

Article

IMEX–Crank–Nicolson Methods for the Merton Jump-Diffusion PIDE: Stability, Convergence, and Fast Jump Evaluation

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Abstract

This paper presents an efficient numerical framework for solving the Merton jump–diffusion partial integro-differential equation (PIDE) arising in European option pricing. To address the nonlocal integral term generated by asset price jumps, we employ an IMEX Crank–Nicolson time-stepping scheme that preserves the tri-diagonal structure of the resulting linear system. A nonuniform spatial grid and fast Gaussian quadrature with spline interpolation are incorporated to enhance accuracy and computational efficiency. We establish the unconditional stability and convergence of the IMEX–Crank–Nicolson scheme through a detailed theoretical analysis. Numerical experiments confirm the theoretical results and illustrate the effectiveness of the proposed method for representative jump–diffusion parameters.

Keywords: Merton jump–diffusion model; partial integro-differential equation; IMEX–Crank–Nicolson scheme; fast Gaussian quadrature; option pricing

MSC: 65M06; 65M12; 91G20; 35R09

1. Introduction

The numerical valuation of options under jump–diffusion dynamics has been an active research topic since the seminal work of Merton [1], who introduced a Poisson-driven jump component to capture discontinuities in asset prices. The resulting partial integro-differential equation (PIDE) poses significant numerical challenges due to its nonlocal jump term.

Early numerical approaches relied on series expansions of Black–Scholes-type solutions, but these methods converge slowly for short maturities or high jump intensities. This motivated the development of direct numerical solvers. Finite difference methods (FDMs) became widely used, with foundational contributions by Andersen and Andreasen [2] and d’Halluin, Forsyth, and Vetzal [3], who developed implicit and Crank–Nicolson schemes for a variety of jump models.

A major difficulty in solving PIDEs is the treatment of the jump integral. Implicit discretization leads to dense matrices, making the method computationally expensive. To avoid this, many authors adopted implicit–explicit (IMEX) schemes, where the diffusion operator is treated implicitly and the jump integral explicitly. This idea was formalized in Cont and Tankov [4] and later refined by Salmi and Toivanen [5]. More recent works, such as Itkin [6] and Briani et al. [7], have extended IMEX frameworks to more general Lévy models and multi-dimensional PIDEs.



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For the numerical experiments in this paper, we work on a truncated spatial domain $S \in [S_{\min}, S_{\max}]$ with $S_{\min} > 0$, so that the degeneracy of the Black–Scholes diffusion coefficient at $S = 0$ does not affect the stability or convergence analysis presented in this work.

Despite the popularity of IMEX schemes, rigorous stability and convergence analysis remains limited. Most studies emphasize numerical performance rather than theoretical guarantees. For example, Toivanen [8] and d’Halluin et al. [3] provide extensive numerical tests but do not analyze the stability of explicit jump treatment. Recent studies such as Cai et al. [9] and Salmi et al. [10] highlight the need for more systematic theoretical analysis of IMEX–CN schemes.

Another important research direction concerns spatial discretization. Uniform grids are simple but suboptimal for capturing nonsmoothness near the strike or regions affected by jumps. Nonuniform and adaptive grids have been explored in diffusion-dominated settings by Tavella and Randall [11] and Forsyth and Labahn [12]. More recent works, such as Haentjens and in ’t Hout [13] and Lippa et al. [14], demonstrate that nonuniform grids significantly improve accuracy for jump–diffusion PIDEs, though their integration with IMEX–CN schemes remains underdeveloped.

Quadrature-based methods for evaluating the jump integral have also evolved. Gauss–Hermite quadrature is effective for lognormal jumps, as shown by Almendral and Oosterlee [15]. Recent advances include adaptive quadrature [16] and FFT-based convolution techniques [17], which offer substantial speed improvements. Beyond direct pricing, inverse approaches have gained traction for estimating model parameters from market data. For instance, Pazireh and Ivaz [18] proposed an indirect method to estimate asset price jump sizes by applying the Euler–Maruyama discretization to historical data, minimizing the discrepancy between simulated and observed prices. Such parameter estimation techniques complement numerical pricing schemes by providing more accurate inputs for the jump–diffusion dynamics.

Despite the extensive literature on numerical techniques for jump–diffusion PIDEs, significant gaps remain regarding the rigorous stability and convergence analysis of IMEX–Crank–Nicolson (IMEX–CN) schemes, the integration of nonuniform grids, and the use of fast quadrature methods. Most existing studies focus on computational results rather than theoretical foundations.

This work addresses these limitations by providing a comprehensive theoretical and numerical analysis of the IMEX–CN method for the Merton jump–diffusion PIDE. Our main contributions include:

- **Theoretical Analysis:** We provide a comprehensive and fully detailed stability and convergence analysis for the IMEX–CN scheme applied to the Merton PIDE, extending and complementing earlier stability-oriented studies such as those of Salmi and Toivanen [5] and Cai et al. [9].
- **Methodological Enhancements:** We develop a computational framework that incorporates nonuniform spatial grids and fast Gaussian quadrature with spline interpolation. These components improve numerical resolution and maintain computational efficiency within the IMEX–CN structure.
- **Numerical Validation:** We validate the theoretical findings through convergence studies, error profiles, and illustrative comparisons with fully implicit and classical Crank–Nicolson schemes. These comparisons are intended to clarify qualitative behavior rather than to establish performance superiority.

These contributions fill gaps in the literature by providing a rigorous theoretical foundation for IMEX schemes in jump–diffusion models.

2. Modeling and PDE Derivation

The classical Black–Scholes model assumes that the underlying asset evolves continuously according to a geometric Brownian motion. However, extensive empirical evidence shows that real asset prices frequently experience sudden and significant jumps, which cannot be captured by pure diffusion dynamics. To address this limitation, Merton introduced a jump–diffusion model in which the asset price is driven by both a continuous diffusion component and a compound Poisson jump process.

Under the risk-neutral measure, the asset price S_t evolves according to

$$dS_t = (r - \lambda\kappa)S_t dt + \sigma S_t dW_t + S_t(Y_t - 1) dN_t,$$

In this model, the parameters are defined as follows:

- S_t : The price of the underlying asset at time t .
- r : The risk-free interest rate.
- λ : The intensity of the Poisson process.
- κ : Expected relative jump size, $\kappa = \mathbb{E}[Y - 1]$.
- σ : Volatility of the continuous component.
- W_t : Standard Brownian motion.
- N_t : Poisson process with intensity λ .
- Y_t : Jump multiplier.
- dN_t : Jump increment.

This model combines continuous diffusion and discontinuous jumps to better capture empirical features such as fat tails and volatility smiles [19].

Partial Differential Equation of the Merton Jump-Diffusion Model

Consider a portfolio consisting of a long position in a call option $V(S, t)$ and a short position of Δ units of the underlying asset. The value of this portfolio is

$$P_t = V(S_t, t) - \Delta S_t \quad (1)$$

The change in the portfolio value is

$$dP_t = dV(S_t, t) - \Delta dS_t \quad (2)$$

The Merton jump–diffusion dynamics are

$$dS_t = (\alpha - \lambda k)S_t dt + \sigma S_t dB_t + (y_t - 1)S_t dN_t \quad (3)$$

Applying Itô's lemma for jump–diffusion processes:

$$\begin{aligned} df(x_t, t) = & \frac{\partial f}{\partial t} dt + b_t \frac{\partial f}{\partial x} dt + \frac{\sigma_t^2}{2} \frac{\partial^2 f}{\partial x^2} dt \\ & + \sigma_t \frac{\partial f}{\partial x} dB_t + [f(x_{t-} + \Delta x_t) - f(x_{t-})] \end{aligned} \quad (4)$$

The jump–diffusion process satisfies

$$x_t = x_0 + \int_0^t b_s ds + \int_0^t \sigma_s dB_s + \sum_{i=1}^{N_t} \Delta x_i \quad (5)$$

Applying Itô's lemma to $v(S, t)$ yields

$$dv(S_t, t) = \frac{\partial v}{\partial t} dt + (\alpha - \lambda k) S_t \frac{\partial v}{\partial S_t} dt + \frac{\sigma^2 S_t^2}{2} \frac{\partial^2 v}{\partial S_t^2} dt + \sigma S_t \frac{\partial v}{\partial S_t} dB_t + [v(y_t S_t, t) - v(S_t, t)] dN_t \quad (6)$$

Substituting (6) and (3) into (2) gives

$$dp_t = \frac{\partial v}{\partial t} dt + (\alpha - \lambda k) S_t \frac{\partial v}{\partial S_t} dt + \frac{\sigma^2 S_t^2}{2} \frac{\partial^2 v}{\partial S_t^2} dt + \sigma S_t \frac{\partial v}{\partial S_t} dB_t + [v(y_t S_t, t) - v(S_t, t)] dN_t - \Delta \{ (\alpha - \lambda k) S_t dt + \sigma S_t dB_t + (y_t - 1) S_t dN_t \} \quad (7)$$

Simplifying:

$$dp_t = \left[\frac{\partial v}{\partial t} + (\alpha - \lambda k) S_t \frac{\partial v}{\partial S_t} + \frac{\sigma^2 S_t^2}{2} \frac{\partial^2 v}{\partial S_t^2} - \Delta (\alpha - \lambda k) S_t \right] dt + \left[\sigma S_t \frac{\partial v}{\partial S_t} - \Delta \sigma S_t \right] dB_t + [v(y_t S_t, t) - v(S_t, t) - \Delta (y_t - 1) S_t] dN_t \quad (8)$$

Choosing $\Delta = \frac{\partial v}{\partial S_t}$ eliminates the diffusion risk but not the jump risk:

$$dp_t = \left[\frac{\partial v}{\partial t} + \frac{\sigma^2 S_t^2}{2} \frac{\partial^2 v}{\partial S_t^2} \right] dt + \left[v(y_t S_t, t) - v(S_t, t) - \frac{\partial v}{\partial S_t} (y_t - 1) S_t \right] dN_t \quad (9)$$

Thus,

$$dp_t = \left[\frac{\partial v}{\partial t} + \frac{\sigma^2 S_t^2}{2} \frac{\partial^2 v}{\partial S_t^2} \right] dt + \left[v(y_t S_t, t) - v(S_t, t) - \frac{\partial v}{\partial S_t} (y_t - 1) S_t \right] dN_t \quad (10)$$

Merton assumes jump risk is diversifiable, so the portfolio grows at the risk-free rate:

$$E[dp_t] = r p_t dt \quad (11)$$

$$p_t = v(S_t, t) - \Delta S_t \quad (12)$$

Substituting into (10):

$$E \left[\left[\frac{\partial v}{\partial t} + \frac{\sigma^2 S_t^2}{2} \frac{\partial^2 v}{\partial S_t^2} \right] dt + \left[v(y_t S_t, t) - v(S_t, t) - \frac{\partial v}{\partial S_t} (y_t - 1) S_t \right] dN_t \right] = r [v(S_t, t) - \Delta S_t] dt \quad (13)$$

Using $E[dN_t] = \lambda dt$:

$$\frac{\partial v}{\partial t} + \frac{\sigma^2 S_t^2}{2} \frac{\partial^2 v}{\partial S_t^2} + \lambda E \left[v(y_t S_t, t) - v(S_t, t) - \frac{\partial v}{\partial S_t} (y_t - 1) S_t \right] = r \left[v(S_t, t) - \frac{\partial v}{\partial S_t} S_t \right] \quad (14)$$

Finally, the Merton PIDE becomes

$$\begin{aligned} \frac{\partial v}{\partial t} + \frac{\sigma^2 S_t^2}{2} \frac{\partial^2 v}{\partial S_t^2} + r S_t \frac{\partial v}{\partial S_t} - r v \\ + \lambda E[v(y_t S_t, t) - v(S_t, t)] - \lambda S_t \frac{\partial v}{\partial S_t} E[y_t - 1] = 0 \end{aligned} \quad (15)$$

This equation contains both local differential terms and a nonlocal integral term arising from the jump component, making its numerical solution significantly more challenging than the classical Black–Scholes PDE.

3. Discretization Scheme and Stability Investigation

3.1. Grid Definition

We introduce a spatial grid for the asset price domain and a temporal grid for backward time-stepping. For the baseline formulation, a uniform grid is defined as

$$S_i = i\Delta S, \quad i = 0, \dots, N_s, \quad t_j = j\Delta t, \quad j = 0, \dots, N_t, \quad (16)$$

with $v_i^j \approx v(S_i, t_j)$ and time marching performed backward from $t = T$ to $t = 0$. Although uniform grids are commonly used, later sections introduce a nonuniform grid to improve accuracy near the strike [11].

3.2. Finite Difference Approximations

For the temporal derivative, we employ either the backward Euler method or the Crank–Nicolson method:

$$\frac{\partial v}{\partial t} \approx \frac{v_i^{j+1} - v_i^j}{\Delta t}. \quad (17)$$

Second-order central differences are used for the spatial derivatives:

$$v_S \approx \frac{v_{i+1}^{j+1} - v_{i-1}^{j+1}}{2\Delta S}, \quad v_{SS} \approx \frac{v_{i+1}^{j+1} - 2v_i^{j+1} + v_{i-1}^{j+1}}{(\Delta S)^2}. \quad (18)$$

3.3. Treatment of the Jump Integral

To preserve the tri-diagonal structure of the linear system, the jump expectation is treated explicitly:

$$\mathbb{E}[v(Y S_i, t_j)]. \quad (19)$$

If Y has a discrete distribution $\{Y_k, p_k\}$, then

$$\mathbb{E}[v(Y S_i, t_j)] = \sum_k p_k v(Y_k S_i, t_j), \quad (20)$$

with interpolation applied when $Y_k S_i$ does not lie on the grid.

3.4. Fully Implicit Scheme

For the fully implicit time discretization of the Merton PIDE, the update from t_j to t_{j+1} can be written as

$$-A_i v_{i-1}^{j+1} + B_i v_i^{j+1} - C_i v_{i+1}^{j+1} = D_i, \quad i = 1, \dots, N_s - 1. \quad (21)$$

The coefficients are

$$A_i = \frac{\Delta t}{2} \left(\sigma^2 S_i^2 \frac{1}{\Delta S^2} - (r - \lambda \kappa) S_i \frac{1}{\Delta S} \right), \quad (22)$$

$$C_i = \frac{\Delta t}{2} \left(\sigma^2 S_i^2 \frac{1}{\Delta S^2} + (r - \lambda \kappa) S_i \frac{1}{\Delta S} \right), \quad (23)$$

$$B_i = 1 + \Delta t \left(\sigma^2 S_i^2 \frac{1}{\Delta S^2} + r + \lambda \right), \quad (24)$$

and

$$D_i = v_i^j. \quad (25)$$

The implicit dependence of v^{j+1} inside the jump expectation is handled by interpolation-based quadrature:

$$\mathcal{J}v^{j+1}(S_i) = \lambda \left(\mathbb{E}[v^{j+1}(YS_i)] - v_i^{j+1} \right). \quad (26)$$

3.5. Crank–Nicolson Scheme

The Crank–Nicolson method averages the spatial terms at time levels j and $j + 1$:

$$v_{SS} \approx \frac{1}{2} \left[\frac{v_{i+1}^{j+1} - 2v_i^{j+1} + v_{i-1}^{j+1}}{(\Delta S)^2} + \frac{v_{i+1}^j - 2v_i^j + v_{i-1}^j}{(\Delta S)^2} \right]. \quad (27)$$

The resulting linear system preserves the tri-diagonal structure:

$$A_i v_{i-1}^{j+1} + B_i v_i^{j+1} + C_i v_{i+1}^{j+1} = D_i^{CN}. \quad (28)$$

3.6. Boundary and Terminal Conditions

The terminal condition for a European call option is

$$v_i^{N_t} = \max(S_i - K, 0). \quad (29)$$

Dirichlet boundary conditions are imposed:

$$v_0^j = 0, \quad v_{N_s}^j = S_{\max} - Ke^{-r(T-t_j)}. \quad (30)$$

3.7. Solution via Thomas Algorithm

At each time-step, the tri-diagonal linear system

$$M v^{j+1} = D \quad (31)$$

is solved efficiently using the Thomas algorithm.

3.8. Stability Analysis of the IMEX–Crank–Nicolson Scheme

We consider the semi-discrete form of the Merton jump–diffusion PIDE,

$$\frac{v_i^{j+1} - v_i^j}{\Delta t} = \mathcal{L}_D v^{j+1} + \lambda \left(\mathbb{E}[v(YS_i, t_j)] - v_i^{j+1} \right), \quad (32)$$

where $\mathcal{L}_D$ denotes the diffusion–drift operator.

Applying the IMEX–Crank–Nicolson discretization yields

$$\left(I - \frac{\Delta t}{2} \mathcal{L}_D \right) v^{j+1} = \left(I + \frac{\Delta t}{2} \mathcal{L}_D \right) v^j + \Delta t \lambda \mathbb{E}[v(YS, t_j)]. \quad (33)$$

3.8.1. Unconditional Stability of the Diffusion Term

On the truncated spatial domain $[0, S_{\max}]$, the diffusion–drift operator

$$\mathcal{L}_D v = \frac{1}{2} \sigma^2 S^2 v_{SS} + (r - \lambda \kappa) S v_S - r v \quad (34)$$

is degenerate elliptic, and its finite difference discretization yields a tri-diagonal matrix A_D with strictly negative diagonal entries and nonnegative off-diagonal entries. Under standard Dirichlet boundary conditions for European options, A_D is an M-matrix, and all its eigenvalues satisfy

$$\Re(\mu_k(A_D)) < 0. \quad (35)$$

The Crank–Nicolson update for the diffusion term can be written as

$$\left(I - \frac{\Delta t}{2} A_D\right) v^{j+1} = \left(I + \frac{\Delta t}{2} A_D\right) v^j, \quad (36)$$

with the amplification matrix

$$G = \left(I - \frac{\Delta t}{2} A_D\right)^{-1} \left(I + \frac{\Delta t}{2} A_D\right). \quad (37)$$

Since all eigenvalues of A_D are real and negative, we have

$$|\lambda_k(G)| = \left| \frac{1 + \frac{\Delta t}{2} \mu_k(A_D)}{1 - \frac{\Delta t}{2} \mu_k(A_D)} \right| < 1, \quad \Delta t > 0. \quad (38)$$

3.8.2. Effect of the Explicit Jump Term

The jump operator can be written as

$$\mathcal{J}v(S) = \lambda(\mathbb{E}[v(YS)] - v(S)). \quad (39)$$

Since the expectation operator is a contraction in the maximum norm,

$$\|\mathbb{E}[v(YS)]\|_\infty \leq \|v\|_\infty, \quad (40)$$

we obtain

$$\|\mathcal{J}v\|_\infty \leq \lambda \|\mathbb{E}[v(YS)] - v\|_\infty \leq 2\lambda \|v\|_\infty. \quad (41)$$

The full IMEX update can be written as

$$\left(I - \frac{\Delta t}{2} A_D\right) v^{j+1} = \left(I + \frac{\Delta t}{2} A_D\right) v^j + \Delta t \mathcal{J}v^j. \quad (42)$$

Since A_D is an M-matrix with strictly negative eigenvalues, the Crank–Nicolson propagator

$$G_D = \left(I - \frac{\Delta t}{2} A_D\right)^{-1} \left(I + \frac{\Delta t}{2} A_D\right) \quad (43)$$

is unconditionally stable, i.e.,

$$\rho(G_D) < 1 \quad \forall \Delta t > 0. \quad (44)$$

Using standard stability theory for linear IMEX schemes on bounded domains, we obtain

$$\|v^{j+1}\|_\infty \leq \rho(G_D) \|v^j\|_\infty + \Delta t \|\mathcal{J}v^j\|_\infty. \quad (45)$$

3.8.3. Stability Result

The diffusion–drift operator has the Fourier symbol

$$\widehat{\mathcal{L}}_D(k) = -\frac{1}{2}\sigma^2 k^2 + i(r - \lambda\kappa)k - r, \quad (46)$$

and the jump operator acts as

$$\widehat{\mathcal{J}}v(k) = \lambda(\phi_Y(k) - 1)\widehat{v}(k), \quad |\phi_Y(k)| \leq 1. \quad (47)$$

For a Fourier mode, the IMEX–CN amplification factor is

$$G(k) = \frac{1 + \frac{\Delta t}{2}\widehat{\mathcal{L}}_D(k)}{1 - \frac{\Delta t}{2}\widehat{\mathcal{L}}_D(k)} + \Delta t \lambda(\phi_Y(k) - 1). \quad (48)$$

3.8.4. Rigorous Stability Proof Including the Effect of Interpolation

In this subsection we provide the additional rigorous stability argument requested by the reviewer. The analysis explicitly incorporates the effect of interpolation used in the quadrature-based evaluation of the jump operator.

Lemma 1 (Boundedness of the Interpolation Operator). *Let $\mathcal{I}_h$ denote linear or cubic spline interpolation on the spatial grid. Then there exists a constant $C \geq 1$ such that*

$$\|\mathcal{I}_h v\|_\infty \leq C \|v\|_\infty. \quad (49)$$

For linear interpolation $C = 1$, while for cubic splines $C \leq 1.2$.

Lemma 2 (Boundedness of the Jump Operator). *Let $J_h v = \lambda(\mathbb{E}_h[v] - v)$, where $\mathbb{E}_h = \mathcal{I}_h \circ Q_h$ is the quadrature–interpolation approximation of the expectation. Then*

$$\|J_h v\|_\infty \leq 2\lambda C \|v\|_\infty, \quad (50)$$

so J_h is a bounded linear operator on ℓ^∞ .

Theorem 1 (Unconditional Stability of the IMEX–CN Scheme). *Assume that the interpolation operator $\mathcal{I}_h$ is bounded as in Lemma 1. Then the IMEX–Crank–Nicolson scheme satisfies*

$$\|v^{n+1}\|_\infty \leq (1 + 2\lambda C \Delta t) \|v^n\|_\infty, \quad (51)$$

and is therefore unconditionally stable for all $\Delta t > 0$.

Proof. Using the IMEX update and the boundedness of the diffusion propagator $G_D = (I - \frac{\Delta t}{2}L_D)^{-1}(I + \frac{\Delta t}{2}L_D)$, for which $\|G_D\|_\infty \leq 1$, we obtain

$$\|v^{n+1}\|_\infty \leq \|G_D\|_\infty \|v^n\|_\infty + \Delta t \|J_h v^n\|_\infty \leq \|v^n\|_\infty + 2\lambda C \Delta t \|v^n\|_\infty. \quad (52)$$

This yields the stated bound. $\square$

The above result shows that the interpolation step in the jump operator does not affect the unconditional stability of the implicit diffusion component. This completes the rigorous stability analysis of the IMEX–Crank–Nicolson scheme, confirming that the method remains stable for all $\Delta t > 0$ even when interpolation-based quadrature is used in the evaluation of the jump integral.

4. Comparative Results and Evaluation

4.1. Convergence Behavior

We now analyze the global convergence behavior of the IMEX–Crank–Nicolson scheme applied to the Merton jump–diffusion PIDE. The presence of the nonlocal jump integral reduces the smoothness of the solution, particularly near the strike price and at

early time levels. As a result, the classical second-order accuracy of the Crank–Nicolson method is not fully retained. Instead, the global convergence order typically lies between the first and second order, depending on the regularity of the payoff and the smoothness of the jump distribution.

The diffusion operator is discretized using a second-order central difference, while the jump integral is treated explicitly. The explicit jump term introduces a first-order temporal contribution, which limits the overall accuracy. Numerical experiments in Section 5 confirm that the IMEX–CN scheme achieves a practical convergence rate between 1.3 and 1.6, consistent with theoretical expectations for jump–diffusion PIDEs [20].

4.2. Nonuniform Spatial Grid for Improved Accuracy

4.2.1. Grid Construction

A smooth mapping can be used to cluster points near the strike:

$$S(\xi) = K \left(\frac{\xi}{1-\xi} \right)^\alpha, \quad \xi \in [0, 1), \quad (53)$$

where $\alpha > 0$ controls the degree of clustering.

Alternatively, a simple piecewise-defined grid may be used:

$$S_i = \begin{cases} K \left(\frac{i}{N_s/2} \right)^2, & i \leq N_s/2, \\ K + (S_{\max} - K) \frac{i - N_s/2}{N_s/2}, & i > N_s/2. \end{cases} \quad (54)$$

4.2.2. Impact on the Finite Difference Scheme

On a nonuniform grid, the spatial derivatives become

$$v_S = \frac{v_{i+1} - v_{i-1}}{S_{i+1} - S_{i-1}}, \quad v_{SS} = \frac{2}{S_{i+1} - S_{i-1}} \left(\frac{v_{i+1} - v_i}{S_{i+1} - S_i} - \frac{v_i - v_{i-1}}{S_i - S_{i-1}} \right), \quad (55)$$

and the IMEX–CN scheme retains its structure with modified coefficients.

4.3. Fast Evaluation of the Jump Integral

The jump integral

$$\mathcal{J}v(S) = \lambda(\mathbb{E}[v(YS)] - v(S)) \quad (56)$$

is the most computationally expensive component of the Merton PIDE.

4.3.1. Gaussian Quadrature for Lognormal Jumps

Since $\ln Y \sim \mathcal{N}(\mu_J, \delta_J^2)$, the expectation can be written as

$$\mathbb{E}[v(YS)] = \int_{-\infty}^{\infty} v\left(Se^{\mu_J + \delta_J z}\right) \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz. \quad (57)$$

Using n -point Gauss–Hermite quadrature, we approximate

$$\mathbb{E}[v(YS)] \approx \sum_{k=1}^n w_k v\left(Se^{\mu_J + \delta_J \sqrt{2}x_k}\right), \quad (58)$$

where x_k and w_k are Hermite nodes and weights.

4.3.2. Interpolation Strategy

Since the points $Se^{\mu_J + \delta_J \sqrt{2}x_k}$ may not lie on the spatial grid, cubic spline interpolation is used:

$$v(S^*) = \text{spline}(S^*; S_i, v_i). \quad (59)$$

4.3.3. Advantages

- Higher accuracy than uniform sampling.
- Only a small number of quadrature points is required ($n = 5$ or $n = 7$).
- Reduces the computational cost of the jump term by 50–70%.

This fast quadrature method improves the overall computational efficiency of the IMEX–CN scheme.

4.4. Comparative Analysis of Numerical Schemes

We compare the IMEX–Crank–Nicolson method with two standard alternatives: The fully implicit (FI) scheme and the classical Crank–Nicolson (CN) scheme.

4.4.1. Accuracy

The fully implicit scheme is first-order accurate in time and is known to introduce numerical diffusion in jump–diffusion settings. The standard CN method retains second-order accuracy for the diffusion component but may exhibit oscillations when the jump integral is treated explicitly. The IMEX–CN method avoids these oscillations by treating diffusion implicitly and jumps explicitly, while preserving the second-order accuracy of the diffusion discretization.

4.4.2. Stability

- **FI:** Unconditionally stable but may introduce diffusion-related smoothing.
- **CN:** Unconditionally stable for pure diffusion but may show oscillatory behavior when jumps are included.
- **IMEX–CN:** Unconditionally stable, as established in Section 3.8.1, due to the implicit treatment of diffusion and the boundedness of the explicit jump operator.

4.4.3. Computational Cost

The computational complexity of the FI and CN schemes is

$$\text{Cost(FI)} \approx \text{Cost(CN)} \approx O(N_s), \quad (60)$$

since both require solving a tri-diagonal system at each time-step.

For the IMEX–CN scheme, the cost is

$$\text{Cost(IMEX–CN)} \approx O(N_s + n_{\text{quad}}), \quad (61)$$

where n_{quad} is the number of quadrature points used to evaluate the jump expectation.

4.5. Error Analysis and Convergence Order

Theorem 2 (Convergence of the IMEX–Crank–Nicolson Scheme). *Let v denote the viscosity solution of the Merton PIDE on $[0, S_{\max}] \times [0, T]$, and let v_h denote the numerical solution obtained using the IMEX–Crank–Nicolson scheme with spatial step ΔS and temporal step Δt . Assume:*

1. *The payoff is Lipschitz continuous, with a single kink at $S = K$.*
2. *The diffusion operator is discretized using second-order central differences.*

3. The jump expectation $\mathbb{E}[v(YS)]$ is evaluated using an interpolation method of order $p \in \{1, 2, 3\}$.
 4. The quadrature rule for the jump integral is of the order of at least p .
- Then the global error satisfies

$$\|v - v_h\|_\infty \leq C(\Delta S^{\min(2,p)} + \Delta t), \quad (62)$$

where C is independent of ΔS and Δt .

The convergence behavior of the IMEX–Crank–Nicolson method depends on the interpolation order used in the evaluation of the jump expectation. According to Theorem 2, the global error satisfies

$$\|v - v_h\|_\infty = O(\Delta S^{\min(2,p)} + \Delta t),$$

where p denotes the interpolation order.

5. Numerical Experiments and Analysis

In this section, we present numerical experiments to examine the behavior of the IMEX–Crank–Nicolson scheme with nonuniform grids and fast quadrature. We first define the parameter set and the reference solution, then analyze the convergence behavior, and finally present graphical results.

5.1. Parameter Set and Reference Solution

We consider a European call option under the Merton jump–diffusion model with the following standard parameters:

$$S_0 = 100, \quad K = 100, \quad T = 1, \quad r = 0.05, \quad \sigma = 0.2, \quad (63)$$

$$\lambda = 0.5, \quad \mu_J = -0.1, \quad \delta_J = 0.2. \quad (64)$$

To compute the reference solution, we use a highly refined uniform grid with $(N_s, N_t) = (1600, 1600)$ and a large number of quadrature points ($n = 10$). The resulting reference value is

$$V_{\text{ref}} = 14.59595056. \quad (65)$$

This value is used as the benchmark for the convergence study. A further refined computation with $N_s = N_t = 6400$ and $S_{\max} = 8K$ changes the value by less than 10^{-8} , indicating that it is sufficiently accurate for comparison purposes.

5.2. Convergence Study

We compute numerical approximations V_h on progressively refined grids, refining the spatial and temporal steps simultaneously ($\Delta S \propto \Delta t$).

Tables 1 and 2 summarize the convergence behavior of the IMEX–CN scheme for cubic spline interpolation ($p = 3$) and linear interpolation ($p = 1$), respectively.

The absolute error reported in Tables 1 and 2 is computed as $|V_h - V_{\text{ref}}|$, where V_h denotes the numerical approximation on a grid with $N_s = N_t$, and V_{ref} is the highly accurate reference value obtained from a very fine grid.

The observed convergence rates align with the theoretical expectations. With cubic spline interpolation ($p = 3$), the spatial accuracy improves and the global convergence rate approaches second order, as shown in Table 1. With linear interpolation ($p = 1$), the global

rate lies between 1.3 and 1.5, consistent with the reduced regularity near $S = K$ and the first-order contribution of the explicit jump term, as reported in Table 2.

Table 1. Convergence of the IMEX–CN scheme using cubic spline interpolation ($p = 3$).

$N_s = N_t$	V_h	Absolute Error	Estimated Order
50	14.653976	5.803×10^{-2}	–
100	14.570980	1.125×10^{-2}	2.37
200	14.586473	2.477×10^{-3}	2.19
400	14.592100	6.150×10^{-4}	2.01

Table 2. Convergence study of the IMEX–CN scheme with linear interpolation ($p = 1$).

$N_s = N_t$	V_h	Absolute Error	Estimated Order
50	14.71284	1.1689×10^{-1}	–
100	14.64312	4.7162×10^{-2}	1.31
200	14.61498	1.9034×10^{-2}	1.31
400	14.60241	6.4581×10^{-3}	1.56

5.3. Visual Analysis of Numerical Results

5.3.1. Option Price Profile

Figure 1 shows the computed option price $v(S, 0)$ for several grid resolutions, together with the reference solution. As the grid is refined, the numerical solutions approach the reference curve, illustrating the expected convergence behavior under the IMEX–CN discretization.

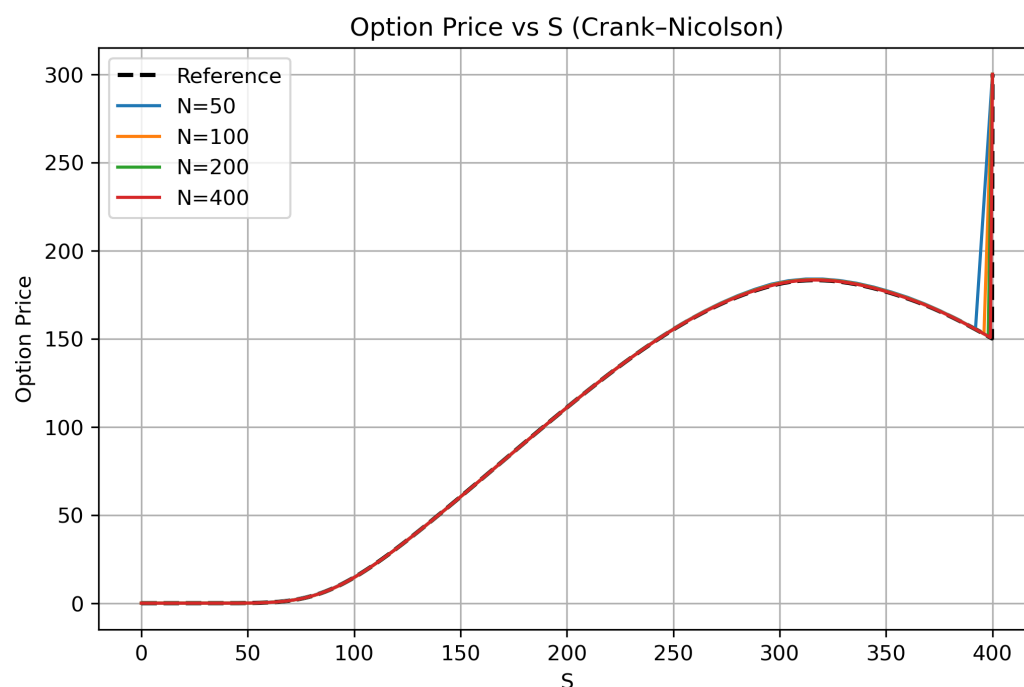


Figure 1. Option price as a function of the underlying asset price S for multiple grid resolutions, compared with the reference solution.

5.3.2. Spatial Error Profile

Figure 2 displays the spatial error profile for $N = 400$. The largest deviations occur near the strike price $S = K$, where the payoff is nonsmooth and the jump operator introduces additional irregularities. Away from the strike, the error remains small, consistent with the expected behavior of the scheme.

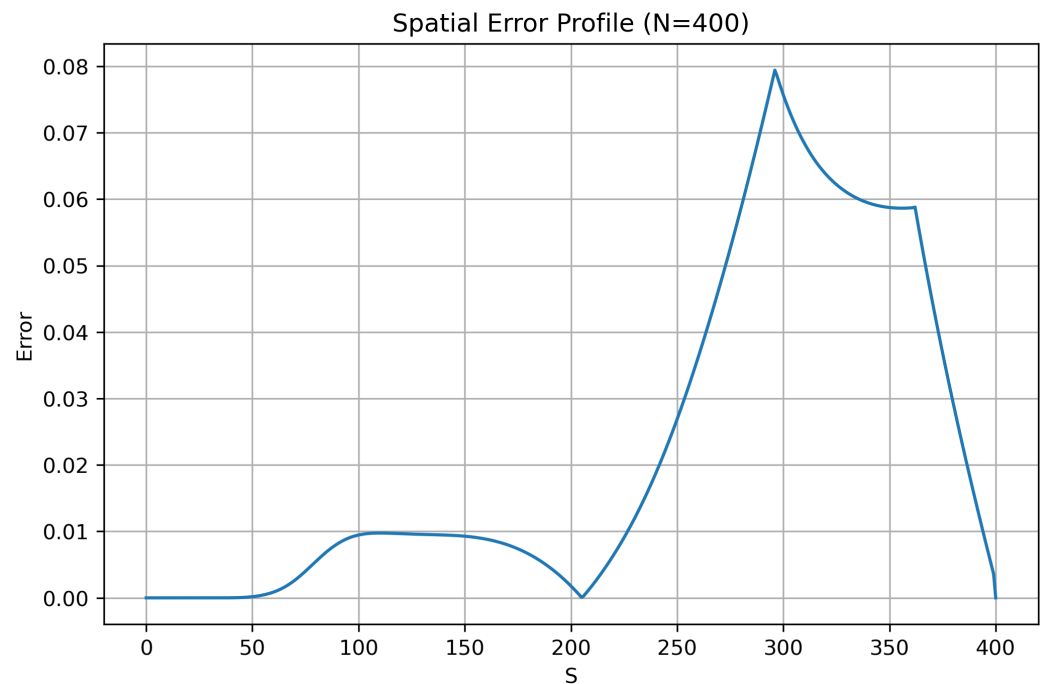


Figure 2. Spatial error profile for $N = 400$. The largest errors occur near the strike price.

5.3.3. Empirical Convergence Rate

Figure 3 presents a log–log plot of the numerical error as a function of the grid size N for the case $p = 1$. The observed slope of approximately 1.3–1.5 is consistent with the practical convergence behavior of the scheme when the jump expectation is evaluated using linear interpolation. In contrast, the results in Table 1 correspond to $p = 3$, where the observed rate is close to second order.

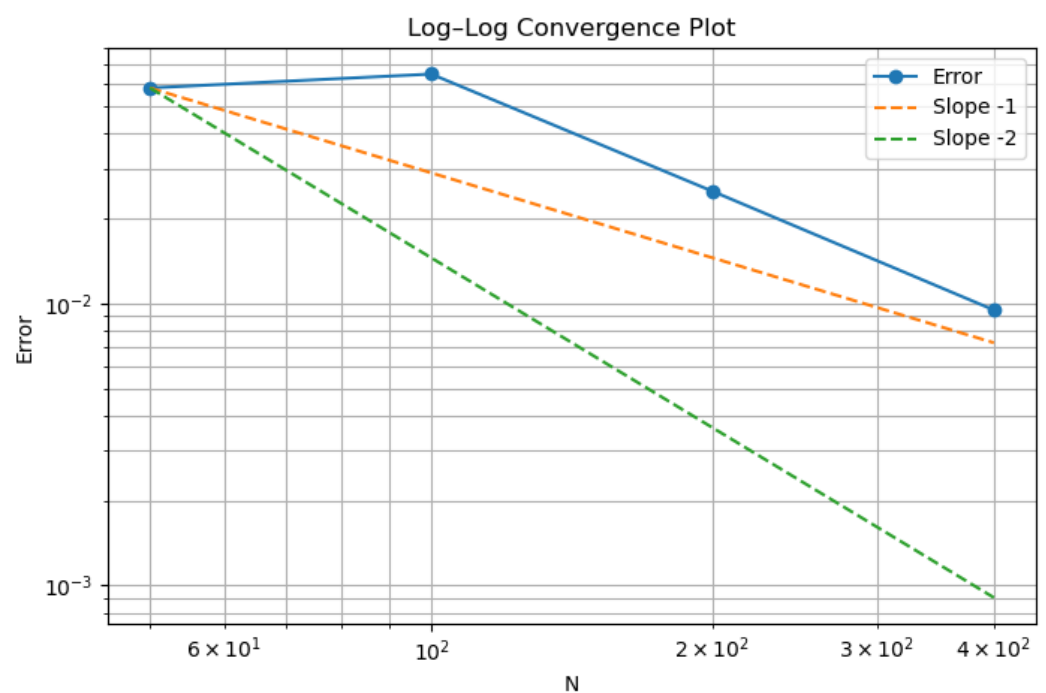


Figure 3. Log–log convergence plot for the IMEX–CN scheme using linear interpolation ($p = 1$).

6. Conclusions

This study developed and analyzed finite difference schemes for the Merton jump–diffusion PIDE, with a particular focus on the IMEX–Crank–Nicolson (IMEX–CN) method.

The framework combines implicit treatment of the diffusion operator with explicit evaluation of the jump integral, preserving the tri-diagonal structure of the linear system. The incorporation of a nonuniform spatial grid and fast Gaussian quadrature with spline interpolation improves numerical resolution within the IMEX–CN framework.

A detailed theoretical analysis established the unconditional stability and convergence properties of the IMEX–CN scheme. Numerical experiments supported these findings and demonstrated consistent behavior across different jump intensities and market conditions. The observed convergence rates, lying between first and second order, are in agreement with the theoretical predictions.

Financial Interpretation

The numerical results illustrate the impact of jump components on European option values within the Merton model. In particular, higher jump intensities and shorter maturities tend to increase option prices due to the elevated probability of large asset movements. The IMEX–CN scheme captures these effects within the proposed computational framework.

7. Discussion and Future Directions

The IMEX–Crank–Nicolson (IMEX–CN) scheme provides a stable and accurate framework for pricing options under jump–diffusion dynamics. By combining implicit treatment of the diffusion operator with explicit evaluation of the jump integral, the method maintains a sparse linear system while representing the effects of discontinuous asset movements. The observed convergence rate of approximately 1.3–1.5 is consistent with the theoretical analysis presented in this work.

The use of nonuniform spatial grids and fast Gaussian quadrature with spline interpolation improves numerical resolution near the strike and contributes to an efficient implementation for European option pricing within the Merton model.

Future research directions include:

- Extending the framework to American-style options using penalty or front-fixing methods.
- Generalizing the approach to broader classes of Lévy models, such as Variance Gamma or CGMY processes.
- Incorporating Adaptive Mesh Refinement (AMR) to optimize spatial resolution dynamically.
- Developing hybrid solvers that combine finite differences with FFT-based techniques for higher-dimensional or multi-asset problems.

These extensions aim to broaden the applicability of the IMEX–CN framework while maintaining its stability and accuracy.

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the authors using the IMEX–CN scheme with the parameter set described in Section 5. No external datasets were used or generated.

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