,x}(1,t) = EI_2\varphi_{2,x}(0,t) + EI_3\varphi_{3,x}(0,t), \\ K_2[\omega_{2,x}(1,t) - \varphi_2(1,t)] = -\alpha_2\omega_{2,t}(1,t),\ K_3[\omega_{3,x}(1,t) - \varphi_3(1,t)] = -\alpha_3\omega_{3,t}(1,t), \\ EI_2\varphi_{2,x}(1,t) = -\beta_2\varphi_{2,t}(1,t),\ EI_3\varphi_{3,x}(1,t) = -\beta_3\varphi_{3,t}(1,t), \\ \omega_i(x,0) = \omega_i^0(x),\ \omega_{i,t}(x,0) = \omega_i^1(x), \\ \varphi_i(x,0) = \varphi_i^0(x),\ \varphi_{i,t}(x,0) = \varphi_i^1(x), \end{cases} \tag{11}$$

where $\rho_i = \rho$, $I_{\rho_i} = I_\rho$, $K_i = K$, $EI_i = EI$, $i = 1,2,3$.

Differentiating $V_i(t)$ with respect to time yields

$$\begin{aligned} \dot{V}_i(t) &= \int_0^1 [2p_{1i}(x)\xi_{1i}(x,t)\xi_{1i,t}(x,t) + 2p_{2i}(x)\xi_{2i}(x,t)\xi_{2i,t}(x,t)]dx \\ &\quad + \int_0^1 [2p_{3i}(x)\xi_{3i}(x,t)\xi_{3i,t}(x,t) + 2p_{4i}(x)\xi_{4i}(x,t)\xi_{4i,t}(x,t)]dx \\ &= 2\int_0^1 p_{1i}(x)\xi_{1i}(x,t)[a\xi_{1i,x}(x,t) - b\varphi_{i,t}(x,t)]dx \\ &\quad + 2\int_0^1 p_{2i}(x)\xi_{2i}(x,t)[-a\xi_{2i,x}(x,t) + b\varphi_{i,t}(x,t)]dx \\ &\quad + 2\int_0^1 p_{3i}(x)\xi_{3i}(x,t)[c\xi_{3i,x}(x,t) + d(\omega_{i,x}(x,t) - \varphi_i(x,t))]dx \\ &\quad + 2\int_0^1 p_{4i}(x)\xi_{4i}(x,t)[-c\xi_{4i,x}(x,t) + d(\omega_{i,x}(x,t) - \varphi_i(x,t))]dx \end{aligned}$$

$$
\begin{aligned}
&= a\Big[p_{1i}(1)\xi_{1i}^2(1,t) - p_{2i}(1)\xi_{2i}^2(1,t) - p_{1i}(0)\xi_{1i}^2(0,t) + p_{2i}(0)\xi_{2i}^2(0,t)\Big] \\
&\quad + c\Big[p_{3i}(1)\xi_{3i}^2(1,t) - p_{4i}(1)\xi_{4i}^2(1,t) - p_{3i}(0)\xi_{3i}^2(0,t) + p_{4i}(0)\xi_{4i}^2(0,t)\Big] \\
&\quad - a\int_0^1 \Big[\frac{\gamma_{1i}}{a}p_{1i}(x)\xi_{1i}^2(x,t) + \frac{\gamma_{2i}}{a}p_{2i}(x)\xi_{2i}^2(x,t)\Big]dx \\
&\quad - c\int_0^1 \Big[\frac{\gamma_{3i}}{c}p_{3i}(x)\xi_{3i}^2(x,t) + \frac{\gamma_{4i}}{c}p_{4i}(x)\xi_{4i}^2(x,t)\Big]dx \\
&\quad + 2\int_0^1 [-bp_{1i}(x)\xi_{1i}(x,t)\varphi_{i,t}(x,t) + bp_{2i}(x)\xi_{2i}(x,t)\varphi_{i,t}(x,t)] \\
&\quad + 2\int_0^1 [dp_{3i}(x)\xi_{3i}(x,t)(\omega_{i,x}(x,t) - \varphi_i(x,t))]dx \\
&\quad + 2\int_0^1 [dp_{4i}(x)\xi_{4i}(x,t)(\omega_{i,x}(x,t) - \varphi_i(x,t))]dx \\
&\le -\min\{\gamma_{1i},\gamma_{2i},\gamma_{3i},\gamma_{4i}\}V_i(t) + A_i(t) \\
&\quad + a\Big[p_{1i}(1)\xi_{1i}^2(1,t) - p_{2i}(1)\xi_{2i}^2(1,t) - p_{1i}(0)\xi_{1i}^2(0,t) + p_{2i}(0)\xi_{2i}^2(0,t)\Big] \\
&\quad + c\Big[p_{3i}(1)\xi_{3i}^2(1,t) - p_{4i}(1)\xi_{4i}^2(1,t) - p_{3i}(0)\xi_{3i}^2(0,t) + p_{4i}(0)\xi_{4i}^2(0,t)\Big].
\end{aligned}
$$

Here,

$$
\begin{aligned}
A_i(t) =& 2\int_0^1 [-bp_{1i}(x)\xi_{1i}(x,t)\varphi_{i,t}(x,t) + bp_{2i}(x)\xi_{2i}(x,t)\varphi_{i,t}(x,t)] \\
&+ 2\int_0^1 [dp_{3i}(x)\xi_{3i}(x,t)(\omega_{i,x}(x,t) - \varphi_i(x,t))]dx \\
&+ 2\int_0^1 [dp_{4i}(x)\xi_{4i}(x,t)(\omega_{i,x}(x,t) - \varphi_i(x,t))]dx.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\dot{V}(t) &= \sum_{i=1}^{3}\dot{V}_i(t) \\
&\le -\sum_{i=1}^{3}\min\{\gamma_{1i},\gamma_{2i},\gamma_{3i},\gamma_{4i}\}V_i(t) + \sum_{i=1}^{3}A_i(t) \\
&\quad + \sum_{i=1}^{3}a\Big[p_{1i}(1)\xi_{1i}^2(1,t) - p_{2i}(1)\xi_{2i}^2(1,t) - p_{1i}(0)\xi_{1i}^2(0,t) + p_{2i}(0)\xi_{2i}^2(0,t)\Big] \\
&\quad + \sum_{i=1}^{3}c\Big[p_{3i}(1)\xi_{3i}^2(1,t) - p_{4i}(1)\xi_{4i}^2(1,t) - p_{3i}(0)\xi_{3i}^2(0,t) + p_{4i}(0)\xi_{4i}^2(0,t)\Big] \\
&= -\sum_{i=1}^{3}\min\{\gamma_{1i},\gamma_{2i},\gamma_{3i},\gamma_{4i}\}V_i(t) + \sum_{i=1}^{3}A_i(t) + R_1(t) + R_2(t),
\end{aligned}
$$

where

$$
R_1(t) = \sum_{i=1}^{3}a\Big[p_{1i}(1)\xi_{1i}^2(1,t) - p_{2i}(1)\xi_{2i}^2(1,t) - p_{1i}(0)\xi_{1i}^2(0,t) + p_{2i}(0)\xi_{2i}^2(0,t)\Big],
$$

$$
R_2(t) = \sum_{i=1}^{3}c\Big[p_{3i}(1)\xi_{3i}^2(1,t) - p_{4i}(1)\xi_{4i}^2(1,t) - p_{3i}(0)\xi_{3i}^2(0,t) + p_{4i}(0)\xi_{4i}^2(0,t)\Big].
$$

3.2. Exponential Stability of the Closed-Loop System

Theorem 2. *The closed-loop system (11) is exponentially stable with respect to the energy, and its exponential decay rate is at least* $\frac{\gamma}{2}$*, where*

$$0<\gamma<\min\left\{a\ln\frac{5}{2},\min_{i=2,3}\frac{a}{2}\ln\left[\frac{\left(\sqrt{\rho}+\frac{\alpha_i}{\sqrt{K}}\right)^2}{3\left(\sqrt{\rho}-\frac{\alpha_i}{\sqrt{K}}\right)^2}\right],c\ln\frac{5}{2},\min_{i=2,3}\frac{c}{2}\ln\left[\frac{\left(\sqrt{I_\rho}+\frac{\beta_i}{\sqrt{EI}}\right)^2}{3\left(\sqrt{I_\rho}-\frac{\beta_i}{\sqrt{EI}}\right)^2}\right]\right\}$$
$$-\sum_{i=1}^{3}\delta_{1i}-\sum_{i=1}^{3}\delta_{2i}.$$

Proof. We first consider $R_1(t)$:

$$\begin{aligned}
&a\sum_{i=1}^{3}\Big[p_{1i}(1)\xi_{1i}^2(1,t)-p_{2i}(1)\xi_{2i}^2(1,t)-p_{1i}(0)\xi_{1i}^2(0,t)+p_{2i}(0)\xi_{2i}^2(0,t)\Big]\\
&=a\Big[p_{11}(1)\xi_{11}^2(1,t)-p_{21}(1)\xi_{21}^2(1,t)-p_{11}(0)\xi_{11}^2(0,t)+p_{21}(0)\xi_{21}^2(0,t)\Big]\\
&\quad+a\sum_{i=2}^{3}\Big[p_{1i}(1)\xi_{1i}^2(1,t)-p_{2i}(1)\xi_{2i}^2(1,t)-p_{1i}(0)\xi_{1i}^2(0,t)+p_{2i}(0)\xi_{2i}^2(0,t)\Big]\\
&=ap_{11}e^{\frac{\gamma_{11}}{a}}\Big[\sqrt{\rho}\omega_{1,t}(1,t)+\sqrt{K}(\omega_{1,x}(1,t)-\varphi_1(1,t))\Big]^2\\
&\quad-ap_{21}e^{\frac{-\gamma_{21}}{a}}\Big[\sqrt{\rho}\omega_{1,t}(1,t)-\sqrt{K}(\omega_{1,x}(1,t)-\varphi_1(1,t))\Big]^2\\
&\quad+a\sum_{i=2}^{3}p_{1i}e^{\frac{\gamma_{1i}}{a}}\Big[\sqrt{\rho}\omega_{i,t}(1,t)+\sqrt{K}\big(\omega_{i,x}(1,t)-\varphi_i(1,t)\big)\Big]^2\\
&\quad-a\sum_{i=2}^{3}p_{2i}e^{\frac{-\gamma_{2i}}{a}}\Big[\sqrt{\rho}\omega_{i,t}(1,t)-\sqrt{K}\big(\omega_{i,x}(1,t)-\varphi_i(1,t)\big)\Big]^2\\
&\quad-a\sum_{i=2}^{3}p_{1i}\Big[\sqrt{\rho}\omega_{i,t}(0,t)+\sqrt{K}\big(\omega_{i,x}(0,t)-\varphi_i(0,t)\big)\Big]^2\\
&\quad+a\sum_{i=2}^{3}p_{2i}\Big[\sqrt{\rho}\omega_{i,t}(0,t)-\sqrt{K}\big(\omega_{i,x}(0,t)-\varphi_i(0,t)\big)\Big]^2\\
&\quad+ap_{21}\Big[\sqrt{\rho}\omega_{1,t}(0,t)-\sqrt{K}(\omega_{1,x}(0,t)-\varphi_1(0,t))\Big]^2\\
&\quad-ap_{11}\Big[\sqrt{\rho}\omega_{1,t}(0,t)+\sqrt{K}(\omega_{1,x}(0,t)-\varphi_1(0,t))\Big]^2\\
&=a\sum_{i=2}^{3}p_{1i}e^{\frac{\gamma_{1i}}{a}}\left[\sqrt{\rho}\omega_{i,t}(1,t)+\frac{u_i(t)}{\sqrt{K}}\right]^2-a\sum_{i=2}^{3}p_{2i}e^{\frac{-\gamma_{2i}}{a}}\left[\sqrt{\rho}\omega_{i,t}(1,t)-\frac{u_i(t)}{\sqrt{K}}\right]^2\\
&\quad+ap_{11}e^{\frac{\gamma_{11}}{a}}\Big[\sqrt{\rho}\omega_{1,t}(1,t)+\sqrt{K}(\omega_{1,x}(1,t)-\varphi_1(1,t))\Big]^2\\
&\quad-ap_{21}e^{\frac{-\gamma_{21}}{a}}\Big[\sqrt{\rho}\omega_{1,t}(1,t)-\sqrt{K}(\omega_{1,x}(1,t)-\varphi_1(1,t))\Big]^2\\
&\quad-a\sum_{i=2}^{3}p_{1i}\Big[\sqrt{\rho}\omega_{i,t}(0,t)+\sqrt{K}\big(\omega_{i,x}(0,t)-\varphi_i(0,t)\big)\Big]^2\\
&\quad+a\sum_{i=2}^{3}p_{2i}\Big[\sqrt{\rho}\omega_{i,t}(0,t)-\sqrt{K}\big(\omega_{i,x}(0,t)-\varphi_i(0,t)\big)\Big]^2+aK(p_{21}-p_{11})\omega_{1,x}^2(0,t)\\
&=a\sum_{i=2}^{3}p_{1i}e^{\frac{\gamma_{1i}}{a}}\left[\sqrt{\rho}\omega_{i,t}(1,t)+\frac{u_i(t)}{\sqrt{K}}\right]^2-a\sum_{i=2}^{3}p_{2i}e^{\frac{-\gamma_{2i}}{a}}\left[\sqrt{\rho}\omega_{i,t}(1,t)-\frac{u_i(t)}{\sqrt{K}}\right]^2\\
&\quad+ap_{11}e^{\frac{\gamma_{11}}{a}}\Big[\sqrt{\rho}\omega_{1,t}(1,t)+\sqrt{K}(\omega_{1,x}(1,t)-\varphi_1(1,t))\Big]^2\\
&\quad-ap_{21}e^{\frac{-\gamma_{21}}{a}}\Big[\sqrt{\rho}\omega_{1,t}(1,t)-\sqrt{K}(\omega_{1,x}(1,t)-\varphi_1(1,t))\Big]^2\\
&\quad-a\sum_{i=2}^{3}p_{1i}\Big[\sqrt{\rho}\omega_{1,t}(1,t)+\sqrt{K}\big(\omega_{i,x}(0,t)-\varphi_i(0,t)\big)\Big]^2\\
&\quad+a\sum_{i=2}^{3}p_{2i}\Big[\sqrt{\rho}\omega_{1,t}(1,t)-\sqrt{K}\big(\omega_{i,x}(0,t)-\varphi_i(0,t)\big)\Big]^2,
\end{aligned}$$

where $p_{11} = p_{21}$.

We choose the parameters as follows:

$$\begin{cases} p_{12} = p_{13} = 3p_{11} = 3p, \\ p_{22} = p_{23} = p_{21} = p, \\ \gamma_{1i} = \gamma_{2i} = \gamma, \ i = 2, 3. \end{cases}$$

Remark 2. *The choices $p_{12} = p_{13} = 3p_{11}$, $p_{22} = p_{23} = p_{21}$, and their rotational analogues, are sufficient conditions adopted for analytical convenience. They ensure that the cross-coupling terms arising from the junction conditions can be absorbed by the dissipative boundary terms via Young's inequality. The factor 3 corresponds to the three branches meeting at the junction. These selections are not optimal; other choices are possible and would yield different decay rate estimates. Our present choice is made to obtain a transparent explicit expression for the decay rate while preserving the negativity of the Lyapunov derivative.*

Substituting the above parameter choices into the expression for $R_1(t)$ gives

$$\begin{aligned} R_1(t) = {} & 3ap \sum_{i=2}^{3} e^{\frac{\gamma}{a}} \left[\sqrt{\rho}\omega_{i,t}(1,t) + \frac{-\alpha_i \omega_{i,t}(1,t)}{\sqrt{K}} \right]^2 \\ & - ap \sum_{i=2}^{3} e^{-\frac{\gamma}{a}} \left[\sqrt{\rho}\omega_{i,t}(1,t) - \frac{-\alpha_i \omega_{i,t}(1,t)}{\sqrt{K}} \right]^2 \\ & + ape^{\frac{\gamma_{11}}{a}} \left[\sqrt{\rho}\omega_{1,t}(1,t) + \sqrt{K}(\omega_{1,x}(1,t) - \varphi_1(1,t)) \right]^2 \\ & - ape^{\frac{-\gamma_{11}}{a}} \left[\sqrt{\rho}\omega_{1,t}(1,t) - \sqrt{K}(\omega_{1,x}(1,t) - \varphi_1(1,t)) \right]^2 \\ & - 3ap \sum_{i=2}^{3} \left[\sqrt{\rho}\omega_{1,t}(1,t) + \sqrt{K}(\omega_{i,x}(0,t) - \varphi_i(0,t)) \right]^2 \\ & + ap \sum_{i=2}^{3} \left[\sqrt{\rho}\omega_{1,t}(1,t) - \sqrt{K}(\omega_{i,x}(0,t) - \varphi_i(0,t)) \right]^2 \\ = {} & apB_1(t), \end{aligned}$$

where

$$\begin{aligned} B_1(t) = {} & 3 \sum_{i=2}^{3} e^{\frac{\gamma}{a}} \left[\left(\sqrt{\rho} - \frac{\alpha_i}{\sqrt{K}} \right) \omega_{i,t}(1,t) \right]^2 - \sum_{i=2}^{3} e^{-\frac{\gamma}{a}} \left[\left(\sqrt{\rho} + \frac{\alpha_i}{\sqrt{K}} \right) \omega_{i,t}(1,t) \right]^2 \\ & + e^{\frac{\gamma_{11}}{a}} \left[\sqrt{\rho}\omega_{1,t}(1,t) + \sqrt{K}(\omega_{1,x}(1,t) - \varphi_1(1,t)) \right]^2 \\ & - e^{-\frac{\gamma_{11}}{a}} \left[\sqrt{\rho}\omega_{1,t}(1,t) - \sqrt{K}(\omega_{1,x}(1,t) - \varphi_1(1,t)) \right]^2 \\ & - 3 \sum_{i=2}^{3} \left[\sqrt{\rho}\omega_{1,t}(1,t) + \sqrt{K}(\omega_{i,x}(0,t) - \varphi_i(0,t)) \right]^2 \\ & + \sum_{i=2}^{3} \left[\sqrt{\rho}\omega_{1,t}(1,t) - \sqrt{K}(\omega_{i,x}(0,t) - \varphi_i(0,t)) \right]^2. \end{aligned}$$

For convenience, we define

$$I_1(t) = 3 \sum_{i=2}^{3} e^{\frac{\gamma}{a}} \left[\left(\sqrt{\rho} - \frac{\alpha_i}{\sqrt{K}} \right) \omega_{i,t}(1,t) \right]^2 - \sum_{i=2}^{3} e^{-\frac{\gamma}{a}} \left[\left(\sqrt{\rho} + \frac{\alpha_i}{\sqrt{K}} \right) \omega_{i,t}(1,t) \right]^2,$$

$$
\begin{aligned}
I_2(t) = e^{\frac{\gamma_{11}}{a}} &\left[\sqrt{\rho}\omega_{1,t}(1,t) + \sqrt{K}(\omega_{1,x}(1,t) - \varphi_1(1,t))\right]^2 \\
&- e^{-\frac{\gamma_{11}}{a}}\left[\sqrt{\rho}\omega_{1,t}(1,t) - \sqrt{K}(\omega_{1,x}(1,t) - \varphi_1(1,t))\right]^2 \\
&- 3\sum_{i=2}^{3}\left[\sqrt{\rho}\omega_{1,t}(1,t) + \sqrt{K}(\omega_{i,x}(0,t) - \varphi_i(0,t))\right]^2 \\
&+ \sum_{i=2}^{3}\left[\sqrt{\rho}\omega_{1,t}(1,t) - \sqrt{K}(\omega_{i,x}(0,t) - \varphi_i(0,t))\right]^2.
\end{aligned}
$$

To ensure $I_1(t) \leq 0$, we require

$$
3e^{\frac{\gamma}{a}}\left[\left(\sqrt{\rho} - \frac{\alpha_i}{\sqrt{K}}\right)\omega_{i,t}(1,t)\right]^2 - e^{-\frac{\gamma}{a}}\left[\left(\sqrt{\rho} + \frac{\alpha_i}{\sqrt{K}}\right)\omega_{i,t}(1,t)\right]^2 \leq 0,
$$

$$
3e^{\frac{\gamma}{a}}\left[\left(\sqrt{\rho} - \frac{\alpha_i}{\sqrt{K}}\right)\omega_{i,t}(1,t)\right]^2 \leq e^{-\frac{\gamma}{a}}\left[\left(\sqrt{\rho} + \frac{\alpha_i}{\sqrt{K}}\right)\omega_{i,t}(1,t)\right]^2,
$$

$$
3e^{\frac{2\gamma}{a}} \leq \frac{\left(\sqrt{\rho} + \frac{\alpha_i}{\sqrt{K}}\right)^2}{\left(\sqrt{\rho} - \frac{\alpha_i}{\sqrt{K}}\right)^2},
$$

which simplifies to

$$
e^{\frac{2\gamma}{a}} \leq \frac{\left(\sqrt{\rho} + \frac{\alpha_i}{\sqrt{K}}\right)^2}{3\left(\sqrt{\rho} - \frac{\alpha_i}{\sqrt{K}}\right)^2}.
$$

Hence,

$$
\gamma \leq \frac{a}{2}\ln\left[\frac{\left(\sqrt{\rho} + \frac{\alpha_i}{\sqrt{K}}\right)^2}{3\left(\sqrt{\rho} - \frac{\alpha_i}{\sqrt{K}}\right)^2}\right].
$$

We next consider $I_2(t)$. Applying the transmission conditions yields

$$
\begin{aligned}
I_2(t) = e^{\frac{\gamma_{11}}{a}} &\left[\sqrt{\rho}\omega_{1,t}(1,t) + \sqrt{K}(\omega_{1,x}(1,t) - \varphi_1(1,t))\right]^2 \\
&- e^{-\frac{\gamma_{11}}{a}}\left[\sqrt{\rho}\omega_{1,t}(1,t) - \sqrt{K}(\omega_{1,x}(1,t) - \varphi_1(1,t))\right]^2 \\
&- 3\sum_{i=2}^{3}\left[\sqrt{\rho}\omega_{1,t}(1,t) + \sqrt{K}(\omega_{i,x}(0,t) - \varphi_i(0,t))\right]^2 \\
&+ \sum_{i=2}^{3}\left[\sqrt{\rho}\omega_{1,t}(1,t) - \sqrt{K}(\omega_{i,x}(0,t) - \varphi_i(0,t))\right]^2 \\
= \rho\omega_{1,t}^2(1,t)&\left[e^{\frac{\gamma_{11}}{a}} - e^{-\frac{\gamma_{11}}{a}} - 6 + 2\right] \\
&+ 2\sqrt{\rho K}\omega_{1,t}(1,t)\left[e^{\frac{\gamma_{11}}{a}}(\omega_{1,x}(1,t) - \varphi_1(1,t)) + e^{-\frac{\gamma_{11}}{a}}(\omega_{1,x}(1,t) - \varphi_1(1,t))\right] \\
&+ 2\sqrt{\rho K}\omega_{1,t}(1,t)\left[\sum_{i=2}^{3}(\omega_{i,x}(0,t) - \varphi_i(0,t)) - \sum_{i=2}^{3}(\omega_{i,x}(0,t) - \varphi_i(0,t))\right] \\
&+ \left(e^{\frac{\gamma_{11}}{a}} - e^{-\frac{\gamma_{11}}{a}}\right)K[\omega_{1,x}(1,t) - \varphi_1(1,t)]^2 - 2K\sum_{i=2}^{3}[\omega_{i,x}(0,t) - \varphi_i(0,t)]^2 \\
= \rho\omega_{1,t}^2(1,t)&\left(e^{\frac{\gamma_{11}}{a}} - e^{-\frac{\gamma_{11}}{a}}\right) \\
&+2\sqrt{\rho K}\omega_{1,t}(1,t)\left[\left(e^{\frac{\gamma_{11}}{a}} + e^{-\frac{\gamma_{11}}{a}}\right)(\omega_{1,x}(1,t) - \varphi_1(1,t)) - 4\sum_{i=2}^{3}(\omega_{i,x}(0,t) - \varphi_i(0,t))\right]
\end{aligned}
$$

$$
\begin{aligned}
&+K\left(e^{\frac{\gamma_{11}}{a}}-e^{-\frac{\gamma_{11}}{a}}\right)[\omega_{1,x}(1,t)-\varphi_1(1,t)]^2-2K\sum_{i=2}^{3}[\omega_{i,x}(0,t)-\varphi_i(0,t)]^2\\
&=\rho\omega_{1,t}^2(1,t)\left(2\sinh\frac{\gamma_{11}}{a}-4\right)+2\sqrt{\rho K}\omega_{1,t}(1,t)\left(2\cosh\frac{\gamma_{11}}{a}-4\right)[\omega_{1,x}(1,t)-\varphi_1(1,t)]\\
&\quad+2K\sinh\frac{\gamma_{11}}{a}[\omega_{1,x}(1,t)-\varphi_1(1,t)]^2-2K\sum_{i=2}^{3}[\omega_{i,x}(0,t)-\varphi_i(0,t)]^2\\
&\le 2\rho\omega_{1,t}^2(1,t)\left(\sinh\frac{\gamma_{11}}{a}-2\right)+2\sqrt{\rho K}\left(\cosh\frac{\gamma_{11}}{a}-2\right)\left[\omega_{1,t}^2(1,t)+(\omega_{1,x}(1,t)-\varphi_1(1,t))^2\right]\\
&\quad+2K\sinh\frac{\gamma_{11}}{a}[\omega_{1,x}(1,t)-\varphi_1(1,t)]^2-2K\sum_{i=2}^{3}[\omega_{i,x}(0,t)-\varphi_i(0,t)]^2\\
&\le 2\omega_{1,t}^2(1,t)\left(e^{\frac{\gamma_{11}}{a}}-4\right)+4\sum_{i=2}^{3}[\omega_{i,x}(0,t)-\varphi_i(0,t)]^2\left(e^{\frac{\gamma_{11}}{a}}-2\right)-2\sum_{i=2}^{3}[\omega_{i,x}(0,t)-\varphi_i(0,t)]^2\\
&=2\omega_{1,t}^2(1,t)\left(e^{\frac{\gamma_{11}}{a}}-4\right)+2\sum_{i=2}^{3}[\omega_{i,x}(0,t)-\varphi_i(0,t)]^2\left(2e^{\frac{\gamma_{11}}{a}}-5\right).
\end{aligned}
$$

For $I_2(t)\le 0$, it is sufficient that

$$
\begin{cases}
e^{\frac{\gamma_{11}}{a}}-4\le 0,\\
2e^{\frac{\gamma_{11}}{a}}-5\le 0,
\end{cases}
$$

which gives $\gamma_{11}\le a\ln\frac{5}{2}$.

Summarizing the analysis of $R_1(t)$, we obtain

$$
\begin{cases}
\gamma_{11}\le a\ln\frac{5}{2},\\
\gamma_{1i}\le\frac{a}{2}\ln\left[\frac{\left(\sqrt{\rho}+\frac{\alpha_i}{\sqrt{K}}\right)^2}{3\left(\sqrt{\rho}-\frac{\alpha_i}{\sqrt{K}}\right)^2}\right],\ i=2,3.
\end{cases}
$$

Next, we analyze $R_2(t)$. Proceeding in a completely analogous way, we obtain

$$
\begin{aligned}
&c\sum_{i=1}^{3}\left[p_{3i}(1)\xi_{3i}^2(1,t)-p_{4i}(1)\xi_{4i}^2(1,t)-p_{3i}(0)\xi_{3i}^2(0,t)+p_{4i}(0)\xi_{4i}^2(0,t)\right]\\
&=c\left[p_{31}(1)\xi_{31}^2(1,t)-p_{41}(1)\xi_{41}^2(1,t)-p_{31}(0)\xi_{31}^2(0,t)+p_{41}(0)\xi_{41}^2(0,t)\right]\\
&\quad+c\sum_{i=2}^{3}\left[p_{3i}(1)\xi_{3i}^2(1,t)-p_{4i}(1)\xi_{4i}^2(1,t)-p_{3i}(0)\xi_{3i}^2(0,t)+p_{4i}(0)\xi_{4i}^2(0,t)\right]\\
&=cp_{31}e^{\frac{\gamma_{31}}{c}}\left[\sqrt{I_\rho}\varphi_{1,t}(1,t)+\sqrt{EI}\varphi_{1,x}(1,t)\right]^2-cp_{41}e^{\frac{-\gamma_{41}}{c}}\left[\sqrt{I_\rho}\varphi_{1,t}(1,t)-\sqrt{EI}\varphi_{1,x}(1,t)\right]^2\\
&\quad+c\sum_{i=2}^{3}p_{3i}e^{\frac{\gamma_{3i}}{c}}\left[\sqrt{I_\rho}\varphi_{i,t}(1,t)+\sqrt{EI}\varphi_{i,x}(1,t)\right]^2-c\sum_{i=2}^{3}p_{4i}e^{\frac{-\gamma_{4i}}{c}}\left[\sqrt{I_\rho}\varphi_{i,t}(1,t)-\sqrt{EI}\varphi_{i,x}(1,t)\right]^2\\
&\quad-c\sum_{i=2}^{3}p_{3i}\left[\sqrt{I_\rho}\varphi_{i,t}(0,t)+\sqrt{EI}\varphi_{i,x}(0,t)\right]^2+c\sum_{i=2}^{3}p_{4i}\left[\sqrt{I_\rho}\varphi_{i,t}(0,t)-\sqrt{EI}\varphi_{i,x}(0,t)\right]^2\\
&\quad+cp_{41}\left[\sqrt{I_\rho}\varphi_{1,t}(0,t)-\sqrt{EI}\varphi_{1,x}(0,t)\right]^2-cp_{31}\left[\sqrt{I_\rho}\varphi_{1,t}(0,t)+\sqrt{EI}\varphi_{1,x}(0,t)\right]^2\\
&=c\sum_{i=2}^{3}p_{3i}e^{\frac{\gamma_{3i}}{c}}\left[\sqrt{I_\rho}\varphi_{i,t}(1,t)+\frac{v_i(t)}{\sqrt{EI}}\right]^2-c\sum_{i=2}^{3}p_{4i}e^{\frac{-\gamma_{4i}}{c}}\left[\sqrt{I_\rho}\varphi_{i,t}(1,t)-\frac{v_i(t)}{\sqrt{EI}}\right]^2\\
&\quad+cp_{31}e^{\frac{\gamma_{31}}{c}}\left[\sqrt{I_\rho}\varphi_{1,t}(1,t)+\sqrt{EI}\varphi_{1,x}(1,t)\right]^2-cp_{41}e^{\frac{-\gamma_{41}}{c}}\left[\sqrt{I_\rho}\varphi_{1,t}(1,t)-\sqrt{EI}\varphi_{1,x}(1,t)\right]^2\\
&\quad-c\sum_{i=2}^{3}p_{3i}\left[\sqrt{I_\rho}\varphi_{i,t}(0,t)+\sqrt{EI}\varphi_{i,x}(0,t)\right]^2+c\sum_{i=2}^{3}p_{4i}\left[\sqrt{I_\rho}\varphi_{i,t}(0,t)-\sqrt{EI}\varphi_{i,x}(0,t)\right]^2\\
&\quad+cEI(p_{41}-p_{31})\varphi_{1,x}^2(0,t)\\
&=c\sum_{i=2}^{3}p_{3i}e^{\frac{\gamma_{3i}}{c}}\left[\sqrt{I_\rho}\varphi_{i,t}(1,t)+\frac{v_i(t)}{\sqrt{EI}}\right]^2-c\sum_{i=2}^{3}p_{4i}e^{\frac{-\gamma_{4i}}{c}}\left[\sqrt{I_\rho}\varphi_{i,t}(1,t)-\frac{v_i(t)}{\sqrt{EI}}\right]^2\\
&\quad+cp_{31}e^{\frac{\gamma_{31}}{c}}\left[\sqrt{I_\rho}\varphi_{1,t}(1,t)+\sqrt{EI}\varphi_{1,x}(1,t)\right]^2-cp_{41}e^{\frac{-\gamma_{41}}{c}}\left[\sqrt{I_\rho}\varphi_{1,t}(1,t)-\sqrt{EI}\varphi_{1,x}(1,t)\right]^2\\
&\quad-c\sum_{i=2}^{3}p_{3i}\left[\sqrt{I_\rho}\varphi_{1,t}(1,t)+\sqrt{EI}\varphi_{i,x}(0,t)\right]^2+c\sum_{i=2}^{3}p_{4i}\left[\sqrt{I_\rho}\varphi_{1,t}(1,t)-\sqrt{EI}\varphi_{i,x}(0,t)\right]^2,
\end{aligned}
$$

where $p_{31} = p_{41}$.

With the following choices

$$\begin{cases} p_{32} = p_{33} = 3p_{31} = 3q, \\ p_{42} = p_{43} = p_{41} = q, \\ \gamma_{3i} = \gamma_{4i} = s,\ i = 2, 3, \end{cases}$$

we obtain

$$\begin{aligned} R_2(t) = {} & 3cq \sum_{i=2}^{3} e^{\frac{s}{c}} \left[\sqrt{I_\rho} \varphi_{i,t}(1,t) + \frac{-\beta_i \varphi_{i,t}(1,t)}{\sqrt{EI}} \right]^2 - cq \sum_{i=2}^{3} e^{-\frac{s}{c}} \left[\sqrt{I_\rho} \varphi_{i,t}(1,t) - \frac{-\beta_i \varphi_{i,t}(1,t)}{\sqrt{EI}} \right]^2 \\ & + cqe^{\frac{\gamma_{31}}{c}} \left[\sqrt{I_\rho} \varphi_{1,t}(1,t) + \sqrt{EI} \varphi_{1,x}(1,t) \right]^2 - cqe^{\frac{-\gamma_{31}}{c}} \left[\sqrt{I_\rho} \varphi_{1,t}(1,t) - \sqrt{EI} \varphi_{1,x}(1,t) \right]^2 \\ & - 3cq \sum_{i=2}^{3} \left[\sqrt{I_\rho} \varphi_{1,t}(1,t) + \sqrt{EI} \varphi_{i,x}(0,t) \right]^2 + cq \sum_{i=2}^{3} \left[\sqrt{I_\rho} \varphi_{1,t}(1,t) - \sqrt{EI} \varphi_{i,x}(0,t) \right]^2 \\ = {} & cqB_2(t), \end{aligned}$$

where

$$\begin{aligned} B_2(t) = {} & 3 \sum_{i=2}^{3} e^{\frac{s}{c}} \left[\left(\sqrt{I_\rho} - \frac{\beta_i}{\sqrt{EI}} \right) \varphi_{i,t}(1,t) \right]^2 - \sum_{i=2}^{3} e^{-\frac{s}{c}} \left[\left(\sqrt{I_\rho} + \frac{\beta_i}{\sqrt{EI}} \right) \varphi_{i,t}(1,t) \right]^2 \\ & + e^{\frac{\gamma_{31}}{c}} \left[\sqrt{I_\rho} \varphi_{1,t}(1,t) + \sqrt{EI} \varphi_{1,x}(1,t) \right]^2 - e^{-\frac{\gamma_{31}}{c}} \left[\sqrt{I_\rho} \varphi_{1,t}(1,t) - \sqrt{EI} \varphi_{1,x}(1,t) \right]^2 \\ & - 3 \sum_{i=2}^{3} \left[\sqrt{I_\rho} \varphi_{1,t}(1,t) + \sqrt{EI} \varphi_{i,x}(0,t) \right]^2 + \sum_{i=2}^{3} \left[\sqrt{I_\rho} \varphi_{1,t}(1,t) - \sqrt{EI} \varphi_{i,x}(0,t) \right]^2. \end{aligned}$$

For simplicity, we define

$$I_3(t) = 3 \sum_{i=2}^{3} e^{\frac{s}{c}} \left[\left(\sqrt{I_\rho} - \frac{\beta_i}{\sqrt{EI}} \right) \varphi_{i,t}(1,t) \right]^2 - \sum_{i=2}^{3} e^{-\frac{s}{c}} \left[\left(\sqrt{I_\rho} + \frac{\beta_i}{\sqrt{EI}} \right) \varphi_{i,t}(1,t) \right]^2,$$

$$\begin{aligned} I_4(t) = {} & e^{\frac{\gamma_{31}}{c}} \left[\sqrt{I_\rho} \varphi_{1,t}(1,t) + \sqrt{EI} \varphi_{1,x}(1,t) \right]^2 - e^{-\frac{\gamma_{31}}{c}} \left[\sqrt{I_\rho} \varphi_{1,t}(1,t) - \sqrt{EI} \varphi_{1,x}(1,t) \right]^2 \\ & - 3 \sum_{i=2}^{3} \left[\sqrt{I_\rho} \varphi_{1,t}(1,t) + \sqrt{EI} \varphi_{i,x}(0,t) \right]^2 + \sum_{i=2}^{3} \left[\sqrt{I_\rho} \varphi_{1,t}(1,t) - \sqrt{EI} \varphi_{i,x}(0,t) \right]^2. \end{aligned}$$

To ensure $I_3(t) \leq 0$, the equivalent condition is

$$3e^{\frac{s}{c}} \left[\left(\sqrt{I_\rho} - \frac{\beta_i}{\sqrt{EI}} \right) \varphi_{i,t}(1,t) \right]^2 - e^{-\frac{s}{c}} \left[\left(\sqrt{I_\rho} + \frac{\beta_i}{\sqrt{EI}} \right) \varphi_{i,t}(1,t) \right]^2 \leq 0,$$

$$3e^{\frac{s}{c}} \left[\left(\sqrt{I_\rho} - \frac{\beta_i}{\sqrt{EI}} \right) \varphi_{i,t}(1,t) \right]^2 \leq e^{-\frac{s}{c}} \left[\left(\sqrt{I_\rho} + \frac{\beta_i}{\sqrt{EI}} \right) \varphi_{i,t}(1,t) \right]^2,$$

$$3e^{\frac{2s}{c}} \leq \frac{\left(\sqrt{I_\rho} + \frac{\beta_i}{\sqrt{EI}} \right)^2}{\left(\sqrt{I_\rho} - \frac{\beta_i}{\sqrt{EI}} \right)^2},$$

$$e^{\frac{2s}{c}} \leq \frac{\left(\sqrt{I_\rho} + \frac{\beta_i}{\sqrt{EI}} \right)^2}{3\left(\sqrt{I_\rho} - \frac{\beta_i}{\sqrt{EI}} \right)^2},$$

which simplifies to

$$s \leq \frac{c}{2} \ln \left[\frac{\left(\sqrt{I_\rho} + \frac{\beta_i}{\sqrt{EI}} \right)^2}{3\left(\sqrt{I_\rho} - \frac{\beta_i}{\sqrt{EI}} \right)^2} \right].$$

Then, we have

$$\begin{aligned}
I_4(t) &= e^{\frac{\gamma_{31}}{c}} \left[\sqrt{I_\rho}\varphi_{1,t}(1,t) + \sqrt{EI}\varphi_{1,x}(1,t) \right]^2 - e^{-\frac{\gamma_{31}}{c}} \left[\sqrt{I_\rho}\varphi_{1,t}(1,t) - \sqrt{EI}\varphi_{1,x}(1,t) \right]^2 \\
&\quad - 3\sum_{i=2}^{3} \left[\sqrt{I_\rho}\varphi_{1,t}(1,t) + \sqrt{EI}\varphi_{i,x}(0,t) \right]^2 + \sum_{i=2}^{3} \left[\sqrt{I_\rho}\varphi_{1,t}(1,t) - \sqrt{EI}\varphi_{i,x}(0,t) \right]^2 \\
&= e^{\frac{\gamma_{31}}{c}} \left[I_\rho\varphi_{1,t}^2(t) + 2\sqrt{I_\rho EI}\varphi_{1,t}(1,t)\varphi_{1,x}(1,t) + EI\varphi_{1,x}^2(1,t) \right] \\
&\quad - e^{-\frac{\gamma_{31}}{c}} \left[I_\rho\varphi_{1,t}^2(1,t) - 2\sqrt{I_\rho EI}\varphi_{1,t}(1,t)\varphi_{1,x}(1,t) + EI\varphi_{1,x}^2(1,t) \right] \\
&\quad - 3\sum_{i=2}^{3} \left[I_\rho\varphi_{1,t}^2(1,t) + 2\sqrt{I_\rho EI}\varphi_{1,t}(1,t)\varphi_{i,x}(0,t) + EI\varphi_{i,x}^2(0,t) \right] \\
&\quad + \sum_{i=2}^{3} \left[I_\rho\varphi_{1,t}^2(1,t) - 2\sqrt{I_\rho EI}\varphi_{1,t}(1,t)\varphi_{i,x}(0,t) + EI\varphi_{i,x}^2(0,t) \right] \\
&= 2\sqrt{I_\rho EI}\varphi_{1,t}(1,t) \left[\sum_{i=2}^{3} \varphi_{i,x}(0,t) - \sum_{i=2}^{3} \varphi_{i,x}(0,t) \right] \\
&\quad + \left(e^{\frac{\gamma_{31}}{c}} - e^{-\frac{\gamma_{31}}{c}} \right) EI\varphi_{1,x}^2(1,t) - 2EI\sum_{i=2}^{3} \varphi_{i,x}^2(0,t) \\
&\quad + 2\sqrt{I_\rho EI}\varphi_{1,t}(1,t) \left[e^{\frac{\gamma_{31}}{c}}\varphi_{1,x}(1,t) + e^{-\frac{\gamma_{31}}{c}}\varphi_{1,x}(1,t) \right] \\
&\quad + I_\rho\varphi_{1,t}^2(1,t) \left[e^{\frac{\gamma_{31}}{c}} - e^{-\frac{\gamma_{31}}{c}} - 6 + 2 \right] \\
&= 2\sqrt{I_\rho EI}\varphi_{1,t}(1,t) \left[\left(e^{\frac{\gamma_{31}}{c}} + e^{-\frac{\gamma_{31}}{c}} \right) \varphi_{1,x}(1,t) - 4\sum_{i=2}^{3} \varphi_{i,x}(0,t) \right] \\
&\quad + EI\left(e^{\frac{\gamma_{31}}{c}} - e^{-\frac{\gamma_{31}}{c}} \right) \varphi_{1,x}^2(1,t) - 2EI\sum_{i=2}^{3} \varphi_{i,x}^2(0,t) \\
&\quad + I_\rho\varphi_{1,t}^2(1,t) \left(e^{\frac{\gamma_{31}}{c}} - e^{-\frac{\gamma_{31}}{c}} \right) \\
&= I_\rho\varphi_{1,t}^2(1,t) \left(2\sinh\frac{\gamma_{31}}{c} - 4 \right) + 2\sqrt{I_\rho EI}\varphi_{1,t}(1,t) \left(2\cosh\frac{\gamma_{31}}{c} - 4 \right) \varphi_{1,x}(1,t) \\
&\quad + 2EI\sinh\frac{\gamma_{31}}{c}\varphi_{1,x}^2(1,t) - 2EI\sum_{i=2}^{3} \varphi_{i,x}^2(0,t) \\
&\leq 2I_\rho\varphi_{1,t}^2(1,t) \left(\sinh\frac{\gamma_{31}}{c} - 2 \right) + 2\sqrt{I_\rho EI} \left(\cosh\frac{\gamma_{31}}{c} - 2 \right) \left[\varphi_{1,t}^2(1,t) + \varphi_{1,x}^2(1,t) \right] \\
&\quad + 2EI\sinh\frac{\gamma_{31}}{c}\varphi_{1,x}^2(1,t) - 2EI\sum_{i=2}^{3} \varphi_{i,x}^2(0,t) \\
&\leq 2\varphi_{1,t}^2(1,t) \left(e^{\frac{\gamma_{31}}{c}} - 4 \right) + 4\sum_{i=2}^{3} \varphi_{i,x}^2(0,t) \left(e^{\frac{\gamma_{31}}{c}} - 2 \right) - 2\sum_{i=2}^{3} \varphi_{i,x}^2(0,t) \\
&= 2\varphi_{1,t}^2(1,t) \left(e^{\frac{\gamma_{31}}{c}} - 4 \right) + 2\sum_{i=2}^{3} \varphi_{i,x}^2(0,t) \left(2e^{\frac{\gamma_{31}}{c}} - 5 \right).
\end{aligned}$$

For $I_4(t) \leq 0$, it is sufficient that

$$\begin{cases} e^{\frac{\gamma_{31}}{c}} - 4 \leq 0, \\ 2e^{\frac{\gamma_{31}}{c}} - 5 \leq 0, \end{cases}$$

which yields $\gamma_{31} \leq c \ln \frac{5}{2}$.

Hence, for the parameters in $R_2(t)$, we obtain

$$\begin{cases} \gamma_{31} \leq c \ln \frac{5}{2}, \\ \gamma_{3i} \leq \frac{c}{2} \ln \left[\frac{\left(\sqrt{I_\rho} + \frac{\beta_i}{\sqrt{EI}}\right)^2}{3\left(\sqrt{I_\rho} - \frac{\beta_i}{\sqrt{EI}}\right)^2} \right], \ i = 2, 3, \\ \gamma_{4i} \leq \frac{c}{2} \ln \left[\frac{\left(\sqrt{I_\rho} + \frac{\beta_i}{\sqrt{EI}}\right)^2}{3\left(\sqrt{I_\rho} - \frac{\beta_i}{\sqrt{EI}}\right)^2} \right], \ i = 2, 3. \end{cases}$$

Finally, we consider the remaining term $A(t) = \sum_{i=1}^{3} A_i(t)$.

We define the auxiliary quantities

$$\begin{cases} S_{1i} = 2b \int_0^1 [p_{2i}(x)\xi_{2i}(x,t) - p_{1i}(x)\xi_{1i}(x,t)]\varphi_{i,t}(x,t)dx, \\ S_{2i} = 2d \int_0^1 [p_{3i}(x)\xi_{3i}(x,t) + p_{4i}(x)\xi_{4i}(x,t)][\omega_{i,x}(x,t) - \varphi_i(x,t)]dx. \end{cases}$$

Using the identities

$$\begin{cases} \omega_{i,x}(x,t) - \varphi_i(x,t) = \frac{\xi_{1i}(x,t) - \xi_{2i}(x,t)}{2\sqrt{K}}, \\ \varphi_{i,t}(x,t) = \frac{\xi_{3i}(x,t) + \xi_{4i}(x,t)}{2\sqrt{I_\rho}}, \end{cases}$$

we can rewrite S_{1i} as

$$S_{1i} = \frac{b}{\sqrt{I_\rho}} \int_0^1 [p_{2i}(x)\xi_{2i}(x,t) - p_{1i}(x)\xi_{1i}(x,t)][\xi_{3i}(x,t) + \xi_{4i}(x,t)]dx.$$

Defining $M_{1i} = \max\{p_{1i}(1),\ p_{2i}(0)\}$, we obtain

$$\begin{cases} |p_{2i}\xi_{2i} - p_{1i}\xi_{1i}| \leq p_{2i}|\xi_{2i}| + p_{1i}|\xi_{1i}| \leq M_{1i}(|\xi_{1i}| + |\xi_{2i}|), \\ |\xi_{3i} + \xi_{4i}| \leq |\xi_{3i}| + |\xi_{4i}|, \end{cases}$$

and hence

$$S_{1i} \leq \frac{bM_{1i}}{\sqrt{I_\rho}} \int_0^1 (|\xi_{1i} + \xi_{2i}||\xi_{3i} + \xi_{4i}|)dx.$$

Using Young's inequality, for any $\varepsilon_{1i} > 0$, we have

$$\begin{aligned} &(|\xi_{1i}| + |\xi_{2i}|)(|\xi_{3i}| + |\xi_{4i}|) \\ &= |\xi_{1i}\xi_{3i}| + |\xi_{1i}\xi_{4i}| + |\xi_{2i}\xi_{3i}| + |\xi_{2i}\xi_{4i}| \\ &\leq \varepsilon_{1i}\left(\xi_{1i}^2 + \xi_{2i}^2\right) + \frac{1}{4\varepsilon_{1i}}\left(\xi_{3i}^2 + \xi_{4i}^2\right). \end{aligned}$$

With $N_i = \min\{p_{1i}(0),\ p_{2i}(1),\ p_{3i}(0),\ p_{4i}(1)\}$, we have the integral bounds

$$\begin{cases} \int_0^1 (\xi_{1i}^2 + \xi_{2i}^2)dx \leq \frac{V_i}{N_i}, \\ \int_0^1 (\xi_{3i}^2 + \xi_{4i}^2)dx \leq \frac{V_i}{N_i}. \end{cases}$$

Substituting $\varepsilon_{1i} = \frac{N_i\sqrt{I_\rho}}{4|b|M_{1i}}$ into the above estimate yields $S_{1i} \leq \delta_{1i}V_i$, where $\delta_{1i} = \frac{N_i^2 I_\rho + M_{1i}^2 b^2}{4N_i^2 I_\rho}$. Similarly, with $M_{2i} = \max\{p_{3i}(1),\ p_{4i}(0)\}$, we obtain $\delta_{2i} = \frac{N_i^2 K + M_{2i}^2 d^2}{4N_i^2 K}$.

Remark 3. *The parameter* $\epsilon_{1i} = \frac{N_i\sqrt{I_\rho}}{2|b|M_{1i}}$ *is chosen as a sufficient condition to absorb the coupling terms via Young's inequality:*

$$2|b|M_{1i}|\xi_{1i}||\varphi_{i,t}| \leq \epsilon_{1i}N_i^2|\xi_{1i}|^2 + \frac{|b|^2 M_{1i}^2}{\epsilon_{1i}N_i^2} I_\rho |\psi_i|^2.$$

The resulting constants

$$\delta_{1i} = \frac{N_i^2 I_\rho + M_{1i}^2 b^2}{4N_i^2 I_\rho}, \quad \delta_{2i} = \frac{N_i^2 K + M_{2i}^2 d^2}{4N_i^2 K}$$

can be made arbitrarily small by making N_i sufficiently large relative to M_{1i} and M_{2i}.

Therefore $A(t) = \sum_{i=1}^{3} A_i(t) \leq \sum_{i=1}^{3} (\delta_{1i} + \delta_{2i}) V_i(t)$.

From the above derivations, we have

$$\dot{V}(t) = -\mu V(t) + \sum_{i=1}^{3} A_i(t),$$

where

$$\mu = \min\left\{ \sum_{i=1}^{3} \left(\frac{a \ln 5}{2}, \frac{a}{2} \ln \frac{(\sqrt{\rho} + \alpha_i/\sqrt{K})^2}{3(\sqrt{\rho} - \alpha_i/\sqrt{K})^2}, \frac{c \ln 5}{2}, \frac{c}{2} \ln \frac{(\sqrt{I_\rho} + \beta_i/\sqrt{EI})^2}{3(\sqrt{I_\rho} - \beta_i/\sqrt{EI})^2} \right) \right\}$$

and the terms $A_i(t)$ satisfy

$$A_i(t) \leq (\delta_{1i} + \delta_{2i}) V_i(t),$$

with

$$\delta_{1i} = \frac{N_i^2 I_\rho + M_{1i}^2 b^2}{4N_i^2 I_\rho}, \qquad \delta_{2i} = \frac{N_i^2 K + M_{2i}^2 d^2}{4N_i^2 K}.$$

Consequently,

$$\dot{V}(t) \leq -\mu V(t) + \sum_{i=1}^{3} (\delta_{1i} + \delta_{2i}) V_i(t) \leq -\gamma V(t),$$

where

$$\gamma = \mu - \sum_{i=1}^{3} (\delta_{1i} + \delta_{2i}) > 0.$$

Applying Gronwall's inequality gives

$$V(t) \leq V(0) e^{-\gamma t}.$$

Since $V(t)$ is equivalent to the system energy $E(t)$, it follows that

$$E(t) \leq C E(0) e^{-\gamma t},$$

which proves that the closed-loop system is exponentially stable with a decay rate of at least $\gamma/2$ in terms of the original energy $E(t)$.

Combining all the estimates, we finally found that the closed-loop system (11) is exponentially stable with an exponential decay rate of at least $\gamma/2$, where γ satisfies

$$0 < \gamma < \min\left\{ a \ln \frac{5}{2}, \min_{i=2,3} \frac{a}{2} \ln \left[\frac{\left(\sqrt{\rho} + \frac{\alpha_i}{\sqrt{K}}\right)^2}{3\left(\sqrt{\rho} - \frac{\alpha_i}{\sqrt{K}}\right)^2} \right], c \ln \frac{5}{2}, \min_{i=2,3} \frac{c}{2} \ln \left[\frac{\left(\sqrt{I_\rho} + \frac{\beta_i}{\sqrt{EI}}\right)^2}{3\left(\sqrt{I_\rho} - \frac{\beta_i}{\sqrt{EI}}\right)^2} \right] \right\} - \sum_{i=1}^{3} \delta_{1i} - \sum_{i=1}^{3} \delta_{2i}.$$

□

To illustrate the practical relevance of the proposed theoretical framework, we consider a lightweight flexible robotic manipulator composed of three interconnected beams arranged in a tree-shaped topology, with a payload mounted at the end effector. This configuration is representative of structural systems found in aerospace engineering and advanced manufacturing. Using realistic material parameters for steel, the derived explicit decay rate γ in Theorem 2 directly guides the selection of feedback gains (α_i, β_i).

4. Numerical Simulation

To validate the exponential stability of the proposed system, we perform numerical simulations based on the theoretical analysis presented above. The system is solved numerically using a finite difference method.

The parameters used in the simulations are as follows, as summarized in Table 1.

Table 1. Key Parameters of Timoshenko Beam System.

Category	Symbol	Name	Value	Unit
Material	ρ	Mass Density	7.80×10^3	kg/m^3
Material	K	Shear Stiffness	6.67×10^8	N/m
Material	EI	Bending Stiffness	1.75×10^5	$N \cdot m^2$
Material	I_ρ	Mass Inertia	6.50×10^{-3}	$kg \cdot m$
Control	α_2	Beam 2 Lateral Gain	1.00×10^3	–
Control	α_3	Beam 3 Lateral Gain	1.00×10^3	–
Control	β_2	Beam 2 Rotational Gain	1.00×10^2	–
Control	β_3	Beam 3 Rotational Gain	1.00×10^2	–

The numerical simulation results are shown in Figure 2.

Figure 2 shows the decay of the system energy over time in a semi-logarithmic coordinate system. Numerical results indicate that the time required for the system energy to decay to 10% of its initial value is 4.7 s, while the time to decay to 1% is 9.3 s. In the semi-logarithmic plot, the theoretical decay curve is strictly linear, thereby confirming the exponential stability of the system. Furthermore, these simulation results are consistent with the conclusion of Theorem 2 in the theoretical analysis of Section 3, validating the effectiveness and feasibility of the designed boundary velocity feedback control strategy.

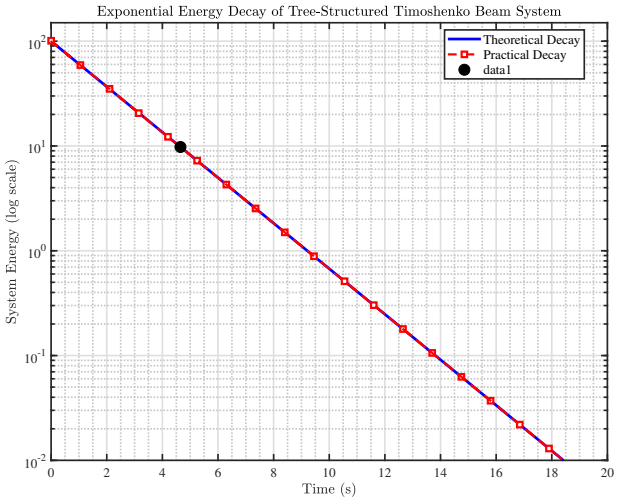


Figure 2. Exponential Energy Decay of Tree-Structured Timoshenko Beam System.

Moreover, to quantitatively assess the sharpness of the decay estimate derived in Theorem 2, we extract the observed decay rate from the numerical energy decay curve presented in Figure 2. Applying a least-squares linear regression to $\ln E(t)$ over the exponential decay regime yields $\gamma_{\text{obs}} \approx 0.493$. This value exceeds the theoretical lower bound γ_{th} established in Theorem 2, confirming that the analytical estimate is valid while being conservative-a feature inherent to Lyapunov-based stability analyses. The comparison validates the theoretical result and quantifies the gap between the worst-case analytical guarantee and the actual numerical performance.

To further explore the influence of control parameters on the decay performance, we

conducted a series of numerical experiments with four gain combinations: no control $(0, 0)$, $(500, 50)$, $(1000, 100)$, and $(2000, 200)$.

Figure 3 shows the system energy $E(t)$ in a semi-logarithmic plot for different control gain combinations. The black solid line corresponds to the uncontrolled case, while the blue dashed, red dash-dotted, and green dotted lines correspond to $(\alpha, \beta) = (500, 50)$, $(1000, 100)$, and $(2000, 200)$, respectively. The nearly horizontal black curve indicates that the system is not exponentially stable without control. In contrast, the linear decay observed in the semi-log scale for the controlled cases confirms exponential stability. Moreover, the increasing slopes with larger gains demonstrate the effectiveness of the proposed boundary feedback strategy.

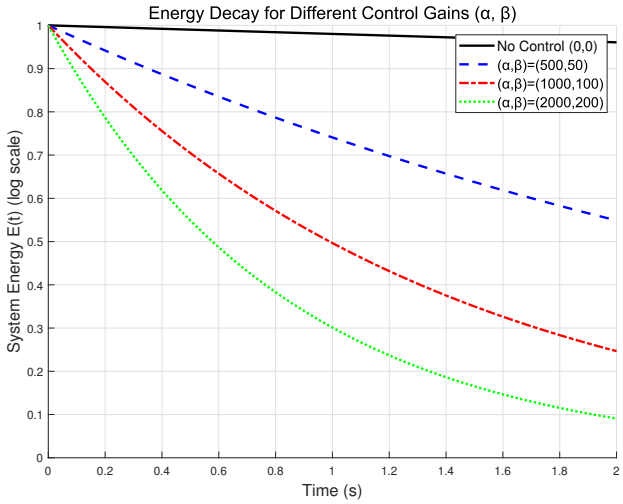


Figure 3. Energy Decay for Different Control Gains (α,β).

5. Conclusions

In this paper, we estimate the energy decay rate of the Timoshenko beam system by constructing and employing a weighted energy functional. Specifically, for the case where a concentrated end load is attached to the right end of the beam, we adopt a boundary-velocity feedback control law. We rigorously prove the exponential stability of the system and derive an explicit expression for the decay rate γ of the weighted energy functional, from which the actual energy decay rate of the system is obtained as $\frac{\gamma}{2}$.

The appropriate selection of control and structural parameters is crucial for the effectiveness of the proposed method. Based on the explicit decay rate formula in Theorem 2 and the numerical results in Section 4, we provide a preliminary qualitative assessment of parameter sensitivity. The material parameters (ρ, K, EI, I_ρ) influence the decay rate through the wave speeds $a = \sqrt{K/\rho}$, $c = \sqrt{EI/I_\rho}$, as well as the auxiliary terms δ_{1i} and δ_{2i}. Among these, the shear stiffness K and bending stiffness EI have a more pronounced effect than the density ρ or the mass inertia I_ρ. Meanwhile, increasing the control gains α_i and β_i accelerates dissipation, although excessively large values risk saturating the actuators and exciting high-frequency dynamics.

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