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Generalized Convexity via Clarke Subdifferential of Interval Mappings and Its Optimization Applications

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Abstract

Clarke subdifferential and directional derivative theories are well developed for real functions. However, the nonsmooth analysis framework for interval mappings is still incomplete. Few works reveal the inherent relationship between Clarke directional derivatives of interval mappings and their endpoint functions. This paper addresses these gaps. We first explore the relationships between Clarke directional derivatives of interval mappings and their endpoint functions. The Clarke subdifferential of interval mappings is defined by means of directional derivatives and interval order relations, with its fundamental properties established. A new generalized convexity concept for interval mappings is further introduced based on the proposed Clarke subdifferential. Finally, sufficient conditions for efficient solutions to interval nonsmooth optimization are derived under the new generalized convexity framework.

Keywords: Clarke directional derivative; Clarke subdifferential; generalized convexity; interval optimization

MSC: 49J52; 90C29; 90C48

1. Introduction

The foundations of interval analysis were laid by Moore [1], who formalized interval arithmetic and its basic algebraic properties. Since then, interval mappings have served as a powerful means of uncertainty modeling. They are widely applied in optimization, control theory and other areas [2,3]. However, a large class of practical optimization problems contain nonsmooth functions, which violate the differentiability assumptions underlying classical interval optimization theory. Therefore, developing theories for nonsmooth interval optimization has become a major research task [4–12].

When dealing with nonsmooth interval mappings, studying directional derivatives of interval mappings becomes important. Ghosh et al. [13] established three classes of directional derivatives for interval mappings. It turns out that gH-Fréchet differentiability implies gH-Gâteaux differentiability, and the latter further ensures the existence of the gH-directional derivative. Stefanini et al. [14] established a more general framework of gH-differentiability. Within this framework, directional differentiability was formally defined for multivariable interval and fuzzy mappings. Using gradient and directional derivative characteristics under gH difference for interval functions, Villanueva et al. [15] derived both necessary and sufficient optimality conditions. Subsequently, relevant studies [16–18] further investigated directional derivative theories for interval-valued functions.

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Subdifferentials are essential for addressing optimization problems involving nonsmooth functions. Karaman [19] proposed the notions of subdifferentials and weak subdifferentials for interval mappings and established their fundamental properties, including existence conditions. Ghosh et al. [20] discussed several key properties of the gH-subdifferential for interval mappings and clarified the relationships between gH-subgradients and efficient solutions of the corresponding optimization problem. Anshika et al. [21] established calculus rules for the gH-subdifferential of convex interval mappings and illustrated the applicability of such rules to nonconvex composite interval optimization models. With the advancement of optimization theory, increasing attention has been focused on the subdifferential theory of interval mappings [22–24]. However, subdifferential existence relies on function convexity, so subdifferentials cannot fully handle nonconvex optimization problems. To address this limitation, the concept of the Clarke subdifferential for interval mappings was introduced. Thus, nonconvex and nonsmooth optimization problems that cannot be solved by existing gH-subdifferential techniques can be effectively addressed.

As a fundamental property, convexity is critical to the theoretical research of interval mappings. Therefore, the investigation of convexity and generalized convexity for interval mappings [25–28] ranks among the most vital components of optimization theory. Li [29] established three distinct total order relations on interval spaces. For each ordering, the convexity concept was introduced in the framework of interval mappings, and the associated Karush–Kuhn–Tucker optimality conditions were derived for interval optimization models. Chauhan et al. [30] proved that an element of a convex interval mapping is efficient if its gH-Hadamard derivative fails to dominate the zero interval at that point. To further address complex optimization problems, we introduce a new generalized convexity via the Clarke subdifferential in this paper.

The subsequent content of the study is organized as follows. First, some necessary preliminaries are presented, including fundamental definitions and classical results concerning intervals and interval mappings. Section 3 first defines the Clarke directional derivative and Clarke subdifferential for interval mappings; it then establishes its relation to the gH-subdifferential under convexity. Building on this subdifferential, we further propose a new generalized convexity notion for interval mappings. In Section 4, with the introduced generalized convexity and Clarke subdifferential, optimality conditions are obtained. Meanwhile, an illustrative example is presented to verify the applicability of the corresponding theoretical conclusions.

2. Interval Mappings: Basic Theory

2.1. Basic Definitions of Intervals

Throughout this work, $I(R)$ denotes the family of real intervals, and $I(R)^n$ refers to the n-dimensional interval vector space.

The fundamental arithmetic operations for intervals are formulated in [1,31], while the gH-difference operation is defined as follows.

Definition 1 (see [31]). *Let $A, B \in I(R)$. The gH-difference $A \ominus_{gH} B$ of A and B is expressed as $A \ominus_{gH} B = [\min\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\}, \max\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\}]$.*

Remark 1 (see [31]). *Let $A, B \in I(R)$ and $k > 0$. The scaling law $k(A \ominus_{gH} B) = kA \ominus_{gH} kB$ holds true.*

Proposition 1. *Suppose that $A, B, C, D \in I(R)$ are real intervals and $k \in R$. If the interval widths satisfy $\overline{a} - \underline{a} \geq \overline{b} - \underline{b}$ and $\overline{c} - \underline{c} \geq \overline{d} - \underline{d}$, or $\overline{a} - \underline{a} \leq \overline{b} - \underline{b}$ and $\overline{c} - \underline{c} \leq \overline{d} - \underline{d}$, then $(A + C) \ominus_{gH} (B + D) = (A \ominus_{gH} B) + (C \ominus_{gH} D)$.*

Proof. If $\overline{a} - \underline{a} \geq \overline{b} - \underline{b}$ and $\overline{c} - \underline{c} \geq \overline{d} - \underline{d}$, then

$$(A + C) \ominus_{gH} (B + D) = [\underline{a} + \underline{c} - \underline{b} - \underline{d}, \overline{a} + \overline{c} - \overline{b} - \overline{d}],$$

and

$$(A \ominus_{gH} B) + (C \ominus_{gH} D) = [\underline{a} - \underline{b}, \overline{a} - \overline{b}] + [\underline{c} - \underline{d}, \overline{c} - \overline{d}] = [\underline{a} - \underline{b} + \underline{c} - \underline{d}, \overline{a} - \overline{b} + \overline{c} - \overline{d}].$$

Therefore, $(A + C) \ominus_{gH} (B + D) = (A \ominus_{gH} B) + (C \ominus_{gH} D)$.

If $\overline{a} - \underline{a} \leq \overline{b} - \underline{b}$ and $\overline{c} - \underline{c} \leq \overline{d} - \underline{d}$, the equality follows similarly. □

Remark 2. *The width conditions in Proposition 1 ensure the validity of the distributive property of the gH-difference. They hold for common classes of interval mappings with monotonic or constant widths, which are relevant in practical interval optimization problems.*

It is widely acknowledged that order relations occupy a central position in optimization theory. The partial order $\preceq$ on intervals is defined as follows [32]:

(i) An interval A is said to be less than or equal to B if $\underline{a} \leq \underline{b}$ and $\overline{a} \leq \overline{b}$, which is written as $A \preceq B$;

(ii) The strict ordering $A \prec B$ holds if $A \preceq B$ and $A \neq B$.

Proposition 2. *Let $A, B, C, D \in I(R)$ and $k \in R$.*

(i) For any $k > 0$, the inequality $kA \preceq kB$ is equivalent to $A \preceq B$;

(ii) If the two interval order relations $A \preceq B$ and $C \preceq D$ hold simultaneously, then $A + C \preceq B + D$;

(iii) If $A \prec B$ and $C \preceq D$, then $A + C \prec B + D$.

Proof. This follows directly from the definitions of interval arithmetic and interval ordering. □

Proposition 3 (see [4,14])**.** *Let $A, B, C \in I(R)$.*

(i) A necessary and sufficient condition for $0 \preceq (\prec) A \ominus_{gH} B$ is that $B \preceq (\prec) A$;

(ii) If $A \preceq B$, then $A \ominus_{gH} C \preceq B \ominus_{gH} C$.

Definition 2 (see [13])**.** *Let $P \subseteq I(R)$ and $A \in I(R)$. If $B \preceq A$ for any $B \in P$, then A is said to be an upper bound of P. Suppose that $C \in I(R)$ is an upper bound of P. If $A \preceq C$ for every upper bound C of P, then A is called the supremum of P, denoted as sup P.*

In particular, let Γ denote an index set and $P = \{[a_i, b_i] \in I(R) : i \in \Gamma\}$. In accordance with Definition 2, the supremum of the set P satisfies

$$\sup P = [\sup_{i \in \Gamma} a_i, \sup_{i \in \Gamma} b_i].$$

Given $A \in I(R)$, the corresponding norm is defined in [1] as

$$\|A\|_{I(R)} = \max\{|\underline{a}|, |\overline{a}|\}. \tag{1}$$

The Pompeiu–Hausdorff distance for $A, B \in I(R)$ is formulated in [32] as

$$D(A, B) = \max[\max_{a \in A} d(a, B), \max_{b \in B} d(b, A)],$$

where $d(a, B) = \min_{b \in B} |a - b|$. As stated in [31], this distance satisfies $D(A, B) = \|A \ominus_{gH} B\|_{I(R)}$.

Proposition 4. *Let $A, B \in I(R)$. Then,*

(i) $\|A + B\|_{I(R)} \leq \|A\|_{I(R)} + \|B\|_{I(R)}$;

(ii) $\|kA\|_{I(R)} = k\|A\|_{I(R)}$ *for any* $k > 0$.

Proof. This follows directly from the definition of the interval norm on $I(R)$. □

Suppose that $A = (A_1, A_2, \ldots, A_n)$, $B = (B_1, B_2, \ldots, B_n)$ belong to $I(R)^n$, and let $k \in R$. The algebraic operations for A and B are defined below [33]:

(i) $A + B = (A_1 + B_1, A_2 + B_2, \ldots, A_n + B_n)$;

(ii) $kA = (kA_1, kA_2, \ldots, kA_n)$.

The ordering for A and B is defined as follows: $A \preceq (\prec) B$ if $A_i \preceq (\prec) B_i$ for every index $i = 1, 2, \ldots, n$.

2.2. Lipschitz Condition of Interval Mappings

Suppose that $K \subseteq R^n$ denotes a convex subset of the n-dimensional Euclidean space. An interval mapping $F : K \to I(R)$ is expressed as $F(x) = [\underline{F}(x), \overline{F}(x)]$, in which $\underline{F}(x)$ and $\overline{F}(x)$ stand for real-valued functions on K satisfying $\underline{F}(x) \leq \overline{F}(x)$ for all $x \in K$.

This work focuses on interval mappings satisfying the conditions given in Proposition 1.

Definition 3 (see [13]). *Let $F : K \to I(R)$ denote an interval mapping. If there exists $L > 0$ such that*

$$\|F(x) \ominus_{gH} F(y)\|_{I(R)} \leq L\|x - y\|,$$

for all $x, y \in K$, then F is said to be Lipschitz on K.

In particular, given $x_0 \in K$, suppose that there exists $\varepsilon > 0$ such that F is Lipschitz on $x_0 + \varepsilon\mathscr{B}$, where $\mathscr{B} = \{x \in R^n \mid \|x\| \leq 1\}$. In this case, F satisfies the Lipschitz condition at x_0.

Proposition 5. *Let $F, G : K \to I(R)$ denote interval mappings, and let $x_0 \in K$. If F and G are Lipschitz at x_0, then*

(i) *kF is Lipschitz at x_0, for any $k > 0$;*

(ii) *$F + G$ is Lipschitz at x_0.*

Proof. (i) As F is Lipschitz at x_0, one can find $\varepsilon > 0$ and $L > 0$ that satisfy

$$\|F(x) \ominus_{gH} F(y)\|_{I(R)} \leq L\|x - y\|,$$

for any $x, y \in x_0 + \varepsilon\mathscr{B}$. According to Remark 1 and Proposition 4, we have

$$\|kF(x) \ominus_{gH} kF(y)\|_{I(R)} = \|k(F(x) \ominus_{gH} F(y))\|_{I(R)} = k\|F(x) \ominus_{gH} F(y)\|_{I(R)},$$

for any $k > 0$. Then,

$$\|kF(x) \ominus_{gH} kF(y)\|_{I(R)} \leq kL\|x - y\|,$$

for any $x, y \in x_0 + \varepsilon\mathscr{B}$ and $k > 0$. Therefore, kF is Lipschitz at x_0.

(ii) As F is Lipschitz at x_0, we can find $\varepsilon_1 > 0$ and $L_1 > 0$, which satisfy the relation

$$\|F(x_1) \ominus_{gH} F(y_1)\|_{I(R)} \leq L_1\|x_1 - y_1\|, \tag{2}$$

for any $x_1, y_1 \in x_0 + \varepsilon_1\mathscr{B}$. Similarly, there also exist $\varepsilon_2 > 0$ and $L_2 > 0$ satisfying

$$\|G(x_2) \ominus_{gH} G(y_2)\|_{I(R)} \leq L_2\|x_2 - y_2\|, \tag{3}$$

for any $x_2, y_2 \in x_0 + \varepsilon_2\mathscr{B}$. Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ and $L = L_1 + L_2$. According to Proposition 1 and Proposition 4, we have

$$\begin{aligned}
&\|(F(x_3) + G(x_3)) \ominus_{gH} (F(y_3) + G(y_3))\|_{I(R)} \\
=&\|((F(x_3) \ominus_{gH} F(y_3)) + (G(x_3) \ominus_{gH} G(y_3)\|_{I(R)} \\
\leq&\|(F(x_3) \ominus_{gH} F(y_3)\|_{I(R)} + \|G(x_3) \ominus_{gH} G(y_3)\|_{I(R)},
\end{aligned}$$

for any $x_3, y_3 \in x_0 + \varepsilon\mathscr{B}$. From Equations (2) and (3), we obtain

$$\begin{aligned}
&\|(F(x_3) \ominus_{gH} F(y_3)\|_{I(R)} + \|G(x_3) \ominus_{gH} G(y_3)\|_{I(R)} \\
\leq&L_1\|x_3 - y_3\| + L_2\|x_3 - y_3\| \\
=&L\|x_3 - y_3\|,
\end{aligned}$$

for any $x_3, y_3 \in x_0 + \varepsilon\mathscr{B}$. Accordingly, $F + G$ is Lipschitz at x_0. □

Lemma 1 (see [18]). *Let $F : K \to I(R)$ denote an interval mapping, and let $x_0 \in K$. If F is Lipschitz at x_0, then $\overline{F}$ and $\underline{F}$ are also Lipschitz at x_0.*

3. Directional Derivative, Subdifferential and Generalized Convexity for Interval Mappings

This section investigates the fundamental properties of directional derivatives for interval mappings. The notion of the Clarke subdifferential for interval mappings is introduced. This work also establishes the correspondence between the Clarke subdifferential and the gH-subdifferential within the framework of convex interval mappings. In addition, a new generalized convexity notion relying on the Clarke subdifferential is defined, and its fundamental properties are analyzed.

3.1. Directional Derivatives of Interval Mappings

As classical differentiability fails for nonsmooth interval mappings, directional derivatives serve as essential substitutes for describing local variation trends. Based on Lipschitz continuity, the existence, structural characteristics and linear operations of Clarke directional derivatives are analyzed. The inherent connections between directional derivatives and the convexity and regularity conditions of interval mappings are established.

Definition 4 (see [32]). *Let $F : K \to I(R)$ denote an interval mapping, and let $A \in I(R)$ and $x, x_0 \in K$. For every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$D(F(x), A) < \varepsilon,$$

for all $x \in x_0 + \delta\mathscr{B}$. Then, A is called the limit of F as $x \to x_0$, denoted as $\lim_{x\to x_0} F(x) = A$.

Theorem 1. *Let $F : K \to I(R)$ denote an interval mapping, and let $A, B \in I(R)$, $x_0 \in K$ and $\delta > 0$. For any $x \in x_0 + \delta\mathscr{B}$, if $F(x) \preceq A$ holds and $\lim_{x\to x_0} F(x) = B$ is satisfied, it can be concluded that $B \preceq A$.*

Proof. According to Definition 4 and $\lim_{x\to x_0} F(x) = B$, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $D(F(x), B) < \varepsilon$ for all $x \in x_0 + \delta\mathscr{B}$. Consequently, $\|F(x) \ominus_{gH} B\|_{I(R)} < \varepsilon$. It follows from Equation (1) that

$$|\underline{F}(x) - \underline{b}| < \varepsilon \quad \text{and} \quad |\overline{F}(x) - \overline{b}| < \varepsilon,$$

for any $x \in x_0 + \delta\mathscr{B}$. Then,

$$-\varepsilon + \underline{b} < \underline{F}(x) < \underline{b} + \varepsilon \quad \text{and} \quad -\varepsilon + \overline{b} < \overline{F}(x) < \overline{b} + \varepsilon,$$

for any $x \in x_0 + \delta\mathscr{B}$. Since $F(x) \preceq A$, it holds that $\underline{F}(x) \leq \underline{a}$ and $\overline{F}(x) \leq \overline{a}$ for all $x \in x_0 + \delta\mathscr{B}$. Then, $-\varepsilon + \underline{b} < \underline{a}$ and $-\varepsilon + \overline{b} < \overline{a}$. Therefore, $\underline{b} \leq \underline{a}$ and $\overline{b} \leq \overline{a}$, which implies that $B \preceq A$. □

Definition 5 (see [18]). *Let an interval mapping $F : K \to I(R)$ satisfy the Lipschitz condition at $x_0 \in K$, and let $x \in K$, $d \in R^n$, $t \in R$ satisfy $x + td \in K$. Whenever the superior limit*

$$\limsup_{\substack{x\to x_0 \\ t\to 0^+}} \frac{F(x+td) \ominus_{gH} F(x)}{t}$$

is well defined, this supremum limit is defined as the Clarke directional derivative of F at x_0 with respect to the direction d, which is written as $F^0(x_0; d)$.

Lemma 2 (see [18]). *Let an interval mapping $F : K \to I(R)$ satisfy the Lipschitz condition at $x_0 \in K$. Then, the Clarke directional derivative of F at x_0 with respect to the direction $d \in R^n$ always exists.*

Theorem 2. *Let an interval mapping $F : K \to I(R)$ satisfy the Lipschitz condition at $x_0 \in K$, and let $d \in R^n$. Then,*

$$F^0(x_0; d) \preceq [\min\{\underline{F}^0(x_0; d), \overline{F}^0(x_0; d)\}, \max\{\underline{F}^0(x_0; d), \overline{F}^0(x_0; d)\}].$$

Proof. According to Definition 5 and Lemma 2, we can obtain

$$\begin{aligned}
F^0(x_0; d) &= \limsup_{\substack{x\to x_0 \\ t\to 0^+}} \frac{F(x+td) \ominus_{gH} F(x)}{t} \\
&= \limsup_{\substack{x\to x_0 \\ t\to 0^+}} [\min\{\frac{\underline{F}(x+td) - \underline{F}(x)}{t}, \frac{\overline{F}(x+td) - \overline{F}(x)}{t}\}, \\
&\quad \max\{\frac{\underline{F}(x+td) - \underline{F}(x)}{t}, \frac{\overline{F}(x+td) - \overline{F}(x)}{t}\}] \\
&= [\limsup_{\substack{x\to x_0 \\ t\to 0^+}} \min\{\frac{\underline{F}(x+td) - \underline{F}(x)}{t}, \frac{\overline{F}(x+td) - \overline{F}(x)}{t}\}, \\
&\quad \limsup_{\substack{x\to x_0 \\ t\to 0^+}} \max\{\frac{\underline{F}(x+td) - \underline{F}(x)}{t}, \frac{\overline{F}(x+td) - \overline{F}(x)}{t}\}].
\end{aligned}$$

Let $f(x,t) = \dfrac{\underline{F}(x+td) - \underline{F}(x)}{t}$, $g(x,t) = \dfrac{\overline{F}(x+td) - \overline{F}(x)}{t}$, $f^* = \limsup_{\substack{x\to x_0 \\ t\to 0^+}} f(x,t)$ and $g^* = \limsup_{\substack{x\to x_0 \\ t\to 0^+}} g(x,t)$. Combined with the Lipschitz condition of F at x_0 and the results in Lemma 1, we have $f^* = \underline{F}^0(x_0; d)$ and $g^* = \overline{F}^0(x_0; d)$.

It is obvious that $\min\{f(x,t), g(x,t)\} \leq f(x,t)$; then,

$$\limsup_{\substack{x\to x_0\\ t\to 0^+}} \min\{f(x,t), g(x,t)\} \leq \limsup_{\substack{x\to x_0\\ t\to 0^+}} f(x,t).$$

Similarly, we can get $\limsup_{\substack{x\to x_0\\ t\to 0^+}} \min\{f(x,t), g(x,t)\} \leq \limsup_{\substack{x\to x_0\\ t\to 0^+}} g(x,t)$. Therefore,

$$\limsup_{\substack{x\to x_0\\ t\to 0^+}} \min\{f(x,t), g(x,t)\} \leq \min\{\limsup_{\substack{x\to x_0\\ t\to 0^+}} f(x,t), \limsup_{\substack{x\to x_0\\ t\to 0^+}} g(x,t)\}.$$

That is, the inequality $\limsup_{\substack{x\to x_0\\ t\to 0^+}} \min\{f(x,t), g(x,t)\} \leq \min\{f^*, g^*\}$ can be derived accordingly. Similarly, we have $\limsup_{\substack{x\to x_0\\ t\to 0^+}} \max\{f(x,t), g(x,t)\} \geq \max\{f^*, g^*\}$.

On the other hand, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$f(x,t) \leq f^* + \varepsilon \quad \text{and} \quad g(x,t) \leq g^* + \varepsilon$$

hold for all x and t with $\|x - x_0\| < \delta$ and $0 < t < \delta$. Then,

$$\max\{f(x,t), g(x,t)\} \leq \max\{f^*, g^*\} + \varepsilon,$$

for any x, t with $\|x - x_0\| < \delta, 0 < t < \delta$. Then,

$$\limsup_{\substack{x\to x_0\\ t\to 0^+}} \max\{f(x,t), g(x,t)\} \leq \max\{f^*, g^*\} + \varepsilon.$$

Thus, $\limsup_{\substack{x\to x_0\\ t\to 0^+}} \max\{f(x,t), g(x,t)\} \leq \max\{f^*, g^*\}$. Therefore, $\limsup_{\substack{x\to x_0\\ t\to 0^+}} \max\{f(x,t), g(x,t)\} = \max\{f^*, g^*\}$.

Combining the above, we have $F^0(x_0; d) \preceq [\min\{f^*, g^*\}, \max\{f^*, g^*\}]$; that is to say,

$$F^0(x_0; d) \preceq [\min\{\underline{F}^0(x_0; d), \overline{F}^0(x_0; d)\}, \max\{\underline{F}^0(x_0; d), \overline{F}^0(x_0; d)\}].$$

□

Example 1. *Let an interval mapping*

$$F(x) = \begin{cases} [0, x], & x \geq 0, \\ [2x, x], & x < 0. \end{cases}$$

In the case $d > 0$, three separate scenarios are identified to be discussed later.

(i) If $x > 0$, then $x + td > 0$. We have

$$\begin{aligned} &\limsup_{\substack{x\to 0\\ t\to 0^+}} \frac{F(x+td) \ominus_{gH} F(x)}{t} \\ =&\limsup_{\substack{x\to 0^+\\ t\to 0^+}} \frac{[0, x+td] \ominus_{gH} [0, x]}{t} \\ =&[0, d]. \end{aligned}$$

(ii) *If $x < 0$ with $x + td > 0$, we have*

$$\begin{aligned}&\limsup_{\substack{x\to 0\\ t\to 0^+}} \frac{F(x+td)\ominus_{gH} F(x)}{t}\\ =&\limsup_{\substack{x\to 0^-\\ t\to 0^+}} \frac{[0, x+td]\ominus_{gH}[2x, x]}{t}\\ =&\limsup_{\substack{x\to 0^-\\ t\to 0^+}} [\min\{d, \frac{-2x}{t}\}, \max\{d, \frac{-2x}{t}\}].\end{aligned}$$

It follows from $x + td > 0$ that $\frac{-2x}{t} < 2d$; therefore, $\limsup_{\substack{x\to 0^-\\ t\to 0^+}} [\min\{d, \frac{-2x}{t}\}, \max\{d, \frac{-2x}{t}\}] \preceq [d, 2d]$.

(iii) *If $x < 0$ with $x + td < 0$, we have*

$$\begin{aligned}&\limsup_{\substack{x\to 0\\ t\to 0^+}} \frac{F(x+td)\ominus_{gH} F(x)}{t}\\ =&\limsup_{\substack{x\to 0^-\\ t\to 0^+}} \frac{[2x+2td, x+td]\ominus_{gH}[2x, x]}{t}\\ =&[d, 2d].\end{aligned}$$

Consequently, for $d > 0$, we have $\limsup_{\substack{x\to 0\\ t\to 0^+}} \frac{F(x+td)\ominus_{gH} F(x)}{t} = [d, 2d]$. Similarly, for $d = 0$, we obtain $\limsup_{\substack{x\to 0\\ t\to 0^+}} \frac{F(x+td)\ominus_{gH} F(x)}{t} = [0, 0]$. Likewise, for $d < 0$, we can get $\limsup_{\substack{x\to 0\\ t\to 0^+}} \frac{F(x+td)\ominus_{gH} F(x)}{t} = [d, 0]$. Therefore,

$$F^0(0; d) = \begin{cases} [d, 2d], & d > 0,\\ [0, 0], & d = 0,\\ [d, 0], & d < 0.\end{cases}$$

In addition,

$$\underline{F}(x) = \begin{cases} 0, & x \geq 0,\\ 2x, & x < 0,\end{cases} \quad \text{and} \quad \overline{F}(x) = x.$$

For $d > 0$, we can get $\underline{F}^0(0; d) = 2d$ and $\overline{F}^0(0; d) = d$. Therefore, $\min\{\underline{F}^0(0; d), \overline{F}^0(0; d)\} = d$ and $\max\{\underline{F}^0(0; d), \overline{F}^0(0; d)\} = 2d$. As a result, $F^0(0; d) = [\min\{\underline{F}^0(0; d), \overline{F}^0(0; d)\}, \max\{\underline{F}^0(0; d), \overline{F}^0(0; d)\}]$ for any $d > 0$. Similarly, for $d < 0$ and $d = 0$, we obtain $F^0(0; d) \preceq [\min\{\underline{F}^0(0; d), \overline{F}^0(0; d)\}, \max\{\underline{F}^0(0; d), \overline{F}^0(0; d)\}]$.

Definition 6 (see [13]). *Let $F : K \to I(R)$ denote an interval mapping, and let $x_0 \in K$, $d \in R^n$, $t \in R$ satisfy $x_0 + td \in K$. If the limit*

$$\lim_{t\to 0^+} \frac{F(x_0+td)\ominus_{gH} F(x_0)}{t}$$

is well defined, then this limit is referred to as the gH-directional derivative of F at x_0 with respect to the direction d, which is denoted by $F'(x_0; d)$.

Definition 7 (see [32]). *An interval mapping $F : K \to I(R)$ is said to be convex on K if*

$$F(\lambda x + (1-\lambda)y) \preceq \lambda F(x) + (1-\lambda)F(y)$$

holds for all $x, y \in K$ and every $\lambda \in [0,1]$.

Lemma 3 (see [13]). *For any convex interval mapping $F : K \to I(R)$, the expression $\dfrac{F(x+td) \ominus_{gH} F(x)}{t}$ is monotone nondecreasing with respect to t.*

Theorem 3. *Let $F : K \to I(R)$ denote an interval mapping, and let $x \in K$. Assume that F is convex and admits the gH-directional derivative at x with respect to the direction $y - x$ for arbitrary $y \in K$. Then,*

$$F'(x; y-x) \preceq F(y) \ominus_{gH} F(x).$$

Proof. As F is convex, the relation $F(x+\lambda(y-x)) \preceq \lambda F(y) + (1-\lambda)F(x)$ is valid for any $x, y \in K$ and $\lambda \in [0,1]$. According to Proposition 3, we can obtain

$$F(x+\lambda(y-x)) \ominus_{gH} F(y) \preceq (\lambda F(y) + (1-\lambda)F(x)) \ominus_{gH} F(y).$$

It follows from Definition 1 that

$$\begin{aligned}
(\lambda F(y) + (1-\lambda)F(x)) \ominus_{gH} F(y) &= [\min\{\lambda\underline{F}(y) + (1-\lambda)\underline{F}(x) - \underline{F}(x), \lambda\overline{F}(y) + (1-\lambda)\overline{F}(x) - \overline{F}(x)\},\\
&\quad \max\{\lambda\underline{F}(y) + (1-\lambda)\underline{F}(x) - \underline{F}(x), \lambda\overline{F}(y) + (1-\lambda)\overline{F}(x) - \overline{F}(x)\}]\\
&= [\min\{\lambda\underline{F}(y) - \lambda\underline{F}(x), \lambda\overline{F}(y) - \lambda\overline{F}(x)\},\\
&\quad \max\{\lambda\underline{F}(y) - \lambda\underline{F}(x), \lambda\overline{F}(y) - \lambda\overline{F}(x)\}]\\
&= \lambda[\min\{\underline{F}(y) - \underline{F}(x), \overline{F}(y) - \overline{F}(x)\}, \max\{\underline{F}(y) - \underline{F}(x), \overline{F}(y) - \overline{F}(x)\}]\\
&= \lambda(F(y) \ominus_{gH} F(x)).
\end{aligned}$$

From Proposition 2, it follows that

$$\frac{F(x+\lambda(y-x)) \ominus_{gH} F(x)}{\lambda} \preceq F(y) \ominus_{gH} F(x).$$

Based on Theorem 1, one obtains

$$\lim_{\lambda \to 0^+} \frac{F(x+\lambda(y-x)) \ominus_{gH} F(x)}{\lambda} \preceq F(y) \ominus_{gH} F(x).$$

Hence, $F'(x; y-x) \preceq F(y) \ominus_{gH} F(x)$. □

Definition 8. *Let $F : K \to I(R)$ denote an interval mapping and $x_0 \in K$. If, for any direction d, the Clarke directional derivative and gH-directional derivative of F at x_0 both exist and satisfy $F^0(x_0; d) = F'(x_0; d)$, then F is said to be regular at x_0.*

Theorem 4. *Let $F, G : K \to I(R)$ denote interval mappings, and let $x_0 \in K$. Provided that F and G satisfy the Lipschitz condition at x_0, then*

(i) $kF^0(x_0; d) = (kF)^0(x_0; d)$, *for any $k > 0$;*

(ii) *When F and G are regular at x_0, it can be concluded that $(F+G)^0(x_0; d) = F^0(x_0; d) + G^0(x_0; d)$.*

Proof. (i) According to Definition 5 and Lemma 2, we have

$$F^0(x_0;d)=\limsup_{\substack{x\to x_0\\ t\to 0^+}}\frac{F(x+td)\ominus_{gH}F(x)}{t}.$$

Accordingly, for any $k>0$,

$$kF^0(x_0;d)=k\limsup_{\substack{x\to x_0\\ t\to 0^+}}\frac{F(x+td)\ominus_{gH}F(x)}{t}=\limsup_{\substack{x\to x_0\\ t\to 0^+}}\frac{k[F(x+td)\ominus_{gH}F(x)]}{t}.$$

As shown in Remark 1, we have

$$\limsup_{\substack{x\to x_0\\ t\to 0^+}}\frac{k[F(x+td)\ominus_{gH}F(x)]}{t}=\limsup_{\substack{x\to x_0\\ t\to 0^+}}\frac{kF(x+td)\ominus_{gH}kF(x)}{t}.$$

By virtue of the Lipschitz condition of F at x_0 and Proposition 5 and Lemma 2, we can deduce $kF^0(x_0;d)=(kF)^0(x_0;d)$.

(ii) Assume that F and G are regular at x_0. Then,

$$F^0(x_0;d)=F^{'}(x_0;d)\quad\text{and}\quad G^0(x_0;d)=G^{'}(x_0;d).\tag{4}$$

According to Definition 5 and Lemma 2, we have

$$\begin{aligned}F^0(x_0;d)+G^0(x_0;d)&=\limsup_{\substack{y\to x_0\\ t\to 0^+}}\frac{F(y+td)\ominus_{gH}F(y)}{t}+\limsup_{\substack{y\to x_0\\ t\to 0^+}}\frac{G(y+td)\ominus_{gH}G(y)}{t}\\&=\lim_{t\to 0^+}\frac{F(x_0+td)\ominus_{gH}F(x_0)}{t}+\lim_{t\to 0^+}\frac{G(x_0+td)\ominus_{gH}G(x_0)}{t}\\&=\lim_{t\to 0^+}\frac{(F(x_0+td)\ominus_{gH}F(x_0))+(G(x_0+td)\ominus_{gH}G(x_0))}{t}.\end{aligned}$$

From Proposition 1, we can get

$$\begin{aligned}&\lim_{t\to 0^+}\frac{(F(x_0+td)\ominus_{gH}F(x_0))+(G(x_0+td)\ominus_{gH}G(x_0))}{t}\\=&\lim_{t\to 0^+}\frac{(F(x_0+td)+G(x_0+td))\ominus_{gH}(G(x_0)+F(x_0))}{t}.\end{aligned}$$

By Proposition 5 and Lemma 2, we obtain

$$\begin{aligned}(F+G)^0(x_0;d)&=\limsup_{\substack{y\to x_0\\ t\to 0^+}}\frac{(F(y+td)+G(y+td))\ominus_{gH}(F(y)+G(y))}{t}\\&\succeq\lim_{t\to 0^+}\frac{(F(x_0+td)+G(x_0+td))\ominus_{gH}(F(x_0)+G(x_0))}{t}.\end{aligned}$$

Therefore, $(F+G)^0(x_0;d)\succeq F^0(x_0;d)+G^0(x_0;d)$.

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In addition, by Proposition 1, Proposition 5 and Lemma 2, we get

$$\begin{aligned}(F+G)^0(x_0;d) &= \limsup_{\substack{x\to x_0\\ t\to 0^+}} \frac{(F(x+td)+G(x+td))\ominus_{gH}(F(x)+G(x))}{t}\\ &= \limsup_{\substack{x\to x_0\\ t\to 0^+}} \frac{(F(x+td)\ominus_{gH}F(x))+(G(x+td)\ominus_{gH}G(x))}{t}\\ &\preceq \limsup_{\substack{x\to x_0\\ t\to 0^+}} \frac{F(x+td)\ominus_{gH}F(x)}{t} + \limsup_{\substack{x\to x_0\\ t\to 0^+}} \frac{G(x+td)\ominus_{gH}G(x)}{t}\\ &= F^0(x_0;d)+G^0(x_0;d).\end{aligned}$$

Therefore, $(F+G)^0(x_0;d) \preceq F^0(x_0;d)+G^0(x_0;d)$. Thus, $(F+G)^0(x_0;d) = F^0(x_0;d)+G^0(x_0;d)$. □

Lemma 4 (see [18]). *Let $F: K \to I(R)$ denote a convex interval mapping, and let $x_0 \in K$. If F is Lipschitz at x_0, then $F^0(x_0;d) = F'(x_0;d)$.*

3.2. Clarke Subdifferentials of Interval Mappings

As classical gradient theory is invalid for nonsmooth interval mappings, subdifferentials act as effective tools to tackle such nonsmooth optimization problems. The distinctions and applicability ranges of the Clarke subdifferential and gH-subdifferential are demonstrated via numerical examples. Moreover, the linear operational properties of Clarke subdifferentials are deduced.

Definition 9. *Let $F: K \to I(R)$ denote an interval mapping, and let $x_0 \in K$. For any $d \in R^n$, if there exists an interval vector $A = (A_1, A_2, \ldots, A_n)$ satisfying*

$$\langle A, d\rangle \preceq F^0(x_0;d),$$

then A is referred to as the Clarke subgradient of F at x_0. Here, $\langle A, d\rangle$ denotes $\sum_{i=1}^n d_i A_i$. The collection of all Clarke subgradients is defined as the Clarke subdifferential of F at x_0, designated as $\partial F^0(x_0)$.

Example 2. *The interval mapping $F(x,y) = [x^2+|y|, x^2+4|y|]$ will be considered, where $(x,y) \in R^2$. Let $\mathscr{B} = \{(x,y) \in R^2 \mid \sqrt{x^2+y^2} \le 1\}$. For any $X = (x_1, y_1)$, $Y = (x_2, y_2) \in \mathscr{B}$, we have*

$$\begin{aligned}\|F(X)\ominus_{gH}F(Y)\|_{I(R)} &= \|[\min\{x_1^2-x_2^2+|y_1|-|y_2|, x_1^2-x_2^2+4(|y_1|-|y_2|)\},\\ &\quad \max\{x_1^2-x_2^2+|y_1|-|y_2|, x_1^2-x_2^2+4(|y_1|-|y_2|)\}]\|_{I(R)}\\ &= max\{|x_1^2-x_2^2+|y_1|-|y_2||, |x_1^2-x_2^2+4(|y_1|-|y_2|)|\}.\end{aligned}$$

Furthermore,

$$|x_1^2-x_2^2+|y_1|-|y_2|| \le |x_1^2-x_2^2| + ||y_1|-|y_2|| \le |x_1+x_2||x_1-x_2| + |y_1-y_2|.$$

By $|x_1| \le 1$, $|x_2| \le 1$, we have $|x_1^2-x_2^2+|y_1|-|y_2|| \le 2|x_1-x_2|+|y_1-y_2|$. Combined with the inequalities $|x_1-x_2| \le \|X-Y\|$ and $|y_1-y_2| \le \|X-Y\|$, we obtain $|x_1^2-x_2^2+|y_1|-|y_2|| \le 3\|X-Y\|$. Similarly, we obtain $|x_1^2-x_2^2+4(|y_1|-|y_2|)| \le 6\|X-Y\|$. Consequently, $\|F(X)\ominus_{gH}F(Y)\|_{I(R)} \le 6\|X-Y\|$, which implies that $F(x,y)$ satisfies the Lipschitz condition on $\mathscr{B}$. It follows from Theorem 2 that

$$F^0((0,0);(d_1,d_2)) \preceq [\min\{\underline{F}^0((0,0);(d_1,d_2)), \overline{F}^0((0,0);(d_1,d_2))\}, \max\{\underline{F}^0((0,0);(d_1,d_2)), \overline{F}^0((0,0);(d_1,d_2))\}].$$

Through calculation, we obtain $\underline{F}^0((0,0);(d_1,d_2)) = |d_2|$ *and* $\overline{F}^0((0,0);(d_1,d_2)) = 4|d_2|$. *Therefore,*

$$F^0((0,0);(d_1,d_2)) \preceq [|d_2|, 4|d_2|].$$

On the other hand, for $x = 0$, *we have*

$$F^0((0,0);(d_1,d_2)) = \limsup_{\substack{(0,y)\to(0,0)\\ t\to 0^+}} \frac{F((0,y)+td) \ominus_{gH} F(0,y)}{t} = [|d_2|, 4|d_2|].$$

When $d_2 \geq 0$, *assume that* $A = ([\underline{a}_1, \overline{a}_1], [\underline{a}_2, \overline{a}_2])$ *with* $\underline{a}_1 = \overline{a}_1 = 0$, $\underline{a}_2 \leq 1$ *and* $\overline{a}_2 \leq 4$, *and we have* $\langle A, (d_1,d_2)\rangle \preceq [d_2, 4d_2]$. *When* $d_2 < 0$, *assume that* $A = ([\underline{a}_1, \overline{a}_1], [\underline{a}_2, \overline{a}_2])$ *with* $\underline{a}_1 = \overline{a}_1 = 0$, $\underline{a}_2 \geq -4$ *and* $\overline{a}_2 \geq -1$, *and we have* $\langle A, (d_1,d_2)\rangle \preceq [-d_2, -4d_2]$.

In summary, $\partial^0 F((0,0)) = \{A = ([0,0], [\underline{a}_2, \overline{a}_2]) \mid -4 \leq \underline{a}_2 \leq 1, -1 \leq \overline{a}_2 \leq 4, \underline{a}_2 \leq \overline{a}_2\}$.

Definition 10 (see [20]). *Let* $F : K \to I(R)$ *denote an interval mapping and* $x_0 \in K$. *If there exists* $A = (A_1, A_2, \ldots, A_n) \in (I(R))^n$ *satisfying*

$$\langle A, x - x_0\rangle \preceq F(x) \ominus_{gH} F(x_0),$$

for any $x \in K$, *then* A *is defined as the gH-subgradient of* F *at* x_0. *The collection of all gH-subgradients is defined as the gH-subdifferential of* F *at* x_0, *designated as* $\partial F(x_0)$.

Remark 3. *An interval mapping admits a nonempty Clarke subdifferential, yet its gH-subdifferential may fail to exist for nonconvex cases. Consequently, the Clarke subdifferentials can handle a broader class of interval optimization problems.*

Example 3. *The interval mapping*

$$F(x) = \begin{cases} [0, x], & x \geq 0, \\ [2x, x], & x < 0, \end{cases}$$

will be considered. We first examine the Clarke subdifferential of F *at* $x = 0$. *For* $d > 0$, *assume that* $A = [\underline{a}, \overline{a}]$ *with* $\underline{a} \leq 1$, $\overline{a} \leq 2$. *We have* $\langle A, d\rangle \preceq [d, 2d]$. *For* $d = 0$, *assume that* $A = [\underline{a}, \overline{a}]$ *with* $\underline{a}, \overline{a} \in R$. *We have* $\langle A, d\rangle \preceq [0,0]$. *For* $d < 0$, *assume that* $A = [\underline{a}, \overline{a}]$ *with* $\underline{a} \geq 0$, $\overline{a} \geq 2$. *We have* $\langle A, d\rangle \preceq [d, 0]$. *It follows from Example 1 that* $\partial^0 F(0) = \{A = [\underline{a}, \overline{a}] \mid 0 \leq \underline{a} \leq 1, 1 \leq \overline{a} \leq 2\}$. *Subsequently, we consider the gH-subdifferential of* F *at* $x = 0$. *For* $x > 0$, *assume that* $B = [\underline{b}, \overline{b}]$ *with* $\underline{b} \leq 0$, $\overline{b} \leq 1$. *We have* $\langle B, x - 0\rangle \preceq [0, x] \ominus_{gH} [0,0]$. *For* $x < 0$, *assume that* $B = [\underline{b}, \overline{b}]$ *with* $\underline{b} \geq 1$, $\overline{b} \geq 2$. *We have* $\langle B, x - 0\rangle \preceq [2x, x] \ominus_{gH} [0,0]$. *Therefore,* $\partial F(0) = \{B = [\underline{b}, \overline{b}]\}$ *is empty.*

Theorem 5. *Let* $F : K \to I(R)$ *denote a convex interval mapping. If, for any* $x_0 \in K$, F *is Lipschitz at* x_0 *and the gH-subdifferential and the Clarke subdifferential of* F *at* x_0 *exist, then* $\partial^0 F(x_0) = \partial F(x_0)$.

Proof. (i) If $A \in \partial^0 F(x_0)$, then $\langle A, d\rangle \preceq F^0(x_0; d)$ for any $d \in R^n$. According to Lemma 4, we can get $\langle A, d\rangle \preceq F'(x_0; d)$. Since F is convex, we obtain from Theorem 3 that $F'(x_0; d) \preceq F(x_0 + d) \ominus_{gH} F(x_0)$. Thus, $\langle A, d\rangle \preceq F(x_0 + d) \ominus_{gH} F(x_0)$, i.e., $A \in \partial F(x_0)$. Therefore, $\partial^0 F(x_0) \subseteq \partial F(x_0)$.

(ii) Provided that $A \in \partial F(x_0)$, it follows from Definition 10 that, for any $d \in R^n$ and $t > 0$, one has

$$\langle A, td \rangle \preceq F(x_0 + td) \ominus_{gH} F(x_0).$$

From Proposition 2, we deduce that

$$\langle A, d \rangle \preceq \frac{F(x_0 + td) \ominus_{gH} F(x_0)}{t}.$$

By Lemma 3, the limit $\lim_{t \to 0^+} \frac{F(x_0 + td) \ominus_{gH} F(x_0)}{t}$ exists, which further indicates that

$$F^{'}(x_0; d) = \lim_{t \to 0^+} \frac{F(x_0 + td) \ominus_{gH} F(x_0)}{t}.$$

From Lemma 4, we obtain $\langle A, d \rangle \preceq F^0(x_0; d)$, i.e., $A \in \partial^0 F(x_0)$. Consequently, $\partial F(x_0) \subseteq \partial^0 F(x_0)$.

From conditions (i) and (ii), we conclude that $\partial^0 F(x_0) = \partial F(x_0)$. □

Theorem 6. *Let $F, G : K \to I(R)$ denote interval mappings, $x_0 \in K$. If F and G are regular mappings at x_0 and the Clarke subdifferentials of F and G at x_0 exist, then*

(i) $\alpha \partial^0 F(x_0) = \partial^0 (\alpha F)(x_0)$ *for any* $\alpha > 0$;
(ii) $\partial^0 F(x_0) + \partial^0 G(x_0) \subseteq \partial^0 (F + G)(x_0)$.

Proof. (i) Let $A \in \partial^0 F(x_0)$. By Definition 9, we have

$$\langle A, d \rangle \preceq F^0(x_0; d),$$

for any $d \in R^n$. Then,

$$\alpha \langle A, d \rangle \preceq \alpha F^0(x_0; d),$$

for any $d \in R^n$ and $\alpha > 0$. According to Theorem 4, we get

$$\langle \alpha A, d \rangle \preceq (\alpha F)^0(x_0; d),$$

for any $d \in R^n$ and $\alpha > 0$. Then, by Definition 9, we can obtain $\alpha A \in \partial^0 (\alpha F)(x_0)$. Thus, $\alpha \partial^0 F(x_0) \subseteq \partial^0 (\alpha F)(x_0)$.

Let $\alpha A \in \partial^0 (\alpha F)(x_0)$, for any $\alpha > 0$. Similarly, we can get $A \in \partial^0 F(x_0)$. Thus, $\partial^0 (\alpha F)(x_0) \subseteq \alpha \partial^0 F(x_0)$, for any $\alpha > 0$.

Therefore, $\alpha \partial^0 F(x_0) = \partial^0 (\alpha F)(x_0)$, for any $\alpha > 0$.

(ii) Let $A \in \partial^0 F(x_0)$ and $B \in \partial^0 G(x_0)$. By Definition 9, we have

$$\langle A, d \rangle \preceq F^0(x_0; d) \quad \text{and} \quad \langle B, d \rangle \preceq G^0(x_0; d), \tag{5}$$

for any $d \in R^n$. It follows from Equation (5) and Proposition 2 that we have

$$\langle A, d \rangle + \langle B, d \rangle \preceq F^0(x_0; d) + G^0(x_0; d),$$

for any $d \in R^n$. Then,

$$\langle A + B, d \rangle \preceq F^0(x_0; d) + G^0(x_0; d),$$

for any $d \in R^n$. According to Theorem 4, we obtain

$$\langle A + B, d \rangle \preceq (F + G)^0(x_0; d),$$

for any $d \in R^n$. Based on Definition 9, we get $A + B \in \partial^0(F + G)(x_0)$. Thus, $\partial^0 F(x_0) + \partial^0 G(x_0) \subseteq \partial^0(F + G)(x_0)$. □

3.3. Generalized Convexity of Interval Mappings

The generalized convexity of interval mappings is defined based on Clarke subdifferentials; this serves as a powerful tool for studying optimality conditions in interval optimization. Moreover, the properties of generalized convexity under positive scalar multiplication and addition are proven.

Definition 11. *Let $F : K \to I(R)$ denote an interval mapping, and suppose that its Clarke subdifferential at $x_0 \in K$ exists. If, for any $A \in \partial^0 F(x_0)$ and any $y \in K$, the inequality*

$$\langle A, y - x_0 \rangle \preceq (\prec) F(y) \ominus_{gH} F(x_0)$$

holds, then F is said to be (strictly) generalized convex at x_0.

Theorem 7. *Let $F, G : K \to I(R)$ denote interval mappings and $x_0 \in K$.*

(i) Suppose that F satisfies generalized convexity at x_0. Then, αF is also generalized convex at x_0 for any $\alpha > 0$.

(ii) Suppose that F and G satisfy generalized convexity at x_0 and the inclusion relation $\partial^0(F+G)(x_0) \subseteq \partial^0 F(x_0) + \partial^0 G(x_0)$ holds. Then, $F + G$ is also generalized convex at x_0.

Proof. (i) Since F satisfies generalized convexity at x_0, by Definition 11, we can obtain

$$\langle A, y - x_0 \rangle \preceq F(y) \ominus_{gH} F(x_0),$$

for any $A \in \partial^0 F(x_0)$ and $y \in K$. According to Proposition 2, we have

$$\alpha \langle A, y - x_0 \rangle \preceq \alpha (F(y) \ominus_{gH} F(x_0)),$$

for any $A \in \partial^0 F(x_0)$, $y \in K$ and $\alpha > 0$. Based on Remark 1, we get

$$\langle \alpha A, y - x_0 \rangle \preceq \alpha F(y) \ominus_{gH} \alpha F(x_0),$$

for any $\alpha A \in \alpha \partial^0 F(x_0)$, $y \in K$ and $\alpha > 0$. According to Theorem 6, we have $\alpha \partial^0 F(x_0) = \partial^0(\alpha F)(x_0)$. Then, by Definition 11, we can obtain that αF is generalized convex at x_0.

(ii) Since F and G satisfy generalized convexity at x_0, by Definition 11, we can obtain

$$\langle A, y - x_0 \rangle \preceq F(y) \ominus_{gH} F(x_0) \quad \text{and} \quad \langle B, y - x_0 \rangle \preceq G(y) \ominus_{gH} G(x_0). \tag{6}$$

for any $A \in \partial^0 F(x_0)$, $B \in \partial^0 G(x_0)$ and $y \in K$. It follows from Proposition 2 and Equation (6) that we get

$$\langle A, y - x_0 \rangle + \langle B, y - x_0 \rangle \preceq (F(y) \ominus_{gH} F(x_0)) + (G(y) \ominus_{gH} G(x_0)),$$

for any $A \in \partial^0 F(x_0)$, $B \in \partial^0 G(x_0)$ and $y \in K$. According to Proposition 1, we can obtain

$$\langle A + B, y - x_0 \rangle \preceq (F(y) + G(y)) \ominus_{gH} (F(x_0)) + G(x_0)),$$

for any $A + B \in \partial^0 F(x_0) + \partial^0 G(x_0)$ and $y \in K$. In addition, it follows from $\partial^0(F+G)(x_0) \subseteq \partial^0 F(x_0) + \partial^0 G(x_0)$ and Theorem 6 that we have $\partial^0(F+G)(x_0) = \partial^0 F(x_0) + \partial^0 G(x_0)$. By Definition 11, $F + G$ is generalized convex at x_0. □

4. Interval Optimization by Clarke Subdifferentials and Generalized Convexity

Clarke subdifferentials constitute a fundamental tool for nonsmooth optimization. With the theories of directional derivatives, subdifferentials and generalized convexity established in Section 3, optimality conditions for interval optimization are constructed in Section 4, and numerical examples are presented to verify these theoretical results.

First, we address the unconstrained interval optimization model

$$\min_{x \in K} F(x), \tag{7}$$

where $F : K \to I(R)$ denotes an interval mapping.

Definition 12 (see [13]). *Let $F : K \to I(R)$ denote an interval mapping, $x_0 \in K$. If no $x \in K$ satisfies $F(x) \prec F(x_0)$, then x_0 is said to be an efficient solution of F.*

Definition 13 (see [4]). *Let $F : K \to I(R)$ denote an interval mapping, $x_0 \in K$. If $F(x_0) \preceq F(x)$ holds for all $x \in K$, then x_0 is said to be a weakly efficient solution of F.*

Consider the minimization problem $\min_{x \in [0,2]} F(x)$, where

$$F(x) = \begin{cases} [x, 3-x], & x \in [0,1], \\ [1,2], & x \in (1,2]. \end{cases}$$

It follows from Definitions 12 and 13 that the set of efficient solutions is the interval $[0,2]$, while the set of weakly efficient solutions is empty. Geometrically, weakly efficient solutions stand for minimal nondominated interval elements, i.e., there exists no feasible point whose interval objective strictly dominates theirs. In contrast, efficient solutions only rule out strictly dominated candidates and satisfy a looser optimality criterion, which leads to a much wider solution set in the interval objective space.

Remark 4. *Let $F : K \to I(R)$ denote an interval mapping, $x_0 \in K$. If x_0 is a weakly efficient solution of F, then x_0 is also an efficient solution of F. However, the reverse implication does not hold.*

Theorem 8. *Let the Clarke subdifferential of interval mapping $F : K \to I(R)$ exist at $x_0 \in K$. If F is generalized convex at x_0 and $\mathbf{0} \in \partial^0 F(x_0)$, then x_0 is a weakly efficient solution of F.*

Proof. As F is generalized convex at x_0, it follows that

$$\langle A, y - x_0 \rangle \preceq F(y) \ominus_{gH} F(x_0)$$

for any $A \in \partial^0 F(x_0)$ and $y \in K$. Furthermore, since $\mathbf{0} \in \partial^0 F(x_0)$, we can obtain

$$0 \preceq F(y) \ominus_{gH} F(x_0),$$

for any $y \in K$. Thus, $F(x_0) \preceq F(y)$. As a consequence, x_0 is a weakly efficient solution of F. □

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Next, we investigate the constrained interval optimization model

$$\begin{aligned} \min_{x\in K} \quad & F(x), \\ \text{subject to} \quad & g_i(x) \preceq 0, \quad i = 1, \ldots, p, \end{aligned} \tag{8}$$

where $F, g_1, g_2, \ldots, g_p : K \to I(R)$ denote interval mappings. The feasible set of this optimization model is defined as

$$M = \{x \in K \mid g_i(x) \preceq 0 \quad i = 1, 2, \ldots, p\}.$$

Theorem 9. *Let $F, g_1, g_2, \ldots, g_p : K \to I(R)$ denote generalized convex interval mappings, $x_0 \in M$, $\beta_i \geq 0$. If $\partial^0(F + \sum_{i\in I(x_0)} \beta_i g_i)(x_0) \subseteq \partial^0 F(x_0) + \sum_{i\in I(x_0)} \beta_i \partial^0 g_i(x_0)$, $I(x_0) = \{i \,|g_i(x_0) = 0\}$ is not empty and $\mathbf{0} \in \partial^0 F(x_0) + \sum_{i\in I(x_0)} \beta_i \partial^0 g_i(x_0)$, then x_0 is an efficient solution of the constrained interval optimization model.*

Proof. Since $\mathbf{0} \in \partial^0 F(x_0) + \sum_{i\in I(x_0)} \beta_i \partial^0 g_i(x_0)$, there exist $A \in \partial^0 F(x_0)$ and $B_i \in \partial^0 g_i(x_0)$ $(i \in I(x_0))$ such that

$$\mathbf{0} = A + \sum_{i\in I(x_0)} \beta_i B_i. \tag{9}$$

If x_0 fails to be an efficient solution to the constrained interval optimization model, then there exists an $x \in M$ satisfying

$$F(x) \prec F(x_0). \tag{10}$$

In view of $x \in M$ and $g_i(x_0) = 0$ for all $i \in I(x_0)$, we can obtain $g_i(x) \preceq g_i(x_0) = 0$. Then,

$$\sum_{i\in I(x_0)} \beta_i g_i(x) \preceq \sum_{i\in I(x_0)} \beta_i g_i(x_0), \tag{11}$$

with $\beta_i \geq 0$. It follows from Equations (10) and (11) that

$$F(x) + \sum_{i\in I(x_0)} \beta_i g_i(x) \prec F(x_0) + \sum_{i\in I(x_0)} \beta_i g_i(x_0).$$

Therefore,

$$(F(x) + \sum_{i\in I(x_0)} \beta_i g_i(x)) \ominus_{gH} (F(x_0) + \sum_{i\in I(x_0)} \beta_i g_i(x_0)) \prec 0. \tag{12}$$

According to Theorem 7, $F(x) + \sum_{i\in I(x_0)} \beta_i g_i(x)$ is generalized convex at x_0. Then, we can get

$$\langle A + \sum_{i\in I(x_0)} \beta_i B_i, x - x_0\rangle \preceq (F(x) + \sum_{i\in I(x_0)} \beta_i g_i(x)) \ominus_{gH} (F(x_0) + \sum_{i\in I(x_0)} \beta_i g_i(x_0)),$$

for all $x \in M$. Based on (10), we have

$$0 \preceq (F(x) + \sum_{i\in I(x_0)} \beta_i g_i(x)) \ominus_{gH} (F(x_0) + \sum_{i\in I(x_0)} \beta_i g_i(x_0)).$$

which contradicts (12). Therefore, x_0 is an efficient solution of the constrained interval optimization model. □

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Example 4. *Consider the following optimization problem:*

$$\begin{aligned} \min_{(x,y)\in R^2} \quad & F(x,y) = [x^2+|y|,\ x^2+4|y|], \\ \text{subject to} \quad & g_1(x,y) = [x^2-y, x^2-y] \preceq 0, \quad g_2(x,y) = [x^2+y^2-2, x^2+y^2-1] \preceq 0. \end{aligned}$$

The feasible region is $M = \{(x,y) \in R^2 \mid g_i(x) \preceq 0 \quad i = 1,2\} = \{(x,y) \in R^2 \mid y \geq x^2, x^2+y^2 \leq 1\}$. *When* $x_0 = (0,0)$, *the index set* $I(x_0) = \{i \mid g_i(x_0) = 0\} = \{1\}$ *is nonempty. According to Example 2, we get*

$$\partial^0 F((0,0)) = \{A = ([0,0], [\underline{a}_2, \overline{a}_2]) \mid -4 \leq \underline{a}_2 \leq 1, -1 \leq \overline{a}_2 \leq 4, \underline{a}_2 \leq \overline{a}_2\}.$$

Next, we show that the interval mapping F *is generalized convex at* $(0,0)$. *When* $y \geq 0$, *for any* $A \in \partial^0 F((0,0))$, *we have* $\langle A, (x,y)-(0,0)\rangle = [\underline{a}_2 y, \overline{a}_2 y] \preceq [x^2+y, x^2+4y]$. *Thus,* $\langle A, (x,y)-(0,0)\rangle \preceq F(x,y) \ominus_{gH} F(0,0)$. *When* $y < 0$, *for any* $A \in \partial^0 F((0,0))$, *we have* $\langle A, (x,y)-(0,0)\rangle = [\overline{a}_2 y, \underline{a}_2 y] \preceq [x^2-y, x^2-4y]$. *Thus,* $\langle A, (x,y)-(0,0)\rangle \preceq F(x,y) \ominus_{gH} F(0,0)$. *Therefore,* $F(x,y)$ *is generalized convex at* $(0,0)$.

Similarly, we can get $\partial^0 g_1(0,0) = ([0,0], [-1,-1])$ *and* $g_1(x,y)$ *is also generalized convex at* $(0,0)$. *In addition, there exists* $\beta_1 = 1$ *such that* $\mathbf{0} = ([0,0],[1,1]) + \beta_1([0,0],[-1,-1])$, *i.e.,* $\mathbf{0} \in \partial^0 F(0,0) + \beta_1 \partial^0 g_1(0,0)$. *Furthermore,*

$$F(x,y) + \beta_1 g_1(x,y) = [2x^2+|y|-y, 2x^2+4|y|-y].$$

After calculation, we find that $\partial^0(F+\beta_1 g_1)(0,0) = \{([0,0],[\underline{a}_2,\overline{a}_2]) \mid -5 \leq \underline{a}_2 \leq 0, -2 \leq \overline{a}_2 \leq 3, \underline{a}_2 \leq \overline{a}_2\}$, *and* $\partial^0 F(0,0) + \beta_1 \partial^0 g_1(0,0) = \{([0,0],[\underline{a}_2,\overline{a}_2]) \mid -5 \leq \underline{a}_2 \leq 0, -2 \leq \overline{a}_2 \leq 3, \underline{a}_2 \leq \overline{a}_2\}$. *This implies that* $\partial^0(F+\beta_1 g_1)(0,0) \subseteq \partial^0 F(0,0) + \beta_1 \partial^0 g_1(0,0)$. *According to Theorem 9,* $x_0 = (0,0)$ *is an efficient solution of the constrained optimization model.*

5. Conclusions

In this paper, a nonsmooth analysis framework for interval mappings is established using the Clarke directional derivative and Clarke subdifferential. The relationship between the Clarke directional derivative of interval mappings and the Clarke directional derivative of endpoint functions is established. Moreover, the coincidence of the Clarke subdifferential with the gH-subdifferential for convex functions validates our work as a natural extension of existing convex analyses. Furthermore, the newly proposed generalized convexity successfully serves as a bridge between nonsmooth analysis and interval optimization. From an optimization perspective, the derived sufficient optimality conditions for both unconstrained and constrained models demonstrate the practical utility of our framework. The numerical examples further confirm that the approach can identify efficient solutions where traditional subdifferential methods fail due to nonconvexity. While this work establishes a fundamental nonsmooth theory for interval mappings, full algorithm implementation and practical real-world experiments are not addressed in the present paper. The primary reason is that evaluating the Clarke directional derivative of interval mappings involves complicated computation. Accordingly, developing fast computational schemes for this derivative and carrying out practical case validations will constitute our priorities for future research.

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