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Another Proof of Levitzki's Theorem for Prime Rings and Application to k -skew Commuting Property

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Abstract

In this paper, we provide another proof of Levitzki's theorem for prime rings with $k!$ -torsion-free using the Vandermonde determinant, which states that a prime ring contains no nonzero nilpotent left or right ideal. As an application of this theorem, we prove that the k -skew commuting element in torsion-free prime $*$ -rings with nonsymmetric element must be zero without using any deep structural results from ring theory.

Keywords: Levitzki; prime ring; nilpotent; Vandermonde determinant; k -skew commuting

MSC: 16N60

1. Introduction

A prime ring is defined by the condition that for any two ideals A, B of R , $AB = 0$ implies $A = 0$ or $B = 0$. The study of nilpotent on prime rings has been a central theme in ring theory since the pioneering works of J. Levitzki and I. N. Herstein in the mid-20th century. Levitzki's theorem [1] state that there exists no non-zero nilpotent left or right ideal in prime ring. This theorem provides a powerful tool for handling nilpotent elements in rings without nilpotent ideals, and it plays a crucial role in the structure theory of prime rings, especially in the theory of polynomial identities. The following form is a variation of Levitzki's theorem.

Lemma 1 (Levitzki's Theorem [1]). *Let R be a prime ring, if there exists an integer $k > 0$ such that $(arb)^k = 0$ for all $r \in R$, then $ba = 0$.*

Herstein established the proof via an ideal induction argument in [1]. Here we present another proof for the case of $k!$ -torsion-free prime rings via the Vandermonde determinant. It should be noted that a different short proof of a related result (a nil right ideal of bounded index forces a nonzero nilpotent ideal) was given by A. A. Klein [2] using a minimality argument. Related results are referred to in [3,4].

Parallel to the investigation of ring structure, the study of derivations and related mappings has flourished. Derivations are algebraic analogues of derivatives, and their behavior on prime rings reveals deep connections with commutativity and the center of the ring (refer to [5–7]). The Lie product regarded as a derivation, defined as $[x, y] = xy - yx$ for elements x, y in a ring, captures the essence of non-commutativity and serves as a bridge between ring theory and Lie theory. By Lanski's lemma [8], a noncentral Lie ideal in a prime ring contains a nonzero ordinary ideal. If, in addition, every element of this Lie ideal is nilpotent, then Levitzki's theorem forces that ideal to be zero. This type appears repeatedly in the classification of derivations and centralizers acting on Lie ideals.



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In a prime ring with an involution $*$, the interplay between the Lie product and the $*$ -operation leads to the study of skew-Lie products $*[x, y] = xy - yx^*$ or k -skew Lie products as an analogy of n -Lie derivations. The present paper aims to advance these topics.

Given a ring R with an involution $*$. For any elements a, b and a positive integer k , denote the k -skew Lie product to be

$$*[a, b]_k = [a, *[a, b]_{k-1}],$$

where $*[a, b]_0 = b$ and $*[a, b]_1 = ab - ba^*$ is the skew Lie product. In the field of operator algebras, the skew Lie product is closely related to structures on operator algebras, such as Jordan derivations and ideals, and has interesting connections (see [9]). Šemrl proved in [10] that every Jordan derivation $J : \mathcal{B}(H) \rightarrow \mathcal{B}(H)$ satisfying $J(T^2) = TJ(T) + J(T)T^*$ has the form $J(T) = TA - AT^*$, where $\mathcal{B}(H)$ is the algebra of all bounded operators on a Hilbert space H . In further research, Molnár [11] systematically study such skew Lie product, proving that if $\mathcal{N} \subseteq \mathcal{B}(H)$ is an ideal, then

$$\mathcal{N} = \text{span}\{AB - BA^* \mid A \in \mathcal{N}, B \in \mathcal{B}(H)\} = \text{span}\{AB - BA^* \mid A \in \mathcal{B}(H), B \in \mathcal{N}\}.$$

The k -skew Lie bracket appears naturally in the skew Baker-Campbell-Hausdorff formula, namely

$$e^A B e^{-A^*} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} *[A, B]_k.$$

The skew Baker-Campbell-Hausdorff formula is closely related to conserved quantities in non-Hermitian quantum systems.

In a Hermitian quantum system, the fixed points of the automorphism

$$e^{iH} A e^{-iH} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} [H, A]_k$$

on the algebra of observables constructed from the Hamiltonian H are called conserved quantities, where $[H, A]_k = [H, [H, A]_{k-1}]$ is the k -Lie product, and the equation is established by the well-known Baker-Campbell-Hausdorff formula. Clearly, an observable A satisfying the commutation relation $AH = HA$ is a conserved quantity. In quantum mechanics, the celebrated Noether theorem states that when an observable is conserved, it generates a continuous symmetry of the system, arising from the invariance of physical laws under certain transformations.

In non-Hermitian quantum systems that describe gain or dissipation, the non-self-adjointness of the Hamiltonian H renders the time evolution operator e^{iH} non-unitary. Nevertheless, observables A that skew-commute with the Hamiltonian H , i.e., satisfy the skew-commutation relation $AH = H^{\dagger}A$, have become fundamental tools in the study of PT-symmetric systems, pseudo-Hermitian systems, and open system dynamics [12]. In pseudo-Hermitian systems, there exists a metric operator that skew-commutes with the Hamiltonian, and all conserved quantities in such systems can be constructed from those in Hermitian systems via the metric operator [13]. Therefore, understanding the properties of k -skew-commutation helps in understanding and constructing conserved quantities in non-Hermitian systems.

In this paper we consider the property of s satisfies $*[a, s]_k = 0$ for all $a \in R$. In this case, we say that s is a k -skew commuting element. It's easy to prove that if s is a k -skew commuting element, so does s^* . In this paper, we prove the following theorem without using any deep structural results from ring theory as in [14], which appeared in the work of the author and X.F. Qi ([14], Theorem 2.5). Refer to [15,16] for more result about

skew-commuting mappings. Here we consider $\ast$ -ring R with nonsymmetric element, i.e., there exists $r \in R$ such $r \neq r^\ast$. The necessity of nonsymmetric element in the following theorem is evident from the proof of Theorem 2 at the end, which is ignored in ([14], Theorem 2.5).

Theorem 1. *The k -skew commuting element in torsion-free prime $\ast$ -rings with nonsymmetric element is zero.*

2. Levitzki's Theorem

In this section, we present a new proof of Levitzki's theorem for torsion-free prime rings. Unlike the classical approach, which relies on chain conditions or the structure of nilpotent left ideals, our argument is based on elementary properties of primeness. This self-contained proof makes it particularly suitable for applications in settings where no finiteness assumptions are available in the next section.

The following lemma about prime rings is useful.

Lemma 2 (Theorem A.7 [17]). *Let R be a prime ring and let $a_i, b_i, c_j, d_j \in R$ be such that*

$$\sum_{i=1}^r a_i x b_i = \sum_{j=1}^m c_j x d_j$$

for all $x \in R$. If $a_1, \dots, a_r$ are linearly independent over C , where C is the extended center of R , then each b_i is a C -linear combination of $d_1, \dots, d_m$. Similarly, if $b_1, \dots, b_r$ are linearly independent over C , then each a_i is a C -linear combination of $c_1, \dots, c_m$.

It is well-known that the extended centroid of prime ring is a field [18]. Recall that prime ring is torsion-free if for any $n \in \mathbb{Z}$ and $a \in R$, $na = 0$ implies $a = 0$.

The following is the well-known Levitzki's theorem, here we give a new proof via Vandermonde determinant.

Lemma 3 (Levitzki's theorem [1]). *Let R be a $k!$ -torsion-free prime ring, if there exists an integer $k > 0$ such $(arb)^k = 0$ for all $r \in R$, then $ba = 0$.*

Proof. Since $(arb)^k = 0$, we have $(bar)^k ba = 0$. Set $r = mu + t$ and $s = ba$, where $m \in \mathbb{Z}$ and $u, t \in R$. Then

$$(s(mu + t))^k s = g_0(u, t) + mg_1(u, t) + \dots + m^k g_k(u, t) = 0 \quad (1)$$

where each $g_i(u, t)$, $i = 0, 1, \dots, k$ is a polynomial in the non-commuting variables u and t . In particular,

$$g_1(u, t) = su(st)^{k-1}s + stsu(st)^{k-2}s + \dots + (st)^i su(st)^{k-1-i}s + \dots + (st)^{k-1} sus.$$

The equality holds for all integers m . Let $m = 1, 2, \dots, k$, the coefficient matrix of Equation (1) is a Vandermonde matrix

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ 2 & 2^2 & \dots & 2^k \\ \vdots & \vdots & \ddots & \vdots \\ k & k^2 & \dots & k^k \end{pmatrix}$$

The determinant of the coefficient matrix is equal to a product of positive integers $1, \dots, k$. Since R is $k!$ -torsion-free, hence every coefficient $g_i(u, t)$ is zero. We focus on the monomials containing exactly one factor of m , that is,

$$g_1(u, t) = \sum_{i=0}^{k-1} (st)^i su(st)^{k-1-i} s = 0. \quad (2)$$

We shall prove that

$$(st)^{k-1} s = 0$$

for any $t \in R$. Suppose, to the contrary, that there exist some $t \in R$ such that $(st)^{k-1} s \neq 0$. Note that Equation (2) implies for any $u \in R$,

$$-su(st)^{k-1} s = \sum_{i=1}^{k-1} (st)^i su(st)^{k-1-i} s.$$

Then by Lemma 2 the element s appearing as a coefficient of u is a linear combination, over the extended center of R , of the remaining terms. That is, there exist scalars $\{a_1, a_2, \dots, a_{k-1}\}$ not all zero and depending on t , such that

$$s = \sum_{i=1}^{k-1} a_i (st)^i s. \quad (3)$$

Now note that $s(ts)^k = (st)^k s = 0$. Multiplying Equation (3) on the right $(ts)^{k-1}$ gives

$$s(ts)^{k-1} = \sum_{i=1}^{k-1} a_i (st)^i s(ts)^{k-1} = \sum_{i=1}^{k-1} a_i (st)^i (st)^{k-1} s = 0,$$

and consequently $(st)^{k-1} s = 0$. Thus we have that $(st)^{k-1} s = 0$ for any $t \in R$.

Repeating the above argument iteratively. Eventually, we can obtain $sts = 0$ for all $t \in R$. Since R is prime, this forces $s = 0$, completing the proof. $\square$

3. k -skew Commuting Element

In this section, we aim to prove the following result arose in ([14], Theorem 2.5). Here the proof is although a little long, but elementary. It does not rely on any deep structural results from ring theory. It maybe offers an alternate way to tackle some nilpotent problems for torsion-free prime rings. In what follows R is a prime ring with the involution $*$. We first introduce some subsets of R which is needed in the following lemmas.

Definition 1. An element a in a ring R is regular if it is neither a left nor right zero divisor in R .

Definition 2. For $s \in R$, denote by $L_s = \{x \in R \mid xs = 0\}$ the left annihilator of s , and by $R_s = \{x \in R \mid sx = 0\}$ the right annihilator of s .

Clearly, L_s is a left ideal of R and R_s is a right ideal of R .

Theorem 2. The k -skew commuting element in torsion-free prime $*$ -rings with nonsymmetric element is zero.

A prime $*$ -ring with no nonsymmetric elements is commutative, since for any $a, b \in R$, $ab = (ab)^* = b^* a^* = ba$.

Remark 1. For the case of $k = 1$, i.e., s is a skew commuting element, we give a direct proof that does not rely on the torsion-free assumption.

If $s \neq s^*$, then since $s(s - s^*) = 0$, it follow that

$$(as - sa^*)(s - s^*) = -sa^*(s - s^*) = 0$$

for all $a \in R$, which forces $s = 0$.

If $s = s^*$, then

$$sas - sa^*s^* = sas - sa^*s = s(as - a^*s) = 0$$

for all $a \in R$. Now consider two subcases.

Case I: If $(a - a^*)s = 0$ for all $a \in R$, choose b with $b \neq b^*$ (such b exists), then

$$(b - b^*)as - (b - b^*)sa^* = (b - b^*)as = 0$$

for all $a \in R$, which implies $s = 0$.

Case II: Otherwise, there exists b such that $(b - b^*)s \neq 0$, then

$$as(b - b^*)s - sa^*(b - b^*)s = -sa^*(b - b^*)s = 0$$

for all $a \in R$, again yielding $s = 0$.

In order to prove Theorem 2, three lemmas are required. We start with the first lemma, which deals with the left annihilator of a k -skew commuting element.

Lemma 4. Assume s is a k -skew commuting element. If $x \in L_s$, then either $x^* \in R_s$ or x is nilpotent of index no more than $k - 1$.

Proof. Since s is a k -skew commuting element, for any $x \in L_s, a \in R, m \in \mathbb{Z}$, we have

$$_*[mx + ax, s]_k = \sum_{i=0}^k \binom{k}{i} (-1)^i (mx + ax)^{k-i} s (mx^* + x^*a^*)^i = 0. \quad (4)$$

Since $x \in L_s$, it follows that $(mx + ax)^{k-i} s = 0$ for i from 0 to $k - 1$. Thus by Equation (4) we have

$$s(mx^* + x^*a^*)^k = 0.$$

Since R is torsion-free, treating this as a polynomial in m and applying the Vandermonde argument (by selecting monomials containing m^k and exactly one factor of m), we can obtain

$$s(x^*)^k = 0 \text{ and } \sum_{i=0}^{k-1} s(x^*)^{k-i} a^* (x^*)^i = 0. \quad (5)$$

Since $s(x^*)^k = 0$, substituting a^* with $(x^*)^{k-2}a^*$ in Equation (5) yields

$$\sum_{i=0}^{k-1} s(x^*)^{k-i} (x^*)^{k-2} a^* (x^*)^i = s(x^*)^{k-1} a^* (x^*)^{k-1} = 0.$$

Since R is prime, it follows that either $s(x^*)^{k-1} = 0$ or $(x^*)^{k-1} = 0$.

Now consider two subcases in the following:

Case I: If $(x^*)^{k-1} = 0$, then x is nilpotent of index no more than $k - 1$.

Case II: If $(x^*)^{k-1} \neq 0$, then $s(x^*)^{k-1} = 0$. Replacing a^* by $(x^*)^{k-3}a^*$ in Equation (5), we obtain

$$\sum_{i=0}^{k-1} s(x^*)^{k-i} (x^*)^{k-3} a^* (x^*)^i = s(x^*)^{k-2} a^* (x^*)^{k-1} = 0.$$

Consequently, $s(x^*)^{k-2} = 0$ by the primeness of R . Continuing this process, replace a^* by $(x^*)^{k-4}a^*$ in Equation (5) to get $s(x^*)^{k-3} = 0$. We eventually obtain $s x^* = 0$. This completes the proof. $\square$

Note that if s is a k -skew commuting element, then s^* shares the same property. Consequently, by Lemma 4, it follows that for every $x \in L_{s^*}$, either $x^* \in R_{s^*}$ or x is nilpotent of index no more than $k - 1$.

As a consequence of the above lemma, we can establish the following result regarding nilpotent k -skew commuting elements, which will serve as a crucial step toward the main theorem.

Lemma 5. Assume that s is a k -skew commuting element. If s is nilpotent, then $s = 0$.

Proof. First it is sufficient to prove $s^2 = 0$ under the assumption that $s^n = 0$ for some $n > 2$. Since s is a k -skew commuting element, for any $a \in R, m \in \mathbb{Z}$ we have that

$$*[s(s^*)^{n-1} + ms^*as^{n-1}, s]_k = \sum_{i=0}^k \binom{k}{i} (-1)^i (s(s^*)^{n-1})^{k-i} s(m(s^*)^{n-1}a^*s)^i = 0.$$

Because R is torsion-free, the Vandermonde argument as before forces each coefficient in the expansion in m^i to vanish. Hence

$$\binom{k}{i} (-1)^i (s(s^*)^{n-1})^{k-i} s((s^*)^{n-1}a^*s)^i = 0$$

for each i and any $a \in R$. In particular, for $i = k$,

$$s((s^*)^{n-1}a^*s)^k = 0.$$

Multiplying the above equation on the left by $(s^*)^{n-1}a^*$, consequently, we have

$$(s^*)^{n-1}a^*s((s^*)^{n-1}a^*s)^k = ((s^*)^{n-1}a^*s)^{k+1} = 0$$

for any $a \in R$, and by Lemma 3 we obtain

$$s(s^*)^{n-1} = 0. \quad (6)$$

Since $n > 2$, then for any $a \in R$, we have

$$*[s(s^*)^{n-2} + s^*as^{n-2}, s]_k = (-1)^k (s^*as^{n-2})^k s + s((s^*)^{n-2}a^*s)^k = 0.$$

Combining with Equation (6) (i.e., $s^{n-1}s^* = 0$), we multiply the above equation on the right by s^* gives

$$s((s^*)^{n-2}a^*s)^k s^* = 0.$$

Then, multiplying the above equation on the right by $(s^*)^{n-1}a^*$ gives

$$s((s^*)^{n-2}a^*s)^k s^* (s^*)^{n-1}a^* = (s(s^*)^{n-2}a^*)^{k+1} = 0.$$

Then by Lemma 3 we obtain

$$s(s^*)^{n-2} = 0.$$

Since $n > 2$, then for any $a \in R$ and $m \in \mathbb{Z}$,

$$*[s^{n-1} + sas^{n-2}, s]_k = (sas^{n-2})^k s = 0.$$

Multiplying the above equation on the left by s^{n-2} gives

$$s^{n-2}(sas^{n-2})^k s = (s^{n-1}a)^{k+1} = 0.$$

By Lemma 3 we obtain $s^{n-1} = 0$. Repeating the same argument, we get $s^2 = 0$.

Thus we may assume $s^2 = 0$. Note that the proof of Equation (6) does not require the restriction $n > 2$, thus we have that $ss^* = 0$. Since s is a k -skew commuting element, for any $r, t \in R, m \in \mathbb{Z}$, we have that

$$*[rs + mt, s]_k = \sum_{i=0}^k \binom{k}{i} (-1)^i (rs + mt)^{k-i} s(s^*r^* + mt^*)^i = 0.$$

Since $s^2 = ss^* = 0$, the Vandermonde argument (selecting monomials containing exactly one factor of m) forces

$$(rs)^{k-1}ts + (-1)^k st^*(s^*r^*)^{k-1} = 0$$

for all $r, t \in R$. Multiplying this equation on the left by s gives

$$s(rs)^{k-1}ts = 0$$

for all $r, t \in R$. Therefore, for any r , either $s = 0$ or $s(rs)^{k-1} = 0$. Then Lemma 3 yields $s = 0$. This completes the proof. $\square$

Furthermore, by combining the nilpotent case with the left annihilator characterization, we are able to derive a stronger conclusion concerning non-regular elements, which is the more general case than the nilpotent case.

Lemma 6. Assume s is a k -skew commuting element. If s is not regular, then $s = 0$.

Proof. Since s is not regular, there exists some nonzero $r \in R$ such that $rs = 0$ or $sr = 0$. If $r = s^n$, then Lemma 5 immediately gives $s = 0$. Otherwise, we may assume $r \in L_s$. By Lemma 4, either $rs = sr^* = rs^* = 0$ or r is nilpotent of index no more than $k - 1$.

If every element of L_s are nilpotent, then for any $a \in R$ we have $ar \in L_s$ and $(ar)^{k-1} = 0$, so $r = 0$ by Lemma 3. Contradicting the non-regularity of s . Hence there exists an $r \in L_s$ that is not nilpotent. Lemma 4 implies $r \in R_s$, i.e., $sr^* = 0$.

Now consider two cases. When $k = 1$, the equation $as - sa^* = 0$ holds for all $a \in R$, then $ras = rsa^* = 0$ for all $a \in R$. and primeness of R yields $s = 0$. For $k > 1$, we have for any $a \in R$,

$$*[r^* + mas, s]_k = \sum_{i=0}^k \binom{k}{i} (-1)^i (r^* + mas)^{k-i} s(r + ms^*a^*)^i = 0.$$

Applying the Vandermonde argument (selecting monomials containing exactly one factor of m) we obtain

$$(r^*)^{k-1}as^2 + \sum_{i=1}^{k-1} (-1)^i [(r^*)^{k-1-i}as^2r^i + (r^*)^{k-i}ss^*a^*r^{i-1}] + ss^*a^*r^{k-1} = 0.$$

Since $r^* \in R_s$ i.e., $sr^* = 0$ by Lemma 4, multiplying the above equation on the left by s gives

$$(-1)^{k-1} sas^2 r^{k-1} + s^2 s^* a^* r^{k-1} = 0$$

for all $a \in R$. Now replace a with $r^* a$ in this identity, we have

$$s^2 s^* a^* r^k = 0.$$

Since r is not nilpotent and R is a prime ring, it follows that $s^2 s^* = 0$. By Lemma 4 we then have $s^3 = 0$ or $s^{2k-2} = 0$, and finally Lemma 5 yields $s = 0$. This completes the proof. $\square$

Consequently, in view of Lemma 6, the only remaining possibility is that s is regular. The final step is to characterize such elements.

Corollary 1. Assume s is a k -skew commuting element. If s is regular, then $s = s^*$.

Proof. Since $*[a, s]_k = 0$ and $*[a, s^*]_k = 0$ for all $a \in R$, we have $*[a, s - s^*]_k = 0$. Setting $a = s - s^*$ yields

$$*[a, s - s^*]_k = 2^k (s - s^*)^k = 0.$$

Since R is torsion-free, the element $s - s^*$ is nilpotent. Then by Lemma 5, we obtain $s = s^*$. $\square$

Finally, to complete the proof of the main theorem, we need only treat the regular case, i.e., s is regular.

Proof of Theorem 2. By Lemma 5, we only need to consider the case where s is regular. From Corollary 1 we have $s = s^*$. Since $*[a, s]_k = 0$ for all $a \in R$, we have

$$*[sa, s]_k = \sum_{i=0}^k \binom{k}{i} (-1)^i (sa)^{k-i} s (a^* s)^i = s \sum_{i=0}^k \binom{k}{i} (-1)^i (as)^{k-i} (a^* s)^i = 0.$$

Replace a by $a - a^*$ in the above identity, then $2^k s ((a - a^*)s)^k = 0$. Since R is torsion free and s is regular, we have

$$((a - a^*)s)^k = 0$$

for all $a \in R$. We shall prove that

$$((a - a^*)s)^{k-1} = 0$$

for all $a \in R$. Suppose, to the contrary, that there exists $r \in R$ such that $((r - r^*)s)^{k-1} \neq 0$.

Denote $\Delta = (s(r - r^*))^{k-1} \neq 0$. Then

$$\Delta^2 = 0 \quad \text{and} \quad \Delta s \Delta^* = 0.$$

Now apply the identity $*[a, s]_k = 0$ with $a = m\Delta t + u\Delta$ for any $t, u \in R, m \in \mathbb{Z}$. Since R is torsion-free, the Vandermonde argument (selecting monomials containing exactly one factor of m) yields

$$\Delta t (u\Delta)^{k-1} s + k(u\Delta)^{k-1} s t^* \Delta^* + k\Delta t s (\Delta^* u^*)^{k-1} + s (\Delta^* u^*)^{k-1} t^* \Delta^* = 0.$$

Multiplying this equation on the left by Δ gives, for any $t, u \in R$,

$$\Delta (u\Delta)^{k-1} s t^* \Delta^* = 0.$$

Since $\Delta \neq 0$ and R is a prime ring, then

$$\Delta(u\Delta)^{k-1}s = 0$$

for all $u \in R$. Since s is regular, it follows that $\Delta(u\Delta)^{k-1} = 0$ for all $u \in R$, i.e., $(\Delta u)^k = 0$ for all $u \in R$. By Lemma 3, we conclude $\Delta = 0$, a contradiction. Therefore $((a - a^*)s)^{k-1} = 0$ for all $a \in R$. Repeating this argument inductively, we eventually obtain $(a - a^*)s = 0$ for any $a \in R$. Regularity of s then forces $a = a^*$ for any $a \in R$, which contradicts the assumption that R is $*$ -ring with nonsymmetric element. Consequently, s must be zero. This completes the proof. $\square$

Remark 2. In the paper [16], several authors introduced the λ - k -skew Lie product as a generalization of k -skew Lie product, namely:

$$*[a, \delta_a(b)_\lambda^*]_k =_* [a, \delta_a(b)_\lambda^*]_{k-1}1,$$

where $\lambda \in \mathbb{R}, k \in \mathbb{Z}$,

$$*[a, \delta_a(b)_\lambda^*]_0 = \delta_a(b)_\lambda^* = ab - \lambda ba^*$$

and

$$*[a, \delta_a(b)_\lambda^*]_1 = a\delta_a(b)_\lambda^* - \delta_a(b)_\lambda^*a.$$

For the case of $\lambda = 1$, λ - k -skew Lie product is the k -skew Lie product. We say that s is a λ - k -skew commuting element if s satisfying $*[a, \delta_a(b)_\lambda^*]_k$ for all $a \in R$ as in the proof of Theorem 2 step-by-step. We can also prove that there exists no λ - k -skew commuting element in torsion-free prime $*$ -rings.

In quantum mechanics, it is customary to describe a quantum physical system by the von Neumann algebra generated by its observable algebra. Recall that a von Neumann algebra M is a subalgebra of $B(H)$ satisfying the double commutant property: $M'' = M$, where

$$M' = \{T \in B(H) : TA = AT \text{ for all } A \in M\},$$

and $M'' = (M')'$. In particular, if the center of M is $\mathbb{C}I$, then M is called a factor von Neumann algebra.

It is well known that factor von Neumann algebras are prime. Therefore, the following conclusion is immediate.

Corollary 2. *There exists no k -skew commuting element on a factor von Neumann algebra.*

In [19], Corollary 2.3 proves that if A is a prime C^* -algebra, then its extended center is $C(A) = \mathbb{C}$. As another corollary of Theorem 2, we obtain the following.

Corollary 3. *There exists no k -skew commuting element on prime C^* -algebra.*

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