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# Jacobi–Sobolev Orthogonal Polynomials, Differential Properties and Structural Formulas

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**Abstract**

In this paper, we extend some differential and structural results for monic Jacobi–Sobolev orthogonal polynomials, associated with a general discrete Sobolev inner product, with a Jacobi continuous part. We consider finitely many exterior mass points and a positive semidefinite Sobolev product matrix. Using Christoffel–Darboux kernels, we derive several structure and connection formulas involving two consecutive Jacobi and Jacobi–Sobolev polynomials. This representation leads to lowering and raising operators with rational coefficients; a second-order ordinary differential equation and a three-term recurrence relation, with polynomial coefficients. These coefficients depend on $n$. These results extend several classical structural properties of Jacobi polynomials to a general discrete Sobolev setting.

**Keywords:** orthogonal polynomials; Jacobi polynomials; Sobolev discrete polynomials; second-order differential equation; ladder operators

**MSC:** 30C15; 33C45; 33C47; 42C05

## 1. Introduction

Orthogonal polynomials are a central topic in approximation theory, spectral analysis, and mathematical physics. These families are defined through orthogonality with respect to a positive measure. Among them, the so-called classical orthogonal polynomials are distinguished by the additional property of being eigenfunctions of second-order differential operators. According to Bochner's theorem [1], Jacobi, Hermite, and Laguerre polynomials are essentially the only families satisfying this property. In particular, Jacobi polynomials $p_n^{\alpha,\beta}$ satisfy a second-order differential equation with polynomial coefficients, and their zeros admit an electrostatic interpretation due to Stieltjes.

Beyond this classical Bochner setting, a broad and currently very active line of research has addressed the differential equation characterization of wider classes of generalized, hybrid, and Appell-type polynomial systems, largely through operational, algebraic, and quasi-monomial methods. Ben Cheikh's quasi-monomiality framework [2] and the connection problems solved via lowering operators [3] provided an early operator-theoretic language that has since been widely adopted. Costabile and collaborators developed a determinantal and algebraic approach to Appell and Sheffer polynomial sequences [4,5], from which

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governing differential equations, recurrence relations, and interpolation formulas can be derived systematically. In a closely related direction, Khan and Riyasat, frequently with Wani, applied factorization methods to obtain finite-order differential and integral equations for iterated and hybrid families built upon classical orthogonal polynomials, including the 2-iterated Appell [6], 2-iterated Bernoulli–Euler [7], Laguerre–Appell [8], and Legendre–Laguerre hybrid [9] polynomials; together with Srivastava, the same authors extended this factorization approach to the $q$-setting, deriving recurrence relations and $q$-difference equations for the 2-iterated $q$-Appell and mixed-type $q$-Appell polynomials and explicitly pointing to generalized orthogonality frameworks as a natural direction of further study [10].

At the same time, the wider literature on differential equations for orthogonal and special polynomial systems has evolved well beyond the strictly Sobolev context, and in particular beyond simple orthogonality itself. In the framework of $d$-orthogonality, Ben Cheikh and Gaied introduced Dunkl-classical $d$-symmetric $d$-orthogonal polynomials and established both a $(d+1)$-order recurrence relation and a $(d+1)$-order differential-difference equation [11]; Ben Cheikh and Gam later characterized $L$-classical $d$-orthogonal polynomial sets of Sheffer type and, in the differential setting, obtained the associated $(d+1)$-order differential equations [12]. These results are a useful point of comparison for the present work, since they show that once orthogonality is relaxed or generalized—whether to $d$-orthogonality or, as here, to a discrete Sobolev inner product—the order of the governing recurrence and differential relations typically increases in a structured, parameter-dependent way.

A complementary operational line has been developed for Appell- and Sheffer-type families under fractional and degenerate settings. Cesarano, Ramírez and coauthors introduced and studied degenerate Apostol-type Hermite polynomials and related extensions [13–15], while Bin-Saad, Zayed and their collaborators pursued fractional-order generalizations of classical orthogonal families, obtaining Rodrigues-type formulas, recurrence relations, and differential equations for fractional Laguerre [16], fractional Legendre-type matrix [17], conformable fractional Legendre [18], and Mittag-Leffler-Gegenbauer polynomials [19]. Related developments for generalized Appell- and Hermite-type polynomial families via fractional operators can be found in recent contributions by Zayed and Wani [20,21], and Alyusof and Wani further extended these determinantal and operational techniques to $\Delta_h$-difference hybrid special polynomials associated with Appell sequences [22]. In a complementary direction, Khan and Wani employed fractional-calculus techniques to construct generalized forms of Hermite–Appell and related polynomial families associated with Appell sequences, deriving generating functions, determinant representations, and recurrence relations for these nonclassical systems [23].

Although these works concern hybrid, $d$-orthogonal, $q$-deformed, and fractional-type families rather than discrete Jacobi–Sobolev orthogonality itself, they illustrates a common and recurring phenomenon: nonclassical polynomial families frequently retain a rich differential, recurrence, or operational structure even when the governing equations cease to belong to the classical Bochner framework, and the order of these governing relations tends to grow in a controlled way with the degree of generalization. This observation motivates, and is consistent with, the results obtained below for Jacobi–Sobolev polynomials, where a positive semidefinite discrete Sobolev product leads to structural formulas, ladder operators, and a second-order differential equation whose coefficients depend explicitly on $n$ and on the mass-point data.

On the other hand, in recent decades, increasing attention has been devoted to orthogonal polynomials associated with Sobolev inner products. In these settings, derivatives of the functions are incorporated into the inner-product structure and, consequently, into the orthogonality conditions. These inner products arise naturally in approximation theory, especially in the context of Sobolev spaces and spectral methods for boundary value problems [24].

The introduction of discrete Sobolev terms significantly modifies the structural properties of the associated orthogonal polynomials. In particular, the loss of symmetry with respect to the multiplication operator alters the behavior of the zeros. Moreover, the resulting families are closely related to Krall-type polynomials. These polynomials extend classical orthogonal polynomials by the addition of Dirac masses and may satisfy higher-order differential equations [25]. Within this framework, several authors have investigated Jacobi–Sobolev polynomials under specific assumptions. For instance, Kwon and Littlejohn [26] studied Sobolev orthogonality involving derivatives at the endpoints and derived second-order differential equations in certain cases.

More general discrete Sobolev perturbations of a Jacobi measure were analyzed by Bavinck [27], who showed that, in general, the corresponding orthogonal polynomials satisfy differential equations of infinite order, although finite-order operators arise in special situations. Later, Durán and de la Iglesia [28] developed a constructive approach based on Casorati determinants and proved that, for suitable parameters, Jacobi–Sobolev polynomials are eigenfunctions of explicitly computable finite-order differential operators.

Additional contributions in this area include [29] where the authors analyze the non-diagonal case for a single point. In [30], the authors provide a second-order linear differential equation and study some asymptotics and estimates for the largest zero of the $n$th orthogonal polynomial with respect to a Sobolev product with masses at the endpoints of the support interval. On the other hand , in [31] the authors obtain some connection formulas relating standard and Sobolev families under consideration, the holonomic equation and an electrostatic interpretation of their zeros. For a recent account of these properties, we refer the reader to [32], where the authors derive a second-order differential equation, a polynomial three-term recurrence relation, and ladder operators for a diagonal Sobolev inner product. The electrostatic interpretation of the zeros was obtained when the discrete part of the inner product satisfies a sequentially ordered structure, see [33].

Despite these advances, a general and unified treatment is still lacking. This concerns the differential properties of orthogonal polynomials associated with discrete Sobolev inner products of arbitrary order and multiple mass points. In particular, it remains an open problem to systematically characterize the differential operators. It also includes the second-order ordinary differential equations and the polynomial three-term recurrence relations, together with their corresponding orders. We direct our efforts in this direction. To present our results, we first introduce some technical notation.

In this work, we consider a finite positive Borel measure $\mu$ with support $\operatorname{supp}\mu = [-1,1] \subset \mathbb{R}$. Let $\{(c_1,d_1),\ldots,(c_N,d_N)\} \subset \mathbb{R}\times\mathbb{Z}_{\geq 0}$, with $c_j \notin \operatorname{supp}\mu$ for $j = 1,\ldots,N$. Assume, without loss of generality, that $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_N$. We denote by $\mathbb{P}$ the linear space of polynomials with real coefficients, and by $\mathbb{P}_n \subset \mathbb{P}$, the linear subspace of polynomials of degree less than or equal to $n \in \mathbb{Z}_+$.

Let $d = N + \sum_{j=1}^{N} d_j$ and consider the following bilinear form on $\mathbb{P}$:

$$\langle f,h\rangle_{\mathsf{s}} = \langle f,h\rangle_{\mu} + \sum_{j=1}^{N}\sum_{i,k=0}^{d_j} \lambda_{i,k}^{[j]} f^{(i)}(c_j)h^{(k)}(c_j) = \int_{-1}^{1} f(x)h(x)d\mu(x) + \mathcal{F}^T(\mathcal{C})\Lambda\mathcal{H}(\mathcal{C}). \quad (1)$$

As usual, $f^{(k)}$ is the $k$th derivative of $f$. Note that $d$ is the sum of the maximum derivative terms in (1). The previous matrix representation contains the following elements:

- $\mathcal{C} = [\underbrace{c_1,\ldots,c_1}_{(d_1+1)\text{ times}},\ldots \underbrace{c_j,\ldots,c_j}_{(d_j+1)\text{ times}},\ldots,\underbrace{c_N,\ldots,c_N}_{(d_N+1)\text{ times}}]^T \in \mathbb{R}^d, j = 1,\ldots,N,$

- $\mathcal{F}(\mathcal{C}) = \left[f^{(0)}(c_1), \ldots, f^{(d_1)}(c_1), f^{(0)}(c_2), \ldots, f^{(d_2)}(c_2), \ldots, f^{(0)}(c_N), \ldots, f^{(d_N)}(c_N)\right]^T$, a $d$-dimensional vector,
- $\mathcal{H}(\mathcal{C}) = \left[h^{(0)}(c_1), \ldots, h^{(d_1)}(c_1), h^{(0)}(c_2), \ldots, h^{(d_2)}(c_2), \ldots, h^{(0)}(c_N), \ldots, h^{(d_N)}(c_N)\right]^T$, a $d$-dimensional vector,
- the $d \times d$ block diagonal matrix $\Lambda$, with structure

$$\Lambda = \begin{bmatrix} \Lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \Lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda_N \end{bmatrix},$$

where the $(i,k)$-th entry of the matrix $\Lambda_j$ is $\lambda_{i,k}^{[j]}$. Each diagonal block corresponds to the discrete structure of the product at a mass point. This diagonal form prevents the mixing of derivative terms associated with different mass points.

**Definition 1** (Jacobi-Sobolev discrete inner-product)**.** *Expression* (1) *defines an inner product on $\mathbb{P}$ if and only if the matrix $\Lambda$ (or, equivalently, each of the matrices $\Lambda_j$, $j = 1, \ldots, N$) is positive semidefinite. In case $d\mu(x) = d\mu^{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}dx$, where $\alpha, \beta > -1$ and whose support is $[-1,1]$; this bilinear form will be called a discrete Jacobi–Sobolev inner product.*

Requiring $\Lambda$ to be positive semidefinite has further implications on the structure of the inner product. According to [34] (Obs. 7.1.10), if a diagonal entry $\lambda_{i,i}^{[j]} = 0$, then $\lambda_{i,k}^{[j]} = \lambda_{k,i}^{[j]} = 0$. Consequently, in terms of the inner product (1), the presence of a non-diagonal coefficient $\lambda_{i,k}^{[j]} \neq 0$ implies that both $\lambda_{i,i}^{[j]} \neq 0$ and $\lambda_{k,k}^{[j]} \neq 0$. Thus, without loss of generality, we assume that $\lambda_{i,i}^{d_j} > 0$. In the case where $\lambda_{i,k}^{[j]} = 0$, for all $j = 1, \ldots, N$, $i,k = 0, \ldots, d_j$, then $g_n \equiv p_n$ trivially, and $d = 0$.

Let $g_n$ $(n \in \mathbb{Z}_+)$ denote the nonzero polynomial of lowest degree satisfying

$$\langle g_n, x^v \rangle_{\mathsf{s}} = 0, \quad \text{for } v = 0, 1, \ldots, n-1. \tag{2}$$

From this point onward, the sequence $\{g_n\}_{n \geqslant 0}$ will be referred to as the system of Jacobi–Sobolev orthogonal polynomials associated with the inner product (1). Determining these polynomials amounts to solving a homogeneous system of $n$ linear equations, given by (2), in $(n+1)$ unknowns (the coefficients of $g_n$). Since the number of unknowns exceeds the number of equations, the system necessarily admits a nontrivial solution, which ensures that $g_n$ is not identically zero. Without loss of generality, we may assume that the leading coefficient of $g_n$ is equal to one; that is, $g_n$ is taken to be monic. Observe that $g_n$ is an $n$th-degree polynomial with real coefficients.

If the mass points are inside the support of the measure, the sequence of Jacobi–Sobolev polynomials and the corresponding Jacobi polynomials can coincide for some indices, as proven in the next example.

**Example 1.** *Consider the inner product*

$$\langle f, h \rangle_{\alpha,0} = \int_{-1}^{1} fg \, d\mu^{\alpha,\alpha} + f'(0)h'(0).$$

*The $(2n)$th orthogonal polynomial with respect to the Sobolev inner product is $p_{2n}^{\alpha,\alpha}$, the $2n$th Gegenbauer polynomial (see [35] (§4.7)).From [35] ((4.21.6), (4.21.7)),it is known that $\left(p_n^{\alpha,\alpha}(x)\right)' = n p_{n-1}^{\alpha+1,\alpha+1}(x)$. Then,*

$$\frac{d}{dx}p_{2n}^{\alpha,\alpha}(0) = 2np_{2n-1}^{\alpha+1,\alpha+1}(0) = 0,$$

*because odd degree Gegenbauer polynomials are odd functions. Thus* $\langle x^m, p_{2n}^{\alpha,\alpha}\rangle_{\alpha,0} = 0$, *for all* $0 \leq m < 2n$.

*Then, for a countable set of indices the Jacobi polynomial and the Sobolev polynomial coincide, for a non-null* $\Lambda$.

Moreover, in Theorem 4 of [36], the authors prove that if $\Lambda \in \mathbb{R}_+$, $\mu$ is in the Nevai class and the mass $c \in \operatorname{supp}\mu$, $g_n$ and $p_n^{\alpha,\beta}$ behave asymptotically the same. Thus, we restrict the mass points to the exterior of the support of the measure; i.e., $c_j \in \mathbb{R} \setminus [-1,1]$.

It is well known that classical Jacobi polynomials satisfy a three-term recurrence relation and are solutions of a second-order differential equation that can be derived from the corresponding raising and lowering operators. These relations and properties have been established for certain particular cases of discrete Sobolev inner products; however, a general approach is still missing in the literature.

In this paper, we study these properties within the framework of a general discrete Jacobi–Sobolev inner product, allowing for an arbitrary number of mass points and derivatives, including both diagonal and non-diagonal cases. To this end, we derive explicit differential relations, construct the associated differential operators, and analyze their order and structural properties. Furthermore, we obtain connection formulas with classical Jacobi polynomials and examine the role of the discrete Sobolev terms in shaping the differential behavior of the polynomial sequence. Although the techniques employed are not entirely new, our approach addresses a highly general configuration under minimal assumptions on the Sobolev inner product.

The paper is organized as follows. Section 2 is devoted to the derivation of a fundamental connection formula between Jacobi and Jacobi–Sobolev polynomials using kernel-function techniques combined with the discrete Sobolev structure. This connection formula allows us to derive the corresponding ladder operators in Section 3. Finally, in Section 4 we construct the associated differential operators and extend several structural relations satisfied by classical Jacobi polynomials to their Jacobi–Sobolev counterparts. In this section, we define raising and lowering operators, with rational coefficients, and derive a three-term recurrence relation, and a second-order ordinary differential, whose solutions are the Jacobi–Sobolev orthogonal polynomials. Both of these relations have polynomial coefficients depending on $n$.

## 2. Kernel Properties Under Sobolev Discrete Configuration

In this section we derive several algebraic properties of the kernels associated with Jacobi polynomials induced by the discrete Sobolev structure, which will later allow us to obtain the corresponding connection formulas. Although our main focus is on Jacobi–Sobolev families, the results of this section are independent of the choice of the continuous measure and the discrete configuration, provided that $\Lambda$ remains positive semidefinite and the inner product induced by $\mu$ is a discrete Sobolev inner product.

From the Christoffel–Darboux formula, we obtain the well-known representation for $K_n(x,y)$, the kernel polynomial associated with $\{p_n\}_{n\geqslant 0}$, the orthogonal polynomials with respect to $\mu$,

$$K_{n-1}(x,y) = \sum_{k=0}^{n-1} \frac{p_k(x)p_k(y)}{\|p_k\|_\mu^2} = \begin{cases} \dfrac{p_n(x)p_{n-1}(y) - p_n(y)p_{n-1}(x)}{\|p_{n-1}\|_\mu^2\,(x-y)}, & \text{if } x \neq y; \\ \dfrac{p_n'(x)p_{n-1}(x) - p_n(x)p_{n-1}'(x)}{\|p_{n-1}\|_\mu^2}, & \text{if } x = y. \end{cases}$$

Let $K_n^{(j,k)}(x,y) = \dfrac{\partial^{j+k} K_n(x,y)}{\partial x^j \partial y^k}$, where $i, j \in \mathbb{Z}_+$. Apply Leibniz's rule to obtain

$$K_{n-1}^{(0,k)}(x,y) = \frac{\partial^k K_{n-1}}{\partial y^k}(x,y) = \sum_{i=0}^{n-1} \frac{p_i(x) p_i^{(k)}(y)}{\|p_i\|_\mu^2} = \eta_{1,n}^{[k]}(x,y)\; p_n(x) - \eta_{2,n}^{[k]}(x,y)\; p_{n-1}(x), \quad (3)$$
$$\eta_{1,n}^{[k]}(x,y) = \frac{k!\; \tau_k(x,y;p_{n-1})}{\|p_{n-1}\|_\mu^2\; (x-y)^{k+1}}, \qquad \eta_{2,n}^{[k]}(x,y) = \frac{k!\; \tau_k(x,y;p_n)}{\|p_{n-1}\|_\mu^2\; (x-y)^{k+1}}.$$

Here, $\tau_k(x,y;f) = \displaystyle\sum_{\nu=0}^{k} \frac{f^{(\nu)}(y)}{\nu!}(x-y)^\nu$ denotes the $k$-degree Taylor polynomial of $f$, centered at $y$.

From (2), we have $\langle g_n, p_m \rangle_{\mathsf{s}} = 0$ for all $m < n$. Therefore,

$$\langle g_n, p_m \rangle_\mu = -\sum_{j=1}^{N} \sum_{i,k=0}^{d_j} \lambda_{i,k}^{[j]}\, g_n^{(i)}(c_j)\; p_m^{(k)}(c_j). \quad (4)$$

For each fixed $n$, there are $n$ real constants $\{\vartheta_0, \vartheta_1, \ldots, \vartheta_{n-1}\}$ such that

$$g_n(x) = p_n(x) + \sum_{m=0}^{n-1} \vartheta_m\; p_m(x), \quad (5)$$

where, from the orthogonality of $\{p_n\}_{n\geqslant 0}$ with respect to the inner product $\langle \cdot,\cdot \rangle_\mu$ and (4),

$$\vartheta_m = \frac{\langle g_n, p_m \rangle_\mu}{\langle p_m, p_m \rangle_\mu} = -\sum_{j=1}^{N} \sum_{i,k=0}^{d_j} \lambda_{i,k}^{[j]}\, g_n^{(i)}(c_j)\, \frac{p_m^{(k)}(c_j)}{\|p_m\|_\mu^2}. \quad (6)$$

Combining Formulas (4)–(6), we obtain, for $y = c_j$,

$$\begin{aligned} g_n(x) &= p_n(x) + \sum_{m=0}^{n-1} \langle g_n, p_m \rangle_\mu \frac{p_m(x)}{\|p_m\|_\mu^2} \\ &= p_n(x) - \sum_{j=1}^{N} \sum_{i,k=0}^{d_j} \lambda_{i,k}^{[j]}\, g_n^{(i)}(c_j) \sum_{m=0}^{n-1} \frac{p_m(x) p_m^{(k)}(c_j)}{\|p_m\|_\mu^2} \\ &= p_n(x) - \sum_{j=1}^{N} \sum_{i,k=0}^{d_j} \lambda_{i,k}^{[j]}\, g_n^{(i)}(c_j)\; K_{n-1}^{(0,k)}(x, c_j). \end{aligned} \quad (7)$$

Finally, substituting (3), recall $y = c_j$, into (7), we obtain the connection formula

$$g_n(x) = \mathsf{a}_n^{[1]}(x)\;\; p_n(x) + \mathsf{b}_n^{[1]}(x)\;\; p_{n-1}(x), \quad (8)$$
$$\text{where} \quad \mathsf{a}_n^{[1]}(x) = 1 - \sum_{j=1}^{N} \sum_{i,k=0}^{d_j} \lambda_{i,k}^{[j]} g_n^{(i)}(c_j)\, \eta_{1,n}^{[k]}(x, c_j),$$
$$\text{and} \quad \mathsf{b}_n^{[1]}(x) = \sum_{j=1}^{N} \sum_{i,k=0}^{d_j} \lambda_{i,k}^{[j]} g_n^{(i)}(c_j)\, \eta_{2,n}^{[k]}(x, c_j).$$

Note that (8) allows us to compute $g_n$ from $p_n$, $p_{n-1}$, and the set of values $g_n^{(i)}(c_j)$ for $j = 1, \ldots, N$ and $i = 0, \ldots, d_j$. We now focus on determining the values $g_n^{(i)}(c_j)$ directly, thereby avoiding the explicit computation of the polynomial $g_n(x)$.

To this end, take in (7) the $v$-derivative evaluated at $x = c_w$ for each ordered pair $(w, v)$, with $w = 1, \dots, N$ and $v = 0, \dots, d_w$. We obtain a system of $d$ linear equations in the $d$ unknowns $g_n^{(v)}(c_w)$:

$$\begin{aligned} p_n^{(v)}(c_w) = &\left(1 + \sum_{k=0}^{d_w} \lambda_{v,k}^{[w]} K_{n-1}^{(v,k)}(c_w, c_w)\right) g_n^{(v)}(c_w) + \sum_{\substack{i,k=0 \\ i \neq v}}^{d_w} \lambda_{i,k}^{[w]} g_n^{(i)}(c_w) K_{n-1}^{(v,k)}(c_w, c_w) \\ &+ \sum_{\substack{j=1 \\ j \neq w}}^{N} \left( \sum_{i,k=0}^{d_j} \lambda_{i,k}^{[j]} g_n^{(i)}(c_j) K_{n-1}^{(v,k)}(c_w, c_j) \right). \end{aligned} \tag{9}$$

**Proposition 1.** *For every $n \geq d$, the linear system (9) admits a unique solution.*

**Proof.** Equation (9) can be written as

$$\mathcal{P}_n(\mathcal{C}) = (\mathcal{I}_d + \mathcal{K}(\mathcal{C}, \mathcal{C})\Lambda)\mathcal{G}_n(\mathcal{C}),$$

where

- $\mathcal{I}_d$ is the identity matrix of size $d \times d$,
- $\mathcal{G}_n(\mathcal{C}) = \left[g_n^{(0)}(c_1), \dots, g_n^{(d_1)}(c_1), g_n^{(0)}(c_2), \dots, g^{(d_2)}(c_2), \dots, g_n^{(0)}(c_N), \dots, g_n^{(d_N)}(c_N)\right]^T$, a $d$-dimensional vector,
- $\mathcal{K}(\mathcal{C}, \mathcal{C})$ is the block-kernel matrix of size $d \times d$, whose blocks correspond to the set $\{c_1, \dots, c_N\}^2$:

$$\mathcal{K}(\mathcal{C}, \mathcal{C}) = \begin{bmatrix} \mathcal{K}_{n-1}^{[d_1,d_1]}(c_1, c_1) & \mathcal{K}_{n-1}^{[d_1,d_2]}(c_1, c_2) & \dots & \mathcal{K}_{n-1}^{[d_1,d_N]}(c_1, c_N) \\ \mathcal{K}_{n-1}^{[d_2,d_1]}(c_2, c_1) & \mathcal{K}_{n-1}^{[d_2,d_2]}(c_2, c_2) & \dots & \mathcal{K}_{n-1}^{[d_2,d_N]}(c_2, c_N) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{K}_{n-1}^{[d_N,d_1]}(c_N, c_1) & \mathcal{K}_{n-1}^{[d_N,d_2]}(c_N, c_2) & \dots & \mathcal{K}_{n-1}^{[d_N,d_N]}(c_N, c_N) \end{bmatrix}. \tag{10}$$

- The blocks along the main diagonal, corresponding to the pair $(c_j, c_j)$, are given by

$$\mathcal{K}_{n-1}^{[d_j,d_j]}(c_j, c_j) = \begin{bmatrix} K_{n-1}^{(0,0)}(c_j, c_j) & K_{n-1}^{(0,1)}(c_j, c_j) & \dots & K_{n-1}^{(0,d_j)}(c_j, c_j) \\ K_{n-1}^{(1,0)}(c_j, c_j) & K_{n-1}^{(1,1)}(c_j, c_j) & \dots & K_{n-1}^{(1,d_j)}(c_j, c_j) \\ \vdots & \vdots & \ddots & \vdots \\ K_{n-1}^{(d_1,0)}(c_j, c_j) & K_{n-1}^{(d_1,1)}(c_j, c_j) & \dots & K_{n-1}^{(d_j,d_j)}(c_j, c_j) \end{bmatrix},$$

  a $(d_j + 1) \times (d_j + 1)$ square matrix.
- The off-diagonal blocks, corresponding to the pair $(c_{j_1}, c_{j_2})$ with $j_1 \neq j_2$, for $j_1, j_2 = 1, \dots, N$, are $(d_{j_1} + 1) \times (d_{j_2} + 1)$ rectangular matrices given by

$$\mathcal{K}_{n-1}^{[d_{j_1},d_{j_2}]}(c_{j_1}, c_{j_2}) = \begin{bmatrix} K_{n-1}^{(0,0)}(c_{j_1}, c_{j_2}) & K_{n-1}^{(0,1)}(c_{j_1}, c_{j_2}) & \dots & K_{n-1}^{(0,d_{j_2})}(c_{j_1}, c_{j_2}) \\ K_{n-1}^{(1,0)}(c_{j_1}, c_{j_2}) & K_{n-1}^{(1,1)}(c_{j_1}, c_{j_2}) & \dots & K_{n-1}^{(1,d_{j_2})}(c_{j_1}, c_{j_2}) \\ \vdots & \vdots & \ddots & \vdots \\ K_{n-1}^{(d_{j_1},0)}(c_{j_1}, c_{j_2}) & K_{n-1}^{(d_{j_1},1)}(c_{j_1}, c_{j_2}) & \dots & K_{n-1}^{(d_{j_1},d_{j_2})}(c_{j_1}, c_{j_2}) \end{bmatrix}.$$

  Consider the $n$-dimensional vectors

$$\mathrm{v}_i(c_j)^T = \left( \frac{p_0^{(i)}(c_j)}{\|p_0\|_\mu}, \frac{p_1^{(i)}(c_j)}{\|p_1\|_\mu}, \ldots, \frac{p_{n-1}^{(i)}(c_j)}{\|p_{n-1}\|_\mu} \right)^T$$

Note that

$$K_{n-1}^{(i,k)}(c_{j_1}, c_{j_2}) = \mathrm{v}_i^T(c_{j_1})\mathrm{v}_k(c_{j_2})$$

Then,

$$\begin{aligned}
\mathcal{K}_{n-1}^{[d_{j_1},d_{j_2}]}(c_{j_1}, c_{j_2}) &= \begin{bmatrix} \mathrm{v}_0^T(c_{j_1})\mathrm{v}_0(c_{j_2}) & \mathrm{v}_0^T(c_{j_1})\mathrm{v}_1(c_{j_2}) & \ldots & \mathrm{v}_0^T(c_{j_1})\mathrm{v}_{d_{j_2}}(c_{j_2}) \\ \mathrm{v}_1^T(c_{j_1})\mathrm{v}_0(c_{j_2}) & \mathrm{v}_1^T(c_{j_1})\mathrm{v}_1(c_{j_2}) & \ldots & \mathrm{v}_1^T(c_{j_1})\mathrm{v}_{d_{j_2}}(c_{j_2}) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{v}_{d_{j_1}}^T(c_{j_1})\mathrm{v}_0(c_{j_2}) & \mathrm{v}_{d_{j_1}}^T(c_{j_1})\mathrm{v}_1(c_{j_2}) & \ldots & \mathrm{v}_{d_{j_1}}^T(c_{j_1})\mathrm{v}_{d_{j_2}}(c_{j_2}) \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{v}_0^T(c_{j_1}) \\ \mathrm{v}_1^T(c_{j_1}) \\ \vdots \\ \mathrm{v}_{d_{j_1}}^T(c_{j_1}) \end{bmatrix} \left[ \mathrm{v}_0(c_{j_2}), \mathrm{v}_1(c_{j_2}), \ldots, \mathrm{v}_{d_{j_2}}(c_{j_2}) \right] = \mathcal{V}^T(c_{j_1})\mathcal{V}(c_{j_2}), \qquad (11)
\end{aligned}$$

where $\mathcal{V}(c_j)$ is a $(d_j + 1) \times n$ matrix. Then, substitute (11) in (10) to obtain

$$\mathcal{K}(\mathcal{C},\mathcal{C}) = \begin{bmatrix} \mathcal{V}^T(c_1)\mathcal{V}(c_1) & \mathcal{V}^T(c_1)\mathcal{V}(c_2) & \ldots & \mathcal{V}^T(c_1)\mathcal{V}(c_N) \\ \mathcal{V}^T(c_2)\mathcal{V}(c_1) & \mathcal{V}^T(c_2)\mathcal{V}(c_2) & \ldots & \mathcal{V}^T(c_2)\mathcal{V}(c_N) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{V}^T(c_N)\mathcal{V}(c_1) & \mathcal{V}^T(c_N)\mathcal{V}(c_2) & \ldots & \mathcal{V}^T(c_N)\mathcal{V}(c_N) \end{bmatrix} \qquad (12)$$

$$= \begin{bmatrix} \mathcal{V}^T(c_1) \\ \mathcal{V}^T(c_2) \\ \vdots \\ \mathcal{V}^T(c_N) \end{bmatrix} [\mathcal{V}(c_1), \mathcal{V}(c_2), \ldots, \mathcal{V}(c_N)]. \qquad (13)$$

According to [32] (Lemma 1), the matrix $[\mathcal{V}(c_1), \mathcal{V}(c_2), \ldots, \mathcal{V}(c_N)]$, for $d \leq n$ is full rank. Consequently, $\mathcal{K}(\mathcal{C},\mathcal{C})$ is symmetric and positive definite; see [34] (Th. 7.2.7(c)).Since $\Lambda$ is positive semidefinite and $\mathcal{K}(\mathcal{C},\mathcal{C})$ is positive definite, then $\mathcal{K}(\mathcal{C},\mathcal{C})\Lambda$ is diagonalizable and has nonnegative eigenvalues $\sigma_i(\mathcal{K}(\mathcal{C},\mathcal{C})\Lambda) \in \mathbb{R}_{\geqslant 0}$, see [34] (Cor. 7.6.2(b)). Then, the $i$th eigenvalue of $\mathcal{I}_d + \mathcal{K}(\mathcal{C},\mathcal{C})\Lambda$ is $1 + \sigma_i(\mathcal{K}(\mathcal{C},\mathcal{C})\Lambda) > 0$.

Since all eigenvalues of $\mathcal{I}_d + \Lambda\mathcal{K}(\mathcal{C},\mathcal{C})$ are nonzero, the matrix is invertible, and hence the system (9) has a unique solution. □

Therefore, the connection Formula (19) is established.

## 3. Connection Formulas and Ladder Operators

The main goal of this section is to derive the ladder operators for the Jacobi–Sobolev polynomials. These polynomials are obtained by considering the measure $d\mu(x) = d\mu^{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta dx$, where $\alpha, \beta > -1$ and whose support is $[-1,1]$, in the bilinear form (1). Thus, note that $g_n(x) = g_n^{\alpha,\beta}(x)$ and $p_n(x) = p_n^{\alpha,\beta}(x)$. In order to avoid excessive notation, the parameters $\alpha$ and $\beta$ will be omitted throughout this section whenever no confusion arises. However, the reader should keep in mind that all polynomials depend on these parameters.

For completeness, we summarize below the principal properties of the monic Jacobi polynomials that will be required throughout the paper. See [35] ((4.1.1), (4.3.2), (4.3.3), and (4.5.1)).

$$
\begin{aligned}
p_n^{\alpha,\beta}(x) =& \binom{2n+\alpha+\beta}{n}^{-1} \sum_{\nu=0}^{n} \binom{n+\alpha}{n-\nu}\binom{n+\beta}{\nu}(x-1)^{\nu}(x+1)^{n-\nu}. \\
h_n^{\alpha,\beta} = \left\| p_n^{\alpha,\beta} \right\|^2_{\mu^{\alpha,\beta}} =& 2^{2n+\alpha+\beta+1} \frac{\Gamma(n+1)\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(2n+\alpha+\beta+2)\Gamma(2n+\alpha+\beta+1)}. \\
x p_n^{\alpha,\beta}(x) =& p_{n+1}^{\alpha,\beta}(x) + \gamma_{1,n} p_n^{\alpha,\beta}(x) + \gamma_{2,n} p_{n-1}^{\alpha,\beta}(x);\ p_0^{\alpha,\beta}(x) = 1,\ p_{-1}^{\alpha,\beta}(x) = 0, \quad (14)
\end{aligned}
$$

where

$$
\begin{aligned}
\gamma_{1,n} =& \gamma_{1,n}^{\alpha,\beta} = \frac{\beta^2 - \alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}, \\
\gamma_{2,n} =& \gamma_{2,n}^{\alpha,\beta} = \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta)^2((2n+\alpha+\beta)^2-1)}.
\end{aligned} \tag{15}
$$

We denote by $\mathcal{I}$ the identity operator. The Jacobi ladder differential operators on $\mathbb{P}$ are given by

$$
\begin{aligned}
\widehat{\mathcal{L}}_n :=& -\frac{\widehat{a}_n(x)}{\widehat{b}_n}\, \mathcal{I} + \frac{1-x^2}{\widehat{b}_n}\, \frac{d}{dx} \quad \text{(lowering Jacobi differential operator)}, \\
\widehat{\mathcal{R}}_n :=& -\frac{\widehat{c}_n(x)}{\widehat{d}_n}\, \mathcal{I} + \frac{1-x^2}{\widehat{d}_n}\, \frac{d}{dx} \quad \text{(raising Jacobi differential operator)}.
\end{aligned}
$$

where

$$
\begin{aligned}
\widehat{a}_n(x) =& -\frac{n((2n+\alpha+\beta)x+\beta-\alpha)}{2n+\alpha+\beta}, \quad \widehat{b}_n = \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta)^2(2n+\alpha+\beta-1)}, \\
\widehat{c}_n(x) =& \frac{(n+\alpha+\beta)((2n+\alpha+\beta)x+\alpha-\beta)}{2n+\alpha+\beta} \quad \text{and} \quad \widehat{d}_n = -(2n+\alpha+\beta-1).
\end{aligned} \tag{16}
$$

From [35] ((4.5.5)–(4.5.7) and (4.21.6)), if $n \geqslant 1$, the sequence $\{p_n\}_{n\geqslant 0}$ satisfies the relations

$$
\begin{aligned}
\widehat{\mathcal{L}}_n[p_n(x)] =& -\frac{\widehat{a}_n(x)}{\widehat{b}_n} p_n(x) + \frac{1-x^2}{\widehat{b}_n}(p_n(x))' = p_{n-1}(x), \\
\widehat{\mathcal{R}}_n[p_{n-1}(x)] =& -\frac{\widehat{c}_n(x)}{\widehat{d}_n} p_{n-1}(x) + \frac{1-x^2}{\widehat{d}_n}(p_{n-1}(x))' = p_n(x).
\end{aligned} \tag{17}
$$

From these ladder operators, we deduce that the Jacobi polynomials satisfy the second-order ordinary differential equation

$$
\left(1-x^2\right) p_n''(x) + (\beta - \alpha - (\alpha+\beta+2)x) p_n'(x) = -n(n+\alpha+\beta+1) p_n(x), \tag{18}
$$

see [35] (Th. 4.2.2 and (4.21.6)).

The ladder operators are obtained through a sequence of connection lemmas, which we list below. The proofs of these lemmas are given in the next section. As a general rule, we use bracketed superscripts to indicate the order of the "composition" layer of the polynomial/rational coefficients appearing in the different formulas.

Let us define $\ell(x) = \prod_{j=1}^{N}(x-c_j)^{d_j+1}$, the annihilation polynomial, with $\deg(\ell) = d$, where $\deg(h)$ denotes the degree of the polynomial $h \in \mathbb{P}$,

$$
\ell(c_j) = \ell'(c_j) = \cdots = \ell^{(d_j)}(c_j) = 0, \quad j = 1, 2, \ldots, N.
$$

**Lemma 1** (Connection formula)**.** *For every $n \geq d$, the monic polynomial sequences $\{g_n\}_{n\geq 0}$ and $\{p_n\}_{n\geq 0}$ satisfy the following connection formula:*

$$
g_n(x) = \mathsf{a}_n^{[1]}(x)\, p_n(x) + \mathsf{b}_n^{[1]}(x)\, p_{n-1}(x), \tag{19}
$$

*where*

$$\mathsf{a}_n^{[1]}(x) = 1 - \sum_{j=1}^{N} \sum_{i,k=0}^{d_j} \lambda_{i,k}^{[j]} g_n^{(i)}(c_j)\, \eta_{1,n}^{[k]}(x, c_j),$$

$$\mathsf{b}_n^{[1]}(x) = \sum_{j=1}^{N} \sum_{i,k=0}^{d_j} \lambda_{i,k}^{[j]} g_n^{(i)}(c_j)\, \eta_{2,n}^{[k]}(x, c_j).$$

**Proof.** See (8). □

**Remark 1.** *When $\Lambda = 0$, the Sobolev term in (1) vanishes and the inner product reduces to the classical Jacobi inner product. Therefore, $g_n = p_n$, for all $n \geq 0$. Moreover, the identities $\lambda_{i,k}^{[j]} = 0$ imply that $\mathsf{b}_n^{[1]}(x) = 0$ and $\mathsf{a}_n^{[1]}(x) = 1$, so that the corresponding formulas recover their classical Jacobi counterparts.*

**Lemma 2** (Polynomial connection)**.** *For $n \geqslant d$, then the monic families $\{g_n\}_{n\geqslant 0}$ and $\{p_n\}_{n\geqslant 0}$ are related through the following forwardpolynomial connection formulas,*

$$\ell(x) g_n(x) = \mathsf{a}_n^{[2]}(x)\, p_n(x) + \mathsf{b}_n^{[2]}(x)\, p_{n-1}(x), \tag{20}$$

$$\ell(x) g_{n-1}(x) = \mathsf{c}_n^{[2]}(x)\, p_n(x) + \mathsf{d}_n^{[2]}(x)\, p_{n-1}(x), \tag{21}$$

*and the backwardrational connection formulas*

$$p_n(x) = \frac{\ell(x)}{\Delta_n(x)} \Big( \mathsf{d}_n^{[2]}(x)\, g_n(x) - \mathsf{b}_n^{[2]}(x)\, g_{n-1}(x) \Big), \tag{22}$$

$$p_{n-1}(x) = \frac{\ell(x)}{\Delta_n(x)} \Big( -\mathsf{c}_n^{[2]}(x)\, g_n(x) + \mathsf{a}_n^{[2]}(x)\, g_{n-1}(x) \Big), \tag{23}$$

*where the polynomials $\mathsf{a}_n^{[2]}(x)$, $\mathsf{b}_n^{[2]}(x)$ (with leading coefficient $B_n^{[2]}$), $\mathsf{c}_n^{[2]}(x)$, $\mathsf{d}_n^{[2]}(x)$, and $\Delta_n(x)$ have the expressions, degrees, and leading coefficients, respectively, given in Table 1.*

**Table 1.** Coefficients of Lemma 2.

| | **Expression** | **Degree** | **Leading Coefficient** |
|---|---|---|---|
| $\mathsf{a}_n^{[2]}(x)$ | $\ell(x)\mathsf{a}_n^{[1]}(x)$ | $d$ | $1$ |
| $\mathsf{b}_n^{[2]}(x)$ | $\ell(x)\mathsf{b}_n^{[1]}(x)$ | $d-1$ | $\sum_{j=1}^{N} \sum_{i,k=0}^{d_j} \frac{\lambda_{i,k}^{[j]}}{\lVert p_{n-1} \rVert_\mu^2} p_n^{(i)}(c_j)\, g_n^{(k)}(c_j)$ |
| $\mathsf{c}_n^{[2]}(x)$ | $-\ell(x) \left( \frac{\mathsf{b}_{n-1}^{[1]}(x)}{\gamma_{2,n-1}} \right)$ | $d-1$ | $-\frac{B_{n-1}^{[2]}}{\gamma_{2,n-1}}$ |
| $\mathsf{d}_n^{[2]}(x)$ | $\ell(x) \left( \mathsf{a}_{n-1}^{[1]}(x) + \mathsf{b}_{n-1}^{[1]}(x) \left( \frac{x-\gamma_{1,n-1}}{\gamma_{2,n-1}} \right) \right)$ | $d$ | $1 + \frac{B_{n-1}^{[2]}}{\gamma_{2,n-1}}$ |
| $\Delta_n(x)$ | $\mathsf{a}_n^{[2]}(x)\, \mathsf{d}_n^{[2]}(x) - \mathsf{c}_n^{[2]}(x)\, \mathsf{b}_n^{[2]}(x)$ | $2d$ | $1 + \frac{B_{n-1}^{[2]}}{\gamma_{2,n-1}}$ |

**Proof.** Let us first focus on (20) and (21) and on the corresponding entries in the previous table. Expression (20) is obtained by multiplying (19) by $\ell(x)$, and after grouping terms, the leading coefficient of $\mathsf{b}_n^{[2]}(x)$ becomes

$$B_n^{[2]} = \sum_{j=1}^{N} \sum_{i,k=0}^{d_j} \frac{\lambda_{i,k}^{[j]}}{\lVert p_{n-1} \rVert_\mu^2} p_n^{(i)}(c_j)\, g_n^{(k)}(c_j).$$

On the other hand, (21) is obtained by considering (20) for $n-1$ and applying the well-known three-term recurrence relation (14).

Next, let us prove that $B_n^{[2]} > 0$. According to (2), we have

$$0 = \langle g_n, g_n - p_n \rangle_{\mathsf{s}} = \langle g_n, g_n \rangle_{\mathsf{s}} - \langle g_n, p_n \rangle_{\mathsf{s}}$$
$$= \|g_n\|_\mu^2 - \|p_n\|_\mu^2 + \sum_{j=1}^{N} \sum_{i,k=0}^{d_j} \lambda_{i,k}^{[j]} g_n^{(i)} g_n^k(c_j) - \|p_{n-1}\|_\mu^2 B_n^{[2]}.$$

This implies that

$$B_n^{[2]} = \|p_{n-1}\|_\mu^{-2} \left( \|g_n\|_\mu^2 - \|p_n\|_\mu^2 + \sum_{j=1}^{N} (\mathcal{G}_n(c_j))^T \Lambda_j \mathcal{G}_n(c_j) \right).$$

Since

$$\|g_n\|_\mu^2 - \|p_n\|_\mu^2 \geq 0,$$

by the extremal property of the orthogonal polynomial associated with the measure $\mu$ (see [35] (§2.1(4))),and taking into account the positive semidefiniteness of the matrices $\Lambda_j$, we conclude that

$$B_n^{[2]} \geq 0.$$

Moreover, if $g_n \not\equiv p_n$, then the inequality is strict. The remaining assertions follow from the fact that $\mathsf{a}_n^{[2]}(x)$ is a monic polynomial of degree $d$ together with the sign of $B_n^{[2]}$. On the other hand, (22) and (23) are obtained by applying Cramer's rule. Note that the determinant $\Delta_n(x)$ has degree $2d$; therefore, these relations are valid except at a finite set of points.

If $g_n = p_n$, for some $n$, then $\mathsf{a}_n^{[2]}(x) = \ell(x)$, $\mathsf{b}_n^{[2]}(x) = 0$, $\mathsf{c}_n^{[2]}(x) = 0$ and $\mathsf{d}_2^{[n]}(x) = \ell(x)$, and the assertion holds. □

**Remark 2.** *From (22) (or (23)), we have $\Delta_n(x) = \ell(x)\delta_n(x)$, where $\delta_n$ is*

$$\delta_n(x) = \mathsf{a}_n^{[1]}(x)\,\mathsf{d}_n^{[2]}(x) - \mathsf{b}_n^{[1]}(x)\,\mathsf{c}_n^{[2]}(x).$$

**Lemma 3.** *For $n \geqslant d$, the monic families $\{g_n\}_{n\geqslant 0}$ and $\{p_n\}_{n\geqslant 0}$ are related through the following differential connection formulas,*

$$(1-x^2)(\ell(x)g_n(x))' = \mathsf{a}_n^{[3]}(x)\, p_n(x) + \mathsf{b}_n^{[3]}(x)\, p_{n-1}(x), \tag{24}$$
$$(1-x^2)(\ell(x)g_{n-1}(x))' = \mathsf{c}_n^{[3]}(x)\, p_n(x) + \mathsf{d}_n^{[3]}(x)\, p_{n-1}(x). \tag{25}$$

*where $\mathsf{a}_n^{[3]}(x)$, $\mathsf{b}_n^{[3]}(x)$, $\mathsf{c}_n^{[3]}(x)$, and $\mathsf{d}_n^{[3]}(x)$ are given in Table 2, and $\widehat{a}_n(x)$, $\widehat{b}_n$, $\widehat{c}_n(x)$, and $\widehat{d}_n$ are defined in (16).*

**Proof.** To obtain (24), differentiate (20),

$$(\ell(x)g_n(x))' = \left(\mathsf{a}_n^{[2]}(x)\right)' p_n(x) + \mathsf{a}_n^{[2]}(x)\, p_n'(x) + \left(\mathsf{b}_n^{[2]}(x)\right)' p_{n-1}(x) + \mathsf{b}_n^{[2]}(x)\, p_{n-1}'(x).$$

Use the Jacobi ladder operators (17), to substitute $p'_n$ and $p'_{n-1}(x)$, and group terms in $p_n$ and $p_{n-1}$ to obtain

$$(\ell(x)g_n(x))' = \left(\left(\mathsf{a}_n^{[2]}(x)\right)' + \frac{\widehat{a}_n(x)\mathsf{a}_n^{[2]}(x)}{(1-x^2)} + \frac{\widehat{d}_n\mathsf{b}_n^{[2]}(x)}{(1-x^2)}\right)p_n(x) + \left(\left(\mathsf{b}_n^{[2]}(x)\right)' + \frac{\widehat{b}_n\mathsf{a}_n^{[2]}(x)}{(1-x^2)} + \frac{\widehat{c}_n(x)\mathsf{b}_n^{[2]}(x)}{(1-x^2)}\right)p_{n-1}(x)$$

Finally, multiply by $(1-x^2)$ to obtain (24). The degree and leading coefficient of $\mathsf{a}_n^{[3]}(x)$ is deduced from Lemma 2 and (16)

$$\begin{aligned} \deg\left(\left(1-x^2\right)\left(\mathsf{a}_n^{[2]}(x)\right)'\right) &= d+1 && \text{with leading coefficient } -d \\ \deg\left(\widehat{a}_n(x)\mathsf{a}_n^{[2]}(x)\right) &= d+1 && \text{with leading coefficient } -n \\ \deg\left(\widehat{d}_n\mathsf{b}_n^{[2]}(x)\right) &= d && \text{with leading coefficient } \widehat{d}_n B_n^{[2]}. \end{aligned}$$

This establishes the entries in the first row of the table. The entries in the second row are obtained by a similar argument, from which we deduce

$$\begin{aligned} \deg\left(\left(1-x^2\right)\left(\mathsf{b}_n^{[2]}(x)\right)'\right) &= d && \text{with leading coefficient } -(d-1)B_n^{[2]} \\ \deg\left(\widehat{b}_n\mathsf{a}_n^{[2]}(x)\right) &= d && \text{with leading coefficient } \widehat{b}_n \\ \deg\left(\widehat{c}_n(x)\mathsf{b}_n^{[2]}(x)\right) &= d && \text{with leading coefficient } (n+\alpha+\beta)B_n^{[2]} \end{aligned}$$

which yields the desired result. To obtain (25), consider (24) for $n-1$,

$$(1-x^2)(\ell(x)g_{n-1}(x))' = \mathsf{a}_{n-1}^{[3]}(x)\,p_{n-1}(x) + \mathsf{b}_{n-1}^{[3]}(x)\,p_{n-2}(x)$$

apply the three-term recurrence relation (15) to substitute $p_{n-2}(x)$

$$p_{n-2}(x) = \frac{(x-\gamma_{1,n-1})p_{n-1}(x) - p_n(x)}{\gamma_{2,n-1}}$$

and collect terms in $p_{n-1}$ and $p_n$. The degree and leading coefficient of $\mathsf{c}_n^{[3]}(x)$ is easily deduced from the second row of the table. Finally, the degree and leading coefficient of $\mathsf{d}_n^{[3]}(x)$ follow from the first and second rows of the table evaluated at $n-1$. In this case, however, only an upper bound for the degree can be established. □

**Table 2.** Coefficients of Lemma 3.

| | Expression | Degree | Leading Coefficient |
|---|---|---|---|
| $\mathsf{a}_n^{[3]}(x)$ | $(1-x^2)\left(\mathsf{a}_n^{[2]}(x)\right)' + \widehat{a}_n(x)\mathsf{a}_n^{[2]}(x) + \widehat{d}_n\mathsf{b}_n^{[2]}(x)$ | $d+1$ | $-(d+n)$ |
| $\mathsf{b}_n^{[3]}(x)$ | $(1-x^2)\left(\mathsf{b}_n^{[2]}(x)\right)' + \widehat{b}_n\mathsf{a}_n^{[2]}(x) + \widehat{c}_n(x)\mathsf{b}_n^{[2]}(x)$ | $d$ | $(n+\alpha+\beta-d+1)B_n^{[2]} + \widehat{b}_n$ |
| $\mathsf{c}_n^{[3]}(x)$ | $-\frac{\mathsf{b}_{n-1}^{[3]}(x)}{\gamma_{2,n-1}}$ | $d$ | $-\frac{(n+\alpha+\beta-d)B_{n-1}^{[2]}+\widehat{b}_{n-1}}{\gamma_{2,n-1}}$ |
| $\mathsf{d}_n^{[3]}(x)$ | $\mathsf{a}_{n-1}^{[3]}(x) + \frac{(x-\gamma_{1,n-1})}{\gamma_{2,n-1}}\,\mathsf{b}_{n-1}^{[3]}(x)$ | $\leq d+1$ | $-(d+n-1) + \frac{(n+\alpha+\beta-d)B_{n-1}^{[2]}+\widehat{b}_{n-1}}{\gamma_{2,n-1}}$ |

**Theorem 1** (Ladder differential operators)**.** *Under the hypotheses stated above, we obtain explicit formulas for the corresponding ladder operators:*

$$g_{n-1}(x) = \mathsf{a}_n^{[4]}(x)\, g_n(x) + \mathsf{b}_n^{[4]}(x)\, g_n'(x), \tag{26}$$

$$g_n(x) = \mathsf{c}_n^{[4]}(x)\, g_{n-1}(x) + \mathsf{d}_n^{[4]}(x)\, g_{n-1}'(x), \tag{27}$$

*where*

$$\mathsf{a}_n^{[4]}(x) = \frac{\mathsf{q}_n^{[2]}(x)}{\mathsf{q}_n^{[1]}(x)},\ \mathsf{b}_n^{[4]}(x) = \frac{\mathsf{q}_n^{[0]}(x)}{\mathsf{q}_n^{[1]}(x)},\ \mathsf{c}_n^{[4]}(x) = \frac{\mathsf{q}_n^{[3]}(x)}{\mathsf{q}_n^{[4]}(x)},\ \mathsf{d}_n^{[4]}(x) = \frac{\mathsf{q}_n^{[0]}(x)}{\mathsf{q}_n^{[4]}(x)},$$

*and the expressions of* $\mathsf{q}_n^{[v]}(x)$ *are given in Table 3.*

**Table 3.** Coefficients of Theorem 1.

| | Expression | Degree |
|---|---|---|
| $\mathsf{q}_n^{[0]}(x)$ | $(1-x^2)\Delta_n(x)$ | $2d+2$ |
| $\mathsf{q}_n^{[1]}(x)$ | $\mathsf{b}_n^{[3]}(x)\,\mathsf{a}_n^{[2]}(x) - \mathsf{a}_n^{[3]}(x)\,\mathsf{b}_n^{[2]}(x)$ | $2d$ |
| $\mathsf{q}_n^{[2]}(x)$ | $(1-x^2)\ell'(x)\delta_n(x) + \mathsf{b}_n^{[3]}(x)\,\mathsf{c}_n^{[2]}(x) - \mathsf{a}_n^{[3]}(x)\,\mathsf{d}_n^{[2]}(x)$ | $2d+1$ |
| $\mathsf{q}_n^{[3]}(x)$ | $(1-x^2)\ell'(x)\delta_n(x) + \mathsf{c}_n^{[3]}(x)\,\mathsf{b}_n^{[2]}(x) - \mathsf{d}_n^{[3]}(x)\,\mathsf{a}_n^{[2]}(x)$ | $2d+1$ |
| $\mathsf{q}_n^{[4]}(x)$ | $\mathsf{c}_n^{[3]}(x)\,\mathsf{d}_n^{[2]}(x) - \mathsf{d}_n^{[3]}(x)\,\mathsf{c}_n^{[2]}(x)$ | $2d$ |

**Proof.** Substitute (22) and (23) into (24) and (25) to obtain

$$\begin{aligned}
\left(1-x^2\right)(\ell(x)g_n(x))' &= \frac{\ell(x)}{\Delta_n(x)}\left(\mathsf{a}_n^{[3]}(x)\,\mathsf{d}_n^{[2]}(x) - \mathsf{b}_n^{[3]}(x)\,\mathsf{c}_n^{[2]}(x)\right)g_n(x)\\
&\quad + \frac{\ell(x)}{\Delta_n(x)}\left(\mathsf{a}_n^{[2]}(x)\,\mathsf{b}_n^{[3]}(x) - \mathsf{a}_n^{[3]}(x)\,\mathsf{b}_n^{[2]}(x)\right)g_{n-1}(x),\\
\left(1-x^2\right)(\ell(x)g_{n-1}(x))' &= \frac{\ell(x)}{\Delta_n(x)}\left(\mathsf{c}_n^{[3]}(x)\,\mathsf{d}_n^{[2]}(x) - \mathsf{d}_n^{[3]}(x)\,\mathsf{c}_n^{[2]}(x)\right)g_n(x)\\
&\quad + \frac{\ell(x)}{\Delta_n(x)}\left(\mathsf{b}_n^{[3]}(x)\,\mathsf{a}_n^{[2]}(x) - \mathsf{a}_n^{[3]}(x)\,\mathsf{b}_n^{[2]}(x)\right)g_{n-1}(x).
\end{aligned}$$

Finally, collect terms to obtain (26) and (27). □

## 4. Results on Structural Properties

In this section, we introduce a family of ladder operators associated with the Jacobi–Sobolev-type orthogonal polynomials. Their construction is based on the connection formulas established in the preceding sections. These operators constitute the main analytic tool used throughout the remainder of the paper, allowing us to derive both a second-order differential equation and a structural three-term recurrence relation. In this sense, they provide Sobolev analogues of the corresponding classical constructions for Jacobi polynomials.

**Definition 2** (Ladder Jacobi–Sobolev differential operators)**.** *Denote by* $\mathcal{I}$ *the identity operator. Consider the following differential operators on* $\mathbb{P}$ *given by*

$$\mathcal{L}_n := \mathsf{a}_n^{[4]}(x)\,\mathcal{I} + \mathsf{b}_n^{[4]}(x)\,\frac{d}{dx} \quad \textit{(lowering Jacobi–Sobolev differential operator)},$$

$$\mathcal{R}_n := \mathsf{c}_n^{[4]}(x)\,\mathcal{I} + \mathsf{d}_n^{[4]}(x)\,\frac{d}{dx} \quad \textit{(raising Jacobi–Sobolev differential operator)}.$$

*these operator are the ladder-operators corresponding to the sequence* $\{g_n\}_{n\geqslant d}$.

**Remark 3.** *The previous ladder-operators correspond to the generalization of the classical standard ladder operators. Consider in* (1) *that* $d\mu(x) = d\mu^{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}dx$, *where* $\alpha, \beta > -1$, *whose support is* $[-1,1]$, *and* $\lambda = 0$. *Then, it is straightforward to verify that* $\mathcal{L}_n \equiv \widehat{\mathcal{L}}_n$ *and* $\mathcal{R}_n \equiv \widehat{\mathcal{R}_n}$.

**Theorem 2.** *For every* $n \geq d$, *the nth Jacobi–Sobolev orthogonal polynomial* $g_n$ *is a polynomial solution of the following second-order ordinary differential equation:*

$$\mathsf{P}_n^{[2]}(x)\, g_n''(x) + \mathsf{P}_n^{[1]}(x)\, g_n'(x) + \mathsf{P}_n^{[0]}(x)\, g_n(x) = 0, \tag{28}$$

*where*

$$\begin{aligned}
\mathsf{P}_n^{[2]}(x) =& \mathsf{q}_n^{[1]}(x)\left(\mathsf{q}_n^{[0]}(x)\right)^2,\\
\mathsf{P}_n^{[1]}(x) =& \mathsf{q}_n^{[0]}(x)\left(\mathsf{q}_n^{[1]}(x)\,\mathsf{q}_n^{[2]}(x) + \mathsf{q}_n^{[1]}(x)\,\mathsf{q}_n^{[3]}(x) + \left(\mathsf{q}_n^{[0]}(x)\right)'\mathsf{q}_n^{[1]}(x) - \mathsf{q}_n^{[0]}(x)\left(\mathsf{q}_n^{[1]}(x)\right)'\right),\\
\mathsf{P}_n^{[0]}(x) =& \mathsf{q}_n^{[1]}(x)\,\mathsf{q}_n^{[2]}(x)\,\mathsf{q}_n^{[3]}(x) + \mathsf{q}_n^{[0]}(x)\left(\left(\mathsf{q}_n^{[2]}(x)\right)'\mathsf{q}_n^{[1]}(x) - \mathsf{q}_n^{[2]}(x)\left(\mathsf{q}_n^{[1]}(x)\right)'\right)\\
&- \mathsf{q}_n^{[4]}(x)\left(\mathsf{q}_n^{[1]}(x)\right)^2,\\
&\deg\left(\mathsf{P}_n^{[2]}(x)\right) = 6d+4,\ \deg\left(\mathsf{P}_n^{[1]}(x)\right) \leqslant 6d+3 \ \textit{and}\ \deg\left(\mathsf{P}_n^{[0]}(x)\right) \leqslant 6d+2.
\end{aligned} \tag{29}$$

*are polynomial coefficients.*

**Proof.** First, let us re-arrange the ladder Equations (26) and (27) as

$$\mathcal{L}_n[g_n(x)] = \left(\mathsf{a}_n^{[4]}(x)\,\mathcal{I} + \mathsf{b}_n^{[4]}(x)\,\frac{d}{dx}\right)g_n(x) = g_{n-1}(x), \tag{30}$$

$$\mathcal{R}_n[g_{n-1}(x)] = \left(\mathsf{c}_n^{[4]}(x)\,\mathcal{I} + \mathsf{d}_n^{[4]}(x)\,\frac{d}{dx}\right)g_{n-1}(x) = g_n(x). \tag{31}$$

Using the ladder-operator representation, we derive a second-order ordinary differential equation satisfied by $g_n$ by applying $\mathcal{R}_n$ to both sides of the identity determined by $\mathcal{L}_n[g_n] = g_{n-1}$.

$$\begin{aligned}
0 =& \mathcal{R}_n[\mathcal{L}_n[g_n(x)]] - g_n(x)\\
=& \mathsf{b}_n^{[4]}(x)\,\mathsf{d}_n^{[4]}(x)\,g_n''(x) + \left(\mathsf{a}_n^{[4]}(x)\,\mathsf{d}_n^{[4]}(x) + \mathsf{b}_n^{[4]}(x)\,\mathsf{c}_n^{[4]}(x) + \mathsf{d}_n^{[4]}(x)\,\mathsf{b}_n^{[4]}(x)\,'(x)\right)g_n'(x)\\
&+ \left(\mathsf{a}_n^{[4]}(x)\,\mathsf{c}_n^{[4]}(x) + \mathsf{d}_n^{[4]}(x)\,\mathsf{a}_n^{[4]}(x)\,'(x) - 1\right)g_n(x)\\
=& \frac{\left(\mathsf{q}_n^{[0]}(x)\right)^2}{\mathsf{q}_n^{[1]}(x)\,\mathsf{q}_n^{[4]}(x)}\,g_n''(x)\\
&+ \frac{\mathsf{q}_n^{[0]}(x)\left(\mathsf{q}_n^{[1]}(x)\,\mathsf{q}_n^{[2]}(x) + \mathsf{q}_n^{[1]}(x)\,\mathsf{q}_n^{[3]}(x) + \left(\mathsf{q}_n^{[0]}(x)\right)'\mathsf{q}_n^{[1]}(x) - \mathsf{q}_n^{[0]}(x)\left(\mathsf{q}_n^{[1]}(x)\right)'\right)}{\mathsf{q}_n^{[4]}(x)\left(\mathsf{q}_n^{[1]}(x)\right)^2}\,g_n'(x)\\
&+ \left(\frac{\mathsf{q}_n^{[1]}(x)\,\mathsf{q}_n^{[2]}(x)\,\mathsf{q}_n^{[3]}(x) + \mathsf{q}_n^{[0]}(x)\left(\left(\mathsf{q}_n^{[2]}(x)\right)'\mathsf{q}_n^{[1]}(x) - \mathsf{q}_n^{[2]}(x)\left(\mathsf{q}_n^{[1]}(x)\right)'\right)}{\mathsf{q}_n^{[4]}(x)\left(\mathsf{q}_n^{[1]}(x)\right)^2} - 1\right)g_n(x).
\end{aligned}$$

Which proves (28). The degree of $\mathsf{P}_n^{[2]}(x)$ is a consequence of $\deg\left(\mathsf{q}_n^{[1]}(x)\right) = 2d$ and $\deg\left(\mathsf{q}_n^{[0]}(x)\right) = 2d+2$. This degree is exact. On the other hand, the upper bounds for the degrees of $\mathsf{P}_n^{[1]}(x)$ and $\mathsf{P}_n^{[0]}(x)$ are computed from Theorem 1. To prove that these bounds are sharp, consider the case $p_n = g_n$, where $d = 0$. The corresponding calculations are given in the next Remark. □

**Remark 4** (Reduction to classical Jacobi differential equation)**.** *Under the assumptions of Remark 3, (1) reduces to the classical Jacobi inner product. Therefore, the associated Jacobi–Sobolev orthogonal polynomials coincide with the classical Jacobi polynomials, namely, $g_n(x) \equiv p_n(x)$.*

*In the case, $\mathsf{a}_n^{[1]}(x) \equiv 1$, $\mathsf{b}_n^{[1]}(x) \equiv 0$, and $\ell(x) \equiv 1$. The remaining terms of the differential Equation (28), yield*

$$\begin{aligned}&\ell(x) \equiv 1,\ \mathsf{a}_n^{[1]}(x) \equiv \mathsf{a}_n^{[2]}(x) \equiv \mathsf{d}_n^{[2]}(x) = 1,\ \mathsf{b}_n^{[1]}(x) \equiv \mathsf{b}_n^{[2]}(x) \equiv \mathsf{c}_n^{[2]}(x) \equiv 0,\\ &\Delta_n(x) \equiv 1,\ \mathsf{a}_n^{[3]}(x) = \widehat{a}_n(x), \mathsf{b}_n^{[3]}(x) = \widehat{b}_n,\ \mathsf{c}_n^{[3]}(x) = -\gamma_{2,n-1}^{-1}\widehat{b}_{n-1} \ \textit{and}\\ &\mathsf{d}_n^{[3]}(x) = \widehat{a}_{n-1}(x) + \gamma_{2,n-1}^{-1}\widehat{b}_{n-1}(x-\gamma_{1,n-1}).\end{aligned}$$

*Thus,*

$$\begin{aligned}\mathsf{q}_n^{[0]}(x) &= \left(1-x^2\right),\ \mathsf{q}_n^{[1]}(x) = \widehat{b}_n, \quad \mathsf{q}_n^{[2]}(x) = -\widehat{a}_n(x),\\ \mathsf{q}_n^{[3]}(x) &= -\widehat{a}_{n-1}(x) - \gamma_{2,n-1}^{-1}\widehat{b}_{n-1}(x-\gamma_{1,n-1})\\ &= -(n+\alpha+\beta)x + \frac{(n+\alpha+\beta)\,(\alpha-\beta)}{2n+\beta+\alpha} \ \textit{and}\\ \mathsf{q}_n^{[4]}(x) &= -\gamma_{2,n-1}^{-1}\widehat{b}_{n-1} = -(2n+\alpha+\beta-1).\end{aligned} \tag{32}$$

*Substituting (32) into (29), we have*

$$\begin{array}{lll}\mathsf{P}_n^{[2]}(x) & = \widehat{b_n}\left(1-x^2\right)^2, & \deg \mathsf{P}_n^{[2]}(x) = 4,\\ \mathsf{P}_n^{[1]}(x) & = \widehat{b_n}\left(1-x^2\right)(\beta-\alpha-(\alpha+\beta+2)x), & \deg \mathsf{P}_n^{[1]}(x) = 3,\\ \mathsf{P}_n^{[0]}(x) & = \widehat{b_n}\left(1-x^2\right)n(n+\alpha+\beta+1), & \deg \mathsf{P}_n^{[0]}(x) = 2,\end{array}$$

*and after reducing the factor $\widehat{b_n}(1-x^2)$ we can verify that differential Equation (28) reduces to (18).*

*Note that $\Lambda = 0$ corresponds to the case $d = 0$, which proves that the degrees given in Theorem 2, though not exact, are sharp.*

*Although (28) reduces to (18) in the trivial case, this equation differs substantially from the setting of Bochner's theorem. Classical orthogonal polynomials are eigenfunctions of a fixed second-order differential operator; see [1] and, for the Jacobi family, [35] (Ths. 4.2.1–4.2.2) In contrast, the present work shows that Jacobi–Sobolev polynomials satisfy a second-order differential equation with polynomial coefficients that depend on n. Consequently, the corresponding differential operator is not fixed, but varies with the degree of the polynomial.*

Furthermore, each polynomial of degree $n$ in the sequence $\{g_n\}_{n\geq d}$ can be generated through the repeated application of the raising differential operator to the first Sobolev-type polynomial; namely, the polynomial of degree zero. This characterization may serve as a useful tool for both symbolic and numerical computations, as it provides a recursive procedure for constructing the entire Jacobi–Sobolev polynomial sequence.

**Theorem 3.** *For $d \geq 1$ and $n \geq d$, the nth Jacobi–Sobolev orthogonal polynomial is obtained as*

$$g_n(x) = (\mathcal{R}_n\mathcal{R}_{n-1}\mathcal{R}_{n-2}\cdots\mathcal{R}_d)[g_{d-1}(x)],$$

*where* $g_0(x) = 1$.

**Proof.** Proceed by induction. Note that $\mathcal{R}_d[g_{d-1}(x)] = g_d(x)$. Assume

$$g_{n-1}(x) = (\mathcal{R}_{n-1}\mathcal{R}_{n-2}\cdots\mathcal{R}_d)[g_{d-1}(x)],$$

and apply $\mathcal{R}_n$ to obtain, using Definition 2,

$$g_n(x) = (\mathcal{R}_n\mathcal{R}_{n-1}\mathcal{R}_{n-2}\cdots\mathcal{R}_d)[g_{d-1}(x)],$$

which establishes the formula. □

We now derive a structural three-term recurrence relation satisfied by the monic Jacobi–Sobolev orthogonal polynomials. To this end, we substitute the explicit formulas for the ladder operators into (31) and then perform the index shift $n \mapsto n+1$. This yields

$$\begin{aligned}
\mathsf{c}_n^{[4]}(x)\, g_n(x) + \mathsf{d}_n^{[4]}(x)\, \frac{d}{dx} g_n(x) &= g_{n-1}(x),\\
\mathsf{a}_n^{[4]}(x)\, g_n(x) + \mathsf{b}_n^{[4]}(x)\, \frac{d}{dx} g_n(x) &= g_{n+1}(x).
\end{aligned}$$

Multiplying the two identities by the polynomial factors $-\mathsf{b}_n^{[4]}(x)$ and $\mathsf{d}_n^{[4]}(x)$, respectively, and then adding the resulting equations yields the following structural recurrence relation with polynomial coefficients.

**Theorem 4.** *Consider the hypothesis of Theorem 2. The sequence of Jacobi–Sobolev orthogonal polynomials satisfies the following structural recurrence relation:*

$$\begin{aligned}
\mathsf{q}_{n+1}^{[4]}(x)\,\mathsf{q}_n^{[0]}(x)\, g_{n+1}(x) =& \Big[\mathsf{q}_{n+1}^{[3]}(x)\,\mathsf{q}_n^{[0]}(x) - \mathsf{q}_n^{[2]}(x)\,\mathsf{q}_{n+1}^{[0]}(x)\Big] g_n(x)\\
&+ \mathsf{q}_n^{[1]}(x)\,\mathsf{q}_{n+1}^{[0]}(x)\, g_{n-1}(x),\\
\mathsf{Q}_n^{[1]}(x)\, g_{n+1}(x) =& \mathsf{Q}_n^{[0]}(x)\, g_n(x) + \mathsf{Q}_n^{[-1]}(x)\, g_{n-1}(x).
\end{aligned} \tag{33}$$

*The* $\mathsf{q}$ *coefficients are defined in Theorem 1, and*

$$\deg \mathsf{Q}_n^{[1]}(x) = 4d+2 \qquad \deg \mathsf{Q}_n^{[0]}(x) = 4d+3 \qquad \deg \mathsf{Q}_n^{[-1]}(x) = 4d+2$$

**Proof.** Considering (27) with $n$ replaced by $n+1$ and using (26), we obtain

$$\begin{aligned}
\mathsf{q}_n^{[2]}(x)\, g_n(x) + \mathsf{q}_n^{[0]}(x)\, g_n'(x) &= \mathsf{q}_n^{[1]}(x)\, g_{n-1}(x),\\
\mathsf{q}_{n+1}^{[3]}(x)\, g_n(x) + \mathsf{q}_{n+1}^{[0]}(x)\, g_n'(x) &= \mathsf{q}_{n+1}^{[4]}(x)\,(x) g_{n+1}(x).
\end{aligned}$$

Multiplying the first equation by $\mathsf{q}_{n+1}^{[0]}(x)$ and the second by $\mathsf{q}_n^{[0]}(x)$, and then subtracting the resulting identities, it yields

$$\Big(\mathsf{q}_{n+1}^{[3]}(x)\,\mathsf{q}_n^{[0]}(x) - \mathsf{q}_n^{[2]}(x)\,\mathsf{q}_{n+1}^{[0]}(x)\Big) g_n(x) = \mathsf{q}_{n+1}^{[4]}(x)\,\mathsf{q}_n^{[0]}(x)\, g_{n+1}(x) - \mathsf{q}_n^{[1]}(x)\,\mathsf{q}_{n+1}^{[0]}(x)\, g_{n-1}(x).$$

□

**Remark 5** (The classical Jacobi three-term recurrence relation)**.** *Despite the fact that* (14) *is an structural recurrence relation with rational coefficients, if we consider the Jacobi case* $\Lambda = 0$, *and substitute* (32) *into* (33), *it verifies that*

$$\frac{\mathsf{q}_{n+1}^{[3]}(x)\,\mathsf{q}_n^{[0]}(x)\;-\mathsf{q}_n^{[2]}(x)\,\mathsf{q}_{n+1}^{[0]}(x)}{\mathsf{q}_{n+1}^{[4]}(x)\,\mathsf{q}_n^{[0]}(x)}=x-\gamma_{1,n}\quad \textit{and}\quad \frac{\mathsf{q}_n^{[1]}(x)\,\mathsf{q}_{n+1}^{[0]}(x)}{\mathsf{q}_{n+1}^{[4]}(x)\,\mathsf{q}_n^{[0]}(x)}=-\gamma_{2,n},$$

*which corresponds to the classical Jacobi three-term recurrence relation, see* (14).

*Although* (33) *reduces to the classical three-term recurrence relation for Jacobi polynomials in the trivial case, an important distinction should be emphasized. In the classical setting, only the constants in the recurrence coefficients depends on $n$. By contrast, the recurrence relation* (33) *involves polynomial coefficients that depend explicitly on $n$.*

## 5. Some Illustrative Examples

We conclude the paper with a series of examples illustrating the main results established in the preceding sections. In particular, we compute the coefficient $\mathsf{d}_n^{[3]}(x)$ to show that the derived upper bound for its degree is attained. We also determine the rational coefficients appearing in the ladder operators, as well as the polynomial coefficients of the associated second-order differential equation and three-term recurrence relation. Whenever possible, we verify that the upper bounds obtained for the degrees of the relevant polynomials are sharp. Due to the increasing complexity of the expressions involved and space limitations, we restrict our attention to examples of low degree.

**Example 2.** *Consider the Jacobi–Sobolev inner product*

$$\langle f,g\rangle_{Krall}=\int_{-1}^{1} f\cdot g\; d\mu^{0,0}(x)+f(2)\,g(2).$$

*This is a Krall-type inner product where $c=2$, $\alpha=\beta=0$ and $\lambda=1$. Due to the increasing complexity of the algebraic expressions we show some results for $n=2,3\geq 1=d$.*

$$p_2(x)=x^2-\frac{1}{3},\qquad g_2(x)=x^2-\frac{22}{15}x-\frac{26}{45},$$

$$p_3(x)=x^3-\frac{3}{5}x,\qquad g_3(x)=x^3-\frac{1122}{665}x^2-\frac{2811}{3325}x+\frac{1734}{3325},$$

$$p_4(x)=x^4-\frac{6x^2}{7}+\frac{3}{35},\qquad g_4(x)=x^4-\frac{15062x^3}{8757}-\frac{22387x^2}{20433}+\frac{20378x}{20433}+\frac{16288}{102165}.$$

*Coefficient $\mathsf{d}_3^{[3]}(x)$*

$$\mathsf{d}_3^{[3]}(x)\;=\;\frac{133x^2}{6}-\frac{52x}{5}-\frac{106}{15}$$

*The upper bound* $\deg\left(\mathsf{d}_3^{[3]}(x)\right)=2=d+1$ *given in Lemma 3 is sharp.*

*Rational coefficients of the ladder operators*

$$\mathsf{a}_3^{[4]}(x)\;=\;\frac{294079625x^3-1010926350x^2+875111405x-53185370}{209642580x^2-729656352x+621741168},$$

$$\mathsf{b}_3^{[4]}(x)\;=-\frac{294079625x^4-1176318500x^3+882238875x^2+1176318500x-1176318500}{628927740x^2-2188969056x+1865223504},$$

$$\mathsf{c}_3^{[4]}(x)\;=\;\frac{3980025x^3-13681710x^2+11843613x-719802}{6633375x^2-20886320x+15721930},$$

$$\mathsf{d}_3^{[4]}(x)\;=\;\frac{1995x^4-7980x^3+5985x^2+7980x-7980}{9975x^2-31408x+23642}.$$

*Polynomial coefficients of the second-order differential equation*

$$\begin{aligned}
\mathsf{P}_3^{[2]}(x) &= \frac{1164681}{400}x^{10} - \frac{4178467}{125}x^9 + \frac{153765677}{1000}x^8 - \frac{42327518}{125}x^7 + \frac{523719093}{2000}x^6 \\
&\quad + \frac{42381029}{125}x^5 - \frac{85849823}{100}x^4 + \frac{58934364}{125}x^3 + \frac{1508983}{5}x^2 - \frac{54809408}{125}x + \frac{17270588}{125},
\end{aligned}$$

$$\begin{aligned}
\mathsf{P}_3^{[1]}(x) &= \frac{1164681}{200}x^9 - \frac{32671389}{500}x^8 + \frac{301280951}{1000}x^7 - \frac{177745739}{250}x^6 + \frac{203287047}{250}x^5 \\
&\quad - \frac{56390917}{500}x^4 - \frac{99481232}{125}x^3 + \frac{108140802}{125}x^2 - \frac{40550336}{125}x + \frac{2997644}{125},
\end{aligned}$$

$$\begin{aligned}
\mathsf{P}_3^{[0]}(x) &= -\frac{3494043}{100}x^8 + \frac{45889836}{125}x^7 - \frac{782469177}{500}x^6 + \frac{333315027}{100}x^5 \\
&\quad - \frac{797108997}{250}x^4 - \frac{207524379}{500}x^3 + \frac{963857961}{250}x^2 - \frac{16426101}{5}x + \frac{116610366}{125}.
\end{aligned}$$

*Note that, since $d = 1$, the bounds for the degrees of the polynomial coefficients given in Theorem 2 are sharp.*

*Polynomials coefficients of the three-term recurrence relation*

$$\begin{aligned}
\mathsf{Q}_3^{[1]}(x) &= \frac{20433}{20}x^6 - \frac{10890463}{1425}x^5 + \frac{38604577}{1900}x^4 - \frac{26648357}{1425}x^3 \\
&\quad - \frac{13138696}{1425}x^2 + \frac{2502588}{95}x - \frac{17270588}{1425},
\end{aligned}$$

$$\begin{aligned}
\mathsf{Q}_3^{[0]}(x) &= \frac{20433}{20}x^7 - \frac{10938182}{1425}x^6 + \frac{97535024605201}{4741915500}x^5 - \frac{7652890039294}{395159625}x^4 \\
&\quad - \frac{10203630884164}{1185478875}x^3 + \frac{31587373402828}{1185478875}x^2 - \frac{114009405628}{8781325}x + \frac{470934393584}{1185478875},
\end{aligned}$$

$$\begin{aligned}
\mathsf{Q}_3^{[-1]}(x) &= -\frac{14085009x^6}{45125} + \frac{490462891668x^5}{210056875} - \frac{1303946797329x^4}{210056875} + \frac{1200135405852x^3}{210056875} \\
&\quad + \frac{591714313056x^2}{210056875} - \frac{338119659504x}{42011375} + \frac{777798201168}{210056875}.
\end{aligned}$$

**Example 3.** *Consider the Jacobi–Sobolev inner product*

$$\langle f, g\rangle_{S_1} = \int_{-1}^{1} f \cdot g \; d\mu^{0,0}(x) + f'(2)\, g'(2).$$

*This is a Legendre–Sobolev inner product where $c = 2$, $\alpha = \beta = 0$, $\lambda_{1,1} = 1$, and*

$$\Lambda = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

*Due to the increasing complexity of the algebraic expressions we show some results for $n = 2, 3 \geq 2 = d$.*

$$g_2(x) = x^2 - \frac{12x}{5} - \frac{1}{3},$$

$$g_3(x) = x^3 - \frac{513x^2}{185} - \frac{726x}{925} + \frac{171}{185},$$

$$g_4(x) = x^4 - \frac{57000x^3}{23483} - \frac{176898x^2}{164381} + \frac{237000x}{164381} + \frac{130449}{821905}.$$

*Coefficient* $\mathsf{d}_3^{[3]}(x)$

$$\mathsf{d}_3^{[3]}(x) = 37x^3 - \frac{454x^2}{5} + \frac{106x}{5} + \frac{124}{5}$$

*The upper bound* $\deg\left(\mathsf{d}_3^{[3]}(x)\right) = 3 = d+1$ *given in Lemma 3 is sharp.*

*Rational coefficients of the ladder operators*

$$\mathsf{a}_3^{[4]}(x) = \frac{25326500x^4 - 128549100x^3 + 206846280x^2 - 94400320x - 16491270}{13033065x^3 - 73021536x^2 + 131135634x - 77996526},$$

$$\mathsf{b}_3^{[4]}(x) = -\frac{25326500x^5 - 151959000x^4 + 256105675x^3 - 5681350x^2 - 281432175x + 157640350}{39099195x^3 - 219064608x^2 + 393406902x - 233989578},$$

$$\mathsf{c}_3^{[4]}(x) = \frac{205350x^4 - 1042290x^3 + 1555587x^2 - 765408x - 12168}{342250x^3 - 1571760x^2 + 2268840x - 1101860},$$

$$\mathsf{d}_3^{[4]}(x) = \frac{740x^5 - 4440x^4 + 7483x^3 - 166x^2 - 8223x + 4606}{3700x^3 - 16992x^2 + 24528x - 11912}.$$

*Polynomial coefficients of the second-order differential equation*

$$\begin{aligned}
\mathsf{P}_3^{[2]}(x) = {} & \frac{7819839x^{16}}{100} - \frac{461425059x^{15}}{250} + \frac{19638456093x^{14}}{1000} \\
& - \frac{11456791669917x^{13}}{92500} + \frac{755381837017539x^{12}}{1480000} - \frac{194094418353183711x^{11}}{136900000} \\
& + \frac{11078508177647442x^{10}}{4278125} - \frac{363116702267243829x^9}{136900000} - \frac{339751447385678 1x^8}{273800000} \\
& + \frac{622678617270363951x^7}{136900000} - \frac{942456877423423389x^6}{136900000} + \frac{572598015953782581x^5}{136900000} \\
& + \frac{1706047180019757x^4}{1850000} - \frac{60848492289842673x^3}{17112500} + \frac{11592727351752654x^2}{4278125} \\
& - \frac{8379302890699503x}{8556250} + \frac{620518014851301}{4278125},
\end{aligned}$$

$$\begin{aligned}
\mathsf{P}_3^{[1]}(x) = {} & \frac{7819839x^{15}}{50} - \frac{915084837x^{14}}{250} + \frac{19378720863x^{13}}{500} - \frac{2266364029958 1x^{12}}{92500} \\
& + \frac{151830948965967x^{11}}{148000} - \frac{407046499978332933x^{10}}{136900000} \\
& + \frac{164531417140908189x^9}{27380000} - \frac{1116069565592432979x^8}{136900000} + \frac{428082660324802593x^7}{68450000} \\
& + \frac{30949302613872969x^6}{136900000} - \frac{475690803573192723x^5}{68450000} + \frac{1211498632685085471x^4}{136900000} \\
& - \frac{4066246547259561x^3}{684500} + \frac{3888892525061943 9x^2}{17112500} - \frac{374765137755246x}{855625} \\
& + \frac{1216221745599}{46250},
\end{aligned}$$

$$
\begin{aligned}
\mathsf{P}_3^{[0]}(x) &= -\frac{23459517x^{14}}{25} + \frac{2594270916x^{13}}{125} - \frac{25962783282x^{12}}{125} \\
&+ \frac{28683594666054x^{11}}{23125} - \frac{72490533441381x^{10}}{14800} + \frac{228244936540560111x^{9}}{17112500} \\
&- \frac{137544487122934989x^{8}}{5476000} + \frac{1063424215202357601x^{7}}{34225000} - \frac{2707354087869037131x^{6}}{136900000} \\
&- \frac{239126912729943291x^{5}}{34225000} + \frac{2048386417802576253x^{4}}{68450000} - \frac{277308406250173341x^{3}}{8556250} \\
&+ \frac{16413109089883056x^{2}}{855625} - \frac{26839447754425236x}{4278125} + \frac{763726967839269}{855625}.
\end{aligned}
$$

*Note that, since $d = 2$, the bounds for the degrees of the polynomial coefficients given in Theorem 2 are sharp.*

*Polynomials coefficients of the three-term recurrence relation*

$$
\begin{aligned}
\mathsf{Q}_3^{[1]}(x) &= \frac{164381x^{10}}{20} - \frac{237244217x^{9}}{1850} + \frac{12675313287x^{8}}{14800} - \frac{4329831246321x^{7}}{1369000} \\
&+ \frac{3762194805243x^{6}}{547600} - \frac{5617077727023x^{5}}{684500} + \frac{3859537803597x^{4}}{1369000} \\
&+ \frac{8251282663743x^{3}}{1369000} - \frac{6386116923837x^{2}}{684500} + \frac{1872066189301x}{342250} - \frac{209563665941}{171125},
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{Q}_3^{[0]}(x) &= \frac{164381x^{11}}{20} - \frac{463975981x^{10}}{3700} + \frac{13171234154924769x^{9}}{16074113500} - \frac{23775219363941283x^{8}}{8037056750} \\
&+ \frac{25136526042259317x^{7}}{4018528375} - \frac{114972369924524469x^{6}}{16074113500} + \frac{15536099250540267x^{5}}{8037056750} \\
&+ \frac{47652377684315523x^{4}}{8037056750} - \frac{66657483977587881x^{3}}{8037056750} + \frac{17697246812211883x^{2}}{4018528375} \\
&- \frac{2901670703107246x}{4018528375} - \frac{388814398033572}{4018528375},
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{Q}_3^{[-1]}(x) &= -\frac{4963061601x^{10}}{2738000} + \frac{7162979075757x^{9}}{253265000} - \frac{48073863385233x^{8}}{253265000} \\
&+ \frac{179404285894479x^{7}}{253265000} - \frac{793367256687039x^{6}}{506530000} + \frac{489108450019077x^{5}}{253265000} \\
&- \frac{191958782312457x^{4}}{253265000} - \frac{334121352957621x^{3}}{253265000} + \frac{278723810174103x^{2}}{126632500} \\
&- \frac{85388590507923x}{63316250} + \frac{9965967111387}{31658125}.
\end{aligned}
$$

**Example 4.** *Consider the Jacobi–Sobolev inner product*

$$\langle f, g \rangle_{S_2} = \int_{-1}^{1} f \cdot g \, d\mu^{0,0}(x) + f(2)\,g(2) + f'(2)\,g(2) + f(2)\,g'(2) + f'(2)\,g'(2).$$

*This is a Legendre–Sobolev inner product where $c = 2$, $\alpha = \beta = 0$ and*

$$\Lambda = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

*Due to the increasing complexity of the algebraic expressions we show some results for $n = 2, 3 \geq 2 = d$.*

$$g_2(x) = x^2 - \frac{23x}{10} - \frac{53}{90},$$

$$g_3(x) = x^3 - \frac{897x^2}{395} - \frac{1653x}{1975} + \frac{1443}{1975},$$

$$g_4(x) = x^4 - \frac{6253x^3}{2892} - \frac{51551x^2}{47236} + \frac{60125x}{47236} + \frac{38041}{236180}.$$

*Coefficient* $\mathsf{d}_3^{[3]}(x)$

$$\mathsf{d}_3^{[3]}(x) = \frac{553x^3}{24} - \frac{4781x^2}{60} + \frac{3313x}{120} + \frac{308}{15}.$$

*The upper bound* $\deg\left(\mathsf{d}_3^{[3]}(x)\right) = 3 = d+1$ *given in Lemma 3 is sharp.*

*Rational coefficients of the ladder operators*

$$\mathsf{a}_3^{[4]}(x) = \frac{431409125x^4 - 2261894425x^3 + 3912672105x^2 - 2206430895x - 58403910}{246745440x^3 - 1351035072x^2 + 2436577488x - 1465459056},$$

$$\mathsf{b}_3^{[4]}(x) = -\frac{431409125x^5 - 2588454750x^4 + 4642991950x^3 - 657801400x^2 - 5074401075x + 3246256150}{740236320x^3 - 4053105216x^2 + 7309732464x - 4396377168},$$

$$\mathsf{c}_3^{[4]}(x) = \frac{9829575x^4 - 51536835x^3 + 87592401x^2 - 50273109x + 226368}{16382625x^3 - 83796090x^2 + 139501360x - 77795645},$$

$$\mathsf{d}_3^{[4]}(x) = \frac{8295x^5 - 49770x^4 + 89274x^3 - 12648x^2 - 97569x + 62418}{41475x^3 - 212142x^2 + 353168x - 196951}.$$

*Polynomial coefficients of the second-order differential equation*

$$\begin{aligned}
\mathsf{P}_3^{[2]}(x) = {} & \frac{8396199x^{16}}{400} - \frac{246380379x^{15}}{500} + \frac{4197380929x^{14}}{800} - \frac{52664047578611x^{13}}{1580000} \\
& + \frac{22023106602109x^{12}}{158000} - \frac{618852047643302283x^{11}}{156025000} + \frac{2351114682595897339x^{10}}{312050000} \\
& - \frac{2612051659367855551x^{9}}{312050000} + \frac{427639988143026687x^{8}}{312050000} + \frac{381410843571653337x^{7}}{31205000} \\
& - \frac{1308530414932927083x^{6}}{624100000} + \frac{915962198671739543x^{5}}{624100000} + \frac{300667950299318847x^{4}}{312050000} \\
& - \frac{832596899134658821x^{3}}{78012500} + \frac{708904488952726x^{2}}{780125} - \frac{138528365496130 21x}{39006250} \\
& + \frac{1101357703688791}{19503125}.
\end{aligned}$$

$$\begin{aligned}
\mathsf{P}_3^{[1]}(x) = {} & \frac{8396199x^{15}}{200} - \frac{487255167x^{14}}{500} + \frac{515704189x^{13}}{50} - \frac{103773746560183x^{12}}{1580000} \\
& + \frac{220563912842943x^{11}}{790000} - \frac{2584934927392062 83x^{10}}{312050000} + \frac{108027955104079573x^{9}}{62410000} \\
& - \frac{192363400289842821x^{8}}{78012500} + \frac{647892833604004149x^{7}}{312050000} - \frac{691953345529704 69x^{6}}{312050000} \\
& - \frac{242876666629263 33x^{5}}{12482000} + \frac{1727915153043639989x^{4}}{624100000} - \frac{155232768243170647x^{3}}{78012500} \\
& + \frac{12555885244200279x^{2}}{15602500} - \frac{3141514072785146x}{19503125} + \frac{358319742635413}{39006250},
\end{aligned}$$

$$
\begin{aligned}
\mathsf{P}_3^{[0]}(x) = & -\frac{25188597x^{14}}{100}+\frac{5594093379x^{13}}{1000}-\frac{113117506047x^{12}}{2000}+\frac{542436805224783x^{11}}{1580000} \\
& -\frac{2191074274045143x^{10}}{1580000}+\frac{2428102218551182641x^{9}}{624100000}-\frac{4755623402955933189x^{8}}{624100000} \\
& +\frac{6225608560990832559x^{7}}{624100000}-\frac{4475086392077301879x^{6}}{624100000}-\frac{653351522393611521x^{5}}{624100000} \\
& +\frac{5637080540753530119x^{4}}{624100000}-\frac{42061644334861164x^{3}}{3900625}+\frac{533613846436467867x^{2}}{78012500} \\
& -\frac{93015623457015039x}{39006250}+\frac{14103021301977303}{39006250}.
\end{aligned}
$$

*Note that, since $d = 2$, the bounds for the degrees of the polynomial coefficients given in Theorem 2 are sharp.*

*Polynomials coefficients of the three-term recurrence relation*

$$
\begin{aligned}
\mathsf{Q}_3^{[1]}(x) = & \frac{35427x^{10}}{10}-\frac{108278926x^{9}}{1975}+\frac{8680952441x^{8}}{23700}-\frac{4261728141601x^{7}}{3120500} \\
& +\frac{2835686968373x^{6}}{936150}-\frac{17637273530177x^{5}}{4680750}+\frac{4796615513109x^{4}}{3120500} \\
& +\frac{23519410247473x^{3}}{9361500}-\frac{4028017998347x^{2}}{936150}+\frac{2087796945577x}{780125}-\frac{1482169359958}{2340375},
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{Q}_3^{[0]}(x) = & \frac{35427x^{11}}{10}-\frac{860146147x^{10}}{15800}+\frac{271611800103357x^{9}}{752040500}-\frac{361909485263578 63x^{8}}{27073458000} \\
& +\frac{1768635858017371x^{7}}{601632400}-\frac{48880898116412341x^{6}}{13536729000}+\frac{18980659237380911x^{5}}{13536729000} \\
& +\frac{22531152797197639x^{4}}{9024486000}-\frac{22430278760909033x^{3}}{5414691600}+\frac{34070622152021371x^{2}}{13536729000} \\
& -\frac{1272706352031317x}{2256121500}-\frac{38511114013393}{3384182250},
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{Q}_3^{[-1]}(x) = & \frac{677456784x^{10}}{780125}+\frac{4141151832384x^{9}}{308149375}-\frac{193844810811852x^{8}}{2157045625} \\
& +\frac{144824378512248x^{7}}{431409125}-\frac{1611019505749698x^{6}}{2157045625}+\frac{2014645618130598x^{5}}{2157045625} \\
& -\frac{837838990834182x^{4}}{2157045625}-\frac{265277930397534x^{3}}{431409125}+\frac{2301396739128468x^{2}}{2157045625} \\
& -\frac{1441365921530856x}{2157045625}+\frac{343179736275024}{2157045625}.
\end{aligned}
$$

## 6. Conclusions

In this paper, we have established the following results:

- We derived a connection formula between Jacobi–Sobolev orthogonal polynomials and the classical Jacobi polynomials under minimal assumptions; namely, the positive semidefiniteness of the Sobolev matrix (see Lemma 1). This framework includes both diagonal and non-diagonal Sobolev products and extends several previously known results obtained only in particular cases.
- We introduced raising and lowering ladder operators for Jacobi–Sobolev orthogonal polynomials, see Theorem 1 and Definition 2. These ladder operators constitute the foundation of the subsequent developments in the paper. Furthermore, we show that, in the trivial case (i.e., $d = 0$, $\Lambda = 0$, and $g_n \equiv p_n$), these operators reduce to classical Jacobi ladder operators (see Remark 3).

- We prove that the Jacobi–Sobolev-type polynomials are solutions of a second-order differential equation with polynomial coefficients, see Theorem 2. We prove that this equation reduces to the second-order differential equation of Jacobi polynomials (32) in the trivial case (i.e., $d = 0$, $\Lambda = 0$ and $g_n \equiv p_n$); see Remark 4.
- We derived a structural three-term recurrence relation with polynomial coefficients for Jacobi–Sobolev-type polynomials, see Theorem 33. We prove that this recurrence reduces to the classical Jacobi relation (14) in the trivial case; i.e., $d = 0$ and $\Lambda = 0$.
- Finally, we proved that the $n$-th Jacobi–Sobolev polynomial can be generated through the recursive application of the raising operator; see Theorem 3. This approach may provide an effective framework for the numerical and symbolic computation of Jacobi–Sobolev polynomials.

The results obtained here naturally give rise to several directions for future research. In [32] the authors obtained an electrostatic interpretation of the zeros of Jacobi–Sobolev polynomials for discrete configurations that are sequentially ordered. This condition is restricted to diagonal products, guarantees simple zeros, and allows the authors to obtain the corresponding interpretation. It would be of considerable interest to determine sufficient conditions on positive semidefinite non-diagonal Sobolev matrices ensuring simple zeros. Such conditions would naturally lead to extensions of several approximation-theoretic and asymptotic results currently available only in more restrictive settings.

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