

Article

Strong Convexity-Based Improvements of Jensen–Mercer Inequalities in Functional, Integral and Probabilistic Settings

Humaira Mumtaz Kaka Khel ¹, Slavica Ivelić Bradanović ^{2,*} and Muhammad Adil Khan ¹

¹ Department of Mathematics, University of Peshawar, Peshawar 25000, Pakistan

² Faculty of Civil Engineering, Architecture and Geodesy, University of Split, 21000 Split, Croatia

* Correspondence: sivelic@gradst.hr

Abstract

By using recently established refinements of Jensen and converse Jensen-type inequalities, together with improved characterizations of strongly convex functions, we derive new Jensen–Mercer-type functional inequalities. The obtained results extend and sharpen several known inequalities for convex functions. As applications, we establish new integral inequalities which improve some recently published estimates. In addition, we obtain inequalities involving generalized logarithmic means. Finally, we derive corresponding probabilistic versions of the main results, which lead to refinements of several known results from the literature.

Keywords: Jensen inequality; Jensen–Mercer inequality; Jensen inequality; Jensen–Mercer inequality; convex function; strongly convex function; generalized logarithmic mean; probabilistic inequality

MSC: 26D15; 26D20; 26A51

1. Introduction

The notion of convexity plays an important role in many areas of mathematics, especially in analysis, optimization and inequality theory. Convex functions possess a number of useful properties which make them suitable for studying various analytical and functional problems. Because of their simple geometric interpretation and wide applicability, convexity methods are frequently used in the derivation of different inequalities and estimates.

A function $\Omega: \mathbb{I} \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be convex if

$$\Omega(\lambda\zeta + (1-\lambda)\nu) \leq \lambda\Omega(\zeta) + (1-\lambda)\Omega(\nu) \quad (1)$$

for all $\zeta, \nu \in \mathbb{I}$ and $\lambda \in [0, 1]$.

Inequalities related to convex functions have been studied extensively and have found applications in mathematical analysis, probability theory and optimization. Among the most important results in this area, an essential tool in the study of convex functions is the Jensen inequality:

$$\Omega(\bar{\zeta}) \leq \sum_{i=1}^{\kappa} a_i \Omega(\zeta_i) \quad (2)$$

which holds for every convex function $\Omega: \mathbb{I} \subseteq \mathbb{R} \rightarrow \mathbb{R}$, every vector $\zeta = (\zeta_1, \dots, \zeta_{\kappa}) \in \mathbb{I}^{\kappa}$ and every nonnegative κ -tuple $\mathbf{a} = (a_1, \dots, a_{\kappa})$ satisfying $\sum_{i=1}^{\kappa} a_i = 1$, where $\bar{\zeta} = \sum_{i=1}^{\kappa} a_i \zeta_i$.



Academic Editor: Hari Mohan Srivastava

Received: 20 May 2026

Revised: 27 June 2026

Accepted: 29 June 2026

Published: 1 July 2026

Copyright: © 2026 by the authors.

Licensee MDPI, Basel, Switzerland.

This article is an open access article distributed under the terms and conditions of the [Creative Commons Attribution \(CC BY\)](https://creativecommons.org/licenses/by/4.0/) license.

Jensen's inequality is a classical tool in convex analysis and inequality theory. Its importance has motivated many refinements and extensions, especially in connection with means, optimization and variational analysis. The geometric structure of convex functions also leads naturally to various reflected and generalized forms of Jensen's inequality. Among them, particular attention has been devoted to the Jensen–Mercer inequality, introduced by A. McD. Mercer in [1]. This inequality may be viewed as a reflected counterpart of the classical Jensen inequality and is based on the transformation $x \mapsto m + n - x$.

More precisely, for a convex function $\Omega: \mathbb{I} \rightarrow \mathbb{R}$, numbers $m, n \in \mathbb{I}$ with $m < n$, a vector $\xi = (\xi_1, \dots, \xi_\kappa) \in [m, n]^\kappa$ and a nonnegative κ -tuple $\mathbf{a} = (a_1, \dots, a_\kappa)$ such that $\sum_{i=1}^\kappa a_i = 1$ and $\bar{\xi} = \sum_{i=1}^\kappa a_i \xi_i$, the Jensen–Mercer inequality states that

$$\Omega(m + n - \bar{\xi}) \leq \Omega(m) + \Omega(n) - \sum_{i=1}^\kappa a_i \Omega(\xi_i). \quad (3)$$

The reflected form appearing in Jensen–Mercer inequalities allows a finer control of endpoint contributions than the classical Jensen framework. This becomes particularly effective in the strongly convex setting, a stronger form of convexity.

A function $\Omega: \mathbb{I} \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is called strongly convex with modulus $c > 0$, if

$$\Omega(\lambda \xi + (1 - \lambda)\nu) \leq \lambda \Omega(\xi) + (1 - \lambda)\Omega(\nu) - c\lambda(1 - \lambda)(\xi - \nu)^2 \quad (4)$$

for all $\xi, \nu \in \mathbb{I}$ and $\lambda \in [0, 1]$.

Strong convexity may be viewed as a natural strengthening of ordinary convexity obtained by adding a quadratic correction term $c\lambda(1 - \lambda)(\xi - \nu)^2$ to the defining inequality. This additional term allows better control of the function and often leads to stronger versions of classical inequalities. For this reason, many results involving convex functions have been extended to the strongly convex setting. Strong convexity concept provides sharper estimates in various optimization problems. Together with ordinary convex functions, they are being increasingly studied and applied in the recent literature because of their useful analytical properties.

Throughout the paper, the notation $\Omega \in SC_c(\mathbb{I})$ means that $\Omega: \mathbb{I} \rightarrow \mathbb{R}$ is strongly convex on $\mathbb{I}$ with modulus $c > 0$.

Every strongly convex function is necessarily convex, while the converse statement is generally false. Several useful characterizations of strong convexity are summarized in the following lemmas (see [2–4]).

Lemma 1. *The function $\Omega \in SC_c(\mathbb{I})$ if and only if the mapping $h: \mathbb{I} \rightarrow \mathbb{R}$, defined by $h(t) = \Omega(t) - ct^2$, is convex.*

Lemma 2. *Suppose that $\Omega: \mathbb{I} \rightarrow \mathbb{R}$ is twice differentiable. Then $\Omega \in SC_c(\mathbb{I})$ if and only if $\Omega''(t) \geq 2c$ for every $t \in \mathbb{I}$.*

In Section 3, we focus in particular on the following example of strongly convex functions.

Example 1. *Consider the family of power functions $\Omega(t) = t^q$, $q \in \mathbb{R}$, defined on an interval $[m, n] \subset (0, \infty)$. Since*

$$\Omega''(t) = q(q - 1)t^{q-2},$$

it follows that $\Omega''(t) > 0$ on $[m, n]$ whenever $q \in (-\infty, 0) \cup (1, \infty)$. Consequently, in these cases the function is strongly convex on $[m, n]$. Lemma 2, implies that the modulus of strong convexity is determined by

$$\frac{1}{2}q(q-1) \min_{t \in [m, n]} t^{q-2}$$

with the explicit representation

$$\min_{t \in [m, n]} t^{q-2} = \begin{cases} m^{q-2}, & q \geq 2, \\ n^{q-2}, & q < 2. \end{cases}$$

Consequently, one may choose

$$c = \begin{cases} \frac{1}{2}q(q-1)m^{q-2}, & q \geq 2, \\ \frac{1}{2}q(q-1)n^{q-2}, & q \in (-\infty, 0) \cup (1, 2). \end{cases}$$

Note that the cases $q = 0$ and $q = 1$ correspond to a constant and a linear function, respectively, which are not strongly convex. Moreover, for $0 < q < 1$, the function $\Omega(t) = t^q$ is strongly concave.

Strongly convex functions satisfy the strengthened Jensen inequality

$$\Omega(\bar{\xi}) \leq \sum_{i=1}^{\kappa} a_i \Omega(\xi_i) - c \sum_{i=1}^{\kappa} a_i (\xi_i - \bar{\xi})^2. \quad (5)$$

For $c = 0$, inequality (5) becomes the classical Jensen inequality (2). This means that strong convexity can be seen as a stronger version of ordinary convexity because of the additional quadratic term. Moreover, for $\kappa = 2$, the difference between the two sides of (5) can be written as

$$\Delta_{\Omega}(c; \xi, \nu) = \Omega(\xi) + \Omega(\nu) - 2\Omega\left(\frac{\xi + \nu}{2}\right) - \frac{c}{2}(\nu - \xi)^2. \quad (6)$$

In the special case $c = 0$, we obtain

$$\Delta_{\Omega}(\xi, \nu) = \Omega(\xi) + \Omega(\nu) - 2\Omega\left(\frac{\xi + \nu}{2}\right). \quad (7)$$

The quantities $\Delta_{\Omega}(c; \xi, \nu)$ and $\Delta_{\Omega}(\xi, \nu)$ measure the deviation from linear behavior at the midpoint and naturally arise in refined forms of Jensen type inequalities. Their nonnegativity follows directly from Jensen's inequality for strongly convex and convex functions, respectively.

Both Jensen's inequality (2) and the Jensen–Mercer inequality (3) can be extended to positive linear functionals, which gives a useful framework for deriving different functional inequalities.

Let L be a linear class of real-valued functions defined on a nonempty set T , containing the unit function $1 \in L$, where $1(t) = 1$ for every $t \in T$. We consider a positive normalized linear functional $F: L \rightarrow \mathbb{R}$ (briefly, $F \in \mathcal{L}(L, \mathbb{R})$) characterized by the following properties:

- $F(\alpha g + \beta h) = \alpha F(g) + \beta F(h)$, for all $g, h \in L$ and $\alpha, \beta \in \mathbb{R}$,
- if $h \in L$ and $h(t) \geq 0$ for all $t \in T$, then $F(h) \geq 0$, together with the normalization condition $F(1) = 1$.

This functional approach connects discrete, integral and probabilistic versions of Jensen-type inequalities within one general framework. In particular, if $F \in \mathcal{L}(L, \mathbb{R})$, Ω is continuous and convex on $[m, n]$ and $h \in L$ with $\Omega(h) \in L$, then Jensen's inequality has the form

$$\Omega(F(h)) \leq F(\Omega(h)). \quad (8)$$

Conversely, the associated converse Jensen inequality provides the estimate

$$F(\Omega(h)) \leq \frac{n - F(h)}{n - m} \Omega(m) + \frac{F(h) - m}{n - m} \Omega(n). \quad (9)$$

Hence, inequalities (8) and (9) together yield two-sided bounds for the quantity $F(\Omega(h)) - \Omega(F(h))$, which measures the deviation from equality in Jensen's inequality.

The corresponding functional Jensen–Mercer inequality is given by

$$\Omega(m + n - F(h)) \leq \Omega(m) + \Omega(n) - F(\Omega(h)), \quad (10)$$

and is based on the reflected transformation $h \mapsto m + n - h$. This reflected form gives estimates that complement those obtained from the classical Jensen inequality.

For the purposes of several results below, we additionally assume that L is a lattice, meaning that

- if $h, g \in L$, then $\min\{h, g\} \in L$ and $\max\{h, g\} \in L$.

We will explicitly emphasize when L is assumed to be a lattice, whenever this property is required.

Remark 1. The ordered space $(\mathbb{R}^T, \leq)$, equipped with the pointwise order, forms a lattice. Moreover, a subspace $X \subseteq \mathbb{R}^T$ is a lattice if and only if $t \in X$ implies $|t| \in X$. Indeed, for every $t \in X$, the positive and negative parts t_- and t_+ are determined through lattice operations, where $|t|(s) = |t(s)|$, $t_- = -\min\{0, t(s)\}$, $t_+ = \max\{0, t(s)\}$ for $s \in T$, while $t_+ + t_- = |t|$, $t_+ - t_- = t$. In addition, the lattice operations can be represented by

$$\min\{t, u\} = \frac{1}{2}(t + u - |t - u|), \quad \max\{t, u\} = \frac{1}{2}(t + u + |t - u|).$$

In recent years, numerous strengthened and refined forms of the Jensen–Mercer inequality have been investigated, especially in the setting of positive linear functionals. Special attention has been given to improvements of Jensen's inequality and its converse obtained through Jensen–Mercer-type methods. One important improvement was proved in [5] and is stated below,

$$\begin{aligned} \Omega(m + n - F(h)) &\leq F(\Omega(m + n - h)) \\ &\leq \frac{F(h) - m}{n - m} \Omega(m) + \frac{n - F(h)}{n - m} \Omega(n) \\ &\leq \Omega(m) + \Omega(n) - F(\Omega(h)), \end{aligned} \quad (11)$$

and holds for all continuous convex functions $\Omega : [m, n] \rightarrow \mathbb{R}$, $F \in \mathcal{L}(L, \mathbb{R})$ and $h \in L$ such that $\Omega(h), \Omega(m + n - h) \in L$.

A key example of improvement is another sharp Jensen–Mercer-type inequality [6],

$$\begin{aligned}
& \Omega(m+n-F(h)) \\
& \leq \frac{F(h)-m}{n-m}\Omega(m) + \frac{n-F(h)}{n-m}\Omega(n) - \left(\frac{1}{2} - \frac{1}{n-m} \left|F(h) - \frac{m+n}{2}\right|\right) \Delta_{\Omega}(m,n) \\
& \leq \Omega(m) + \Omega(n) - F(\Omega(h)) \\
& - \left[1 - \frac{1}{n-m} \left(F\left(\left|h - \frac{m+n}{2}\right|\right) + \left|F(h) - \frac{m+n}{2}\right|\right)\right] \Delta_{\Omega}(m,n) \\
& \leq \Omega(m) + \Omega(n) - F(\Omega(h)) - \left[1 - \frac{2}{n-m} F\left(\left|h - \frac{m+n}{2}\right|\right)\right] \Delta_{\Omega}(m,n) \\
& \leq \Omega(m) + \Omega(n) - F(\Omega(h)),
\end{aligned} \tag{12}$$

with

$$\Delta_{\Omega}(m,n) = \Omega(m) + \Omega(n) - 2\Omega\left(\frac{m+n}{2}\right) \geq 0, \tag{13}$$

valid for all continuous convex functions $\Omega : [m,n] \rightarrow \mathbb{R}$, $F \in \mathcal{L}(L, \mathbb{R})$ on lattice L and $h \in L$ such that $\Omega(h), \Omega(m+n-h) \in L$.

Both inequalities (11) and (12) provide the intermediate bounds for the Jessen–Mercer inequality (10).

The classical Jessen (8) and the converse Jessen inequality (9) can be further sharpened by virtue of the stronger properties of strongly convex functions. The first refinement results were obtained in the paper [7]. Subsequently, exploiting the stronger structure of strongly convex functions, further refinements of Jessen’s and converse Jessen’s inequalities were established more recently in [2], which we cite and present here.

Let $\mathbb{I} \subseteq \mathbb{R}$ be an open interval, $\Omega \in SC_c(\mathbb{I})$ and $0 \leq K \leq 1$. Further, let $F \in \mathcal{L}(L, \mathbb{R})$, $h \in L$, such that $h^2, \Omega(h) \in L$. Then the following refinement holds:

$$\begin{aligned}
\Omega(F(h)) & \leq \Omega(F(h)) + K \left[F(\Omega(h)) - \Omega(F(h)) - c \left(F(h^2) - (F(h))^2 \right) \right] \\
& \leq F(\Omega(h)) - c \left(F(h^2) - (F(h))^2 \right) \\
& \leq F(\Omega(h))
\end{aligned} \tag{14}$$

In the same paper [2], the improvement of the converse Jessen inequality (9) was proved. Let $\mathbb{I} \subseteq \mathbb{R}$ be an open interval, $m, n \in \mathbb{I}$ with $m < n$ and $\Omega \in SC_c(\mathbb{I})$. Assume that $F \in \mathcal{L}(L, \mathbb{R})$ on a lattice L , $h \in L$ with values in $[m,n]$ such that $h^2, \Omega(h) \in L$. Then

$$\begin{aligned}
F(\Omega(h)) & \leq \frac{n-F(h)}{n-m}\Omega(m) + \frac{F(h)-m}{n-m}\Omega(n) - cR_{F(h)}(m,n) \\
& - F(h_{\min})\Delta_{\Omega}(c;m,n),
\end{aligned} \tag{15}$$

where $h_{\min}$ is defined by

$$h_{\min} = \min \left\{ \frac{n-h(x)}{n-m}, \frac{h(x)-m}{n-m} \right\} = \frac{1}{2} \left(1 - \frac{1}{n-m} \left| h - \frac{m+n}{2} \right| \right) \geq 0 \tag{16}$$

and

$$F(h_{\min}) = \frac{1}{2} - \frac{1}{n-m} F\left(\left|h - \frac{m+n}{2}\right|\right) \geq 0, \tag{17}$$

$$R_{F(h)}(m,n) = \frac{F(h)-m}{n-m}n^2 + \frac{n-F(h)}{n-m}m^2 - F(h^2) \geq 0, \tag{18}$$

$$\Delta_{\Omega}(c;m,n) = \Omega(m) + \Omega(n) - 2\Omega\left(\frac{m+n}{2}\right) - \frac{c}{2}(m-n)^2 \geq 0. \tag{19}$$

In particular, for $c = 0$, inequality (15) reduces to

$$F(\Omega(h)) \leq \frac{n-F(h)}{n-m} \Omega(m) + \frac{F(h)-m}{n-m} \Omega(n) - F(h_{\min}) \Delta_{\Omega}(m, n), \quad (20)$$

where $\Delta_{\Omega}(m, n)$ and $F(h_{\min})$ are given by (13), (17), respectively. In this reduced case, corresponding to the setting of convex functions, no terms involving $F(h^2)$ appear, and the assumption $h^2 \in L$, which is needed in the strongly convex case, is no longer required.

Remark 2. Note that it holds that

$$\begin{aligned} R_{F(h)}(m, n) &= \frac{F(h)-m}{n-m} n^2 + \frac{n-F(h)}{n-m} m^2 - F(h^2) \\ &= \frac{F(h)n^2 - mn^2 + nm^2 - F(h)m^2 - F(h^2)n - F(h^2)m}{n-m} \\ &= \frac{F(h)(n^2 - m^2) - mn(n-m) - F(h^2)(n-m)}{n-m} \\ &= F(h)(n+m) - mn - F(h^2) \\ &= (n-F(h))(F(h)-m) + (F(h))^2 - F(h^2). \end{aligned}$$

This enables us to write (15) in the equivalent forms

$$\begin{aligned} F(\Omega(h)) &\leq \frac{n-F(h)}{n-m} \Omega(m) + \frac{F(h)-m}{n-m} \Omega(n) \\ &\quad - c \left[(n-F(h))(F(h)-m) + (F(h))^2 - F(h^2) \right] \\ &\quad - F(h_{\min}) \Delta_{\Omega}(c; m, n) \end{aligned} \quad (21)$$

or

$$\begin{aligned} F(\Omega(h)) &\leq \frac{n-F(h)}{n-m} \Omega(m) + \frac{F(h)-m}{n-m} \Omega(n) \\ &\quad - c \left[(m+n)F(h) - mn - F(h^2) \right] \\ &\quad - F(h_{\min}) \Delta_{\Omega}(c; m, n). \end{aligned} \quad (22)$$

Finally, we mention another important result from the paper [2]. For $\Omega \in SC_c(\mathbb{I})$, where $\mathbb{I} \subseteq \mathbb{R}$ is an open interval, $\delta, \rho \geq 0$, $\delta + \rho = 1$, and $m, n \in \mathbb{I}$, $m < n$, we have

$$\begin{aligned} 0 &\leq \min\{\delta, \rho\} \left[\Omega(m) + \Omega(n) - 2\Omega\left(\frac{m+n}{2}\right) - \frac{c}{2}(m-n)^2 \right] \\ &\leq \delta\Omega(m) + \rho\Omega(n) - \Omega(\delta m + \rho n) - c(\delta m^2 + \rho n^2 - (\delta m + \rho n)^2). \end{aligned} \quad (23)$$

In the following remark, we present some key derivations and conclusions that we use in the proofs of the new refinements.

Remark 3. It holds that

$$\begin{aligned} c\delta\rho(n-m)^2 &= c\delta\rho n^2 - 2c\delta\rho nm + c\delta\rho m^2 \\ &= c(\delta m^2 + \rho n^2 - (\delta m + \rho n)^2) \\ &= c(\delta m^2 + \rho n^2 - \delta^2 m^2 - 2\delta\rho mn - \rho^2 n^2) \\ &= c(\delta m^2(1-\delta) + \rho n^2(1-\rho) - 2\delta\rho mn) \\ &= c(\delta\rho m^2 + \delta\rho n^2 - 2\delta\rho mn), \end{aligned}$$

i.e., inequality (23) is equivalent to

$$\begin{aligned} & \Omega(\delta m + \rho n) \\ & \leq \delta \Omega(m) + \rho \Omega(n) - c\delta\rho(n-m)^2 - \min\{\delta, \rho\} \Delta_{\Omega}(c; m, n) \\ & = \delta \Omega(m) + \rho \Omega(n) - c(\delta m^2 + \rho n^2 - (\delta m + \rho n)^2) \\ & \quad - \min\{\delta, \rho\} \Delta_{\Omega}(c; m, n) \end{aligned} \quad (24)$$

Further, we have

$$\begin{aligned} & c\delta\rho(n-m)^2 + \min\{\delta, \rho\} \Delta_{\Omega}(c; m, n) \\ & = c\delta\rho(n-m)^2 \\ & \quad + \min\{\delta, \rho\} \left[\Omega(m) + \Omega(n) - 2\Omega\left(\frac{m+n}{2}\right) - \frac{c}{2}(m-n)^2 \right] \\ & = c\delta\rho(n-m)^2 - \min\{\delta, \rho\} \frac{c}{2}(n-m)^2 \\ & \quad + \min\{\delta, \rho\} \left[\Omega(m) + \Omega(n) - 2\Omega\left(\frac{m+n}{2}\right) \right] \\ & \geq \min\{\delta, \rho\} \left[\Omega(m) + \Omega(n) - 2\Omega\left(\frac{m+n}{2}\right) \right] \\ & = \min\{\delta, \rho\} \Delta_{\Omega}(m, n), \end{aligned} \quad (25)$$

because of

$$\begin{aligned} & c\delta\rho(n-m)^2 - \min\{\delta, \rho\} \frac{c}{2}(n-m)^2 \\ & = \begin{cases} c\delta(n-m)^2 \left(\rho - \frac{1}{2}\right) \geq 0, & \text{for } 0 \leq \delta \leq \frac{1}{2}, \text{ i.e., } \min\{\delta, \rho\} = \delta, \\ c\rho(n-m)^2 \left(\delta - \frac{1}{2}\right) > 0, & \text{for } \frac{1}{2} < \delta \leq 1, \text{ i.e., } \min\{\delta, \rho\} = \rho. \end{cases} \end{aligned}$$

Jensen's inequality has been the subject of extensive research due to its importance in convex analysis, inequality theory and optimization. In 2009, M. A. Khan et al. [8] obtained improved versions of Jensen's inequality for convex and monotone functions and applied them to several classes of means, including the Cauchy mean. Later, D. Choi et al. [9] proposed refinements based on linear interpolation, leading to improved Jensen-type, Young-type and matrix inequalities. In 2015, Matković et al. [6] refined the Jensen–Mercer inequality, established several generalizations for convex functions and introduced a method for constructing exponentially convex functions. Their results were further applied to mean value theorems and Stolarsky-type means. Related developments have also been obtained for broader classes of convexity, with applications in integral and functional analysis. For example, Dragomir [10] derived several Lebesgue integral Jensen-type inequalities for convex functions on real intervals. Operator versions of Jensen's inequality have also attracted considerable attention. Hosseini et al. [11] established new operator Jensen inequalities and their reverse forms for convex functions that are not necessarily operator convex. In 2023, Ivelić Bradanović [2] employed strongly convex functions to refine Jensen's and Jensen's inequalities, including their converse and interpolating forms and applied the obtained results to strongly f -divergences. More recently, strongly convex functions have become an important framework for refining Jensen–Mercer-type inequalities. Adil Khan et al. [12] and Ivelić Bradanović and Lovričević [13] developed refined Jensen–Mercer inequalities for strongly convex functions and applied them to information-theoretic quantities, f -divergences, entropy measures and related inequalities. Further contributions were made by Sayyari et al. [14], who obtained new Jensen-type refinements with applications

to entropy and divergence measures. Alshehri et al. [15] established Hermite–Hadamard–Mercer-type inequalities and several related trapezoid-type inequalities. Most recently, Dragomir et al. [16] introduced characterizations of generalized (m, M, ψ) -convex functions and applied them to majorization-type and mean inequalities.

The main objective of this paper is to establish new refinements of the Jessen–Mercer inequality and its converse by employing the class of strongly convex functions. More precisely, by combining recent improvements of the Jessen inequality (14) and its converse (15) with the newly established structural characterization of strong convexity (24), we derive several new chains of Jessen–Mercer-type inequalities. The obtained results refine inequalities (11) and (12) by producing a sequence of intermediate estimates which do not appear in previously known Jensen–Mercer-type inequalities. In particular, the new inequalities incorporate correction terms arising from strong convexity and recent refinements of Jensen-type inequalities, leading to substantially sharper upper bounds and a finer localization of the quantity $\Omega(m + n - F(h))$.

The first main result is obtained by applying the recently established strongly convex Jessen inequality to the reflected mapping $h \mapsto m + n - h$. Building upon this observation, the subsequent results combine these refinements with converse Jessen inequalities and the structural characterization of strong convexity, yielding new intermediate bounds and refined chains of inequalities that are not present in the previously known Jensen–Mercer-type results.

Furthermore, the developed approach leads to improved integral and probabilistic Jensen–Mercer-type inequalities, as well as to new inequalities involving generalized logarithmic means and their special cases. Our exposition is organized as follows. In Section 2, we establish the main results of the paper by deriving new refined Jessen–Mercer-type inequalities and converse Jessen–Mercer inequalities for strongly convex functions. The obtained results generate finer chains of inequalities containing several new intermediate bounds that strengthen previously known Jensen–Mercer-type estimates. In Section 3, we apply these results in the integral setting and obtain improvements of recent inequalities from [17]. As particular cases, we derive new estimates for generalized logarithmic means and their special instances. Finally, in Section 4, we establish probabilistic versions of the obtained inequalities, leading to refinements and extensions of several results from [7,18,19].

2. Main Results

Our first statement can be viewed as an improved version of the reflected Jessen inequality (11). It is obtained by applying the Jessen inequality for strongly convex functions (14) to the reflected mapping $h \mapsto m + n - h$. The reflected form makes it possible to derive estimates which differ from those in the classical setting and, in some cases, provide more accurate bounds. The influence of strong convexity is reflected through the additional quadratic term appearing in the inequalities.

Throughout the remainder of the paper, $\mathbb{I} \subseteq \mathbb{R}$ denotes an open interval, unless stated otherwise.

Theorem 1. Let $\Omega \in SC_c(\mathbb{I})$, $m, n \in \mathbb{I}$ with $m < n$ and $0 \leq K \leq 1$. Further, let $F \in \mathcal{L}(L, \mathbb{R})$, $h \in L$ takes values in $[m, n]$ and assume $h^2, \Omega(m + n - h) \in L$. Then

$$\begin{aligned} & \Omega(m + n - F(h)) \\ & \leq \Omega(m + n - F(h)) \\ & + K \left[F(\Omega(m + n - h)) - \Omega(m + n - F(h)) - c \left(F(h^2) - (F(h))^2 \right) \right] \\ & \leq F(\Omega(m + n - h)) - c \left(F(h^2) - (F(h))^2 \right) \\ & \leq F(\Omega(m + n - h)). \end{aligned} \tag{26}$$

Proof. By applying (14) to $m + n - h$ instead of h , we get

$$\begin{aligned} & \Omega(F(m + n - h)) \\ &= \Omega(m + n - F(h)) \\ &\leq \Omega(m + n - F(h)) \\ &+ K \left[F(\Omega(m + n - h)) - \Omega(m + n - F(h)) - c \left[F((m + n - h)^2) - (F(m + n - h))^2 \right] \right] \\ &\leq F(\Omega(m + n - h)) - c \left[F((m + n - h)^2) - (F(m + n - h))^2 \right] \\ &\leq F(\Omega(m + n - h)). \end{aligned}$$

Since

$$F((m + n - h)^2) - (F(m + n - h))^2 = F(h^2) - (F(h))^2,$$

the required result holds. $\square$

Combining inequality (26) with the following auxiliary result, which gives several successive upper estimates for $F(\Omega(m + n - h))$, leads to a refined version of inequality (11). In this way, one can observe how the obtained bounds are related to each other and how they can be gradually improved. This also gives a better overview of the structure of the corresponding inequalities.

Theorem 2. Let $\Omega \in SC_c(\mathbb{I})$ and let $m, n \in \mathbb{I}$ satisfy $m < n$. Assume that L is a lattice and $F \in \mathcal{L}(L, \mathbb{R})$. Moreover, let $h \in L$ be a function with values in $[m, n]$ and suppose that $h^2, \Omega(h), \Omega(m + n - h) \in L$. Then the following inequalities hold:

$$\begin{aligned} & F(\Omega(m + n - h)) \\ &\leq \frac{F(h) - m}{n - m} \Omega(m) + \frac{n - F(h)}{n - m} \Omega(n) - cF((h - m)(n - h)) - F(h_{\min}) \Delta_{\Omega}(c; m, n) \\ &\leq \frac{F(h) - m}{n - m} \Omega(m) + \frac{n - F(h)}{n - m} \Omega(n) \\ &\leq \Omega(m) + \Omega(n) - F(\Omega(h)) - cR_{F(h)}(m, n) - F(h_{\min}) \Delta_{\Omega}(c; m, n) \\ &\leq \Omega(m) + \Omega(n) - F(\Omega(h)) - cR_{F(h)}(m, n) \\ &\leq \Omega(m) + \Omega(n) - F(\Omega(h)), \end{aligned} \tag{27}$$

where $F(h_{\min})$, $R_{F(h)}(m, n)$, and $\Delta_{\Omega}(c; m, n)$ are defined by (17), (18) and (19), respectively.

Proof. Define the mappings $\delta, \rho : \mathbb{I} \rightarrow \mathbb{R}$ by

$$\delta(x) = \frac{n - x}{n - m}, \quad \rho(x) = \frac{x - m}{n - m}. \tag{28}$$

Then for every $x \in \mathbb{I}$,

$$\Omega(x) = \Omega\left(\frac{n - x}{n - m}m + \frac{x - m}{n - m}n\right) = \Omega(\delta(x)m + \rho(x)n). \tag{29}$$

Using (29) in (24), we get

$$\begin{aligned} \Omega(x) &= \Omega(\delta(x)m + \rho(x)n) \\ &\leq \delta(x)\Omega(m) + \rho(x)\Omega(n) - c\delta(x)\rho(x)(n - m)^2 \\ &\quad - \min\{\delta(x), \rho(x)\} \Delta_{\Omega}(c; m, n), \end{aligned}$$

i.e.,

$$\begin{aligned}\Omega(x) &\leq \frac{n-x}{n-m}\Omega(m) + \frac{x-m}{n-m}\Omega(n) - c(n-x)(x-m) \\ &\quad - \min\{\delta, \rho\}\Delta_{\Omega}(c; m, n).\end{aligned}\quad (30)$$

Now for $h \in L$, such that $\Omega(h) \in L$ and $\Omega(m+n-h) \in L$, first substituting $m+n-h$ in place of x , and then applying F on both sides of (30), with $\min\{\delta, \rho\} = \frac{1}{2} - \frac{1}{n-m}|h - \frac{m+n}{2}| = h_{\min}$, we obtain

$$\begin{aligned}F(\Omega(m+n-h)) &\leq \frac{F(h)-m}{n-m}\Omega(m) + \frac{n-F(h)}{n-m}\Omega(n) - cF((h-m)(n-h)) - F(h_{\min})\Delta_{\Omega}(c; m, n).\end{aligned}$$

Since h takes values in $[m, n]$, we have $(h-m)(n-h) \geq 0$ and therefore

$$F((h-m)(n-h)) \geq 0.$$

Further, $\Omega \in SC_c(\mathbb{I})$ and then $\Delta_{\Omega}(c; m, n) \geq 0$. Moreover, $h_{\min} \geq 0$ on $[m, n]$, hence

$$F(h_{\min})\Delta_{\Omega}(c; m, n) \geq 0.$$

Consequently,

$$F(\Omega(m+n-h)) \leq \frac{F(h)-m}{n-m}\Omega(m) + \frac{n-F(h)}{n-m}\Omega(n).$$

Furthermore,

$$\begin{aligned}&\frac{F(h)-m}{n-m}\Omega(m) + \frac{n-F(h)}{n-m}\Omega(n) \\ &= \Omega(m) + \Omega(n) + \left(\frac{F(h)-m}{n-m} - 1\right)\Omega(m) + \left(\frac{n-F(h)}{n-m} - 1\right)\Omega(n) \\ &= \Omega(m) + \Omega(n) - \left[\frac{n-F(h)}{n-m}\Omega(m) + \frac{F(h)-m}{n-m}\Omega(n)\right].\end{aligned}$$

Now, applying (15), we obtain

$$\begin{aligned}&\frac{n-F(h)}{n-m}\Omega(m) + \frac{F(h)-m}{n-m}\Omega(n) \\ &\geq F(\Omega(h)) + cR_{F(h)}(m, n) + F(h_{\min})\Delta_{\Omega}(c; m, n)\end{aligned}$$

which implies

$$\begin{aligned}F(\Omega(m+n-h)) &\leq \Omega(m) + \Omega(n) - F(\Omega(h)) - cR_{F(h)}(m, n) - F(h_{\min})\Delta_{\Omega}(c; m, n) \\ &\leq \Omega(m) + \Omega(n) - F(\Omega(h)) - cR_{F(h)}(m, n) \\ &\leq \Omega(m) + \Omega(n) - F(\Omega(h))\end{aligned}$$

which follows directly from (15) together with the nonnegativity of $R_{F(h)}(m, n)$ and $F(h_{\min})\Delta_{\Omega}(c; m, n)$. $\square$

The following theorem presents an additional improvement of the Jessen–Mercer-type inequality (12). More precisely, the result introduces a chain of intermediate inequalities that further clarify the relationship between the involved expressions and yield tighter estimates. This also highlights how intermediate steps can be used to better track the behavior of the corresponding bounds.

Theorem 3. Let $\Omega \in SC_c(\mathbb{I})$ and let $m, n \in \mathbb{I}$ satisfy $m < n$. Assume that L is a lattice and $F \in \mathcal{L}(L, \mathbb{R})$. Moreover, let $h \in L$ be a function with values in $[m, n]$ and suppose that $\Omega(h) \in L$. Then the following inequalities hold:

$$\Omega(m + n - F(h)) \quad (31)$$

$$\begin{aligned} &\leq \frac{F(h) - m}{n - m} \Omega(m) + \frac{n - F(h)}{n - m} \Omega(n) \\ &\quad - c \left[\frac{F(h) - m}{n - m} m^2 + \frac{n - F(h)}{n - m} n^2 - \left(\frac{F(h) - m}{n - m} m + \frac{n - F(h)}{n - m} n \right)^2 \right] \\ &\quad - \left(\frac{1}{2} - \frac{1}{n - m} \left| F(h) - \frac{m + n}{2} \right| \right) \Delta_{\Omega}(c; m, n) \\ &\leq \frac{F(h) - m}{n - m} \Omega(m) + \frac{n - F(h)}{n - m} \Omega(n) \\ &\quad - \left(\frac{1}{2} - \frac{1}{n - m} \left| F(h) - \frac{m + n}{2} \right| \right) \Delta_{\Omega}(m, n) \\ &\leq \Omega(m) + \Omega(n) - F(\Omega(h)) \\ &\quad - \left[1 - \frac{1}{n - m} \left(F \left(\left| h - \frac{m + n}{2} \right| \right) + \left| F(h) - \frac{m + n}{2} \right| \right) \right] \Delta_{\Omega}(m, n) \\ &\leq \Omega(m) + \Omega(n) - F(\Omega(h)) \\ &\quad - \left[1 - \frac{2}{n - m} F \left(\left| h - \frac{m + n}{2} \right| \right) \right] \Delta_{\Omega}(m, n) \\ &\leq \Omega(m) + \Omega(n) - F(\Omega(h)), \end{aligned} \quad (32)$$

where $\Delta_{\Omega}(m, n)$ and $\Delta_{\Omega}(c; m, n)$ are defined by (13) and (19), respectively.

Proof. We again consider the functions $\delta, \rho : \mathbb{I} \rightarrow \mathbb{R}$ introduced in (28). Using (29) in (24), together with the second representation and estimate given in Remark 3, yields

$$\begin{aligned} \Omega(x) &\leq \delta(x) \Omega(m) + \rho(x) \Omega(n) - c \left(\delta(x) m^2 + \rho(x) n^2 - (\delta(x) m + \rho(x) n)^2 \right) \\ &\quad - \min\{\delta(x), \rho(x)\} \Delta_{\Omega}(c; m, n) \\ &\leq \delta(x) \Omega(m) + \rho(x) \Omega(n) - \min\{\delta(x), \rho(x)\} \Delta_{\Omega}(m, n). \end{aligned}$$

By substituting $x = F(h)$, we have

$$\min\{\delta(F(h)), \rho(F(h))\} = \min \left\{ \frac{n - F(h)}{n - m}, \frac{F(h) - m}{n - m} \right\} = \frac{1}{2} - \frac{1}{n - m} \left| F(h) - \frac{n + m}{2} \right|$$

and

$$\begin{aligned}
\Omega(F(h)) &\leq \frac{n-F(h)}{n-m}\Omega(m) + \frac{F(h)-m}{n-m}\Omega(n) \\
&\quad - c \left[\frac{n-F(h)}{n-m}m^2 + \frac{F(h)-m}{n-m}n^2 - \left(\frac{n-F(h)}{n-m}m + \frac{F(h)-m}{n-m}n \right)^2 \right] \\
&\quad - \left(\frac{1}{2} - \frac{1}{n-m} \left| F(h) - \frac{m+n}{2} \right| \right) \Delta_{\Omega}(c; m, n) \\
&\leq \frac{n-F(h)}{n-m}\Omega(m) + \frac{F(h)-m}{n-m}\Omega(n) \\
&\quad - \left(\frac{1}{2} - \frac{1}{n-m} \left| F(h) - \frac{m+n}{2} \right| \right) \Delta_{\Omega}(m, n).
\end{aligned}$$

Replacing h by $m+n-h$ further gives

$$\begin{aligned}
&\Omega(m+n-F(h)) \tag{33} \\
&\leq \frac{F(h)-m}{n-m}\Omega(m) + \frac{n-F(h)}{n-m}\Omega(n) \\
&\quad - c \left[\frac{F(h)-m}{n-m}m^2 + \frac{n-F(h)}{n-m}n^2 - \left(\frac{F(h)-m}{n-m}m + \frac{n-F(h)}{n-m}n \right)^2 \right] \\
&\quad - \left(\frac{1}{2} - \frac{1}{n-m} \left| F(h) - \frac{m+n}{2} \right| \right) \Delta_{\Omega}(c; m, n) \\
&\leq \frac{F(h)-m}{n-m}\Omega(m) + \frac{n-F(h)}{n-m}\Omega(n) \\
&\quad - \left(\frac{1}{2} - \frac{1}{n-m} \left| F(h) - \frac{m+n}{2} \right| \right) \Delta_{\Omega}(m, n) \\
&= \Omega(m) + \Omega(n) - \left[\frac{n-F(h)}{n-m}\Omega(m) + \frac{F(h)-m}{n-m}\Omega(n) \right] \\
&\quad - \left(\frac{1}{2} - \frac{1}{n-m} \left| F(h) - \frac{m+n}{2} \right| \right) \Delta_{\Omega}(m, n) \\
&\leq \Omega(m) + \Omega(n) - F(\Omega(h)) - F(h_{\min})\Delta_{\Omega}(m, n) \\
&\quad - \left(\frac{1}{2} - \frac{1}{n-m} \left| F(h) - \frac{m+n}{2} \right| \right) \Delta_{\Omega}(m, n),
\end{aligned}$$

where the last inequality follows directly from (20), namely from the estimate

$$-\left[\frac{n-F(h)}{n-m}\Omega(m) + \frac{F(h)-m}{n-m}\Omega(n) \right] \leq -F(\Omega(h)) - F(h_{\min})\Delta_{\Omega}(m, n),$$

where $F(h_{\min})$ is defined by (17).

Further, we have

$$\begin{aligned}
&\Omega(m) + \Omega(n) - F(\Omega(h)) - F(h_{\min})\Delta_{\Omega}(m, n) \tag{34} \\
&\quad - \left(\frac{1}{2} - \frac{1}{n-m} \left| F(h) - \frac{m+n}{2} \right| \right) \Delta_{\Omega}(m, n) \\
&= \Omega(m) + \Omega(n) - F(\Omega(h)) \\
&\quad - \left(1 - \frac{1}{n-m} \left(F \left(\left| h - \frac{m+n}{2} \right| \right) + \left| F(h) - \frac{m+n}{2} \right| \right) \right) \Delta_{\Omega}(m, n) \\
&\leq \Omega(m) + \Omega(n) - F(\Omega(h)) - \left(1 - \frac{2}{n-m} F \left(\left| h - \frac{m+n}{2} \right| \right) \right) \Delta_{\Omega}(m, n) \\
&\leq \Omega(m) + \Omega(n) - F(\Omega(h)).
\end{aligned}$$

Note that

$$\left| F(h) - \frac{m+n}{2} \right| = \left| F\left(h - \frac{m+n}{2}\right) \right| \leq F\left(\left|h - \frac{m+n}{2}\right|\right),$$

and $m \leq h \leq n$ implies

$$\left|h - \frac{m+n}{2}\right| \leq \frac{n-m}{2}.$$

Since F is positive and normalized,

$$F\left(\left|h - \frac{m+n}{2}\right|\right) \leq \frac{n-m}{2},$$

and therefore

$$1 - \frac{2}{n-m} F\left(\left|h - \frac{m+n}{2}\right|\right) \geq 0.$$

Moreover, $\Delta_{\Omega}(m, n) \geq 0$ by (13). Hence

$$\left(1 - \frac{2}{n-m} F\left(\left|h - \frac{m+n}{2}\right|\right)\right) \Delta_{\Omega}(m, n) \geq 0,$$

which justifies the last inequality in (34). Now, combining (33) and (34), we get the required result (31). $\square$

Finally, we provide an alternative formulation of the refined Jessen–Mercer inequality in the setting of strongly convex functions. This version emphasizes a different representation of the obtained bounds and complements the previous refinements discussed in this section. In particular, it shows that the same inequality can be expressed in several equivalent forms depending on the chosen approach.

Theorem 4. Let $\Omega \in SC_c(\mathbb{I})$ and let $m, n \in \mathbb{I}$ satisfy $m < n$. Assume that L is a lattice and $F \in \mathcal{L}(L, \mathbb{R})$. Moreover, let $h \in L$ be a function with values in $[m, n]$ and suppose that $h^2, \Omega(h) \in L$. Then the following inequalities hold:

$$\begin{aligned} & \Omega(m+n-F(h)) \\ & \leq \frac{F(h)-m}{n-m} \Omega(m) + \frac{n-F(h)}{n-m} \Omega(n) - c \left[(m+n)F(h) - mn - (F(h))^2 \right] \\ & \quad - \left(\frac{1}{2} - \frac{1}{n-m} \left| F(h) - \frac{m+n}{2} \right| \right) \Delta_{\Omega}(c; m, n) \\ & \leq \Omega(m) + \Omega(n) - F(\Omega(h)) - c \left[2(m+n)F(h) - 2mn - (F(h))^2 - F(h^2) \right] \\ & \quad - \left[1 - \frac{1}{n-m} \left(\left| F(h) - \frac{m+n}{2} \right| + F\left(\left|h - \frac{m+n}{2}\right|\right) \right) \right] \Delta_{\Omega}(c; m, n) \\ & \leq \Omega(m) + \Omega(n) - F(\Omega(h)) - c \left[2(m+n)F(h) - 2mn - (F(h))^2 - F(h^2) \right] \\ & \quad - \left(1 - \frac{2}{n-m} F\left(\left|h - \frac{m+n}{2}\right|\right) \right) \Delta_{\Omega}(c; m, n) \\ & \leq \Omega(m) + \Omega(n) - F(\Omega(h)) - c \left[2(m+n)F(h) - 2mn - (F(h))^2 - F(h^2) \right] \\ & \leq \Omega(m) + \Omega(n) - F(\Omega(h)), \end{aligned} \tag{35}$$

where $\Delta_{\Omega}(c; m, n)$ is defined by (19).

Proof. Replacing x with $F(h)$ in (30) gives

$$\begin{aligned}\Omega(F(h)) &\leq \frac{n-F(h)}{n-m}\Omega(m) + \frac{F(h)-m}{n-m}\Omega(n) - c(n-F(h))(F(h)-m) \\ &\quad - \left(\frac{1}{2} - \frac{1}{n-m} \left| F(h) - \frac{m+n}{2} \right| \right) \Delta_{\Omega}(c; m, n).\end{aligned}$$

After substituting $h \mapsto m+n-h$, we obtain

$$\begin{aligned}\Omega(m+n-F(h)) &\leq \frac{F(h)-m}{n-m}\Omega(m) + \frac{n-F(h)}{n-m}\Omega(n) - c[(m+n)F(h) - mn - (F(h))^2] \\ &\quad - \left(\frac{1}{2} - \frac{1}{n-m} \left| F(h) - \frac{m+n}{2} \right| \right) \Delta_{\Omega}(c; m, n).\end{aligned}$$

Next, we add and subtract the quantities $\Omega(m)$ and $\Omega(n)$,

$$\begin{aligned}\Omega(m+n-F(h)) &\leq \Omega(m) + \Omega(n) + \frac{F(h)-m}{n-m}\Omega(m) - \Omega(m) + \frac{n-F(h)}{n-m}\Omega(n) - \Omega(n) \\ &\quad - c[(m+n)F(h) - mn - (F(h))^2] - \left(\frac{1}{2} - \frac{1}{n-m} \left| F(h) - \frac{m+n}{2} \right| \right) \Delta_{\Omega}(c; m, n) \\ &= \Omega(m) + \Omega(n) - \left[\frac{n-F(h)}{n-m}\Omega(m) + \frac{F(h)-m}{n-m}\Omega(n) \right] \\ &\quad - c[(m+n)F(h) - mn - (F(h))^2] - \left(\frac{1}{2} - \frac{1}{n-m} \left| F(h) - \frac{m+n}{2} \right| \right) \Delta_{\Omega}(c; m, n),\end{aligned}\tag{36}$$

By substituting x with $h(t)$ and applying the functional F to both sides of (30) leads to

$$\begin{aligned}F(\Omega(h)) &\leq \frac{n-F(h)}{n-m}\Omega(m) + \frac{F(h)-m}{n-m}\Omega(n) - c[(m+n)F(h) - mn - F(h^2)] \\ &\quad - \left(\frac{1}{2} - \frac{1}{n-m} F\left(\left| h - \frac{m+n}{2} \right| \right)\right) \Delta_{\Omega}(c; m, n).\end{aligned}$$

Consequently, we arrive at

$$\begin{aligned}-F(\Omega(h)) &- c\left((m+n)F(h) - mn - F(h^2)\right) - \left(\frac{1}{2} - \frac{1}{n-m} F\left(\left| h - \frac{m+n}{2} \right| \right)\right) \Delta_{\Omega}(c; m, n) \\ &\geq -\left[\frac{n-F(h)}{n-m}\Omega(m) + \frac{F(h)-m}{n-m}\Omega(n)\right].\end{aligned}\tag{37}$$

Inserting inequality (37) into (36) finally yields

$$\begin{aligned}\Omega(m+n-F(h)) &\leq \frac{F(h)-m}{n-m}\Omega(m) + \frac{n-F(h)}{n-m}\Omega(n) - c[(m+n)F(h) - mn - (F(h))^2] \\ &\quad - \left(\frac{1}{2} - \frac{1}{n-m} \left| F(h) - \frac{m+n}{2} \right| \right) \Delta_{\Omega}(c; m, n) \\ &\leq \Omega(m) + \Omega(n) - F(\Omega(h)) \\ &\quad - c\left(2(m+n)F(h) - 2mn - F(h^2) - (F(h))^2\right) \\ &\quad - \left(1 - \frac{1}{n-m} \left(F\left(\left| h - \frac{m+n}{2} \right| \right) + \left| F(h) - \frac{m+n}{2} \right| \right)\right) \Delta_{\Omega}(c; m, n).\end{aligned}\tag{38}$$

Using additionally

$$\left| F(h) - \frac{m+n}{2} \right| = \left| F\left(h - \frac{m+n}{2}\right) \right| \leq F\left(\left|h - \frac{m+n}{2}\right|\right),$$

we derive

$$\begin{aligned} & - \left(1 - \frac{1}{n-m} \left(\left| F(h) - \frac{m+n}{2} \right| + F\left(\left|h - \frac{m+n}{2}\right|\right) \right) \right) \Delta_{\Omega}(c; m, n) \\ & \leq - \left(1 - \frac{2}{n-m} F\left(\left|h - \frac{m+n}{2}\right|\right) \right) \Delta_{\Omega}(c; m, n). \end{aligned}$$

Combining this relation with the non-negativity of the quantities

$$\left(1 - \frac{2}{n-m} F\left(\left|h - \frac{m+n}{2}\right|\right) \right) \Delta_{\Omega}(c; m, n)$$

and

$$\begin{aligned} & c \left[2(m+n)F(h) - 2mn - F(h^2) - (F(h))^2 \right] \\ & = c \left[2(m+n)F(h) - 2mn - 2F(h^2) + F(h^2) - (F(h))^2 \right] \\ & = c \left[2F((h-m)(n-h)) + F(h^2) - (F(h))^2 \right] \geq 0, \end{aligned}$$

immediately implies inequality (35). Finally, the proof is complete. $\square$

3. Applications to Integral Inequalities and Generalized Logarithmic Means

In this section, we derive new integral inequalities for strongly convex functions using the previous main results. Since the functional setting based on positive linear functionals provides a general setting, we now pass to its integral version, which can be viewed as a natural realization in terms of a probability measure space. In this setting, the abstract functional F is represented by an integral functional. In the following results we assume that (T, Σ, μ) is a probability measure space, $L = L_1(\mu)$ and for $h \in L_1(\mu)$, we define a linear, positive and normalized functional F given by

$$F(h) = \int_T h(t) d\mu.$$

As a direct consequence of Theorem 1, we obtain the following result.

Corollary 1. *Let $\Omega \in SC_c(\mathbb{I})$ and let $m, n \in \mathbb{I}$ satisfy $m < n$ and $0 \leq K \leq 1$. Assume that $h \in L_1(\mu)$ takes values in $[m, n]$ and suppose that $h^2, \Omega(m+n-h) \in L_1(\mu)$. Then the following inequalities hold:*

$$\begin{aligned} & \Omega(m+n-\bar{h}) \\ & \leq \Omega(m+n-\bar{h}) \\ & + K \left[\int_T \Omega(m+n-h(t)) d\mu - \Omega(m+n-\bar{h}) - c \left(\int_T h^2(t) d\mu - \bar{h}^2 \right) \right] \\ & \leq \int_T \Omega(m+n-h(t)) d\mu - c \left(\int_T h^2(t) d\mu - \bar{h}^2 \right) \\ & \leq \int_T \Omega(m+n-h(t)) d\mu, \end{aligned} \tag{39}$$

where $\bar{h} = \int_T h(t) d\mu$.

The next statement can be deduced directly from Theorem 2.

Corollary 2. Let $\Omega \in SC_c(\mathbb{I})$ and let $m, n \in \mathbb{I}$ satisfy $m < n$. Assume that $h \in L_1(\mu)$ takes values in $[m, n]$ and suppose that $h^2, \Omega(h), \Omega(m + n - h) \in L_1(\mu)$. Then the following inequalities hold:

$$\begin{aligned} & \int_T \Omega(m + n - h(t)) d\mu \\ & \leq \frac{\bar{h} - m}{n - m} \Omega(m) + \frac{n - \bar{h}}{n - m} \Omega(n) - c \int_T (h(t) - m)(n - h(t)) d\mu \\ & - \left[\frac{1}{2} - \frac{1}{n - m} \int_T \left| h - \frac{m + n}{2} \right| d\mu \right] \Delta_\Omega(c; m, n) \\ & \leq \frac{\bar{h} - m}{n - m} \Omega(m) + \frac{n - \bar{h}}{n - m} \Omega(n) \\ & \leq \Omega(m) + \Omega(n) - \int_T \Omega(h(t)) d\mu \\ & - c \left[\frac{\bar{h} - m}{n - m} n^2 + \frac{n - \bar{h}}{n - m} m^2 - \int_T h^2(t) d\mu \right] \\ & - \left[\frac{1}{2} - \frac{1}{n - m} \int_T \left| h - \frac{m + n}{2} \right| d\mu \right] \Delta_\Omega(c; m, n) \\ & \leq \Omega(m) + \Omega(n) - \int_T \Omega(h(t)) d\mu \\ & - c \left[\frac{\bar{h} - m}{n - m} n^2 + \frac{n - \bar{h}}{n - m} m^2 - \int_T h^2(t) d\mu \right] \\ & \leq \Omega(m) + \Omega(n) - \int_T \Omega(h(t)) d\mu, \end{aligned} \quad (40)$$

where $\bar{h} = \int_T h(t) d\mu$, while $\Delta_\Omega(c; m, n)$ is introduced in (19).

Remark 4. Our result (39) together with (40) improves result from [17] (Theorem 5.2).

To illustrate the practical significance of the obtained results (39) together with (40), we consider the following numerical example.

Example 2. In Table 1, we consider the interval $[m, n] = [2, 3]$ equipped with the uniform probability measure $d\mu = \frac{1}{n-m} dt$, together with the function $\Omega(t) = t^4$, which is strongly convex with modulus $c = 24$. Furthermore, the parameter K satisfies $0 \leq K \leq 1$, so that the bound $39.06 + 1.14K$ ranges from 39.06 to 40.20.

The first group of estimates in Table 1 (from the first to the third upper bound) is obtained from Corollary 1, while the second group (from the fourth to the seventh upper bound) follows from Corollary 2. These bounds are compared with the corresponding estimates obtained in [17] (Theorem 5.2) and with the classical integral Jensen–Mercer inequality.

The comparison clearly shows that our results provide sharper upper bounds for the quantity $\Omega(m + n - \bar{h})$. Moreover, combining the estimates from Corollaries 1 and 2 yields several additional intermediate bounds that do not appear in [17] (Theorem 5.2). Consequently, our approach produces a substantially finer chain of inequalities and a more accurate localization of the quantity $\Omega(m + n - \bar{h})$.

The next corollary is a direct consequence of Theorem 3.

Table 1. Comparison of all bounds appearing in Corollaries 1 and 2 with [17] (Theorem 5.2) and classical integral Jensen–Mercer inequality.

Bound	Our Result	Paper [17]	Jensen–Mercer
$\Omega(m + n - \bar{h})$	39.06	39.06	39.06
1. upper bound	$39.06 + 1.14K$	40.20	–
2. upper bound	40.20	43.78	–
3. upper bound	42.20	45.36	–
4. upper bound	42.78	–	–
5. upper bound	48.50	–	–
6. upper bound	49.08	–	–
7. upper bound	50.80	–	–
Final upper bound	54.80	54.80	54.80

Corollary 3. Let $\Omega \in SC_c(\mathbb{I})$ and let $m, n \in \mathbb{I}$ satisfy $m < n$. Assume that $h \in L_1(\mu)$ takes values in $[m, n]$ and suppose that $\Omega(h) \in L_1(\mu)$. Then the following inequalities hold:

$$\begin{aligned}
 & \Omega(m + n - \bar{h}) \\
 & \leq \frac{\bar{h} - m}{n - m} \Omega(m) + \frac{n - \bar{h}}{n - m} \Omega(n) \\
 & \quad - c \left[\frac{\bar{h} - m}{n - m} m^2 + \frac{n - \bar{h}}{n - m} n^2 - \left(\frac{\bar{h} - m}{n - m} m + \frac{n - \bar{h}}{n - m} n \right)^2 \right] \\
 & \quad - \left(\frac{1}{2} - \frac{1}{n - m} \left| \bar{h} - \frac{m + n}{2} \right| \right) \Delta_{\Omega}(c; m, n) \\
 & \leq \frac{\bar{h} - m}{n - m} \Omega(m) + \frac{n - \bar{h}}{n - m} \Omega(n) \\
 & \quad - \left(\frac{1}{2} - \frac{1}{n - m} \left| \bar{h} - \frac{m + n}{2} \right| \right) \Delta_{\Omega}(m, n) \\
 & \leq \Omega(m) + \Omega(n) - \int_T \Omega(h(t)) d\mu \\
 & \quad - \left[1 - \frac{1}{n - m} \left(\int_T \left| h - \frac{m + n}{2} \right| d\mu + \left| \bar{h} - \frac{m + n}{2} \right| \right) \right] \Delta_{\Omega}(m, n) \\
 & \leq \Omega(m) + \Omega(n) - \int_T \Omega(h(t)) d\mu \\
 & \quad - \left[1 - \frac{2}{n - m} \int_T \left| h - \frac{m + n}{2} \right| d\mu \right] \Delta_{\Omega}(m, n) \\
 & \leq \Omega(m) + \Omega(n) - \int_T \Omega(h(t)) d\mu,
 \end{aligned} \tag{41}$$

where $\bar{h} = \int_T h(t) d\mu$ and the quantities $\Delta_{\Omega}(m, n)$, $\Delta_{\Omega}(c; m, n)$ are given by (13), (19), respectively.

Finally, as a special case of Theorem 4, we get the following result.

Corollary 4. Let $\Omega \in SC_c(\mathbb{I})$ and let $m, n \in \mathbb{I}$ satisfy $m < n$. Assume that $h \in L_1(\mu)$ takes values in $[m, n]$ and suppose that $h^2, \Omega(h) \in L_1(\mu)$. Then the following inequalities hold:

$$\begin{aligned} & \Omega(m+n-\bar{h}) \\ & \leq \frac{\bar{h}-m}{n-m}\Omega(m) + \frac{n-\bar{h}}{n-m}\Omega(n) - c \left[(m+n)\bar{h} - mn - \bar{h}^2 \right] \\ & \quad - \left(\frac{1}{2} - \frac{1}{n-m} \left| \bar{h} - \frac{m+n}{2} \right| \right) \Delta_{\Omega}(c; m, n) \\ & \leq \Omega(m) + \Omega(n) - \int_T \Omega(h(t)) d\mu - c \left[2(m+n)\bar{h} - 2mn - \bar{h}^2 - \int_T h^2(t) d\mu \right] \\ & \quad - \left[1 - \frac{1}{n-m} \left(\left| \bar{h} - \frac{m+n}{2} \right| + \int_T \left| h - \frac{m+n}{2} \right| d\mu \right) \right] \Delta_{\Omega}(c; m, n) \\ & \leq \Omega(m) + \Omega(n) - \int_T \Omega(h(t)) d\mu - c \left[2(m+n)\bar{h} - 2mn - \bar{h}^2 - \int_T h^2(t) d\mu \right] \\ & \quad - \left(1 - \frac{2}{n-m} \int_T \left| h - \frac{m+n}{2} \right| d\mu \right) \Delta_{\Omega}(c; m, n) \\ & \leq \Omega(m) + \Omega(n) - \int_T \Omega(h(t)) d\mu \\ & \quad - c \left[2(m+n)\bar{h} - 2mn - \bar{h}^2 - \int_T h^2(t) d\mu \right] \\ & \leq \Omega(m) + \Omega(n) - \int_T \Omega(h(t)) d\mu, \end{aligned}$$

where $\bar{h} = \int_T h(t) d\mu$ with $\Delta_{\Omega}(c; m, n)$ defined in (19).

By applying the previously derived corollaries to a particular family of positive normalized linear functionals, we obtain new inequalities involving the generalized logarithmic means. For $q \in \mathbb{R} \setminus \{-1, 0\}$, these means are defined by

$$L_q(m, n) = \begin{cases} m, & \text{if } m = n, \\ \left[\frac{n^{q+1} - m^{q+1}}{(q+1)(n-m)} \right]^{1/q}, & \text{if } m \neq n. \end{cases}$$

The family can be continuously extended to the excluded case $q = -1$. Indeed, taking the limit as $q \rightarrow -1$, one obtains the logarithmic mean

$$L(m, n) = \begin{cases} m, & \text{if } m = n, \\ \frac{m-n}{\ln m - \ln n}, & \text{if } m \neq n. \end{cases}$$

Accordingly, throughout this section we write $L_q(m, n)$ for $q \in \mathbb{R} \setminus \{0\}$, understanding that $L_{-1}(m, n) = L(m, n)$.

This class of means provides a continuous bridge between different classical averages and plays an important role in various comparison results. In particular, the parameter q allows a smooth transition between different types of generalized means. For $q = 1$, we recover the arithmetic mean

$$L_1(m, n) = A(m, n) = \frac{m+n}{2}.$$

Let $\mathbb{I} \subseteq (0, \infty)$ be an open interval and let $m, n \in \mathbb{I}$ with $m < n$. We consider the interval $T = [m, n] \subseteq (0, \infty)$ equipped with the uniform probability measure

$$d\mu = \frac{1}{n-m} dt.$$

This choice of measure ensures that the associated functional corresponds to the classical average value over the interval. In this setting, the associated positive normalized linear functional F is given by

$$F(h) = \frac{1}{n-m} \int_m^n h(t) dt.$$

The next identities can be viewed as particular cases of the previously established main results, obtained by choosing specific functions and the corresponding positive normalized linear functional. In this way, the general theory developed in the previous sections directly leads to the following expressions.

Assuming that $q \in \mathbb{R} \setminus \{0\}$ and $h, h^q \in L_1(\mu)$, we have

$$\begin{aligned} F(h) &= \frac{1}{n-m} \int_m^n t dt = \frac{m+n}{2}, \\ F(h^q) &= \frac{1}{n-m} \int_m^n t^q dt = L_q^q(m, n), \quad q \in \mathbb{R} \setminus \{-1, 0\}. \end{aligned}$$

For the limiting case $q = -1$, we obtain

$$F(h^{-1}) = \frac{1}{n-m} \int_m^n \frac{dt}{t} = \frac{\ln n - \ln m}{n-m} = \frac{1}{L(m, n)}.$$

Hence,

$$F(h^{-1}) = L^{-1}(m, n) = L_{-1}^{-1}(m, n),$$

in accordance with our notation.

We also have

$$\begin{aligned} F(h_{\min}) &= \frac{1}{2} - \frac{1}{n-m} F\left(\left|h - \frac{m+n}{2}\right|\right) = \frac{1}{4}, \\ F\left(\left|h - \frac{m+n}{2}\right|\right) &= \frac{1}{n-m} \int_m^n \left|t - \frac{m+n}{2}\right| dt = \frac{n-m}{4}, \end{aligned}$$

and for $\Omega(t) = t^q, q \in \mathbb{R}$, we get

$$\begin{aligned} \Omega(m+n-F(h)) &= \left(\frac{m+n}{2}\right)^q = A^q(m, n), \\ F(\Omega(m+n-h)) &= \frac{1}{n-m} \int_m^n (m+n-t)^q dt = \frac{n^{q+1} - m^{q+1}}{(q+1)(n-m)} = L_q^q(m, n), \\ F(\Omega(h)) &= \frac{n^{q+1} - m^{q+1}}{(q+1)(n-m)} = L_q^q(m, n), \\ F((h-m)(n-h)) &= \frac{1}{n-m} \int_m^n (t-m)(n-t) dt = \frac{(n-m)^2}{6} = R_{F(h)}(m, n) \end{aligned}$$

and

$$\begin{aligned} F(h^q) - (F(h))^q &= \frac{1}{n-m} \int_m^n t^q dt - \left(\frac{1}{n-m} \int_m^n t dt\right)^q \\ &= L_q^q(m, n) - A^q(m, n). \end{aligned}$$

Using the above representations, we apply the previously established corollaries to generalized logarithmic means which allows us to express them in terms of classical and generalized means. Accordingly, we consider the family of power functions $\Omega(t) = t^q$, $q \in (-\infty, 0) \cup (1, \infty)$ defined on $[m, n] \subseteq (0, \infty)$. These functions are strongly convex on $[m, n]$ and their modulus of strong convexity is given by

$$c = \begin{cases} \frac{1}{2}q(q-1)m^{q-2}, & q \geq 2, \\ \frac{1}{2}q(q-1)n^{q-2}, & q \in (-\infty, 0) \cup (1, 2). \end{cases}$$

The following examples illustrate the consequences of the preceding corollaries for generalized logarithmic means and their relationships with classical means.

Example 3. Applying Corollary 1 to the above family of strongly convex power functions yields the following inequality:

$$\begin{aligned} & A^q(m, n) \\ & \leq A^q(m, n) \\ & + K \left[L_q^q(m, n) - A^q(m, n) - \frac{1}{24}(n-m)^2q(q-1) \min_{t \in [m, n]} t^{q-2} \right] \\ & \leq L_q^q(m, n) - \frac{1}{24}(n-m)^2q(q-1) \min_{t \in [m, n]} t^{q-2} \\ & \leq L_q^q(m, n). \end{aligned}$$

Example 4. Using Corollary 2 for the above family of strongly convex power functions, we obtain

$$\begin{aligned} & L_q^q(m, n) \\ & \leq \frac{1}{2}A(m^q, n^q) + \frac{1}{2}A^q(m, n) - \frac{1}{48}(n-m)^2q(q-1) \min_{t \in [m, n]} t^{q-2} \\ & \leq A(m^q, n^q) \\ & \leq \frac{3}{2}A(m^q, n^q) + \frac{1}{2}A^q(m, n) - L_q^q(m, n) - \frac{1}{48}(n-m)^2q(q-1) \min_{t \in [m, n]} t^{q-2} \\ & \leq 2A(m^q, n^q) - L_q^q(m, n) - \frac{1}{12}(n-m)^2q(q-1) \min_{t \in [m, n]} t^{q-2} \\ & \leq 2A(m^q, n^q) - L_q^q(m, n). \end{aligned}$$

Example 5. By specializing Corollary 4 to the above family of strongly convex power functions, we arrive at

$$\begin{aligned} & A^q(m, n) \\ & \leq \frac{1}{2}A(m^q, n^q) + \frac{3}{2}A^q(m, n) - L_q^q(m, n) - \frac{1}{48}(n-m)^2q(q-1) \min_{t \in [m, n]} t^{q-2} \\ & \leq A(m^q, n^q) + A^q(m, n) - L_q^q(m, n) - \frac{1}{12}(n-m)^2q(q-1) \min_{t \in [m, n]} t^{q-2} \\ & \leq 2A(m^q, n^q) - L_q^q(m, n) - \frac{5}{24}(n-m)^2q(q-1) \min_{t \in [m, n]} t^{q-2} \\ & \leq 2A(m^q, n^q) - L_q^q(m, n). \end{aligned}$$

4. Application to Probabilistic Inequalities

In this section, we provide a probabilistic perspective on the previously established results. Throughout the discussion, we work with real valued random variables defined on a probability space (T, Σ, μ) and assume that all considered expectations are finite. For a random variable $Y : T \rightarrow \mathbb{R}$, the symbols $\mathbb{E}[Y]$ and $\mathbb{D}^2[Y]$ denote its expectation and variance, respectively.

It is well known that every convex function $\Omega : \mathbb{I} \rightarrow \mathbb{R}$ satisfies the classical Jensen inequality

$$\Omega(\mathbb{E}[Y]) \leq \mathbb{E}[\Omega(Y)] \quad (42)$$

for any random variable Y whose values belong to the interval $\mathbb{I}$.

A strengthened form of inequality (42) was established in [19]. More precisely, for $\Omega \in SC_c(\mathbb{I})$, the following estimate holds:

$$\Omega(\mathbb{E}[Y]) \leq \mathbb{E}[\Omega(Y)] - c\mathbb{D}^2[Y] \leq \mathbb{E}[\Omega(Y)]. \quad (43)$$

This inequality shows that the variance term naturally appears as a measure of deviation between the two expressions. The probabilistic reformulation of our principal results yields further refinements of inequality (43), together with several related inequalities of the Jensen–Mercer type. In this setting, the obtained bounds provide a more detailed description of the deviation between the expectation of a transformed random variable and the transform of its expectation.

Let $\Omega \in SC_c(\mathbb{I})$ and let $m, n \in \mathbb{I}$ satisfy $m < n$. For a random variable $Y : T \rightarrow [m, n]$, under the assumptions $Y^2, \Omega(Y), \Omega(m + n - Y) \in L^1(\mu)$, we consider a positive normalized linear functional F such that

$$\begin{aligned} F(Y) &= \mathbb{E}[Y], \\ F(Y^2) &= \mathbb{E}[Y^2], \\ F(\Omega(Y)) &= \mathbb{E}[\Omega(Y)], \\ F(\Omega(m + n - Y)) &= \mathbb{E}[\Omega(m + n - Y)] \end{aligned}$$

and

$$F(Y^2) - (F(Y))^2 = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 = \mathbb{D}^2[Y]$$

The next corollary follows immediately from Theorem 1 and represents a direct application of the preceding result.

Corollary 5. Let $\Omega \in SC_c(\mathbb{I})$, $m, n \in \mathbb{I}$ with $m < n$ and $0 \leq K \leq 1$. Further let $Y : T \rightarrow [m, n]$ be a random variable such that $Y^2, \Omega(m + n - Y) \in L^1(\mu)$. Then

$$\begin{aligned} &\Omega(m + n - \mathbb{E}[Y]) \\ &\leq \Omega(m + n - \mathbb{E}[Y]) \\ &+ K \left(\mathbb{E}[\Omega(m + n - Y)] - \Omega(m + n - \mathbb{E}[Y]) - c\mathbb{D}^2[Y] \right) \\ &\leq \mathbb{E}[\Omega(m + n - Y)] - c\mathbb{D}^2[Y] \\ &\leq \mathbb{E}[\Omega(m + n - Y)]. \end{aligned} \quad (44)$$

Remark 5. In particular, replacing $m + n - Y$ by Z and $m + n - \mathbb{E}[Y]$ by $\mathbb{E}[Z]$ in (44), leads to the following refinement of inequality (43):

$$\begin{aligned}\Omega(\mathbb{E}[Z]) &\leq \Omega(\mathbb{E}[Z]) + K\left(\mathbb{E}[\Omega(Z)] - \Omega(\mathbb{E}[Z]) - c\mathbb{D}^2[Z]\right) \\ &\leq \mathbb{E}[\Omega(Z)] - c\mathbb{D}^2[Z] \\ &\leq \mathbb{E}[\Omega(Z)].\end{aligned}\quad (45)$$

An immediate consequence of Theorem 2 is given in the following statement.

Corollary 6. Let $\Omega \in SC_c(\mathbb{I})$ and $m, n \in \mathbb{I}$ with $m < n$. Assume that $Y : T \rightarrow [m, n]$ is a random variable satisfying $\Omega(Y), \Omega(m + n - Y) \in L^1(\mu)$. Then the following inequalities hold:

$$\begin{aligned}\mathbb{E}[\Omega(m + n - Y)] &\leq \frac{\mathbb{E}[Y] - m}{n - m}\Omega(m) + \frac{n - \mathbb{E}[Y]}{n - m}\Omega(n) - c\mathbb{E}[(Y - m)(n - Y)] \\ &\quad - \left(\frac{1}{2} - \frac{1}{n - m}\mathbb{E}\left[\left|Y - \frac{m + n}{2}\right|\right]\right)\Delta_\Omega(c; m, n) \\ &\leq \frac{\mathbb{E}[Y] - m}{n - m}\Omega(m) + \frac{n - \mathbb{E}[Y]}{n - m}\Omega(n) \\ &\leq \Omega(m) + \Omega(n) - \mathbb{E}[\Omega(Y)] - c\left(\frac{\mathbb{E}[Y] - m}{n - m}n^2 + \frac{n - \mathbb{E}[Y]}{n - m}m^2 - \mathbb{E}[Y^2]\right) \\ &\quad - \left(\frac{1}{2} - \frac{1}{n - m}\mathbb{E}\left[\left|\frac{m + n}{2} - Y\right|\right]\right)\Delta_\Omega(c; m, n) \\ &\leq \Omega(m) + \Omega(n) - \mathbb{E}[\Omega(Y)] - c\left(\frac{\mathbb{E}[Y] - m}{n - m}n^2 + \frac{n - \mathbb{E}[Y]}{n - m}m^2 - \mathbb{E}[Y^2]\right) \\ &\leq \Omega(m) + \Omega(n) - \mathbb{E}[\Omega(Y)],\end{aligned}$$

where $\Delta_\Omega(c; m, n)$ is defined by relation (19).

Remark 6. By introducing the substitutions $m + n - Y = Z$ and $m + n - \mathbb{E}[Y] = \mathbb{E}[Z]$ into the previous inequality, we obtain the corresponding reformulated expression:

$$\begin{aligned}\mathbb{E}[\Omega(Z)] &\leq \frac{n - \mathbb{E}[Z]}{n - m}\Omega(m) + \frac{\mathbb{E}[Z] - m}{n - m}\Omega(n) - c\mathbb{E}[(Z - m)(n - Z)] \\ &\quad - \left(\frac{1}{2} - \frac{1}{n - m}\mathbb{E}\left[\left|Z - \frac{m + n}{2}\right|\right]\right)\Delta_\Omega(c; m, n) \\ &\leq \frac{n - \mathbb{E}[Z]}{n - m}\Omega(m) + \frac{\mathbb{E}[Z] - m}{n - m}\Omega(n) \\ &\leq \Omega(m) + \Omega(n) - \mathbb{E}[\Omega(m + n - Z)] - c\left(\frac{n - \mathbb{E}[Z]}{n - m}n^2 + \frac{\mathbb{E}[Z] - m}{n - m}m^2 - \mathbb{E}[(m + n - Z)^2]\right) \\ &\quad - \left(\frac{1}{2} - \frac{1}{n - m}\mathbb{E}\left[\left|\frac{m + n}{2} - Z\right|\right]\right)\Delta_\Omega(c; m, n) \\ &\leq \Omega(m) + \Omega(n) - \mathbb{E}[\Omega(m + n - Z)] - c\left(\frac{n - \mathbb{E}[Z]}{n - m}n^2 + \frac{\mathbb{E}[Z] - m}{n - m}m^2 - \mathbb{E}[(m + n - Z)^2]\right) \\ &\leq \Omega(m) + \Omega(n) - \mathbb{E}[\Omega(m + n - Z)].\end{aligned}\quad (46)$$

Combining inequalities (45) and (46) yields a refinement of the results established in [7] (Corollary 7) and [18] (Corollary 9), leading to sharper bounds in the considered setting.

Author Contributions: All authors contributed to the study conception and design. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: No new data were created or analyzed in this study.

Acknowledgments: The authors would like to thank the anonymous reviewers whose valuable comments helped improve the manuscript. The second author acknowledges institutional support from the project “Geoinformatics framework for sustainable coastal and island development” (GEORAZ, IP-UNIST-17), funded through the Institutional Research Projects Programme of the University of Split, financed by the Recovery and Resilience Facility of the European Union and approved by the Ministry of Science, Education and Youth of the Republic of Croatia. This support is provided for research development and training activities of the author and was not used to fund the research presented in this article.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Mercer, A.M. A variant of Jensen’s inequality. *J. Inequal. Pure Appl. Math.* **2003**, *4*, 73.
2. Ivelić Bradanović, S. Improvements of Jensen’s inequality and its converse for strongly convex functions with applications to strongly f -divergences. *J. Math. Anal. Appl.* **2024**, *531*, 127866. [\[CrossRef\]](#)
3. Nikodem, K. On Strongly Convex Functions and Related Classes of Functions. In *Handbook of Functional Equations*; Springer: New York, NY, USA, 2014; pp. 365–405.
4. Roberts, A.W.; Varberg, D.E. *Convex Functions*; Academic Press: New York, NY, USA, 1973.
5. Cheung, W.S.; Matković, A.; Pečarić, J. A variant of Jessen’s inequality and generalized means. *J. Inequal. Pure Appl. Math.* **2006**, *7*, 10.
6. Matković, A.; Pečarić, J.; Perić, J. A refinement of the Jessen-Mercer inequality and a generalization on convex hulls in $\mathbb{R}^n$. *J. Math. Inequal.* **2015**, *9*, 1093–1114. [\[CrossRef\]](#)
7. Klaričić Bakula, M.; Nikodem, K. On the converse Jensen inequality for strongly convex functions. *J. Math. Anal. Appl.* **2016**, *434*, 516–522. [\[CrossRef\]](#)
8. Adil Khan, M.; Anwar, M.; Jakšetić, J.; Pečarić, J. On some improvements of the Jensen inequality with some applications. *J. Inequal. Appl.* **2009**, *2009*, 323615. [\[CrossRef\]](#)
9. Choi, D.; Krnić, M.; Pečarić, J. Improved Jensen-type inequalities via linear interpolation and applications. *J. Math. Inequal.* **2017**, *11*, 301–322. [\[CrossRef\]](#)
10. Dragomir, S.S. Lebesgue Integral Inequalities of Jensen Type for λ -Convex Functions. *Armen. J. Math.* **2018**, *10*, 1–19. [\[CrossRef\]](#)
11. Hosseini, M.S.; Moradi, H.R.; Moosavi, B. Operator Jensen’s type inequalities for convex functions. *J. Math. Ext.* **2021**, *15*, 1–16. [\[CrossRef\]](#)
12. Adil Khan, M.; Ivelić Bradanović, S.; Mahmoud, H.A. New Improvements of the Jensen-Mercer Inequality for Strongly Convex Functions with Applications. *Axioms* **2024**, *13*, 553. [\[CrossRef\]](#)
13. Ivelić Bradanović, S.; Lovričević, N. Generalized Jensen and Jensen-Mercer inequalities for strongly convex functions with applications. *J. Inequal. Appl.* **2024**, *2024*, 112. [\[CrossRef\]](#)
14. Sayyari, Y.; Donganont, M.; Dehghanian, M.; Jahanshahi, M.A. Strongly convex functions and extensions of related inequalities with applications to entropy. *AIMS Math.* **2024**, *9*, 10997–11006. [\[CrossRef\]](#)
15. Alshehri, M.G.A.; Hyder, A.-A.; Budak, H.; Barakat, M.A. Some New Improvements for Fractional Hermite-Hadamard Inequalities by Jensen-Mercer Inequalities. *J. Funct. Spaces* **2024**, *2024*, 6691058. [\[CrossRef\]](#)
16. Dragomir, S.S.; Ivelić Bradanović, S.; Lovričević, N. Majorization-type inequalities for (m, M, ψ) -convex functions with applications. *Open Math.* **2025**, *23*, 20250130. [\[CrossRef\]](#)
17. Rubab, F.; Nabi, H.; Asif, R. Generalization and refinements of Jensen inequality. *J. Math. Anal.* **2021**, *12*, 1–27. [\[CrossRef\]](#)
18. Nikodem, K.; Rajba, T. Ohlin and Levin-Stečkin-type results for strongly convex functions. *Ann. Math. Silesianae* **2020**, *34*, 123–132. [\[CrossRef\]](#)
19. Rajba, T.; Wasowicz, S. Probabilistic characterization of strong convexity. *Opusc. Math.* **2011**, *31*, 97–103. [\[CrossRef\]](#)

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.