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Stabilization of a Logarithmic Viscoelastic Wave Equation with the Not Necessarily Decreasing Kernel and Distributed Delay

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Abstract

This paper is concerned with the asymptotic behavior of a viscoelastic wave equation involving distributed delay, logarithmic nonlinearity and dynamic Wentzell boundary conditions. In general, when the memory kernel function $g(t)$ is monotonically decreasing, the system energy's decay is similar to that of the kernel function. However, this work addresses the case where the kernel function does not necessarily decay; thus, at this point, whether the system energy can still decay, and especially maintain exponential decay, is a very interesting question. Assuming that the kernel function is not necessarily decreasing, which means that it may oscillate, under some proper conditions, utilizing the Lyapunov functional method and constructing auxiliary functions, an exponential decay result is attained. To some extent, the result extends and improves several earlier related results in the literature.

Keywords: exponential decay; viscoelastic wave equation; distributed delay; not necessarily decreasing kernel

MSC: 35L20; 35Q40; 93C20; 93D15

1. Introduction

For a bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ with two closed and disjoint smooth boundaries Γ_0 and Γ_1, consider the following system:

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds - \mu_1 \Delta u_t \\ -\int_{\tau_1}^{\tau_2} |\mu_2(\tau)|\Delta u_t(x,t-\tau)d\tau = k|u|^{\gamma-2}u\ln|u|, & \text{in } \Omega \times (0,\infty), \\ u = 0, & \text{on } \Gamma_0 \times (0,\infty), \\ u_{tt} - \Delta_\Gamma u + \partial_\nu u - \int_0^t g(t-s)\partial_\nu u(s)ds \\ +\mu_1 \partial_\nu u_t + \int_{\tau_1}^{\tau_2} |\mu_2(\tau)|\partial_\nu u_t(x,t-\tau)d\tau = 0, & \text{on } \Gamma_1 \times (0,\infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & \text{in } \Omega, \\ u_t(x,-t) = f_0(x,t), & \text{in } \Omega \times (0,\tau_2), \end{cases} \tag{1}$$

where Δ and Δ_Γ are the Laplace–Beltrami operators on Ω and Γ_1, respectively. The symbol ∇_Γ stands for the Levi–Civita connection on Γ_1, while ν denotes the unit outward normal vector to the boundary Γ. The constants γ, μ_1, and k satisfy the following conditions:

Check for updates

Academic Editors: Branislav M. Ranđelović and Branislav Vlahovic

Received: 28 April 2026
Revised: 4 June 2026
Accepted: 12 June 2026
Published: 13 June 2026

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$\gamma > 2$, $\mu_1 > 0$, and $k > 0$. $\int_{\tau_1}^{\tau_2} |\mu_2(\tau)|\Delta u_t(x, t-\tau)d\tau$ represents the distributed delay, and $\mu_2(t) : [\tau_1, \tau_2] \to \mathbb{R}$ is a bounded function, where $\tau_1 \in \mathbb{R}$ and $\tau_2 \in \mathbb{R}$ satisfying $0 \leq \tau_1 < \tau_2$, and $g(t)$ is the memory kernel function. This work addresses the stabilization problem of system (1).

Logarithmic nonlinearity stems from inflationary cosmology and supersymmetric field theories [1,2]. Logarithmic nonlinearities feature prominently in numerous physical branches, for instance, nuclear physics, the field of optics, and quantum field theory. Many researchers have investigated hyperbolic equations with logarithmic nonlinearity terms in nonlinear models [3–17]. Choucha et al. [8] studied a logarithmic viscoelastic wave equation involving a time-varying delay on a fractional boundary condition, and obtained a blow-up result. An exponential decay estimate and finite time blow-up was obtained for a fractional Kirchhoff logarithmic wave system [16]. General decay and blow-up was established for a nonlinear logarithmic wave model possessing fractional dissipation imposed on the boundary [17]. In dynamic elasticity models, the Wentzell boundary condition emerges when the vibrating body possesses a thin boundary layer of high stiffness. Goldstein [18] clearly provided the physical interpretation of the Wentzell boundary condition in the wave equation.

Numerous studies have addressed delays in the velocity u_t, including constant delay τ, time-varying delay $\tau(t)$ and distributed forms [7,10–13,19,20]. Distributed delays arise widely in practical problems, as numerous physical phenomena rely on their historical states. Lv and Hao [10] considered a viscoelastic wave system that featured a distributed delay term of u_t and logarithmic nonlinearity, and they proved that the solution exists globally and established general decay results. In the context of a coupled Lame system that contained a distributed delay related to the strong damping Δu_t and logarithmic source $(u \ln |u|)$, Choucha et al. [5] obtained an exponential decay with a decreasing kernel. When the kernel is not necessarily decreasing, Choucha et al. [6] studied a nonlinear logarithmic Kirchhoff equation that contained a distributed delay related to Δu_t and Balakrishnan–Taylor damping, and they derived a similar exponential decay conclusion.

Regarding the stability of viscoelastic wave equations, most of the studies have been conducted based on Dirichlet boundary conditions or other static boundary conditions. Hao and Du [9] studied a viscoelastic wave system subject to variable coefficients, a logarithmic term, and a dynamical boundary condition, and they established explicit general decay alongside a finite-time blow-up. A blow-up criterion was established for a viscoelastic wave system that contained logarithmic nonlinearity, velocity delay, and an acoustic boundary condition by Park [12]. Liu and Fang [21] obtained general decay estimates for viscoelastic waves' transmission problem featuring a kernel not necessarily decreasing. Guo and Zhang [14,15] discussed wave equations subjected to several dynamic boundary conditions, and they obtained exponential or general decay results. Li et al. [22] investigated wave equations' asymptotic behavior featuring a generalized positive definite kernel.

Many related works such as [9,10,14,15,23] have been completed assuming that the kernel function $g(t)$ has the form of $g'(t) \leq -\zeta(t)H(g(t))$, in which H and $\zeta(t)$ meet certain conditions and $g(t)$ must be decreasing particularly. Medjden and Tatar [24] first considered viscoelastic wave equations with the kernel not necessarily decreasing, where the proof was achieved by using the Lyapunov functional method. Mesloub and Boulaaras [25] improved [24] and derived some more general decay results. There are some other viscoelastic wave equations with the kernel not necessarily decreasing, such as [6,21,26].

As far as we know, there are few studies on a logarithmic wave system with a delay term where the associated kernel function need not be monotonically decreasing. Therefore, motivated by [6,25], an exponential decay property is achieved for a wave system that contains distributed delay, a logarithmic source, a kernel function not necessarily decreasing,

and dynamic Wentzell boundary conditions. The studies of [6,10,25] belong to Dirichlet boundary conditions. There is no delay or logarithmic term in [22]. Although [9,14,15] discussed the wave equation with the dynamic Wentzell boundary condition, the memory kernel functions are decreasing. The study reported in this paper extends and improves them to some extent.

The rest of this paper is arranged as follows. The main result, together with some preliminaries, is given in Section 2. Section 3 provides a technical lemma, and Section 4 completes Theorem 2's proof.

2. Preliminaries and the Main Result

Denote $H_0^1(\Omega) = \{u \in H^1(\Omega) : u|_{\Gamma_0} = 0\}$. Let $\|\cdot\|$ be the norm in $L^2(\Omega)$ and $\|\cdot\|_{\Gamma_1}$ be the norm in $L^2(\Gamma_1)$. Let $\|\cdot\|_p$ be the norm in $L^p(\Omega)$. $C > 0$ is a generic constant.

The main assumptions are listed below.

Assumption 1. *Assume the following conditions for γ , $g(t)$, and $\mu_2(t)$:*

(A1) For γ it fulfills

$$2 < \gamma < \infty, \text{ for } n = 1, 2; \quad 2 < \gamma < \frac{2(n-1)}{n-2}, \text{ for } n \geq 3.$$

(A2) The function $g(t) : [0, \infty) \to (0, \infty)$ is C^1 satisfying

$$\int_0^\infty g(s)ds < 1. \tag{2}$$

(A3) Let $\xi(t)$ be positive differentiable satisfying $g'(t) + \xi(t)g(t) \geq 0$; moreover, $e^{\alpha t}[g'(t) + \xi(t)g(t)] \in L^1(0, \infty)$ for $\alpha > 0$, and for $\beta > 0$ and $L > 0$,

$$\xi(t) \geq \beta > 0, \left|\frac{\xi'(t)}{\xi(t)}\right| \leq L, \xi'(t) \leq 0, \int_0^\infty \xi(s)ds = \infty. \tag{3}$$

(A4) The function $\mu_2(t) : [\tau_1, \tau_2] \to \mathbb{R}$ is bounded satisfying

$$\frac{3}{4}\mu_1 < \int_{\tau_1}^{\tau_2} |\mu_2(s)|ds < \mu_1. \tag{4}$$

Remark 1. *Assumption (A3) indicates that the sign of $g'(t)$ cannot be determined; it may vary, which means the kernel function $g(t)$ may oscillate. There is a concrete example of $g(t)$ in Liu and Fang [21].*

$l(t)$, $\bar{l}$, $l_\alpha(t)$, $\bar{l_\alpha}$, and $\bar{g}$ are defined as follows:

$$\begin{aligned} &\xi(t)l(t) := g'(t) + \xi(t)g(t), \\ &\bar{l} := \int_0^\infty l(s)ds, \ l_\alpha(t) := e^{\alpha t}l(t), \\ &\bar{l_\alpha} := \int_0^\infty l_\alpha(s)ds, \ \bar{g} := \int_0^\infty g(s)ds. \end{aligned} \tag{5}$$

Define

$$z(x, \rho, \tau, t) = u_t(x, t - \rho\tau),$$

Thus, the following holds:

$$\begin{cases} \tau z_t(x,\rho,\tau,t) + z_\rho(x,\rho,\tau,t) = 0, \\ z(x,0,\tau,t) = u_t(x,t). \end{cases} \tag{6}$$

Therefore, system (1) turns into

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds - \mu_1 \Delta u_t \\ - \int_{\tau_1}^{\tau_2} |\mu_2(\tau)| \Delta z(x,1,\tau,t)d\tau = k|u|^{\gamma-2}u \ln|u|, & \text{in } \Omega \times (0,\infty), \quad (7a) \\ \tau z_t(x,\rho,\tau,t) + z_\rho(x,\rho,\tau,t) = 0, & \text{for } (x,\rho,\tau,t) \in \Omega \times (0,1) \\ & \times(\tau_1,\tau_2) \times (0,\infty), \quad (7b) \\ u = 0, & \text{on } \Gamma_0 \times (0,\infty), \quad (7c) \\ u_{tt} - \Delta_\Gamma u + \partial_\nu u - \int_0^t g(t-s)\partial_\nu u(s)ds \\ + \mu_1 \partial_\nu u_t + \int_{\tau_1}^{\tau_2} |\mu_2(\tau)| \partial_\nu z(x,1,\tau,t)d\tau = 0, & \text{on } \Gamma_1 \times (0,\infty), \quad (7d) \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & \text{in } \Omega, \quad (7e) \\ z(x,\rho,\tau,0) = f_0(x,\rho\tau), & \text{in } \Omega \times (0,1) \times (0,\tau_2). \quad (7f) \end{cases}$$

Denote the following:

$$(g * v)(t) = \int_0^t g(t-s)v(s)ds,$$

$$(g \square v)(t) = \int_0^t g(t-s)\|v(x,t) - v(x,s)\|^2 ds.$$

For system (7), its energy is given by

$$\begin{aligned} E(t) = \quad & \frac{1}{2}\|u_t\|^2 + \frac{1}{2}(1 - \int_0^t g(s)ds)\|\nabla u\|^2 + \frac{1}{2}(g \square \nabla u) \\ & + \frac{1}{2}\int_0^1 \int_{\tau_1}^{\tau_2} \tau |\mu_2(\tau)| \|\nabla z(x,\rho,\tau,t)\|^2 d\tau d\rho \\ & - \frac{k}{\gamma}\int_\Omega |u|^\gamma \ln|u| dx + \frac{k}{\gamma^2}\|u\|_\gamma^\gamma + \frac{1}{2}\|u_t\|_{\Gamma_1}^2 + \frac{1}{2}\|\nabla_\Gamma u\|_{\Gamma_1}^2. \end{aligned} \tag{8}$$

For the sake of completeness, existence together with regularity is given in what follows, and for more details please refer to [3,6,27].

Theorem 1. *Given assumptions (A1)–(A4), let* $(u_0, u_1, f_0) \in H_0^1(\Omega) \times L^2(\Omega) \times H^1(\Omega \times (0,1) \times (\tau_1, \tau_2))$. *System* (7) *admits a weak solution* (u, z) *satisfying*

$$\begin{aligned} & u \in C(0,\infty; H_0^1(\Omega)) \cap C^1(0,\infty; L^2(\Omega)), \\ & z \in L^\infty(0,\infty; H^1(\Omega \times (0,1) \times (\tau_1,\tau_2))). \end{aligned}$$

Theorem 2. *Assuming assumptions (A1)–(A4) hold, provided that* $E(0) > 0$, *and* $\bar{l_\alpha}$ *is given in* (35), *system* (7)*'s energy approaches zero in an exponential manner. Namely, for positive constants* ω_1 *and* ω_2 *it holds that*

$$E(t) \le \omega_1 e^{-\omega_2 \int_0^t \xi(s)ds}.$$

3. A Technical Lemma

Regarding the energy functional $E(t)$, the following holds:

Lemma 1. *Under the assumptions (A1)–(A4),*

$$E'(t) \leq -(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(\tau)|d\tau)\|\nabla u_t\|^2 + \frac{1}{2}(g'\Box\nabla u)(t) - \frac{1}{2}g(t)\|\nabla u\|^2. \tag{9}$$

Proof. Multiplying (7a) by u_t, integrating by parts in Ω, and making use of (7c), we have

$$\begin{aligned}&(u_{tt}, u_t) - (\Delta u, u_t) + (g * \Delta u, u_t) - \mu_1(\Delta u_t, u_t)\\ &- \int_{\tau_1}^{\tau_2} |\mu_2(\tau)|(\Delta z(x, 1, \tau, t), u_t)d\tau = k(u|u|^{\gamma-2}\ln|u|, u_t).\end{aligned} \tag{10}$$

Noting that

$$(u_{tt}, u_t) = \frac{1}{2}\frac{d}{dt}\|u_t\|^2, \tag{11}$$

and applying integration by parts, we have

$$\begin{aligned}-(\Delta u, u_t) &= \frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 - \int_{\Gamma_1} \partial_\nu u u_t d\Gamma,\\ -\mu_1(\Delta u_t, u_t) &= \mu_1\|\nabla u_t\|^2 - \mu_1\int_{\Gamma_1} \partial_\nu u_t u_t d\Gamma.\end{aligned} \tag{12}$$

For the term $(g * \Delta u, u_t)$, we have

$$\begin{aligned}&(g * \Delta u, u_t)\\ =\ & \int_0^t g(t-s)\int_\Omega [\frac{1}{2}\frac{d}{dt}|\nabla u(x,s) - \nabla u(x,t)|^2]dxds\\ &- \int_0^t g(t-s)\int_\Omega [\frac{1}{2}\frac{d}{dt}|\nabla u(x,t)|^2]dxds\\ &+ \int_0^t g(t-s)(\partial_\nu u(s), u_t(t))_{\Gamma_1}ds\\ =\ & \frac{1}{2}\int_0^t g(t-s)\frac{d}{dt}\|\nabla u(x,s) - \nabla u(x,t)\|^2 ds\\ &- \frac{1}{2}\int_0^t g(t-s)\frac{d}{dt}\|\nabla u(x,t)\|^2 ds\\ &+ \int_0^t g(t-s)(\partial_\nu u(s), u_t(t))_{\Gamma_1}ds\\ =\ & \frac{d}{dt}\left[\frac{1}{2}(g\Box\nabla u) - \frac{1}{2}(\int_0^t g(s)ds)\|\nabla u(x,t)\|^2\right]\\ &- \frac{1}{2}(g'\Box\nabla u) + \frac{1}{2}g(t)\|\nabla u(x,t)\|^2\\ &+ \int_0^t g(t-s)(\partial_\nu u(s), u_t(t))_{\Gamma_1}ds.\end{aligned} \tag{13}$$

Noting that $|u|^{\gamma-2}uu_t = \frac{1}{\gamma}\frac{d}{dt}|u|^\gamma$, we have

$$\begin{aligned}&k(u|u|^{\gamma-2}\ln|u|, u_t)\\ =\ & \frac{d}{dt}\left[\frac{k}{\gamma}\int_\Omega |u|^\gamma \ln|u|dx - \frac{k}{\gamma^2}\|u\|_\gamma^\gamma\right].\end{aligned} \tag{14}$$

Finally, the term $-\int_{\tau_1}^{\tau_2}|\mu_2(\tau)|(\Delta z(x,1,\tau,t),u_t)d\tau$ is handled as follows. Using (6), we have

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\tau|\mu_2(\tau)|\|\nabla z(x,\rho,\tau,t)\|^2d\tau d\rho\right]\\
&=-\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\tau)|\nabla z\cdot\nabla z_\rho d\tau d\rho dx\\
&=-\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\tau)|\frac{d}{d\rho}|\nabla z|^2d\tau d\rho dx\\
&=\frac{1}{2}(\int_{\tau_1}^{\tau_2}|\mu_2(\tau)|d\tau)\|\nabla u_t\|^2-\frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_2(\tau)|\|\nabla z(x,1,\tau,t)\|^2d\tau,
\end{aligned}\tag{15}$$

and

$$\begin{aligned}
&-\int_{\tau_1}^{\tau_2}|\mu_2(\tau)|(\Delta z(x,1,\tau,t),u_t)d\tau\\
&\le\frac{1}{2}(\int_{\tau_1}^{\tau_2}|\mu_2(\tau)|d\tau)\|\nabla u_t\|^2+\frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_2(\tau)|\|\nabla z(x,1,\tau,t)\|^2d\tau\\
&-\int_{\tau_1}^{\tau_2}|\mu_2(\tau)|(\partial_\nu z(x,1,\tau,t),u_t)_{\Gamma_1}d\tau,
\end{aligned}\tag{16}$$

where Young's inequality is used.

Substituting (11)–(16) into (10), and using (7d) and (8), (9) is obtained. □

4. Proof of Theorem 2

In the following, an exponential decay result is proved for system (7) using the Lyapunov functional method.

First, define the following functionals:

$$F(t):=\int_\Omega uu_t dx+\int_{\Gamma_1}uu_t d\Gamma,\tag{17}$$

$$G(t):=(L_\alpha\Box\nabla u)(t),\tag{18}$$

where

$$L_\alpha(t):=e^{-\alpha t}\int_t^\infty l(s)e^{\alpha s}ds,\tag{19}$$

and

$$\Theta(t):=\int_0^1\int_{\tau_1}^{\tau_2}\tau e^{-\rho\tau}|\mu_2(\tau)|\|\nabla z(x,\rho,\tau,t)\|^2d\tau d\rho.\tag{20}$$

Now, letting $\tilde{k}>1$, introduce the functional

$$\begin{aligned}
H(t):=&E(t)+\epsilon\xi(t)F(t)+\frac{\epsilon\mu_1}{2}\xi(t)\|\nabla u(t)\|^2+\lambda\xi(t)\Theta(t)\\
&+\eta\xi(t)G(t)-\eta\xi(t)(\int_0^t L_\alpha(s)ds)\|\nabla u\|^2+2\eta\xi(t)(\nabla u,L_\alpha*\nabla u)\\
&-\tilde{k}k_1\xi(0)(\int_0^t[\|u_s(s)\|^2+\|u_s(s)\|^2_{\Gamma_1}]ds)\\
&-\tilde{k}k_2\xi(0)(\int_0^t\|\nabla u(s)\|^2ds)-\tilde{k}k_3\xi(0)(\int_0^t\|u(s)\|^\gamma_\gamma ds)\\
&-\tilde{k}k_4\xi(0)(\int_0^t\int_0^1\int_{\tau_1}^{\tau_2}\tau|\mu_2(\tau)|\|\nabla z(x,\rho,\tau,s)\|^2d\tau d\rho ds),
\end{aligned}\tag{21}$$

for constants $\epsilon>0$, $\eta>0$, $\lambda>0$, and $k_i>0, i=1,\dots,4$, to be specified later.

Remark 2. *Here, $\tilde{k} = 2$ is not directly set as in [5,6]; it only needs to meet the condition $\tilde{k} > 1$.*

Proposition 1. *Let*

$$\begin{aligned} M(t) &:= H(t) + \tilde{k}k_1\xi(0)\Big(\int_0^t [\|u_s(s)\|^2 + \|u_s(s)\|_{\Gamma_1}^2]ds\Big) \\ &\quad + \tilde{k}k_2\xi(0)\Big(\int_0^t \|\nabla u(s)\|^2 ds\Big) + \tilde{k}k_3\xi(0)\Big(\int_0^t \|u(s)\|_\gamma^\gamma ds\Big) \\ &\quad + \tilde{k}k_4\xi(0)\Big(\int_0^t \int_0^1 \int_{\tau_1}^{\tau_2} \tau|\mu_2(\tau)|\|\nabla z(x,\rho,\tau,s)\|^2 d\tau d\rho ds\Big) \\ &:= E(t) + \epsilon\xi(t)F(t) + \frac{\epsilon\mu_1}{2}\xi(t)\|\nabla u(t)\|^2 + \lambda\xi(t)\Theta(t) + \eta\xi(t)G(t) \\ &\quad - \eta\xi(t)\Big(\int_0^t L_\alpha(s)ds\Big)\|\nabla u\|^2 + 2\eta\xi(t)(\nabla u, L_\alpha * \nabla u). \end{aligned} \tag{22}$$

Then, for some $C_1 > 0$ and $C_2 > 0$, the following holds:

$$C_1 E(t) \le M(t) \le C_2(E(t) + G(t)). \tag{23}$$

Proof. Recalling (17), and applying the Hölder inequality, Young's inequality, the Poincaré inequality, and the trace theorem, we have

$$\begin{aligned} F(t) &\le \|u_t\|\|u\| + \|u_t\|_{\Gamma_1}\|u\|_{\Gamma_1} \\ &\le \frac{\epsilon_1}{2}\|u_t\|^2 + \frac{1}{2\epsilon_1}\|u\|^2 + \frac{\epsilon_1}{2}\|u_t\|_{\Gamma_1}^2 + \frac{1}{2\epsilon_1}\|u\|_{\Gamma_1}^2 \\ &\le \frac{\epsilon_1}{2}\|u_t\|^2 + \frac{C_* + B_*}{2\epsilon_1}\|\nabla u\|^2 + \frac{\epsilon_1}{2}\|u_t\|_{\Gamma_1}^2, \end{aligned} \tag{24}$$

and recalling that $0 < \xi(t) \le \xi(0)$, we observe that

$$\epsilon\xi(t)F(t) \le \frac{\epsilon\xi(0)\epsilon_1}{2}(\|u_t\|^2 + \|u_t\|_{\Gamma_1}^2) + \frac{\epsilon\xi(0)(C_* + B_*)}{2\epsilon_1}\|\nabla u\|^2. \tag{25}$$

And the following holds:

$$\frac{\epsilon\mu_1}{2}\xi(t)\|\nabla u(t)\|^2 \le \frac{\epsilon\mu_1}{2}\xi(0)\|\nabla u(t)\|^2, \tag{26}$$

$$\lambda\xi(t)\Theta(t) \le \lambda\xi(0)\int_0^1 \int_{\tau_1}^{\tau_2} \tau|\mu_2(\tau)|\|\nabla z(x,\rho,\tau,t)\|^2 d\tau d\rho. \tag{27}$$

Noting (5), (19) and (A3), we have

$$\begin{aligned} &- \eta\xi(t)\int_0^t L_\alpha(s)ds\|\nabla u\|^2 \\ &\le \eta\xi(0)\frac{1}{\alpha}\int_0^\infty l(s)e^{\alpha s}ds\|\nabla u\|^2 \\ &\le \frac{\eta\xi(0)\bar{l_\alpha}}{\alpha}\|\nabla u\|^2. \end{aligned} \tag{28}$$

By applying Young's inequality and (18), we obtain

$$\begin{aligned}
&(\nabla u(t), L_\alpha * \nabla u)\\
&= \int_0^t L_\alpha(t-s)(\nabla u(s) - \nabla u(t), \nabla u(t))ds + \int_0^t L_\alpha(t-s)\|\nabla u(t)\|^2 ds\\
&\le \frac{1}{4\epsilon_2}\int_0^t L_\alpha(t-s)\|\nabla u(s) - \nabla u(t)\|^2 ds + \epsilon_2 \int_0^t L_\alpha(t-s)\|\nabla u(t)\|^2 ds\\
&\quad + (\int_0^t L_\alpha(s)ds)\|\nabla u(t)\|^2\\
&= \frac{1}{4\epsilon_2}G(t) + (1+\epsilon_2)(\int_0^t L_\alpha(s)ds)\|\nabla u(t)\|^2,
\end{aligned} \tag{29}$$

Thus, recalling (A3) again and $\int_0^t L_\alpha(s)ds \le \frac{\bar{l_\alpha}}{\alpha}$, we have

$$\begin{aligned}
&2\eta\xi(t)(\nabla u(t), L_\alpha * \nabla u)\\
\le\ & \frac{\eta}{2\epsilon_2}\xi(t)G(t) + \frac{2\eta(1+\epsilon_2)\xi(0)\bar{l_\alpha}}{\alpha}\|\nabla u(t)\|^2.
\end{aligned} \tag{30}$$

Substituting (25)–(28) and (30) into (22) and combining (5) and (8), we obtain

$$\begin{aligned}
M(t) \ge& (\frac{1}{2} - \frac{\epsilon\xi(0)\epsilon_1}{2})(\|u_t\|^2 + \|u_t\|_{\Gamma_1}^2)\\
&+ \Big[\frac{1-\bar{g}}{2} - \frac{\epsilon\xi(0)(C_* + B_*)}{2\epsilon_1} - \frac{\epsilon\mu_1\xi(0)}{2} - \frac{\eta\xi(0)\bar{l_\alpha}}{\alpha}\\
&- \frac{2\eta(1+\epsilon_2)\xi(0)\bar{l_\alpha}}{\alpha}\Big]\|\nabla u\|^2\\
&+ (\frac{1}{2} - \lambda\xi(0))\int_0^1\int_{\tau_1}^{\tau_2}\tau|\mu_2(\tau)|\|\nabla z(x,\rho,\tau,s)\|^2 d\tau d\rho\\
&+ \eta(1 - \frac{1}{2\epsilon_2})\xi(t)G(t) + \frac{1}{2}(g\Box\nabla u)(t)\\
&- \frac{k}{\gamma}(\int_\Omega |u|^\gamma \ln|u| dx) + \frac{k}{\gamma^2}\|u\|_\gamma^\gamma + \frac{1}{2}\|\nabla_\Gamma u\|_{\Gamma_1}^2.
\end{aligned} \tag{31}$$

Let $\epsilon_1 = \frac{1}{2\epsilon\xi(0)}$ and $\epsilon_2 = 1$, and select suitable parameters ϵ, η and λ satisfying

$$\begin{cases}
\epsilon < \min\{\dfrac{\sqrt{1-\bar{g}}}{2\xi(0)\sqrt{6(C_* + B_*)}}, \dfrac{1-\bar{g}}{2\mu_1\xi(0)}\},\\
\eta < \dfrac{\alpha(1-\bar{g})}{30\xi(0)\bar{l_\alpha}},\\
\lambda < \dfrac{1}{4\xi(0)}.
\end{cases} \tag{32}$$

Utilizing (32) in (31),

$$\begin{aligned}
M(t) \ge& \frac{1}{4}(\|u_t\|^2 + \|u_t\|_{\Gamma_1}^2) + \frac{1-\bar{g}}{24}\|\nabla u\|^2\\
&+ \frac{1}{4}\int_0^1\int_{\tau_1}^{\tau_2}\tau|\mu_2(\tau)|\|\nabla z(x,\rho,\tau,s)\|^2 d\tau d\rho\\
&+ \frac{1}{2}(g\Box\nabla u)(t)\\
&- \frac{k}{\gamma}(\int_\Omega |u|^\gamma \ln|u| dx) + \frac{k}{\gamma^2}\|u\|_\gamma^\gamma + \frac{1}{2}\|\nabla_\Gamma u\|_{\Gamma_1}^2.
\end{aligned} \tag{33}$$

Consequently, it holds for some $C_1 > 0$ that

$$M(t) \geq C_1 E(t).$$

On the other hand, similarly, using (8), (25)–(28), and (30) in (22), and taking $\epsilon_1 = 1$ and $\epsilon_2 = 1$,

$$\begin{aligned} M(t) \leq\ & (\frac{1+\epsilon\xi(0)}{2})(\|u_t\|^2 + \|u_t\|^2_{\Gamma_1}) \\ & + (\frac{1+\epsilon\xi(0)(C_* + B_* + \mu_1)}{2} + \frac{5\eta\xi(0)\bar{l}_\alpha}{\alpha})\|\nabla u\|^2 \\ & + (\frac{1}{2} + \lambda\xi(0))\int_0^1 \int_{\tau_1}^{\tau_2} \tau|\mu_2(\tau)|\|\nabla z(x,\rho,\tau,s)\|^2 d\tau d\rho \\ & + \frac{3\eta}{2}\xi(0)G(t) + \frac{1}{2}(g\Box\nabla u)(t) \\ & - \frac{k}{\gamma}(\int_\Omega |u|^\gamma \ln|u| dx) + \frac{k}{\gamma^2}\|u\|^\gamma_\gamma + \frac{1}{2}\|\nabla_\Gamma u\|^2_{\Gamma_1}. \end{aligned} \tag{34}$$

So it holds for some $C_2 > 0$ that

$$M(t) \leq C_2(E(t) + G(t)).$$

□

Then, setting

$$\begin{cases} \epsilon := \min\{\dfrac{\sqrt{1-\bar{g}}}{2\xi(0)\sqrt{6(C_* + B_*)}}, \dfrac{1-\bar{g}}{2\mu_1\xi(0)}, \dfrac{1-\bar{g}}{4\bar{g}}\}, \\ \bar{l}_\alpha \leq \min\{\dfrac{\alpha^2(1-\bar{g})\beta}{4(L+2\alpha)(2L+3\alpha)\xi(0)}\epsilon, \dfrac{\alpha(1-\bar{g})\beta}{30\xi^2(0)}\}, \end{cases} \tag{35}$$

Theorem 2 is obtained.

Proof. Take the derivative of (21) to establish

$$\begin{aligned} & \frac{d}{dt}\Big[H(t) + (\tilde{k}-1)k_1\xi(0)(\int_0^t [\|u_s(s)\|^2 + \|u_s(s)\|^2_{\Gamma_1}]ds) \\ & + (\tilde{k}-1)k_2\xi(0)(\int_0^t \|\nabla u(s)\|^2 ds) + (\tilde{k}-1)k_3\xi(0)(\int_0^t \|u(s)\|^\gamma_\gamma ds) \\ & + (\tilde{k}-1)k_4\xi(0)(\int_0^t \int_0^1 \int_{\tau_1}^{\tau_2} \tau|\mu_2(\tau)|\|\nabla z(x,\rho,\tau,s)\|^2 d\tau d\rho ds)\Big] \\ = & \frac{d}{dt}\Big[E(t) + \epsilon\xi(t)F(t) + \frac{\epsilon\mu_1}{2}\xi(t)\|\nabla u(t)\|^2\Big] + \frac{d}{dt}(\lambda\xi(t)\Theta(t)) \\ & + \frac{d}{dt}\Big[\eta\xi(t)G(t) - \eta\xi(t)(\int_0^t L_\alpha(s)ds)\|\nabla u\|^2 + 2\eta\xi(t)(\nabla u, L_\alpha * \nabla u)\Big] \\ & - k_1\xi(0)[\|u_t(t)\|^2 + \|u_t(t)\|^2_{\Gamma_1}] - k_2\xi(0)\|\nabla u(t)\|^2 - k_3\xi(0)\|u(t)\|^\gamma_\gamma \\ & - k_4\xi(0)\int_0^1 \int_{\tau_1}^{\tau_2} \tau|\mu_2(\tau)|\|\nabla z(x,\rho,\tau,t)\|^2 d\tau d\rho. \end{aligned} \tag{36}$$

Similarly, use (17) to obtain

$$\frac{d}{dt}[\xi(t)F(t)] = \xi'(t)F(t) + \xi(t)[\|u_t\|^2 + \|u_t\|^2_{\Gamma_1} + (u, u_{tt}) + (u, u_{tt})_{\Gamma_1}]. \tag{37}$$

Multiplying (7a) by u,

$$\begin{aligned}
&\xi(t)(u,u_{tt})\\
=&\underbrace{\xi(t)(u,\Delta u)}_{I_1}\underbrace{-\xi(t)(u,g*\Delta u)}_{I_2}+\underbrace{\xi(t)(u,\mu_1\Delta u_t)}_{I_3}\\
&+\underbrace{\xi(t)(u,\int_{\tau_1}^{\tau_2}|\mu_2(\tau)|\Delta z(x,1,\tau,t)d\tau)}_{I_4}+\underbrace{\xi(t)(u,k|u|^{\gamma-2}u\ln|u|)}_{I_5},
\end{aligned}\tag{38}$$

and now the terms I_i, $i=1,\ldots,5$, are dealt with. Utilizing integration by parts and (7c),

$$I_1=-\xi(t)\|\nabla u\|^2+\xi(t)(u,\partial_\nu u)_{\Gamma_1},\tag{39}$$

$$I_2=\xi(t)(\nabla u,g*\nabla u)-\xi(t)(u,g*\partial_\nu u)_{\Gamma_1},\tag{40}$$

$$\begin{aligned}
I_3=&\xi(t)(u,\mu_1\Delta u_t)\\
=&-\mu_1\xi(t)(\nabla u,\nabla u_t)+\mu_1\xi(t)(u,\partial_\nu u_t)_{\Gamma_1}\\
=&-\frac{\mu_1}{2}\frac{d}{dt}[\xi(t)\|\nabla u\|^2]+\frac{\mu_1}{2}\xi'(t)\|\nabla u\|^2\\
&+\mu_1\xi(t)(u,\partial_\nu u_t)_{\Gamma_1},
\end{aligned}\tag{41}$$

$$\begin{aligned}
I_4=&\ \xi(t)(u,\int_{\tau_1}^{\tau_2}|\mu_2(\tau)|\Delta z(x,1,\tau,t)d\tau)\\
=&-\xi(t)(\nabla u,\int_{\tau_1}^{\tau_2}|\mu_2(\tau)|\nabla z(x,1,\tau,t)d\tau)\\
&+\xi(t)(u,\int_{\tau_1}^{\tau_2}|\mu_2(\tau)|\partial_\nu z(x,1,\tau,t)d\tau)_{\Gamma_1},
\end{aligned}\tag{42}$$

and obviously

$$I_5=k\xi(t)\int_\Omega|u|^\gamma\ln|u|dx.\tag{43}$$

Substituting (38)–(43) into (37), using (7d), and integrating by parts,

$$\begin{aligned}
\frac{d}{dt}[\xi(t)F(t)]=&\ \xi'(t)F(t)+\xi(t)[\|u_t\|^2+\|u_t\|_{\Gamma_1}^2]\\
&-\xi(t)\|\nabla u\|^2-\xi(t)\|\nabla_\Gamma u\|_{\Gamma_1}^2\\
&+\xi(t)(\nabla u,g*\nabla u)\\
&-\frac{\mu_1}{2}\frac{d}{dt}[\xi(t)\|\nabla u\|^2]+\frac{\mu_1}{2}\xi'(t)\|\nabla u\|^2\\
&-\xi(t)(\nabla u,\int_{\tau_1}^{\tau_2}|\mu_2(\tau)|\nabla z(x,1,\tau,t)d\tau)\\
&+k\xi(t)\int_\Omega|u|^\gamma\ln|u|dx,
\end{aligned}$$

so by moving the term $-\frac{\mu_1}{2}\frac{d}{dt}[\xi(t)\|\nabla u\|^2]$ to the left in the above equation, we have

$$\begin{aligned}
&\frac{d}{dt}[\xi(t)F(t)+\frac{\mu_1}{2}\xi(t)\|\nabla u\|^2]\\
=&\ \xi'(t)F(t)+\xi(t)[\|u_t\|^2+\|u_t\|_{\Gamma_1}^2]\\
&-\xi(t)\|\nabla u\|^2-\xi(t)\|\nabla_\Gamma u\|_{\Gamma_1}^2\\
&+\xi(t)(\nabla u,g*\nabla u)+\frac{\mu_1}{2}\xi'(t)\|\nabla u\|^2\\
&-\xi(t)(\nabla u,\int_{\tau_1}^{\tau_2}|\mu_2(\tau)|\nabla z(x,1,\tau,t)d\tau)\\
&+k\xi(t)\int_\Omega|u|^\gamma\ln|u|dx.
\end{aligned}\tag{44}$$

Now, $\frac{d}{dt}[\xi(t)G(t)]$ is treated. Using (19),

$$\begin{aligned}
&\frac{d}{dt}[\xi(t)G(t)]\\
&=\xi'(t)G(t)+\xi(t)\frac{d}{dt}G(t)\\
&=\xi'(t)G(t)+\xi(t)\Big[\int_\Omega\int_0^t\frac{d}{dt}[L_\alpha(t-s)]|\nabla u(t)-\nabla u(s)|^2dsdx\\
&\quad+\int_\Omega\int_0^t L_\alpha(t-s)\frac{d}{dt}|\nabla u(t)-\nabla u(s)|^2dsdx\Big]\\
&=\xi'(t)G(t)-\alpha\xi(t)G(t)-\xi(t)(l\Box\nabla u)(t)\\
&\quad+2(\int_0^t L_\alpha(s)ds)\xi(t)\int_\Omega\nabla u_t\cdot\nabla udx-2\xi(t)(\nabla u_t,L_\alpha*\nabla u).
\end{aligned}\tag{45}$$

Next, the last two terms of (45) are dealt with. By a direct calculation and recalling that $\frac{d}{dt}[L_\alpha(t)]=-\alpha L_\alpha(t)-l(t)$,

$$\begin{aligned}
&2(\int_0^t L_\alpha(s)ds)\xi(t)\int_\Omega\nabla u_t\cdot\nabla udx\\
&=\frac{d}{dt}\Big[(\int_0^t L_\alpha(s)ds)\xi(t)\|\nabla u\|^2\Big]\\
&\quad-\Big[L_\alpha(t)\xi(t)+(\int_0^t L_\alpha(s)ds)\xi'(t)\Big]\|\nabla u\|^2,
\end{aligned}\tag{46}$$

$$\begin{aligned}
&-2\xi(t)(\nabla u_t,L_\alpha*\nabla u)\\
&=-2\xi(t)\frac{d}{dt}(\nabla u,L_\alpha*\nabla u)+2\xi(t)(\nabla u,\frac{d}{dt}[L_\alpha*\nabla u])\\
&=-2\frac{d}{dt}[\xi(t)(\nabla u,L_\alpha*\nabla u)]+2\xi'(t)(\nabla u,L_\alpha*\nabla u)\\
&\quad+2L_\alpha(0)\xi(t)\|\nabla u\|^2-2\alpha\xi(t)(\nabla u,L_\alpha*\nabla u)-2\xi(t)(\nabla u,l*\nabla u).
\end{aligned}\tag{47}$$

Combine (46) and (47) with (45) to get

$$\begin{aligned}
&\frac{d}{dt}\Big[\xi(t)G(t)-(\int_0^t L_\alpha(s)ds)\xi(t)\|\nabla u\|^2+2\xi(t)(\nabla u,L_\alpha*\nabla u)\Big]\\
&=[\xi'(t)-\alpha\xi(t)]G(t)-\xi(t)(l\Box\nabla u)(t)\\
&\quad-\Big[L_\alpha(t)\xi(t)+(\int_0^t L_\alpha(s)ds)\xi'(t)-2L_\alpha(0)\xi(t)\Big]\|\nabla u\|^2\\
&\quad+2[\xi'(t)-\alpha\xi(t)](\nabla u,L_\alpha*\nabla u)-2\xi(t)(\nabla u,l*\nabla u).
\end{aligned}\tag{48}$$

Next, computing the derivative of $\Theta(t)$, using (6), noting that for any $0<\rho<1$, $e^{-\tau}\le e^{-\rho\tau}\le 1$, and setting $\eta_1=e^{-\tau_2}$,

$$\begin{aligned}
&\frac{d}{dt}\Big[\int_0^1\int_{\tau_1}^{\tau_2}\tau e^{-\rho\tau}|\mu_2(\tau)|\|\nabla z(x,\rho,\tau,t)\|^2d\tau d\rho\Big]\\
&=-\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}e^{-\rho\tau}|\mu_2(\tau)|\frac{d}{d\rho}|\nabla z|^2d\tau d\rho dx\\
&=-\int_0^1\int_{\tau_1}^{\tau_2}\tau e^{-\rho\tau}|\mu_2(\tau)|\|\nabla z\|^2d\tau d\rho\\
&\quad+(\int_{\tau_1}^{\tau_2}|\mu_2(\tau)|d\tau)\|\nabla u_t\|^2-\int_{\tau_1}^{\tau_2}e^{-\tau}|\mu_2(\tau)|\|\nabla z(x,1,\tau,t)\|^2d\tau\\
&\le-\eta_1\int_0^1\int_{\tau_1}^{\tau_2}\tau|\mu_2(\tau)|\|\nabla z\|^2d\tau d\rho\\
&\quad-\eta_1\int_{\tau_1}^{\tau_2}|\mu_2(\tau)|\|\nabla z(x,1,\tau,t)\|^2d\tau+(\int_{\tau_1}^{\tau_2}|\mu_2(\tau)|d\tau)\|\nabla u_t\|^2,
\end{aligned}\tag{49}$$

Using the righthand side of (4),

$$\xi(t)\Theta'(t) \leq -\eta_1\xi(t)\int_0^1\int_{\tau_1}^{\tau_2}\tau|\mu_2(\tau)|\|\nabla z\|^2 d\tau d\rho - \eta_1\xi(t)\int_{\tau_1}^{\tau_2}|\mu_2(\tau)|\|\nabla z(x,1,\tau,t)\|^2 d\tau + \mu_1\xi(t)\|\nabla u_t\|^2. \tag{50}$$

Combining (9), (44), (50), and (48) with (36), we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big[H(t) + (\tilde{k}-1)k_1\xi(0)(\int_0^t[\|u_s(s)\|^2 + \|u_s(s)\|^2_{\Gamma_1}]ds)\\
&+ (\tilde{k}-1)k_2\xi(0)(\int_0^t\|\nabla u(s)\|^2 ds) + (\tilde{k}-1)k_3\xi(0)(\int_0^t\|u(s)\|^\gamma_\gamma ds)\\
&+ (\tilde{k}-1)k_4\xi(0)(\int_0^t\int_0^1\int_{\tau_1}^{\tau_2}\tau|\mu_2(\tau)|\|\nabla z(x,\rho,\tau,s)\|^2 d\tau d\rho ds)\Big]\\
=\ &\frac{d}{dt}\Big[E(t) + \epsilon\xi(t)F(t) + \frac{\epsilon\mu_1}{2}\xi(t)\|\nabla u(t)\|^2\Big] + \frac{d}{dt}(\lambda\xi(t)\Theta(t))\\
&+ \frac{d}{dt}\Big[\eta\xi(t)G(t) - \eta\xi(t)(\int_0^t L_\alpha(s)ds)\|\nabla u\|^2 + 2\eta\xi(t)(\nabla u, L_\alpha * \nabla u)\Big]\\
&- k_1\xi(0)[\|u_t(t)\|^2 + \|u_t(t)\|^2_{\Gamma_1}] - k_2\xi(0)\|\nabla u(t)\|^2 - k_3\xi(0)\|u(t)\|^\gamma_\gamma\\
&- k_4\xi(0)\int_0^1\int_{\tau_1}^{\tau_2}\tau|\mu_2(\tau)|\|\nabla z(x,\rho,\tau,t)\|^2 d\tau d\rho\\
\leq\ &\Big[-(\mu_1 - \int_{\tau_1}^{\tau_2}|\mu_2(\tau)|d\tau)\|\nabla u_t\|^2 + \frac{1}{2}(g'\Box\nabla u)(t) - \frac{1}{2}g(t)\|\nabla u\|^2\Big]\\
&+ \Big[\epsilon\xi'(t)F(t) + \epsilon\xi(t)[\|u_t\|^2 + \|u_t\|^2_{\Gamma_1}]\\
&- \epsilon\xi(t)\|\nabla u\|^2 - \epsilon\xi(t)\|\nabla_\Gamma u\|^2_{\Gamma_1}\\
&+ \epsilon\xi(t)(\nabla u, g * \nabla u) + \epsilon\frac{\mu_1}{2}\xi'(t)\|\nabla u\|^2\\
&- \epsilon\xi(t)(\nabla u, \int_{\tau_1}^{\tau_2}|\mu_2(\tau)|\nabla z(x,1,\tau,t)d\tau)\\
&+ \epsilon k\xi(t)\int_\Omega |u|^\gamma \ln|u| dx\Big]\\
&+ \Big[\eta[\xi'(t) - \alpha\xi(t)]G(t) - \eta\xi(t)(l\Box\nabla u)(t)\\
&- \eta\big[L_\alpha(t)\xi(t) + (\int_0^t L_\alpha(s)ds)\xi'(t) - 2L_\alpha(0)\xi(t)\big]\|\nabla u\|^2\\
&+ 2\eta[\xi'(t) - \alpha\xi(t)](\nabla u, L_\alpha * \nabla u) - 2\eta\xi(t)(\nabla u, l * \nabla u)\Big]\\
&+ \Big[\lambda\xi'(t)\Theta(t) - \lambda\eta_1\xi(t)\int_0^1\int_{\tau_1}^{\tau_2}\tau|\mu_2(\tau)|\|\nabla z\|^2 d\tau d\rho\\
&- \lambda\eta_1\xi(t)\int_{\tau_1}^{\tau_2}|\mu_2(\tau)|\|\nabla z(x,1,\tau,t)\|^2 d\tau + \lambda\mu_1\xi(t)\|\nabla u_t\|^2\Big]\\
&- k_1\xi(0)[\|u_t(t)\|^2 + \|u_t(t)\|^2_{\Gamma_1}] - k_2\xi(0)\|\nabla u(t)\|^2 - k_3\xi(0)\|u(t)\|^\gamma_\gamma\\
&- k_4\xi(0)\int_0^1\int_{\tau_1}^{\tau_2}\tau|\mu_2(\tau)|\|\nabla z(x,\rho,\tau,t)\|^2 d\tau d\rho.
\end{aligned} \tag{51}$$

Making use of (3) and similar to (24),

$$\epsilon\xi'(t)F(t) \leq \frac{\epsilon L\delta_1}{2}\xi(t)(\|u_t\|^2 + \|u_t\|^2_{\Gamma_1}) + \frac{\epsilon L(C_* + B_*)}{2\delta_1}\xi(t)\|\nabla u\|^2. \tag{52}$$

Making use of Young's inequality and noting that $\int_0^t g(s)ds \leq \bar{g}$,

$$\begin{aligned}
&\epsilon\xi(t)(\nabla u, g * \nabla u)\\
&\leq \frac{\epsilon\delta_2}{2}\xi(t)(g\Box\nabla u)(t) + \epsilon\bar{g}(\frac{1}{2\delta_2} + 1)\xi(t)\|\nabla u(t)\|^2.
\end{aligned} \tag{53}$$

Using (3), $\int_0^t L_\alpha(s)ds \le \frac{\bar{l}_\alpha}{\alpha}$, and similar to (29), we have

$$\begin{aligned} &2\eta[\xi'(t)-\alpha\xi(t)](\nabla u, L_\alpha * \nabla u) \\ &\le \frac{\eta(L+\alpha)}{2\delta_3}\xi(t)G(t) + \frac{2\eta(L+\alpha)(1+\delta_3)\bar{l}_\alpha}{\alpha}\xi(t)\|\nabla u(t)\|^2. \end{aligned} \tag{54}$$

Similar to (29) and utilizing Young's inequality and (4), we have

$$\begin{aligned} &-2\eta\xi(t)(\nabla u(t), l * \nabla u) \\ &\le \frac{\eta}{2\delta_4}\xi(t)(l\square\nabla u)(t) + 2\eta(1+\delta_4)\bar{l}\xi(t)\|\nabla u(t)\|^2, \end{aligned} \tag{55}$$

and

$$\begin{aligned} &-\epsilon\xi(t)(\nabla u(t), \int_{\tau_1}^{\tau_2}|\mu_2(\tau)|\nabla z(x,1,\tau,t)d\tau) \\ &\le \frac{\epsilon\delta_5}{2}\xi(t)\int_{\tau_1}^{\tau_2}|\mu_2(\tau)|\|\nabla z(x,1,\tau,t)\|^2 d\tau + \epsilon\frac{\mu_1}{2\delta_5}\xi(t)\|\nabla u(t)\|^2. \end{aligned} \tag{56}$$

Using (3) and noting that $e^{-\rho\tau} < 1$,

$$\lambda\xi'(t)\Theta(t) \le \lambda L\xi(t)\int_0^1\int_{\tau_1}^{\tau_2}\tau|\mu_2(\tau)|\|\nabla z(x,\rho,\tau,t)\|^2 d\tau d\rho. \tag{57}$$

Combing (52)–(57) with (51), and noting (5), $g'(t) = \xi(t)l(t) - \xi(t)g(t)$, and (A3), it is deduced that

$$\begin{aligned} &\frac{d}{dt}\Big[H(t) + (\tilde{k}-1)k_1\xi(0)(\int_0^t[\|u_s(s)\|^2 + \|u_s(s)\|^2_{\Gamma_1}]ds) \\ &+ (\tilde{k}-1)k_2\xi(0)(\int_0^t\|\nabla u(s)\|^2 ds) + (\tilde{k}-1)k_3\xi(0)(\int_0^t\|u(s)\|_\gamma^\gamma ds) \\ &+ (\tilde{k}-1)k_4\xi(0)(\int_0^t\int_0^1\int_{\tau_1}^{\tau_2}\tau|\mu_2(\tau)|\|\nabla z(x,\rho,\tau,s)\|^2 d\tau d\rho ds)\Big] \\ \le &-(k_1 - \frac{\epsilon L\delta_1}{2} - \epsilon)\xi(t)[\|u_t(t)\|^2 + \|u_t(t)\|^2_{\Gamma_1}] \\ &-(\frac{1}{2} - \frac{\epsilon\delta_2}{2})\xi(t)(g\square\nabla u)(t) - [\eta(1-\frac{1}{2\delta_4}) - \frac{\xi(0)}{2\beta}]\xi(t)(l\square\nabla u)(t) \\ &-\Big[k_2 + \epsilon[(1-\bar{g}) - \frac{\mu_1 L}{2} - \frac{L(C_*+B_*)}{2\delta_1} - \frac{\bar{g}}{2\delta_2} - \frac{\mu_1}{2\delta_5}] \\ &-\frac{\eta\bar{l}_\alpha L}{\alpha} - 2\eta\bar{l}_\alpha - \frac{2\eta(L+\alpha)(1+\delta_3)\bar{l}_\alpha}{\alpha} - 2\eta(1+\delta_4)\bar{l}\Big]\xi(t)\|\nabla u\|^2 \\ &-\epsilon\xi(t)\|\nabla_\Gamma u\|^2_{\Gamma_1} - [\mu_1 - \int_{\tau_1}^{\tau_2}|\mu_2(\tau)|d\tau - \lambda\mu_1\xi(0)]\|\nabla u_t\|^2 \\ &+\epsilon k\xi(t)\int_\Omega |u|^\gamma \ln|u|dx - k_3\xi(t)\|u\|_\gamma^\gamma \\ &-\eta(\alpha - \frac{L+\alpha}{2\delta_3})\xi(t)G(t) \\ &-(k_4 + \lambda\eta_1 - \lambda L)\xi(t)\int_0^1\int_{\tau_1}^{\tau_2}\tau|\mu_2(\tau)|\|\nabla z\|^2 d\tau d\rho \\ &-(\lambda\eta_1 - \frac{\epsilon\delta_5}{2})\xi(t)\int_{\tau_1}^{\tau_2}|\mu_2(\tau)|\|\nabla z(x,1,\tau,t)\|^2 d\tau. \end{aligned} \tag{58}$$

Taking $\delta_1 = \frac{2L(C_*+B_*)}{1-\bar{g}}$, $\delta_2 = \frac{2\bar{g}}{1-\bar{g}}$, $\delta_3 = \frac{L+\alpha}{\alpha}$, $\delta_4 = 1$, and $\delta_5 = \frac{2\lambda\eta_1}{\epsilon}$, choose

$$\begin{cases} k_1 = \epsilon(L\delta_1 + 2), \\ k_2 = 4\bar{l}\eta + \dfrac{\epsilon\mu_1 L}{2} + \dfrac{\epsilon^2\mu_1}{4\lambda\eta_1}, \\ \eta = \dfrac{\xi(0)}{\beta}, \epsilon < \dfrac{1-\bar{g}}{4\bar{g}}, \\ k_4 = \lambda L, \lambda = \dfrac{\mu_1 - \int_{\tau_1}^{\tau_2}|\mu_2(\tau)|d\tau}{\mu_1\xi(0)}. \end{cases} \tag{59}$$

Using (4), λ in (59) satisfies (32).

Therefore, if $\bar{l_\alpha} \leq \min\{\frac{\alpha^2(1-\bar{g})\beta}{4(L+2\alpha)(2L+3\alpha)\xi(0)}\epsilon, \frac{\alpha(1-\bar{g})\beta}{30\xi^2(0)}\}$, from (58),

$$\begin{aligned}
&\frac{d}{dt}\Big[H(t) + (\tilde{k}-1)k_1\xi(0)(\int_0^t [\|u_s(s)\|^2 + \|u_s(s)\|^2_{\Gamma_1}]ds)\\
&+ (\tilde{k}-1)k_2\xi(0)(\int_0^t \|\nabla u(s)\|^2 ds) + (\tilde{k}-1)k_3\xi(0)(\int_0^t \|u(s)\|^\gamma_\gamma ds)\\
&+ (\tilde{k}-1)k_4\xi(0)(\int_0^t \int_0^1 \int_{\tau_1}^{\tau_2} \tau|\mu_2(\tau)|\|\nabla z(x,\rho,\tau,s)\|^2 d\tau d\rho ds)\Big]\\
\leq& -\frac{k_1}{2}\xi(t)[\|u_t(t)\|^2 + \|u_t(t)\|^2_{\Gamma_1}] - \frac{1}{4}\xi(t)(g\Box\nabla u)(t)\\
&- \epsilon\frac{1-\bar{g}}{4}\xi(t)\|\nabla u\|^2 - \epsilon\xi(t)\|\nabla_\Gamma u\|^2_{\Gamma_1}\\
&+ \epsilon k\xi(t)\int_\Omega |u|^\gamma \ln|u| dx - k_3\xi(t)\|u\|^\gamma_\gamma\\
&- \lambda\eta_1\xi(t)\int_0^1 \int_{\tau_1}^{\tau_2} \tau|\mu_2(\tau)|\|\nabla z\|^2 d\tau d\rho\\
&- \frac{\alpha}{2}\xi(t)G(t).
\end{aligned} \tag{60}$$

Making use of (8), there exists $C_3 > 0$ fulfilling

$$\begin{aligned}
&\frac{d}{dt}\Big[H(t) + (\tilde{k}-1)k_1\xi(0)(\int_0^t [\|u_s(s)\|^2 + \|u_s(s)\|^2_{\Gamma_1}]ds)\\
&+ (\tilde{k}-1)k_2\xi(0)(\int_0^t \|\nabla u(s)\|^2 ds) + (\tilde{k}-1)k_3\xi(0)(\int_0^t \|u(s)\|^\gamma_\gamma ds)\\
&+ (\tilde{k}-1)k_4\xi(0)(\int_0^t \int_0^1 \int_{\tau_1}^{\tau_2} \tau|\mu_2(\tau)|\|\nabla z(x,\rho,\tau,s)\|^2 d\tau d\rho ds)\Big]\\
\leq& -C_3\xi(t)[E(t) + G(t)].
\end{aligned} \tag{61}$$

Utilizing (23) in (61), we observe

$$\begin{aligned}
&\frac{d}{dt}\Big[H(t) + (\tilde{k}-1)k_1\xi(0)(\int_0^t [\|u_s(s)\|^2 + \|u_s(s)\|^2_{\Gamma_1}]ds)\\
&+ (\tilde{k}-1)k_2\xi(0)(\int_0^t \|\nabla u(s)\|^2 ds) + (\tilde{k}-1)k_3\xi(0)(\int_0^t \|u(s)\|^\gamma_\gamma ds)\\
&+ (\tilde{k}-1)k_4\xi(0)(\int_0^t \int_0^1 \int_{\tau_1}^{\tau_2} \tau|\mu_2(\tau)|\|\nabla z(x,\rho,\tau,s)\|^2 d\tau d\rho ds)\Big]\\
\leq& -\frac{C_3}{C_2}\xi(t)\Big[H(t) + (\tilde{k}-1)k_1\xi(0)(\int_0^t [\|u_s(s)\|^2 + \|u_s(s)\|^2_{\Gamma_1}]ds)\\
&+ (\tilde{k}-1)k_2\xi(0)(\int_0^t \|\nabla u(s)\|^2 ds) + (\tilde{k}-1)k_3\xi(0)(\int_0^t \|u(s)\|^\gamma_\gamma ds)\\
&+ (\tilde{k}-1)k_4\xi(0)(\int_0^t \int_0^1 \int_{\tau_1}^{\tau_2} \tau|\mu_2(\tau)|\|\nabla z(x,\rho,\tau,s)\|^2 d\tau d\rho ds)\Big],
\end{aligned} \tag{62}$$

which indicates

$$\begin{aligned}
&H(t) + (\tilde{k}-1)k_1\xi(0)(\int_0^t [\|u_s(s)\|^2 + \|u_s(s)\|^2_{\Gamma_1}]ds)\\
&+ (\tilde{k}-1)k_2\xi(0)(\int_0^t \|\nabla u(s)\|^2 ds) + (\tilde{k}-1)k_3\xi(0)(\int_0^t \|u(s)\|^\gamma_\gamma ds)\\
&+ (\tilde{k}-1)k_4\xi(0)(\int_0^t \int_0^1 \int_{\tau_1}^{\tau_2} \tau|\mu_2(\tau)|\|\nabla z(x,\rho,\tau,s)\|^2 d\tau d\rho ds)\\
\leq& H(0)e^{-\frac{C_3}{C_2}\int_0^t \xi(s)ds}.
\end{aligned} \tag{63}$$

Utilizing the condition $E(0) > 0$ in Theorem 2 and using (22) and (23), it is obvious that $H(0) > 0$. Noting $\tilde{k} > 1$ and using (23) again, the conclusion of Theorem 2 is attained:

$$E(t) \leq \frac{H(0)}{C_1} e^{-\frac{C_3}{C_2}\int_0^t \xi(s)ds}.$$

□

5. Discussion

This paper shows that when the kernel function does not decay necessarily, the logarithmic wave system with distributed delay and dynamic Wentzell boundary conditions can still achieve an exponential decay result. This is relevant in applications such as the modeling of smart materials, vibration control in structures with viscoelastic coatings, and acoustic wave propagation in tubes with elastic walls. Numerical simulations will be considered in the future work. Next, the author will consider the case of variable coefficients and the situation where the domain is time-dependent, which means that the boundary of the domain will move. Finally, inspired by [28–30], if there is an opportunity in the future, it would be very valuable to collaborate with others to explore how the current model could be extended to fractional-order formulations.

Funding: This research was funded by the National Natural Science Foundation of China, grant number 12501616, and by the Natural Science Foundation of Henan Province, grant number 252300423495.

Data Availability Statement: Data sharing is not applicable to this article, as no datasets were generated or analyzed during the current study.

Conflicts of Interest: The author declares no conflicts of interest.

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