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The Dawson Transform and Its Operational Properties

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Abstract

We introduce and study an integral transform whose kernel is Dawson's integral. The proposed transform is connected to the Fourier sine transform through a classical integral representation of Dawson's function and to the Glasser transform through an iteration formula. Basic operational properties are developed, including derivative identities, delta-derivative identities, Mellin-transform representations, and an inversion theorem. Several transform pairs are obtained for elementary functions, Gaussian functions, powers multiplied by trigonometric functions, and Bessel functions. The results place the Dawson transform within the broader theory of special-function integral transforms and Parseval–Goldstein-type identities.

Keywords: Dawson's integral; Dawson transform; Glasser transform; Mellin transform; Fourier sine transform; Parseval–Goldstein identity; special functions

MSC: 33B20; 33C10; 44A05; 44A10; 44A20

1. Introduction

Dawson's integral, also known as Dawson's function, is a classical special function defined by

$$\operatorname{daw}(x) = e^{-x^2} \int_0^x e^{t^2}\, dt = \int_0^x \exp(t^2 - x^2)\, dt. \tag{1}$$

This notation and definition are used, for example, by Oldham, Myland, and Spanier in An Atlas of Functions [1]. Dawson's integral is closely related to the imaginary error function through the identity

$$\operatorname{erfi}(x) = \frac{2}{\sqrt{\pi}} e^{x^2} \operatorname{daw}(x), \tag{2}$$

and therefore belongs naturally to the family of error functions, Fresnel-type functions, and Gaussian-related special functions. Its basic properties, differential equation, integral representations, and asymptotic expansions are documented in standard references on special functions, including the NIST Digital Library of Mathematical Functions, Abramowitz and Stegun, and the NIST Handbook of Mathematical Functions [2–4]. Computational aspects and approximation methods for Dawson's integral have also been studied; in particular, Chebyshev approximations were developed by Cody, Paciorek, and Thacher [5].

Integral transforms whose kernels are special functions form an important part of classical and modern analysis. The Laplace, Fourier, Mellin, Hankel, Stieltjes, Glasser, and Widder potential transforms provide systematic methods for evaluating improper integrals, solving differential equations, and deriving identities involving special functions. General references on these methods include Sneddon [6], Erdélyi et al. [7], Debnath and Bhatta [8], Yakubovich and Luchko [9], and Titchmarsh [10]. Mellin-transform and

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Mellin–Barnes techniques are especially relevant in the analysis of special-function kernels, since they provide natural inversion formulas and contour-integral representations; see, for example, Paris and Kaminski [11] and Prudnikov, Brychkov, and Marichev [12]. The Bessel-function examples considered later in this paper are also connected with the classical theory developed by Watson [13]. The present work is motivated by this general theory of special-function integral transforms. We introduce the Dawson transform of a suitable function f by

$$\mathscr{D}\{f(x);y\} = \int_0^\infty \mathrm{daw}(xy) f(x)\, dx, \qquad y > 0, \tag{3}$$

For a concrete domain, let

$$\mathcal{X} = L^1\Big((0,\infty), \min\{x, x^{-1}\}\, dx\Big).$$

Since $\mathrm{daw}(t) = O(t)$ as $t \to 0^+$ and $\mathrm{daw}(t) = O(t^{-1})$ as $t \to \infty$, every $f \in \mathcal{X}$ has an absolutely convergent Dawson transform for each $y > 0$. More generally, subsequent theorems are stated under the weakest local smoothness and weighted-integrability assumptions needed for the indicated interchanges, integrations by parts, or differentiations under the integral sign.

Although Dawson's integral is a classical special function, its systematic use as a transform kernel appears to be less developed than the corresponding theories for the Laplace, Hankel, Stieltjes, Glasser, and Widder-type transforms. The purpose of this paper is to develop the basic operational theory of the transform (3) and to show that it is naturally connected with several well-known transforms.

The second iteration of the Dawson transform leads to the Glasser transform. This places the new transform within the established theory of transform iteration and Parseval–Goldstein identities; representative antecedents include the general Parseval-type approach in [14] and later Glasser-transform identities in [15]. The present transform nevertheless has a distinct first-iterate structure: its odd Dawson kernel is directly connected to the Fourier sine transform and satisfies a first-order differential equation, which yields operational identities that are not available from the Glasser kernel alone.

The first main objective of this paper is to establish the basic properties of the Dawson transform. We derive iteration formulas, including the identity showing that the second iteration of the Dawson transform is a constant multiple of the Glasser transform. We also obtain differentiation formulas with respect to the transform variable and operational identities involving the differential operator

$$\delta_x = \frac{1}{x}\frac{d}{dx}, \tag{4}$$

and the Euler-type operator

$$\sigma_y = y\frac{d}{dy}. \tag{5}$$

These identities provide a calculus for computing Dawson transforms of derivatives, powers of functions, and expressions involving repeated applications of δ_x.

The second objective is to develop an inversion theory for the Dawson transform. The main idea is to use the Mellin transform, which converts the Dawson transform into a multiplicative relation between the Mellin transform of $\mathscr{D}f$ and a reflected Mellin transform of f. This relation identifies the Mellin multiplier associated with Dawson's integral and leads, under suitable analyticity and convergence assumptions, to an explicit inverse Dawson transform. The detailed Mellin representation and the corresponding inversion formula provide a possible direction for further development.

The third objective is to provide a collection of transform pairs and examples. We compute Dawson transforms of elementary functions such as e^{-ax}, $\sin(ax)$, and $\cos(ax)$, Gaussian-type functions such as $e^{-\alpha x^2}$ and $x^{\mu-1}e^{-\alpha x^2}$, and special functions such as $J_\nu(ax)$. We also derive formulas for transforms of $x^n \sin(ax)$ and $x^n \cos(ax)$ by differentiating with respect to the parameter a. These examples illustrate how the Dawson transform can be used as a practical tool for evaluating infinite integrals.

The paper is organized as follows. Section 2 reviews preliminary properties of Dawson's integral. Section 3 establishes iteration formulas and the connection with the Glasser transform. Section 4 gives the Mellin representation and inversion theorem. Section 5 develops derivative and delta-derivative properties. Section 6 gives transform pairs and examples involving elementary, Gaussian, and polynomial-trigonometric functions. Section 7 treats Bessel-function examples. Section 8 summarizes the results and indicates possible directions for further work.

2. Preliminaries on Dawson's Integral

In this section we recall several basic facts about Dawson's integral that will be used throughout the paper. We begin with its elementary differential equation and then fix the normalizations of the Fourier sine and Fourier cosine transforms. These conventions lead to Fourier sine and Fourier cosine representations of Dawson's integral, which provide useful connections between Dawson's integral and classical integral transforms.

Differentiating the definition (1) of Dawson's integral gives the first-order differential equation

$$\operatorname{daw}'(x) + 2x \operatorname{daw}(x) = 1, \tag{6}$$

or, equivalently,

$$\operatorname{daw}'(x) = 1 - 2x \operatorname{daw}(x). \tag{7}$$

Throughout the paper we use the following normalizations for the Fourier sine and Fourier cosine transforms:

$$\mathcal{F}_s\{f(t);u\} = \int_0^\infty f(t)\sin(ut)\,dt, \qquad u > 0, \tag{8}$$

and

$$\mathcal{F}_c\{f(t);u\} = \int_0^\infty f(t)\cos(ut)\,dt, \qquad u > 0, \tag{9}$$

whenever the corresponding integrals exist. With this convention, Dawson's integral admits the Fourier sine representation

$$\operatorname{daw}(u) = \int_0^\infty e^{-t^2}\sin(2ut)\,dt = \mathcal{F}_s\{e^{-t^2};2u\}, \qquad u > 0. \tag{10}$$

This is equivalent, after the change of variables $s = 2t$, to the standard Fourier–Laplace sine representation of Dawson's integral; see [2] (Section 7.7)

The sine representation also gives a useful connection with the Fourier cosine transform. Differentiating (10) with respect to u, we obtain

$$\operatorname{daw}'(u) = 2\int_0^\infty te^{-t^2}\cos(2ut)\,dt = 2\mathcal{F}_c\{te^{-t^2};2u\}. \tag{11}$$

Combining this identity with (7) yields

$$\mathcal{F}_c\{te^{-t^2};2u\} = \int_0^\infty te^{-t^2}\cos(2ut)\,dt = \frac{1}{2} - u\operatorname{daw}(u), \qquad u > 0. \tag{12}$$

Solving for $\mathrm{daw}(u)$ gives the equivalent Fourier cosine representation

$$\mathrm{daw}(u) = \frac{1}{2u} - \frac{1}{u}\mathcal{F}_c\{te^{-t^2}; 2u\}, \qquad u > 0, \tag{13}$$

or, in integral form,

$$\mathrm{daw}(u) = \frac{1}{2u} - \frac{1}{u}\int_0^\infty te^{-t^2}\cos(2ut)\,dt, \qquad u > 0. \tag{14}$$

We shall also need the following elementary Laplace transform in the next section. Recall that the Laplace transform of a function f is defined by

$$\mathscr{L}\{f(x); y\} = \int_0^\infty e^{-xy} f(x)\,dx, \qquad y > 0, \tag{15}$$

whenever the integral exists. We shall use a Laplace transform of a derivative property

$$\mathscr{L}\{f'(x); y\} = y\mathscr{L}\{f(x); y\} - f(0), \tag{16}$$

and differentiation in the transform variable property

$$\mathscr{L}\{xf(x); y\} = -\frac{d}{dy}\mathscr{L}\{f(x); y\}. \tag{17}$$

The next lemma gives the Laplace transform of Dawson's integral after a linear change of variable.

Lemma 1. *Let $a > 0$. For $y > 0$, one has*

$$F(y) = \mathscr{L}\{\mathrm{daw}(ax); y\} = \frac{1}{4a}\exp\left(\frac{y^2}{4a^2}\right)E_1\left(\frac{y^2}{4a^2}\right), \tag{18}$$

where

$$E_1(z) = \int_z^\infty \frac{e^{-t}}{t}\,dt, \qquad z > 0. \tag{19}$$

Proof. Let

$$F(y) = \mathscr{L}\{\mathrm{daw}(ax); y\}.$$

Since Dawson's integral satisfies (7), we have

$$\frac{d}{dx}\mathrm{daw}(ax) = a\,\mathrm{daw}'(ax) = a - 2a^2x\,\mathrm{daw}(ax).$$

Taking the Laplace transform of both sides and using (16), (17), and $\mathrm{daw}(0) = 0$, we obtain

$$yF(y) = \frac{a}{y} - 2a^2\mathscr{L}\{x\,\mathrm{daw}(ax); y\} = \frac{a}{y} + 2a^2F'(y).$$

Hence

$$F'(y) - \frac{y}{2a^2}F(y) = -\frac{1}{2ay}. \tag{20}$$

Multiplying (20) by the integrating factor

$$\exp\left(-\frac{y^2}{4a^2}\right),$$

gives

$$\frac{d}{dy}\left[\exp\left(-\frac{y^2}{4a^2}\right)F(y)\right] = -\frac{1}{2ay}\exp\left(-\frac{y^2}{4a^2}\right).$$

Since $F(y) \to 0$ as $y \to \infty$, integration from y to ∞ yields

$$\exp\left(-\frac{y^2}{4a^2}\right)F(y) = \frac{1}{2a}\int_y^\infty \frac{1}{s}\exp\left(-\frac{s^2}{4a^2}\right)ds.$$

With the substitution $t = s^2/(4a^2)$, this becomes

$$\exp\left(-\frac{y^2}{4a^2}\right)F(y) = \frac{1}{4a}E_1\left(\frac{y^2}{4a^2}\right).$$

Therefore

$$F(y) = \frac{1}{4a}\exp\left(\frac{y^2}{4a^2}\right)E_1\left(\frac{y^2}{4a^2}\right),$$

which proves (18). □

3. Iteration and Connection with the Glasser Transform

Before proving the iteration formula, we recall the Parseval identity for the Fourier sine transform with the normalization adopted in this paper. If $\mathcal{F}_s\{g(t);s\}$ and $\mathcal{F}_s\{h(t);s\}$ denote the Fourier sine transforms of g and h, respectively, then, under the usual hypotheses ensuring convergence, Parseval's formula is

$$\int_0^\infty g(t)h(t)\,dt = \frac{2}{\pi}\int_0^\infty \mathcal{F}_s\{g(t);s\}\mathcal{F}_s\{h(t);s\}\,ds. \tag{21}$$

Thus, an integral involving the product of two functions on $(0,\infty)$ can be expressed as an integral involving the product of their Fourier sine transforms. This identity will be used to evaluate the kernel appearing in the proof of the following iteration theorem.

Theorem 1 (Iteration of the Dawson transform with itself)**.** *Under conditions which justify the interchange of the order of integration, the iteration of the Dawson transform* (3) *with itself is given by*

$$\mathscr{D}\{\mathscr{D}\{f(x);y\};z\} = \frac{\pi^{3/2}}{8}\int_0^\infty \frac{f(x)}{\sqrt{x^2+z^2}}\,dx, \qquad z>0. \tag{22}$$

Equivalently,

$$\mathscr{D}^2\{f(x);z\} = \frac{\pi^{3/2}}{8}\mathscr{G}\{f(x);z\}, \tag{23}$$

where

$$\mathscr{G}\{f(x);z\} = \int_0^\infty \frac{f(x)}{\sqrt{x^2+z^2}}\,dx, \tag{24}$$

is the Glasser transform.

Proof. By definition (3) of the Dawson transform,

$$\begin{aligned}\mathscr{D}\{\mathscr{D}\{f(x);y\};z\} &= \int_0^\infty \operatorname{daw}(yz)\left[\int_0^\infty \operatorname{daw}(xy)f(x)\,dx\right]dy\\ &= \int_0^\infty f(x)\left[\int_0^\infty \operatorname{daw}(xy)\operatorname{daw}(zy)\,dy\right]dx.\end{aligned} \tag{25}$$

It remains to evaluate the inner kernel

$$K(x,z) = \int_0^\infty \operatorname{daw}(xy)\operatorname{daw}(zy)\,dy.$$

Using the Fourier sine representation (10) of Dawson's integral, we have

$$\mathcal{F}_s\{\operatorname{daw}(ay);s\} = \int_0^\infty \operatorname{daw}(ay)\sin(sy)\,dy = \frac{\pi}{4a}\exp\left(-\frac{s^2}{4a^2}\right), \qquad a>0. \tag{26}$$

Therefore, by Parseval's Formula (21) for the Fourier sine transform, applied with

$$g(y) = \operatorname{daw}(xy), \qquad h(y) = \operatorname{daw}(zy),$$

we obtain

$$\begin{aligned} K(x,z) &= \frac{2}{\pi}\int_0^\infty \mathcal{F}_s\{\operatorname{daw}(xy);s\}\mathcal{F}_s\{\operatorname{daw}(zy);s\}\,ds \\ &= \frac{2}{\pi}\int_0^\infty \frac{\pi}{4x}\exp\left(-\frac{s^2}{4x^2}\right)\frac{\pi}{4z}\exp\left(-\frac{s^2}{4z^2}\right)ds \\ &= \frac{\pi}{8xz}\int_0^\infty \exp\left[-\frac{x^2+z^2}{4x^2z^2}s^2\right]ds \\ &= \frac{\pi}{8xz}\cdot\frac{\sqrt{\pi}xz}{\sqrt{x^2+z^2}}. \end{aligned} \tag{27}$$

Hence

$$K(x,z) = \frac{\pi^{3/2}}{8\sqrt{x^2+z^2}}. \tag{28}$$

Substituting (28) into (25) gives (22). □

Theorem 2 (Third iteration of the Dawson transform)**.** *Let $f \in \mathcal{X}$ and assume, for the fixed $y>0$, that*

$$\int_0^\infty\int_0^\infty \frac{|f(x)|\,|\operatorname{daw}(yz)|}{\sqrt{x^2+z^2}}\,dz\,dx < \infty.$$

Then Tonelli–Fubini applies, and the third iteration of the Dawson transform is

$$\mathscr{D}^3\{f(x);y\} = \frac{\pi^{3/2}}{8}\int_0^\infty f(x)\left[\int_0^\infty \frac{\operatorname{daw}(yz)}{\sqrt{x^2+z^2}}\,dz\right]dx. \tag{29}$$

Equivalently,

$$\mathscr{D}^3\{f(x);y\} = \frac{\pi^{3/2}}{8}\mathscr{D}\{\mathscr{G}\{f(x);z\};y\}. \tag{30}$$

Proof. Applying the Dawson transform once more to the second-iteration Formula (22) gives

$$\begin{aligned} \mathscr{D}^3\{f(x);y\} &= \int_0^\infty \operatorname{daw}(yz)\mathscr{D}^2\{f(x);z\}\,dz \\ &= \frac{\pi^{3/2}}{8}\int_0^\infty \operatorname{daw}(yz)\left[\int_0^\infty \frac{f(x)}{\sqrt{x^2+z^2}}\,dx\right]dz. \end{aligned} \tag{31}$$

Interchanging the order of integration, we obtain

$$\mathscr{D}^3\{f(x);y\} = \frac{\pi^{3/2}}{8}\int_0^\infty f(x)\left[\int_0^\infty \frac{\operatorname{daw}(yz)}{\sqrt{x^2+z^2}}\,dz\right]dx.$$

This proves (29). □

4. Mellin Representation and Inversion

Let $\mathcal{M}\{h;s\} = \int_0^\infty x^{s-1}h(x)\,dx$. The asymptotics of Dawson's integral at zero and infinity show that its Mellin transform converges absolutely in the fundamental strip $-1 < \Re(s) < 1$. In this strip,

$$K(s) := \mathcal{M}\{\mathrm{daw};s\} = 2^{-s-1}\Gamma(s)\Gamma\left(\frac{1-s}{2}\right)\sin\left(\frac{\pi s}{2}\right), \tag{32}$$

where the removable singularity at $s = 0$ is understood by continuity. The formula follows first for $0 < \Re(s) < 1$ from the Fourier-sine representation and then extends to the full fundamental strip by analytic continuation, consistently with absolute convergence.

Theorem 3 (Mellin representation)**.** *Suppose that f has a Mellin transform in a strip containing $1-s$, that $-1 < \Re(s) < 1$, and that Fubini's theorem is applicable. Then*

$$\mathcal{M}_y\{\mathscr{D}f;s\} = K(s)\,\mathcal{M}\{f;1-s\}. \tag{33}$$

Proof. Substitution $u = xy$ gives

$$\int_0^\infty y^{s-1}\,\mathrm{daw}(xy)\,dy = x^{-s}K(s).$$

Multiplying by $f(x)$, integrating in x, and interchanging the integrals yields (33). □

Theorem 4 (Mellin inversion of the Dawson transform)**.** *Let $F(y) = \mathscr{D}\{f(x);y\}$. Choose c with $0 < c < 2$ such that $\mathcal{M}\{f;r\}$ is analytic on a strip containing $\Re(r) = c$, and assume*

$$\int_{-\infty}^{\infty}\left|\frac{\mathcal{M}\{F;1-c-it\}}{K(1-c-it)}\right|dt < \infty.$$

Then, at every continuity point of f,

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\mathcal{M}\{F;1-r\}}{K(1-r)}x^{-r}\,dr. \tag{34}$$

The contour restriction $0 < c < 2$ is exactly the condition $-1 < \Re(1-r) < 1$ required by the kernel multiplier.

Proof. Equation (33) with $s = 1-r$ gives $\mathcal{M}\{f;r\} = \mathcal{M}\{F;1-r\}/K(1-r)$. The stated vertical-line integrability permits application of the classical Mellin inversion theorem. □

5. Derivative and Delta-Derivative Properties

In this section, we develop several operational properties of the Dawson transform. The main purpose is to establish a calculus for the transform which parallels, in a suitable form, the familiar operational rules for classical integral transforms. In particular, we obtain differentiation formulas with respect to the transform parameter and identities involving Euler-type differential operators. These formulas show how multiplication by powers of the original variable and differentiation with respect to the transform variable are related under the Dawson transform.

The operators introduced earlier provide compact forms of these identities and make the structure of the Dawson transform more transparent. Such operational relations are useful not only for deriving new transform pairs, but also for computing Dawson transforms of derivatives and functions involving powers of the independent variable. The

results of this section therefore provide the foundation for the examples and applications considered later.

Theorem 5 (Derivative property with respect to the operator δ_y)**.** *Assume that the integrals below exist and that differentiation under the integral sign is justified. Then*

$$\delta_y \mathscr{D}\{f(x);y\} = \frac{1}{y}\int_0^\infty xf(x)\,dx - 2\mathscr{D}\{x^2 f(x);y\}, \qquad y \neq 0. \tag{35}$$

Proof. Applying the operator $\delta_y = (1/y)d/dy$ and differentiating under the integral sign, we obtain

$$\begin{aligned}\delta_y \mathscr{D}\{f(x);y\} &= \frac{1}{y}\frac{d}{dy}\int_0^\infty \operatorname{daw}(xy)f(x)\,dx \\ &= \frac{1}{y}\int_0^\infty x\operatorname{daw}'(xy)f(x)\,dx.\end{aligned} \tag{36}$$

Using the differential identity (7) with $u = xy$ gives

$$\operatorname{daw}'(xy) = 1 - 2xy\operatorname{daw}(xy).$$

Therefore,

$$\begin{aligned}\delta_y \mathscr{D}\{f(x);y\} &= \frac{1}{y}\int_0^\infty x[1-2xy\operatorname{daw}(xy)]f(x)\,dx \\ &= \frac{1}{y}\int_0^\infty xf(x)\,dx - 2\int_0^\infty x^2\operatorname{daw}(xy)f(x)\,dx.\end{aligned} \tag{37}$$

By the definition of the Dawson transform (3),

$$\int_0^\infty x^2\operatorname{daw}(xy)f(x)\,dx = \mathscr{D}\{x^2 f(x);y\}.$$

Hence

$$\delta_y \mathscr{D}\{f(x);y\} = \frac{1}{y}\int_0^\infty xf(x)\,dx - 2\mathscr{D}\{x^2 f(x);y\},$$

which proves (35). □

Theorem 6 (Dawson transform of a derivative)**.** *Suppose that f is differentiable on $(0,\infty)$, that the relevant integrals exist, and that*

$$\lim_{x\to 0^+}\operatorname{daw}(xy)f(x) = 0, \qquad \lim_{x\to\infty}\operatorname{daw}(xy)f(x) = 0. \tag{38}$$

Then

$$\mathscr{D}\{f'(x);y\} = -y\int_0^\infty f(x)\,dx + 2y^2\mathscr{D}\{xf(x);y\}. \tag{39}$$

Equivalently,

$$\mathscr{D}\{f'(x);y\} = -\sigma_y\mathscr{D}\left\{\frac{f(x)}{x};y\right\}. \tag{40}$$

Proof. By integration by parts,

$$\begin{aligned}\mathscr{D}\{f'(x);y\} &= \int_0^\infty \operatorname{daw}(xy)f'(x)\,dx \\ &= [\operatorname{daw}(xy)f(x)]_0^\infty - \int_0^\infty \frac{\partial}{\partial x}\operatorname{daw}(xy)f(x)\,dx.\end{aligned} \tag{41}$$

By the hypotheses, the boundary term vanishes. Moreover, using (7), we obtain

$$\frac{\partial}{\partial x}\operatorname{daw}(xy) = y\operatorname{daw}'(xy) = y - 2xy^2\operatorname{daw}(xy).$$

Therefore,

$$\begin{aligned}\mathscr{D}\{f'(x);y\} &= -\int_0^\infty \left[y - 2xy^2\operatorname{daw}(xy)\right] f(x)\,dx \\ &= -y\int_0^\infty f(x)\,dx + 2y^2\int_0^\infty \operatorname{daw}(xy)xf(x)\,dx \\ &= -y\int_0^\infty f(x)\,dx + 2y^2\mathscr{D}\{xf(x);y\}. \end{aligned} \tag{42}$$

This proves (39).

Next,

$$\mathscr{D}\left\{\frac{f(x)}{x};y\right\} = \int_0^\infty \operatorname{daw}(xy)\frac{f(x)}{x}\,dx.$$

Differentiating with respect to y, we obtain

$$\begin{aligned}\frac{d}{dy}\mathscr{D}\left\{\frac{f(x)}{x};y\right\} &= \int_0^\infty \operatorname{daw}'(xy)f(x)\,dx \\ &= \int_0^\infty [1 - 2xy\operatorname{daw}(xy)]f(x)\,dx \\ &= \int_0^\infty f(x)\,dx - 2y\mathscr{D}\{xf(x);y\}. \end{aligned} \tag{43}$$

Multiplying both sides by $-y$ gives

$$-y\frac{d}{dy}\mathscr{D}\left\{\frac{f(x)}{x};y\right\} = -y\int_0^\infty f(x)\,dx + 2y^2\mathscr{D}\{xf(x);y\}.$$

By (39), this yields

$$\mathscr{D}\{f'(x);y\} = -y\frac{d}{dy}\mathscr{D}\left\{\frac{f(x)}{x};y\right\}.$$

Finally, (40) follows from the definition of the Euler-type operator σ_y in (5). □

Theorem 7 (Dawson transform of the nth derivative)**.** *Suppose that f is n times differentiable on $(0,\infty)$, that the relevant integrals exist, and that the boundary terms required in the successive integrations by parts vanish. Then*

$$\mathscr{D}\{f^{(n)}(x);y\} = (-1)^n\sigma_y(\sigma_y-1)(\sigma_y-2)\cdots(\sigma_y-n+1)\mathscr{D}\left\{\frac{f(x)}{x^n};y\right\}. \tag{44}$$

Equivalently,

$$\mathscr{D}\{f^{(n)}(x);y\} = (-1)^n(\sigma_y)_n\mathscr{D}\left\{\frac{f(x)}{x^n};y\right\}, \tag{45}$$

where

$$(\sigma_y)_n = \sigma_y(\sigma_y-1)(\sigma_y-2)\cdots(\sigma_y-n+1). \tag{46}$$

Proof. The result follows by induction from the first-derivative Formula (40). For $n = 1$, (45) reduces exactly to (40). Assume that the formula holds for some positive

integer n. Thus, Equation (44) is true. We prove the formula for $n+1$. Applying the first-derivative property to $f^{(n)}(x)$, we obtain

$$\mathscr{D}\{f^{(n+1)}(x);y\} = -\sigma_y \mathscr{D}\left\{\frac{f^{(n)}(x)}{x};y\right\}. \tag{47}$$

Repeatedly rewriting derivatives of quotients gives the identity

$$\mathscr{D}\left\{\frac{f^{(n)}(x)}{x};y\right\} = (-1)^n(\sigma_y-1)(\sigma_y-2)\cdots(\sigma_y-n)\mathscr{D}\left\{\frac{f(x)}{x^{n+1}};y\right\}. \tag{48}$$

Substituting (48) into (47) yields

$$\mathscr{D}\{f^{(n+1)}(x);y\} = (-1)^{n+1}\sigma_y(\sigma_y-1)(\sigma_y-2)\cdots(\sigma_y-n)\mathscr{D}\left\{\frac{f(x)}{x^{n+1}};y\right\}.$$

This is precisely (44) with n replaced by $n+1$. Therefore, by induction, the formula holds for every positive integer n. □

Theorem 8 (General delta-derivative identity for Dawson's integral)**.** *For every nonnegative integer n,*

$$\left(\delta_x + 2a^2\right)\delta_x^n \operatorname{daw}(ax) = a\,\delta_x^n(x^{-1}). \tag{49}$$

Equivalently,

$$\left(\delta_x + 2a^2\right)\delta_x^n \operatorname{daw}(ax) = (-1)^n a(2n-1)!!\,x^{-(2n+1)}. \tag{50}$$

Here $(-1)!! = 1$, so that the formula also holds for $n = 0$.

Proof. Dawson's integral satisfies the differential Equation (6). Taking $u = ax$, we obtain

$$\delta_x \operatorname{daw}(ax) = \frac{a}{x} - 2a^2 \operatorname{daw}(ax).$$

Therefore,

$$\left(\delta_x + 2a^2\right)\operatorname{daw}(ax) = \frac{a}{x}. \tag{51}$$

Applying δ_x^n to both sides of (51), and using the fact that δ_x commutes with the constant-coefficient operator $\delta_x + 2a^2$, gives

$$\left(\delta_x + 2a^2\right)\delta_x^n \operatorname{daw}(ax) = a\,\delta_x^n(x^{-1}).$$

This proves the compact form (49).

It remains only to compute $\delta_x^n(x^{-1})$. Since

$$\delta_x x^m = \frac{1}{x}\frac{d}{dx}x^m = mx^{m-2},$$

we have

$$\delta_x^n x^m = m(m-2)(m-4)\cdots(m-2n+2)x^{m-2n}.$$

Taking $m = -1$, this gives

$$\delta_x^n x^{-1} = (-1)(-3)(-5)\cdots(-(2n-1))x^{-(2n+1)}.$$

Thus

$$\delta_x^n x^{-1} = (-1)^n(2n-1)!!\,x^{-(2n+1)}.$$

Substituting this into (49), we obtain

$$\left(\delta_x + 2a^2\right)\delta_x^n \operatorname{daw}(ax) = (-1)^n a(2n-1)!!\, x^{-(2n+1)}.$$

This proves (50). □

Theorem 9 (Dawson transform of the delta derivative)**.** *Suppose that f is differentiable on $(0,\infty)$, that the relevant integrals exist, and that the boundary term*

$$\left[\frac{\operatorname{daw}(xy)}{x} f(x)\right]_0^\infty,$$

vanishes. Then

$$\mathscr{D}\{\delta_x f(x); y\} = -y\int_0^\infty \frac{f(x)}{x}\,dx + 2y^2\mathscr{D}\{f(x); y\} + \mathscr{D}\left\{\frac{f(x)}{x^2}; y\right\}. \tag{52}$$

Proof. Using the definition of the Dawson transform (3), we have

$$\mathscr{D}\{\delta_x f(x); y\} = \int_0^\infty \frac{\operatorname{daw}(xy)}{x} f'(x)\,dx.$$

Integrating by parts gives

$$\mathscr{D}\{\delta_x f(x); y\} = \left[\frac{\operatorname{daw}(xy)}{x} f(x)\right]_0^\infty - \int_0^\infty f(x)\frac{d}{dx}\left[\frac{\operatorname{daw}(xy)}{x}\right]dx.$$

By assumption, the boundary term vanishes. Using (7), we obtain

$$\frac{d}{dx}\left[\frac{\operatorname{daw}(xy)}{x}\right] = \frac{y}{x} - 2y^2\operatorname{daw}(xy) - \frac{\operatorname{daw}(xy)}{x^2}.$$

Therefore,

$$\begin{aligned}\mathscr{D}\{\delta_x f(x); y\} &= -\int_0^\infty f(x)\left[\frac{y}{x} - 2y^2\operatorname{daw}(xy) - \frac{\operatorname{daw}(xy)}{x^2}\right]dx \\ &= -y\int_0^\infty \frac{f(x)}{x}\,dx + 2y^2\int_0^\infty \operatorname{daw}(xy) f(x)\,dx + \int_0^\infty \operatorname{daw}(xy)\frac{f(x)}{x^2}\,dx.\end{aligned}$$

Hence, the proof is complete. □

6. Transform Pairs and Examples

The derivative and delta-derivative identities developed in the preceding section provide an efficient operational method for evaluating Dawson transforms. We now illustrate how these identities may be used systematically. Once a basic transform pair is known, additional transform pairs can often be generated by applying the differential operators introduced above, rather than by evaluating each defining integral directly. The following examples demonstrate this procedure for trigonometric, exponential, Gaussian, and related functions. Many further transform pairs can be obtained in the same way by combining the operational identities with known seed transforms.

Example 1. *Use the derivative property of the Dawson transform to show that, for $a > 0$ and $y > 0$,*

$$\mathscr{D}\{\sin(ax); y\} = \frac{\pi}{4y}\exp\left(-\frac{a^2}{4y^2}\right), \tag{53}$$

and

$$\mathscr{D}\{\cos(ax);y\} = -\frac{1}{4y}\exp\left(-\frac{a^2}{4y^2}\right)\mathrm{Ei}\left(\frac{a^2}{4y^2}\right). \tag{54}$$

Solution. Let

$$S(a,y) = \mathscr{D}\{\sin(ax);y\}, \qquad C(a,y) = \mathscr{D}\{\cos(ax);y\}.$$

Applying (39) to $f(x) = \cos(ax)$, and using the Abel value

$$\int_0^\infty \cos(ax)\,dx = 0,$$

we obtain

$$-aS(a,y) = 2y^2\frac{\partial S}{\partial a}(a,y).$$

Thus

$$\frac{\partial S}{\partial a}(a,y) = -\frac{a}{2y^2}S(a,y),$$

and hence

$$S(a,y) = A(y)\exp\left(-\frac{a^2}{4y^2}\right).$$

Using

$$\lim_{a\to 0^+} S(a,y) = \frac{1}{2y}\int_0^\infty \frac{\sin(ax)}{x}\,dx = \frac{\pi}{4y},$$

we find $A(y) = \pi/(4y)$. Therefore (53) follows.

Next, applying (39) to $f(x) = \sin(ax)$, and using

$$\int_0^\infty \sin(ax)\,dx = \frac{1}{a},$$

gives

$$aC(a,y) = -\frac{y}{a} + 2y^2\mathscr{D}\{x\sin(ax);y\}.$$

Since

$$\mathscr{D}\{x\sin(ax);y\} = -\frac{\partial C}{\partial a}(a,y),$$

we get

$$\frac{\partial C}{\partial a}(a,y) + \frac{a}{2y^2}C(a,y) = -\frac{1}{2ay}.$$

Multiplying by the integrating factor

$$\exp\left(\frac{a^2}{4y^2}\right),$$

yields

$$\frac{\partial}{\partial a}\left[e^{a^2/(4y^2)}C(a,y)\right] = -\frac{1}{2ay}e^{a^2/(4y^2)}.$$

Hence

$$e^{a^2/(4y^2)}C(a,y) = -\frac{1}{4y}\mathrm{Ei}\left(\frac{a^2}{4y^2}\right),$$

where the integration constant is fixed by the Abel interpretation of the Fourier-cosine transform. Therefore (54) follows. □

The next examples show how the same operational method applies to exponentially decaying functions. In this case the known transform of e^{-ax} serves as the seed transform, and higher powers of x are generated either by σ_y-operations or by differentiation with respect to the parameter a.

Example 2. *Use the derivative property of the Dawson transform to show that*

$$\mathscr{D}\{xe^{-ax};y\} = \frac{1}{2ay}\left[1 - \frac{a^2}{4y^2}\exp\left(\frac{a^2}{4y^2}\right)E_1\left(\frac{a^2}{4y^2}\right)\right], \tag{55}$$

where $a > 0$ and $y > 0$.

Solution. Let

$$f(x) = xe^{-ax}, \qquad a > 0.$$

Then

$$\frac{f(x)}{x} = e^{-ax}, \qquad f'(x) = e^{-ax} - axe^{-ax}.$$

By the derivative property of the Dawson transform,

$$\mathscr{D}\{f'(x);y\} = -\sigma_y \mathscr{D}\left\{\frac{f(x)}{x};y\right\}.$$

Hence

$$\mathscr{D}\{e^{-ax};y\} - a\mathscr{D}\{xe^{-ax};y\} = -\sigma_y \mathscr{D}\{e^{-ax};y\},$$

and therefore

$$\mathscr{D}\{xe^{-ax};y\} = \frac{1}{a}\left[\mathscr{D}\{e^{-ax};y\} + \sigma_y \mathscr{D}\{e^{-ax};y\}\right]. \tag{56}$$

Using the known transform pair

$$\mathscr{D}\{e^{-ax};y\} = \frac{1}{4y}\exp\left(\frac{a^2}{4y^2}\right)E_1\left(\frac{a^2}{4y^2}\right),$$

put

$$q = \frac{a^2}{4y^2}, \qquad H(q) = e^q E_1(q).$$

Then

$$\mathscr{D}\{e^{-ax};y\} = \frac{1}{4y}H(q), \qquad \sigma_y = y\frac{d}{dy}.$$

Since $dq/dy = -2q/y$, we have

$$\sigma_y\left[\frac{1}{4y}H(q)\right] = -\frac{1}{4y}H(q) - \frac{q}{2y}H'(q).$$

Thus

$$\mathscr{D}\{e^{-ax};y\} + \sigma_y \mathscr{D}\{e^{-ax};y\} = -\frac{q}{2y}H'(q).$$

Using

$$H'(q) = \frac{d}{dq}(e^q E_1(q)) = e^q E_1(q) - \frac{1}{q},$$

we obtain

$$\mathscr{D}\{e^{-ax};y\} + \sigma_y \mathscr{D}\{e^{-ax};y\} = \frac{1}{2y}[1 - qe^q E_1(q)].$$

Substitution into (56) gives

$$\mathscr{D}\{xe^{-ax};y\} = \frac{1}{2ay}[1 - qe^q E_1(q)], \qquad q = \frac{a^2}{4y^2}.$$

Therefore (55) follows. □

Example 3. *Let $a > 0$ and $y > 0$, and define*

$$A_n(y) = \mathscr{D}\{x^n e^{-ax};y\}, \qquad n = 0,1,2,\ldots.$$

Show that

$$A_0(y) = \mathscr{D}\{e^{-ax};y\} = \frac{1}{4y}\exp\left(\frac{a^2}{4y^2}\right)E_1\left(\frac{a^2}{4y^2}\right). \tag{57}$$

Then, for $n = 1,2,3,\ldots$, the transforms $A_n(y)$ satisfy the recurrence

$$A_{n+1}(y) = \frac{1}{2y^2}\left[nA_{n-1}(y) - aA_n(y) + \frac{y\,n!}{a^{n+1}}\right]. \tag{58}$$

Moreover, for each nonnegative integer n,

$$\mathscr{D}\{x^n e^{-ax};y\} = (-1)^n \frac{\partial^n}{\partial a^n}\left[\frac{1}{4y}\exp\left(\frac{a^2}{4y^2}\right)E_1\left(\frac{a^2}{4y^2}\right)\right]. \tag{59}$$

Equivalently, if $s = a/(2y)$, then

$$\mathscr{D}\{x^n e^{-ax};y\} = \frac{(-1)^n}{2^{n+2}y^{n+1}}\frac{d^n}{ds^n}\left[e^{s^2}E_1(s^2)\right]. \tag{60}$$

In particular,

$$\mathscr{D}\{xe^{-ax};y\} = \frac{1}{2ay}\left[1 - \frac{a^2}{4y^2}\exp\left(\frac{a^2}{4y^2}\right)E_1\left(\frac{a^2}{4y^2}\right)\right]. \tag{61}$$

Solution. First, using the known Laplace integral for Dawson's function,

$$\int_0^\infty e^{-\lambda t}\,\mathrm{daw}(t)\,dt = \frac{1}{4}\exp\left(\frac{\lambda^2}{4}\right)E_1\left(\frac{\lambda^2}{4}\right), \qquad \lambda > 0,$$

and the change of variables $t = xy$, we obtain

$$A_0(y) = \int_0^\infty \mathrm{daw}(xy)e^{-ax}\,dx = \frac{1}{y}\int_0^\infty \mathrm{daw}(t)e^{-(a/y)t}\,dt.$$

Therefore

$$A_0(y) = \frac{1}{4y}\exp\left(\frac{a^2}{4y^2}\right)E_1\left(\frac{a^2}{4y^2}\right).$$

For $n \geq 1$, let

$$f(x) = x^n e^{-ax}.$$

Then

$$f'(x) = nx^{n-1}e^{-ax} - ax^n e^{-ax}.$$

The boundary terms vanish, since

$$\mathrm{daw}(xy)x^n e^{-ax} \to 0 \qquad (x \to 0^+,\ x \to \infty).$$

Hence, by the derivative theorem for the Dawson transform,

$$\mathscr{D}\{f'(x);y\} = -y\int_0^\infty f(x)\,dx + 2y^2\mathscr{D}\{xf(x);y\}.$$

Thus

$$nA_{n-1}(y) - aA_n(y) = -y\int_0^\infty x^n e^{-ax}\,dx + 2y^2A_{n+1}(y).$$

Since

$$\int_0^\infty x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}},$$

we get

$$A_{n+1}(y) = \frac{1}{2y^2}\left[nA_{n-1}(y) - aA_n(y) + \frac{y\,n!}{a^{n+1}}\right].$$

Moreover,

$$\frac{\partial^n}{\partial a^n}e^{-ax} = (-x)^n e^{-ax}.$$

Differentiating under the integral sign gives

$$\frac{\partial^n}{\partial a^n}A_0(y) = (-1)^n A_n(y),$$

and hence

$$A_n(y) = (-1)^n\frac{\partial^n}{\partial a^n}\left[\frac{1}{4y}\exp\left(\frac{a^2}{4y^2}\right)E_1\left(\frac{a^2}{4y^2}\right)\right].$$

If $s = a/(2y)$, then

$$\frac{\partial}{\partial a} = \frac{1}{2y}\frac{d}{ds}, \qquad A_0(y) = \frac{1}{4y}e^{s^2}E_1(s^2),$$

so that

$$A_n(y) = \frac{(-1)^n}{2^{n+2}y^{n+1}}\frac{d^n}{ds^n}\left[e^{s^2}E_1(s^2)\right].$$

Finally, for $n = 1$, using

$$\frac{d}{dz}(e^z E_1(z)) = e^z E_1(z) - \frac{1}{z}, \qquad z = \frac{a^2}{4y^2},$$

we obtain

$$A_1(y) = -\frac{\partial}{\partial a}\left[\frac{1}{4y}e^z E_1(z)\right] = \frac{1}{2ay}\left[1 - \frac{a^2}{4y^2}\exp\left(\frac{a^2}{4y^2}\right)E_1\left(\frac{a^2}{4y^2}\right)\right].$$

This gives (61) and completes the proof. □

Example 4. *Show that, for* $\alpha > 0$, $y > 0$, *and* $\Re(\mu) > 0$,

$$\mathscr{D}\{x^{\mu-1}e^{-\alpha x^2};y\} = \frac{y}{2}\Gamma\left(\frac{\mu+1}{2}\right)(\alpha+y^2)^{-(\mu+1)/2}{}_2F_1\left(\frac{1}{2},\frac{\mu+1}{2};\frac{3}{2};\frac{y^2}{\alpha+y^2}\right). \tag{62}$$

Solution. From the definition of Dawson's integral (1), and the change of variable $s = ut$, we obtain

$$\mathrm{daw}(u) = u\int_0^1 e^{-u^2(1-t^2)}\,dt.$$

Hence, with $u = xy$,

$$\mathrm{daw}(xy) = xy\int_0^1 e^{-x^2y^2(1-t^2)}\,dt.$$

Therefore,

$$\mathscr{D}\{x^{\mu-1}e^{-\alpha x^2};y\} = \int_0^\infty \operatorname{daw}(xy)x^{\mu-1}e^{-\alpha x^2}\,dx$$
$$= y\int_0^1\left[\int_0^\infty x^\mu e^{-[\alpha+y^2(1-t^2)]x^2}\,dx\right]dt.$$

By the standard gamma integral representation [2] (Equation 5.9.1),

$$\int_0^\infty x^\mu e^{-\beta x^2}\,dx = \frac{1}{2}\beta^{-(\mu+1)/2}\Gamma\left(\frac{\mu+1}{2}\right), \qquad \Re(\mu) > -1, \quad \beta > 0.$$

Since

$$\beta = \alpha + y^2(1-t^2) > 0,$$

we obtain

$$\mathscr{D}\{x^{\mu-1}e^{-\alpha x^2};y\} = \frac{y}{2}\Gamma\left(\frac{\mu+1}{2}\right)\int_0^1\left[\alpha+y^2(1-t^2)\right]^{-(\mu+1)/2}dt.$$

Now write

$$\alpha + y^2(1-t^2) = (\alpha+y^2)\left(1-\frac{y^2}{\alpha+y^2}t^2\right).$$

Then

$$\mathscr{D}\{x^{\mu-1}e^{-\alpha x^2};y\} = \frac{y}{2}\Gamma\left(\frac{\mu+1}{2}\right)(\alpha+y^2)^{-(\mu+1)/2}$$
$$\times\int_0^1\left(1-\frac{y^2}{\alpha+y^2}t^2\right)^{-(\mu+1)/2}dt.$$

Using the Euler integral representation of the Gauss hypergeometric function [2] (Equation 15.6.1), equivalently

$$\int_0^1(1-\lambda t^2)^{-\rho}\,dt = {}_2F_1\left(\frac{1}{2},\rho;\frac{3}{2};\lambda\right), \qquad |\lambda| < 1,$$

and taking

$$\rho = \frac{\mu+1}{2}, \qquad \lambda = \frac{y^2}{\alpha+y^2},$$

we obtain (62). Since $\alpha > 0$ and $y > 0$, we have $0 < \lambda < 1$, so the hypergeometric representation is valid. This completes the proof. □

Remark 1. *Taking* $\mu = 1$ *in* (62) *gives*

$$\mathscr{D}\{e^{-\alpha x^2};y\} = \frac{y}{2(\alpha+y^2)}{}_2F_1\left(\frac{1}{2},1;\frac{3}{2};\frac{y^2}{\alpha+y^2}\right).$$

Using the elementary identity

$${}_2F_1\left(\frac{1}{2},1;\frac{3}{2};z\right) = \frac{\operatorname{arctanh}\sqrt{z}}{\sqrt{z}}, \qquad 0 < z < 1,$$

with

$$z = \frac{y^2}{\alpha+y^2},$$

we obtain

$$\mathscr{D}\{e^{-\alpha x^2};y\} = \frac{1}{2\sqrt{\alpha+y^2}}\operatorname{arctanh}\left(\frac{y}{\sqrt{\alpha+y^2}}\right).$$

Equivalently,

$$\mathscr{D}\{e^{-\alpha x^2};y\} = \frac{1}{4\sqrt{\alpha+y^2}}\log\left(\frac{\sqrt{\alpha+y^2}+y}{\sqrt{\alpha+y^2}-y}\right), \tag{63}$$

for $\alpha > 0$ *and* $y > 0$.

Before stating the next transform pairs, we recall the normalization of the Hermite polynomials used below. The physicists' Hermite polynomials are defined by Rodrigues' formula

$$H_m(z) = (-1)^m e^{z^2}\frac{d^m}{dz^m}e^{-z^2}, \qquad m = 0,1,2,\ldots. \tag{64}$$

Equivalently,

$$\frac{d^m}{dz^m}e^{-z^2} = (-1)^m H_m(z)e^{-z^2}. \tag{65}$$

In the formulas below, "Abel-regularized" means that an oscillatory integral that is not absolutely convergent is interpreted as the limit

$$\lim_{\varepsilon\to 0^+}\int_0^\infty e^{-\varepsilon x}\,\mathrm{daw}(xy)f(x)\,dx,$$

provided that this limit exists. Thus, the exponential factor $e^{-\varepsilon x}$ is first introduced to ensure convergence, the transform is evaluated for $\varepsilon > 0$, and the limit $\varepsilon \to 0^+$ is then taken.

Example 5. *In the operational or Abel-regularized sense,*

$$\mathscr{D}\{x^{2n}\sin(ax);y\} = \frac{\pi(-1)^n}{2^{2n+2}y^{2n+1}}H_{2n}\left(\frac{a}{2y}\right)\exp\left(-\frac{a^2}{4y^2}\right), \tag{66}$$

where $a > 0$, $y > 0$, *and* $n = 0,1,2,\ldots$.

Furthermore,

$$\mathscr{D}\{x^{2n+1}\sin(ax);y\} = \frac{(-1)^n}{2^{2n+3}y^{2n+2}} \times\left[-H_{2n+1}\left(\frac{a}{2y}\right)\exp\left(-\frac{a^2}{4y^2}\right)\mathrm{Ei}\left(\frac{a^2}{4y^2}\right) + R_{2n+1}\left(\frac{a}{2y}\right)\right], \tag{67}$$

where the rational functions $R_m(z)$ *are defined recursively by*

$$R_0(z) = 0, \tag{68}$$

and

$$R_{m+1}(z) = \frac{d}{dz}R_m(z) + \frac{2(-1)^m H_m(z)}{z}, \qquad m = 0,1,2,\ldots. \tag{69}$$

Proof. We use the basic Dawson transform (53). Since

$$\frac{\partial^{2n}}{\partial a^{2n}}\sin(ax) = (-1)^n x^{2n}\sin(ax),$$

we obtain

$$\mathscr{D}\{x^{2n}\sin(ax);y\} = (-1)^n\frac{\partial^{2n}}{\partial a^{2n}}\mathscr{D}\{\sin(ax);y\}.$$

Using

$$\frac{d^m}{da^m}\exp\left(-\frac{a^2}{4y^2}\right) = \frac{(-1)^m}{(2y)^m}H_m\left(\frac{a}{2y}\right)\exp\left(-\frac{a^2}{4y^2}\right),$$

with $m = 2n$, gives

$$\mathscr{D}\{x^{2n}\sin(ax); y\} = \frac{\pi(-1)^n}{2^{2n+2}y^{2n+1}} H_{2n}\left(\frac{a}{2y}\right)\exp\left(-\frac{a^2}{4y^2}\right).$$

This proves (66).

For the odd-power case, we start from (54). Since

$$\frac{\partial^{2n+1}}{\partial a^{2n+1}}\cos(ax) = (-1)^{n+1}x^{2n+1}\sin(ax),$$

we have

$$\mathscr{D}\{x^{2n+1}\sin(ax); y\} = (-1)^{n+1}\frac{\partial^{2n+1}}{\partial a^{2n+1}}\mathscr{D}\{\cos(ax); y\}.$$

Substituting (54) gives

$$\mathscr{D}\{x^{2n+1}\sin(ax); y\} = \frac{(-1)^n}{4y}\frac{\partial^{2n+1}}{\partial a^{2n+1}}\left[\exp\left(-\frac{a^2}{4y^2}\right)\mathrm{Ei}\left(\frac{a^2}{4y^2}\right)\right].$$

Put

$$z = \frac{a}{2y}.$$

Then

$$\frac{\partial}{\partial a} = \frac{1}{2y}\frac{d}{dz},$$

and therefore

$$\frac{\partial^{2n+1}}{\partial a^{2n+1}} = \frac{1}{(2y)^{2n+1}}\frac{d^{2n+1}}{dz^{2n+1}}.$$

Hence

$$\mathscr{D}\{x^{2n+1}\sin(ax); y\} = \frac{(-1)^n}{2^{2n+3}y^{2n+2}}\frac{d^{2n+1}}{dz^{2n+1}}\left[e^{-z^2}\,\mathrm{Ei}(z^2)\right].$$

The repeated derivatives of $e^{-z^2}\,\mathrm{Ei}(z^2)$ have the form

$$\frac{d^m}{dz^m}\left[e^{-z^2}\,\mathrm{Ei}(z^2)\right] = -H_m(z)e^{-z^2}\,\mathrm{Ei}(z^2) + R_m(z),$$

where

$$R_0(z) = 0,$$

and

$$R_{m+1}(z) = \frac{d}{dz}R_m(z) + \frac{2(-1)^m H_m(z)}{z}.$$

Taking $m = 2n+1$, we obtain

$$\mathscr{D}\{x^{2n+1}\sin(ax); y\} = \frac{(-1)^n}{2^{2n+3}y^{2n+2}}\left[-H_{2n+1}(z)e^{-z^2}\,\mathrm{Ei}(z^2) + R_{2n+1}(z)\right].$$

Replacing z by $a/(2y)$ gives (67). The proof is complete. □

Remark 2. *For the first few values of m, the rational functions $R_m(z)$ are*

$$R_1(z) = \frac{2}{z},$$
$$R_3(z) = 8z - \frac{4}{z} + \frac{4}{z^3},$$
$$R_5(z) = 32z^3 - 128z + \frac{24}{z} - \frac{8}{z^3} + \frac{48}{z^5}.$$

These give, in particular,

$$\mathscr{D}\{x\sin(ax); y\} = \frac{1}{ay} - \frac{a}{4y^3}\exp\left(-\frac{a^2}{4y^2}\right)\mathrm{Ei}\left(\frac{a^2}{4y^2}\right),$$

and

$$\mathscr{D}\{x^3\sin(ax); y\} = \frac{a(a^2-6y^2)}{32y^7}\exp\left(-\frac{a^2}{4y^2}\right)\mathrm{Ei}\left(\frac{a^2}{4y^2}\right) - \frac{a^4-2a^2y^2+8y^4}{8a^3y^5}.$$

Thus the general formulas obtained above yield explicit closed forms for each odd power $x^{2n+1}\sin(ax)$ by substituting the corresponding rational function R_{2n+1}.

Example 6. *Analogously, and with the same Abel-regularized interpretation, one obtains the corresponding cosine formulas. For $a > 0$, $y > 0$, and $n = 0, 1, 2, \ldots$,*

$$\mathscr{D}\{x^{2n}\cos(ax); y\} = \frac{(-1)^{n+1}}{2^{2n+2}y^{2n+1}} \times \left[H_{2n}\left(\frac{a}{2y}\right)\exp\left(-\frac{a^2}{4y^2}\right)\mathrm{Ei}\left(\frac{a^2}{4y^2}\right) + R_{2n}\left(\frac{a}{2y}\right)\right], \tag{70}$$

Moreover,

$$\mathscr{D}\{x^{2n+1}\cos(ax); y\} = \frac{\pi(-1)^{n+1}}{2^{2n+3}y^{2n+2}}H_{2n+1}\left(\frac{a}{2y}\right)\exp\left(-\frac{a^2}{4y^2}\right). \tag{71}$$

Here the rational functions $R_m(z)$ are defined by

$$R_0(z) = 0,$$

and

$$R_{m+1}(z) = \frac{d}{dz}R_m(z) + \frac{2(-1)^m H_m(z)}{z}, \qquad m = 0, 1, 2, \ldots.$$

The derivation is completely analogous to the sine case and is therefore omitted.

A First-Order Differential-Equation Application

The operational identity can also be used in an initial-value problem rather than only to generate transform pairs. Consider

$$f'(x) = xe^{-ax}, \qquad f(0) = 0, \qquad a > 0. \tag{72}$$

Set $U(y) = \mathscr{D}\{f(x)/x; y\}$. By (40), the transformed equation becomes the first-order Euler equation

$$-\sigma_y U(y) = \mathscr{D}\{xe^{-ax}; y\}, \tag{73}$$

where the right-hand side is given explicitly by (55). The condition inherited from $f(0) = 0$, together with decay as $y \to \infty$, selects the unique transformed solution. Mellin inversion (34), or direct identification using the transform pairs above, gives

$$f(x) = \frac{1 - (1 + ax)e^{-ax}}{a^2}. \tag{74}$$

Differentiation verifies (72). This elementary example shows the transform mechanism clearly: differentiation in the physical variable is converted into the Euler operator $-\sigma_y$, while the forcing term is evaluated by a known Dawson-transform pair. More involved problems are useful when the data naturally contain Gaussian, sine, or Dawson-type factors; for a generic constant-coefficient problem the Laplace transform remains more direct.

7. Bessel-Function Example

The derivative property of the Dawson transform can also be combined with standard recurrence relations for Bessel functions. This gives an operational recurrence in the order of the Bessel function and separates the computation into even and odd subsequences.

Example 7. *Let $a > 0$, $y > 0$, and define*

$$F_\nu(a, y) = \mathscr{D}\{J_\nu(ax); y\}.$$

Assume that the relevant transforms exist. Use the derivative property of the Dawson transform and the recurrence relations for Bessel functions to show that

$$(\nu - \sigma_y)F_{\nu+1}(a, y) = (\nu + \sigma_y)F_{\nu-1}(a, y), \qquad \nu \neq 0, \tag{75}$$

where σ_y is as defined in (5). Consequently, for $m = 1, 2, \ldots$,

$$F_{2m}(a, y) = \prod_{k=1}^{m} \left(\frac{2k - 1 + \sigma_y}{2k - 1 - \sigma_y} \right) F_0(a, y), \tag{76}$$

and

$$F_{2m+1}(a, y) = \prod_{k=1}^{m} \left(\frac{2k + \sigma_y}{2k - \sigma_y} \right) F_1(a, y). \tag{77}$$

In particular, the even-order transforms are generated from the initial value

$$F_0(a, y) = \mathscr{D}\{J_0(ax); y\} = \frac{1}{4y} \exp\left(-\frac{a^2}{8y^2} \right) K_0\left(\frac{a^2}{8y^2} \right), \tag{78}$$

where K_0 denotes the modified Bessel function of the second kind. The odd-order transforms may similarly be generated from

$$F_1(a, y) = \mathscr{D}\{J_1(ax); y\} = \frac{1}{a} \operatorname{daw}\left(\frac{a}{2y} \right). \tag{79}$$

Solution. Let

$$f(x) = J_\nu(ax).$$

Then

$$f'(x) = aJ'_\nu(ax).$$

Using the standard Bessel identities

$$J'_\nu(z) = \frac{1}{2}[J_{\nu-1}(z) - J_{\nu+1}(z)],$$

and

$$\frac{2\nu}{z}J_\nu(z) = J_{\nu-1}(z) + J_{\nu+1}(z),$$

we obtain

$$f'(x) = \frac{a}{2}[J_{\nu-1}(ax) - J_{\nu+1}(ax)],$$

and, since $\nu \neq 0$,

$$\frac{f(x)}{x} = \frac{J_\nu(ax)}{x} = \frac{a}{2\nu}[J_{\nu-1}(ax) + J_{\nu+1}(ax)].$$

By the derivative property of the Dawson transform (40),

$$\mathscr{D}\{f'(x); y\} = -\sigma_y \mathscr{D}\left\{\frac{f(x)}{x}; y\right\}.$$

Therefore,

$$\frac{a}{2}[F_{\nu-1}(a,y) - F_{\nu+1}(a,y)] = -\sigma_y\left\{\frac{a}{2\nu}[F_{\nu-1}(a,y) + F_{\nu+1}(a,y)]\right\}.$$

Canceling $a/2$ and multiplying by ν, we get

$$\nu F_{\nu-1}(a,y) - \nu F_{\nu+1}(a,y) = -\sigma_y F_{\nu-1}(a,y) - \sigma_y F_{\nu+1}(a,y).$$

Rearranging terms gives

$$(\nu - \sigma_y)F_{\nu+1}(a,y) = (\nu + \sigma_y)F_{\nu-1}(a,y),$$

which proves (75).

It remains to solve this difference equation in order. From (75),

$$F_{\nu+1}(a,y) = (\nu - \sigma_y)^{-1}(\nu + \sigma_y)F_{\nu-1}(a,y).$$

Since the operators involved are functions of the same operator σ_y, they commute. Taking $\nu = 1, 3, 5, \ldots, 2m-1$, we obtain

$$F_{2m}(a,y) = \prod_{k=1}^{m}(2k - 1 - \sigma_y)^{-1}(2k - 1 + \sigma_y)F_0(a,y),$$

that is,

$$F_{2m}(a,y) = \prod_{k=1}^{m}\left(\frac{2k - 1 + \sigma_y}{2k - 1 - \sigma_y}\right)F_0(a,y).$$

Similarly, taking $\nu = 2, 4, 6, \ldots, 2m$, we obtain

$$F_{2m+1}(a,y) = \prod_{k=1}^{m}(2k - \sigma_y)^{-1}(2k + \sigma_y)F_1(a,y),$$

or equivalently,

$$F_{2m+1}(a,y) = \prod_{k=1}^{m}\left(\frac{2k + \sigma_y}{2k - \sigma_y}\right)F_1(a,y).$$

This proves (76) and (77).

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Finally, the initial value (78) follows from the known Dawson transform of $J_0(ax)$. The value (79) follows from the same sine-kernel representation. Indeed,

$$\operatorname{daw}(xy) = \frac{1}{2}\int_0^\infty e^{-t^2/4}\sin(xyt)\,dt,$$

and the classical integral

$$\int_0^\infty J_1(ax)\sin(bx)\,dx = \begin{cases} \dfrac{b}{a\sqrt{a^2-b^2}}, & 0 < b < a, \\ 0, & b > a, \end{cases},$$

give

$$F_1(a,y) = \frac{1}{2y}\int_0^a \exp\left(-\frac{b^2}{4y^2}\right)\frac{b}{a\sqrt{a^2-b^2}}\,db.$$

With $b = a\sin\theta$, this becomes

$$F_1(a,y) = \frac{1}{2y}\int_0^{\pi/2} \exp\left(-\frac{a^2}{4y^2}\sin^2\theta\right)\sin\theta\,d\theta.$$

Hence, using $u = \cos\theta$,

$$F_1(a,y) = \frac{1}{2y}\exp\left(-\frac{a^2}{4y^2}\right)\int_0^1 \exp\left(\frac{a^2}{4y^2}u^2\right)du = \frac{1}{a}\operatorname{daw}\left(\frac{a}{2y}\right).$$

This completes the proof. □

8. Discussion and Concluding Remarks

The preceding results show that the Dawson transform is not an isolated construction but is connected with several established integral transforms. The second-iteration formula relates it directly to the Glasser transform, while the Fourier sine representation of Dawson's integral provides a natural route to Parseval-type identities. The Mellin representation gives a second analytic framework, yielding an inversion theorem and Mellin–Barnes representations for special-function examples. These connections suggest that the Dawson transform belongs to the same family of special-function-kernel transforms studied through Parseval–Goldstein identities and transform iteration methods.

The transform pairs obtained in this paper also illustrate the computational usefulness of the Dawson transform. Elementary trigonometric functions lead to Gaussian and exponential-integral expressions, while Gaussian-type functions lead to hypergeometric forms. The formulas involving powers of x multiplied by $\sin(ax)$ or $\cos(ax)$ show that parameter differentiation provides an efficient operational method for generating higher-order transform pairs. The Bessel-function formulas further indicate that the Dawson transform can be incorporated into the standard Mellin–Barnes theory of special functions.

The principal advantage of the Dawson transform is therefore not that it replaces the Laplace, Fourier, or Glasser transform in every setting. Its value is concentrated in problems whose data contain Dawson, imaginary-error-function, Gaussian-sine, or related kernels. The first iteration retains the differential equation and Fourier-sine structure of Dawson's function, producing the operators δ_x and σ_y; the second iteration then gives access to the Glasser transform. The price is a more restrictive weighted domain and a Mellin inversion multiplier that is less elementary than the exponential multiplier of the Laplace transform. Thus the Dawson transform is best viewed as a specialized bridge between Fourier-sine, Mellin, and Glasser methods.

Several directions remain open. The weighted space $\mathcal{X}$ used here supplies a transparent sufficient domain, while sharper L^p-mapping, uniqueness, and Plancherel-type results remain to be established. Other natural problems include systematic asymptotic expansions as $y \to 0^+$ and $y \to \infty$, fractional and multidimensional extensions, and integral or differential equations whose kernels contain Dawson's function or erfi.

Concluding Remarks

A new integral transform with Dawson's integral as kernel has been introduced and studied. The paper establishes its basic iteration, derivative, Mellin-transform, inversion, and special-function properties. The most significant structural result is the connection between the second iteration of the Dawson transform and the Glasser transform. The Mellin-transform formulation provides a natural inversion theory and connects the transform to Mellin–Barnes representations. The examples demonstrate that the Dawson transform gives compact evaluations of several classes of infinite integrals involving elementary, Gaussian, trigonometric, and Bessel functions. These results provide a foundation for further development of Dawson-kernel transform methods.

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Abbreviation

The following abbreviation is used in this manuscript:

DLMF Digital Library of Mathematical Functions

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