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A Unified Matrix-Based Approach to the Explicit Iterations and Global Dynamics of Linear Rational Functions

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Abstract

The iterations of general linear rational functions are investigated. The explicit formulas of iterations are derived for all possible parameters. This allows for an explanation of the global dynamics of linear rational functions. A unification method is presented regarding recent results in this direction. In addition, possible further directions and procedures are outlined.

Keywords: iterations; explicit solutions; linear rational functions

MSC: 26A18; 26C15; 37E99

1. Introduction

A class of nonlinear rational difference equations with delayed terms is studied in [1], which often arise in various mathematical models. Qualitative behaviour is analysed by showing how their iterations can be explicitly represented through second-order linear recurrence relations, leading to a connection with generalized balancing sequences. A method derived from [1] is ad hoc. The purpose of this note is to show a general direct approach by considering not iterations of points but rather iterations of functions; see Formulae (24) and (25) for more details. This leads to a complete analysis of iterations for all parametric cases: hyperbolic, elliptic, and parabolic.

A general theory of difference equations is given simply in Ref. [2]. The global behavior and stability of rational systems are studied in [3–5]. An introduction to applications of difference equations is given in [6]. Difference equations on time scales are studied in [7]. The periodicity, oscillatory and boundedness of nonlinear systems of difference equations are given in [8–10]. Higher-order and coupled systems of difference equations, including results with variable or periodic coefficients, are investigated in [11,12]. Generalized sequences like the Fibonacci [13–15], Padovan [16–18], Lucas [19], and Pell sequences [20], are used to construct explicit solutions for various nonlinear and higher-order systems, while a new generalized balancing sequence is applied to rational difference equations in our paper [1]. These papers strongly motivated us to write this article.

The paper is organized as follows: Section 2 introduces the studied problem, which is then solved in Sections 3 and 4. Section 5 discusses related issues and possible extensions. Section 6 summarizes our main results.

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Academic Editor: Nicolae Lupa

Received: 26 April 2026
Revised: 2 June 2026
Accepted: 5 June 2026
Published: 7 June 2026

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2. Preliminaries

If we consider the real linear rational functions

$$f(x) = \frac{ax+b}{cx+d}, \quad g(x) = \frac{px+q}{rx+s}, \tag{1}$$

then

$$g(f(x)) = \frac{(ap+cq)x+bp+dq}{(ar+cs)x+br+ds}. \tag{2}$$

So, the set G_r of linear rational functions is a group under an operation of composition. By setting

$$f = (a,b,c,d), \quad g = (p,q,r,s), \tag{3}$$

we obtain

$$\begin{aligned} g \circ f &= (p,q,r,s) \circ (a,b,c,d) \\ &= (ap+cq, bp+dq, ar+cs, br+ds). \end{aligned} \tag{4}$$

Note that G_r is the set of Möbious transformations [3] and is nonabelian.

Now, we consider the iterations

$$f, f^2, \ldots$$

Since

$$f'(x) = \frac{ad-bc}{(cx+d)^2}$$

and

$$f''(x) = \frac{2c(bc-ad)}{(cx+d)^3},$$

we suppose that

$$c(ad-bc) \neq 0 \tag{5}$$

when concentrating on a nonlinear f.

Remark 1. *If (5) fails, then we arrive at the linear case $f(x) = ax+b$, whose dynamics are well-understood; see, for instance [2] (p. 4).*

We set

$$x_n = f^n(x_0) = \frac{a_n x_0 + b_n}{c_n x_0 + d_n}. \tag{6}$$

Then

$$\begin{aligned} a_{n+1} &= aa_n + bc_n, \\ b_{n+1} &= ab_n + bd_n, \\ c_{n+1} &= ca_n + dc_n, \\ d_{n+1} &= cb_n + dd_n. \end{aligned} \tag{7}$$

(7) is block-diagonal, so it is split into two independent similar planar linear difference equations, as follows:

$$\begin{aligned} a_{n+1} &= aa_n + bc_n, \\ c_{n+1} &= ca_n + dc_n, \end{aligned} \tag{8}$$

and

$$\begin{aligned} b_{n+1} &= ab_n + bd_n, \\ d_{n+1} &= cb_n + dd_n, \end{aligned} \tag{9}$$

i.e.,

$$A_{n+1} = AA_n$$

for

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad A_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}.$$

Remark 2. *Since $f(x)$ is invariant under a scaling*

$$(a, b, c, d) \to r(a, b, c, d), \quad r \in \mathbb{R},$$

we can consider a projective space $P\mathbb{R}^4$ and the projective linear group $PGL(2, \mathbb{R})$, but we do not adopt this method.

Next, the condition (5) means $\det A \neq 0$. Clearly,

$$A_n = A^n.$$

Then, by (5), we have

$$a_n d_n - b_n c_n = \det A_n = (\det A)^n = (ad - bc)^n \neq 0,$$

which gives that either $c_n \neq 0$ or $d_n \neq 0$. This implies that $f^n(x)$ is well defined for any $n \in \mathbb{N}$ and that there is, at most, many countable $x_0 \in \mathbb{R}$, such that $c_n x_0 + d_n = 0$ for some $n \in \mathbb{N}$.

Thus, generically, i.e., for almost all $x_0 \in \mathbb{R}$, the iteration $\{f^n(x_0)\}_{n=1}^{\infty}$ exists. We call this iteration generic. Note that a nongeneric iteration means that blows up in a finite step. For instance, if $a, b, c, d \in \mathbb{Q}$ are all rational and $x_0 \in \mathbb{R} \setminus \mathbb{Q}$ is irrational, then its iteration is generic. Indeed, then $a_n, b_n, c_n, d_n \in \mathbb{Q}$ for all $n \in \mathbb{N}$, so certainly $c_n x_0 + d_n \neq 0$ for any $n \in \mathbb{N}$. Clearly, if $a > 0$, $b > 0$, $c > 0$, $d > 0$ are all positive and $x_0 > 0$ is also positive, then its iteration is generic.

In the rest of the paper, we apply a standard theory of [2] to solve (8) and (9) for all particular cases. This allows us to find the explicit form of solutions, leading to their asymptotic properties.

3. A Generic Case: Hyperbolic and Elliptic Dynamics

Assuming that

$$(a - d)^2 + 4bc \neq 0, \tag{10}$$

the eigenvalues of A are

$$\lambda_1 = \frac{1}{2}\left(a + d - \sqrt{(a-d)^2 + 4bc}\right),$$
$$\lambda_2 = \frac{1}{2}\left(a + d + \sqrt{(a-d)^2 + 4bc}\right),$$

with the corresponding eigenvectors

$$\xi_1 = \begin{pmatrix} a - d - \sqrt{(a-d)^2 + 4bc} \\ 2c \end{pmatrix},$$
$$\xi_2 = \begin{pmatrix} a - d + \sqrt{(a-d)^2 + 4bc} \\ 2c \end{pmatrix}.$$

Then,

$$\begin{pmatrix} a_n \\ c_n \end{pmatrix} = \alpha\lambda_1^n\xi_1 + \beta\lambda_2^n\xi_2,$$
$$\begin{pmatrix} b_n \\ d_n \end{pmatrix} = \gamma\lambda_1^n\xi_1 + \delta\lambda_2^n\xi_2,$$

for α, β, γ and δ determined by

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \alpha\xi_1 + \beta\xi_2,$$
$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \gamma\xi_1 + \delta\xi_2.$$

Then,

$$\alpha = -\beta = -\frac{1}{2\sqrt{(a-d)^2+4bc}},$$
$$\gamma = \frac{\sqrt{(a-d)^2+4bc}+a-d}{4c\sqrt{(a-d)^2+4bc}},$$
$$\delta = \frac{\sqrt{(a-d)^2+4bc}-a+d}{4c\sqrt{(a-d)^2+4bc}}.$$

Consequently, we get an explicit solution of (6)

$$x_n = \frac{(\alpha\lambda_1^n\xi_{11} + \beta\lambda_2^n\xi_{21})x_0 + \gamma\lambda_1^n\xi_{11} + \delta\lambda_2^n\xi_{21}}{2c\big((\alpha\lambda_1^n + \beta\lambda_2^n)x_0 + \gamma\lambda_1^n + \delta\lambda_2^n\big)}, \tag{11}$$

where

$$\xi_{11} = a - d - \sqrt{(a-d)^2+4bc},$$
$$\xi_{21} = a - d + \sqrt{(a-d)^2+4bc}.$$

3.1. The Case $|\lambda_1| \neq |\lambda_2|$: Hyperbolic Dynamics

Now, (10) means

$$(a-d)^2 + 4bc > 0, \tag{12}$$

so we have real $\lambda_{1,2}$ and $\xi_{1,2}$. Then, f has two fixed points

$$\bar{x}_1 = -\frac{\sqrt{(a-d)^2+4bc}-a+d}{2c} = \frac{\xi_{11}}{2c},$$
$$\bar{x}_2 = \frac{\sqrt{(a-d)^2+4bc}+a-d}{2c} = \frac{\xi_{21}}{2c},$$

with

$$f'(\bar{x}_1) = \frac{4ad-4bc}{\left(-\sqrt{(a-d)^2+4bc}+a+d\right)^2} = \frac{ad-bc}{\lambda_1^2} = \frac{\lambda_2}{\lambda_1},$$
$$f'(\bar{x}_2) = \frac{4ad-4bc}{\left(\sqrt{(a-d)^2+4bc}+a+d\right)^2} = \frac{ad-bc}{\lambda_2^2} = \frac{\lambda_1}{\lambda_2}.$$

3.1.1. The Case $|\lambda_1| > |\lambda_2|$: Hyperbolic Attractor

For this case, we have the following result:

Theorem 1. *Suppose* (5), (12) *and*

$$a + d < 0. \tag{13}$$

Then, generic iterations (6) *of* f *are explicitly given by* (11). *Moreover,* $\bar{x}_1$ *is a global exponential attractor of* f.

Proof. Since

$$\lambda_1^2 - \lambda_2^2 = -(a+d)\sqrt{(a-d)^2 + 4bc},$$

$|\lambda_1| > |\lambda_2|$ is equivalent to (13). Then, (11) implies

$$\lim_{n\to\infty} x_n = \frac{\alpha\xi_{11}x_0 + \gamma\xi_{11}}{2c(\alpha x_0 + \gamma)} = \frac{\xi_{11}}{2c} = \bar{x}_1.$$

Moreover, we have

$$|f'(\bar{x}_1)| = \left|\frac{\lambda_2}{\lambda_1}\right| < 1,$$
$$|f'(\bar{x}_2)| = \left|\frac{\lambda_1}{\lambda_2}\right| > 1.$$

The proof is completed. □

Remark 3. (11) *implies*

$$|\bar{x}_1 - x_n| = \left|\frac{\lambda_2^n(\beta x_0(\xi_{21} - \xi_{11}) - \delta(\xi_{11} - 1)\xi_{21})}{2c\left(\lambda_2^n(\alpha x_0 + \gamma) + \lambda_1^n(\beta x_0 + \delta\xi_{21})\right)}\right|$$
$$= \left|\frac{\lambda_2}{\lambda_1}\right|^n \left|\frac{\beta x_0(\xi_{21} - \xi_{11}) - \delta(\xi_{11} - 1)\xi_{21}}{2c\left(\left(\frac{\lambda_2}{\lambda_1}\right)^n(\alpha x_0 + \gamma) + \beta x_0 + \delta\xi_{21}\right)}\right|,$$

which gives an explicit exponential convergence rate for the attractor $\bar{x}_1$.

3.1.2. The Case $|\lambda_1| < |\lambda_2|$: Hyperbolic Attractor

Similarly, like in the previous subsection, we obtain the following result:

Theorem 2. *Suppose* (5), (12) *and*

$$a + d > 0. \tag{14}$$

Then, generic iterations (6) *of* f *are explicitly given by* (11). *Moreover,* $\bar{x}_2$ *is a global exponential attractor of* f.

Of course, we have the following dual version of Remark 3.

Remark 4. (11) *implies*

$$|\bar{x}_2 - x_n| = \left|\frac{\lambda_2^n(\xi_{11} - \xi_{21})(\alpha x_0 + \gamma) - \delta(\xi_{21} - 1)\xi_{21}\lambda_1^n}{2c\left(\lambda_2^n(\alpha x_0 + \gamma) + \lambda_1^n(\beta x_0 + \delta x_{21})\right)}\right|$$
$$= \left|\frac{\lambda_1}{\lambda_2}\right|^n \left|\frac{(\xi_{11} - \xi_{21})(\alpha x_0 + \gamma) - \delta(\xi_{21} - 1)\xi_{21}\left(\frac{\lambda_1}{\lambda_2}\right)^n}{2c\left(\alpha x_0 + \gamma + \left(\frac{\lambda_1}{\lambda_2}\right)^n(\beta x_0 + \delta\xi_{21})\right)}\right|,$$

which gives an explicit exponential convergence rate for the attractor $\bar{x}_2$.

Remark 5. *In this case,*

$$|f'(\bar{x}_1)| \neq 1, \quad |f'(\bar{x}_2)| \neq 1,$$

so fixed points are hyperbolic [21].

3.2. The Case $|\lambda_1| = |\lambda_2|$: Elliptic Dynamics

3.2.1. The Real Valued Case: Elliptic Involution

First, we consider a real valued case when (12) holds. Then, we hold that

$$\lambda_1 = -\lambda_2,$$

which gives

$$a + d = 0, \tag{15}$$

so

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix},$$

and then

$$\lambda_{1,2} = \mp\sqrt{a^2 + bc},$$
$$\xi_{1,2} = 2\begin{pmatrix} a \mp \sqrt{a^2 + bc} \\ c \end{pmatrix},$$
$$f'(\bar{x}_{1,2}) = -1.$$

Clearly, (11) becomes

$$x_n = \frac{(\alpha\xi_{11} + \beta(-1)^n\xi_{21})x_0 + \gamma\xi_{11} + \delta(-1)^n\xi_{21}}{2c((\alpha + \beta(-1)^n)x_0 + \gamma + \delta(-1)^n)}. \tag{16}$$

In summary, we arrive at the following result:

Theorem 3. *Suppose* (5), (12) *and* (15). *Then, generic iterations of f are explicitly given by* (16). *Moreover, all generic iterations of f are* 2*-periodic.*

Remark 6. *Condition*

$$f'(\bar{x}_{1,2}) = -1$$

means that period-double, or flip, bifurcations may occur generally at the fixed points $\bar{x}_{1,2}$ [21]. We see that if we slightly perturb (15), *we get either Theorem 1 or Theorem 2. Hence, Theorem 3 refers to a kind of non-generic, degenerate period-double bifurcation result at the fixed points of $\bar{x}_{1,2}$ by passing transversally through the surface* (15).

Remark 7. *Since*

$$A^2 = \begin{pmatrix} a^2 + bc & b(a + d) \\ c(a + d) & bc + d^2 \end{pmatrix},$$

assumption (15) *is equivalent to*

$$A^2 = (a^2 + bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

when $c \neq 0$. So, the matrix A is a square root of a multiple of the identity matrix.

3.2.2. The Complex Valued Case: Elliptic Quasiperiodicity

Now, we consider a complex valued case when the negation of condition (12) holds that

$$(a - d)^2 + 4bc < 0, \tag{17}$$

meaning we have complex

$$\lambda_1 = \overline{\lambda_2}$$

and

$$\xi_1 = \overline{\xi_2}.$$

Then, f has no fixed point. Setting

$$\lambda_1 = \rho e^{\omega \imath}, \quad \rho > 0, \tag{18}$$

(11) becomes

$$x_n = \frac{(\alpha e^{2\omega n\imath}\xi_{11} + \beta\xi_{21})x_0 + \gamma e^{2\omega n\imath}\xi_{11} + \delta\xi_{21}}{2c((\alpha e^{2\omega n\imath} + \beta)x_0 + \gamma e^{2\omega n\imath} + \delta)}. \tag{19}$$

Summarizing, we get the following result:

Theorem 4. *Suppose* (5) *and* (17). *Then, generic iterations of f are explicitly given by* (19). *Moreover, we have the following possibilities:*

(i) All generic iterations of f are periodic for rational ω.
(ii) All generic iterations of f are quasiperiodic for irrational ω.

Equation (18) means

$$\frac{a+d}{2} = \rho\cos\omega,$$
$$\frac{1}{4}\Big(-(a-d)^2 - 4bc\Big) = \rho^2\sin^2\omega,$$

with a solution

$$d = 2\rho\cos\omega - a,$$
$$b = -\frac{a^2 - 2a\rho\cos\omega + \rho^2}{c}. \tag{20}$$

Scaling (20) by

$$(a,b,c,d) \to \rho(a,b,c,d),$$

we get

$$d = 2\cos\omega - a,$$
$$b = -\frac{a^2 - 2a\cos\omega + 1}{c}.$$

Thus,

$$f(x) = \frac{1}{c}\left(\frac{\rho^2}{a - cx - 2\rho\cos\omega} + a\right).$$

For instance, taking $\omega = \frac{\pi}{4}$, we have the first 4 iterations of $f(x)$

$$f(x) = \frac{\frac{\rho^2}{a-cx-\sqrt{2}\rho} + a}{c},$$
$$f^2(x) = \frac{\frac{\rho\left(a-cx-\sqrt{2}\rho\right)}{\left(-\sqrt{2}\right)a+\sqrt{2}cx+\rho} + a}{c},$$
$$f^3(x) = \frac{\rho\left(\frac{\rho}{a-cx} - \sqrt{2}\right) + a}{c},$$
$$f^4(x) = x.$$

We see that

$$f^4(x) = x,$$

justifying Theorem 4.

4. A Nongeneric Case

Assuming

$$(a-d)^2+4bc=0, \tag{21}$$

the eigenvalues of A are

$$\lambda=\lambda_1=\lambda_2=\frac{a+d}{2}.$$

By (5) and (21), we get

$$b=-\frac{(a-d)^2}{4c},$$

so

$$A=\begin{pmatrix} a & -\frac{(a-d)^2}{4c} \\ c & d \end{pmatrix}$$

with the Jordan decomposition

$$\begin{aligned} A&=\begin{pmatrix} \frac{a-d}{2c} & \frac{1}{c} \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} \frac{a-d}{2c} & \frac{1}{c} \\ 1 & 0 \end{pmatrix}^{-1} \\ &=\begin{pmatrix} \frac{a-d}{2c} & \frac{1}{c} \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} 0 & 1 \\ c & \frac{d-a}{2} \end{pmatrix}. \end{aligned}$$

This gives

$$\begin{aligned} A^n&=\begin{pmatrix} \frac{a-d}{2c} & \frac{1}{c} \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^n\begin{pmatrix} 0 & 1 \\ c & \frac{d-a}{2} \end{pmatrix} \\ &=\lambda^{n-1}\begin{pmatrix} \frac{n(a-d)}{2}+\lambda & -\frac{n(a-d)^2}{4c} \\ cn & \lambda+\frac{(d-a)n}{2} \end{pmatrix}. \end{aligned}$$

Then,

$$\begin{aligned} x_n=f^n(x_0)&=\frac{a_nx_0+b_n}{c_nx_0+d_n} \\ &=\frac{n(a-d)(-a+2cx_0+d)+4c\lambda x_0}{2c(n(-a+2cx_0+d)+2\lambda)}. \end{aligned} \tag{22}$$

Note

$$\begin{aligned} f(x)&=\frac{4acx-(a-d)^2}{4c(cx+d)}, \\ f'(x)&=\frac{(a+d)^2}{4(cx+d)^2}, \\ f''(x)&=-\frac{c(a+d)^2}{2(cx+d)^3} \end{aligned}$$

with a unique fixed point

$$\bar{x}=\frac{a-d}{2c}$$

of $f(x)$. We compute that

$$f'(\bar{x})=1.$$

So, we suppose

$$a+d\neq 0 \tag{23}$$

which is equivalent to (5). We see from (22) that

$$\lim_{n\to\infty}x_n=\bar{x}.$$

Summarizing, we arrive at the following result:

Theorem 5. *Suppose* (21) *and* (23). *Then, generic iterations of* f *are explicitly given by* (22). *Moreover,* $\bar{x}$ *is a global attractor of* f.

Remark 8. (22) *gives*

$$|x_n - \bar{x}| = \left| \frac{\lambda(-a + 2cx_0 + d)}{c(n(-a + 2cx_0 + d) + 2\lambda)} \right| = \frac{1}{n} \left| \frac{\lambda(-a + 2cx_0 + d)}{c((-a + 2cx_0 + d) + 2\frac{\lambda}{n})} \right|,$$

which gives an explicit harmonic convergence rate for the attractor $\bar{x}$.

Remark 9. *Property*

$$f'(\bar{x}) = 1$$

means that a saddle-node bifurcation may occur at the fixed point $\bar{x}$ *in general [21]. We observe that if we slightly perturb* (21), *we get either Theorem 1 or Theorem 2 on one side of* (21) *or Theorem 4 on the other side of* (21). *Hence, there is a saddle-node bifurcation result at the fixed point* $\bar{x}$ *by passing transversally through the surface* (21).

5. Discussion

1. We consider $f, g \in G_r$ given by (1). Then, using identification (3) along with (2), we get (4), which gives

$$g \circ f = \begin{pmatrix} p & 0 & q & 0 \\ 0 & p & 0 & q \\ r & 0 & s & 0 \\ 0 & r & 0 & s \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}. \tag{24}$$

   Relationship (24) between a composition and its representation as a matrix product gives (6)

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \\ d_{n+1} \end{pmatrix} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix} \begin{pmatrix} a_n \\ b_n \\ c_n \\ d_n \end{pmatrix}. \tag{25}$$

   Of course, we derive again from (7). This is a key observation in this paper that iterations of linear rational functions $f \in G_r$ are reduced to iterations of linear one (25). The rest of the paper is a standard analysis of (25).
2. The surfaces (15) and (21) are bifurcation surfaces where dynamics of $f(x)$ given by (1) are changing; see Remarks 6 and 9. Their intersection is a constant map

$$f(x) = \frac{a}{c},$$

   where several bifurcations occur. We observe that assuming (5), we can take

$$u = a/c, \quad v = b/c, \quad z = d/c.$$

   This allows us to visualize the surfaces (15) and (21) (see Figure 1)

$$u + z = 0, \quad 4v + (u - z)^2 = 0 \tag{26}$$

in $\mathbb{R}^3$. The intersection of (26) is a parabolic curve

$$(u, -u^2, -u) \tag{27}$$

along with the corresponding constant function

$$f(x) = u.$$

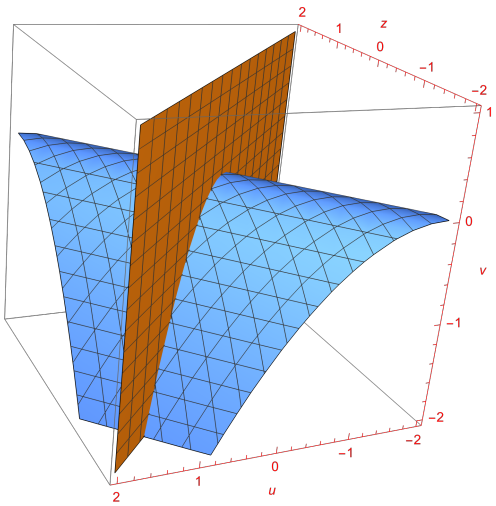


Figure 1. Surfaces (26) with the intersection (27) for $u \in [-2, 2]$, $v \in -2, 1]$ and $z \in [-2, 2]$.

The bifurcation surfaces (15) and (21) split $\mathbb{R}^3$ into 4 domains where dynamics of $f(x)$ are studied in the above sections; see Table 1. We note that this study is also given on these surfaces. We observe that (5) requires that

$$uz - v = 0 \tag{28}$$

does not hold. Moreover, we have

$$u + z = 0 \wedge uz - v = 0 \Leftrightarrow (27)$$

and

$$4v + (u - z)^2 = 0 \wedge uz - v = 0 \Leftrightarrow (27).$$

So, (28) holds on the bifurcation surfaces (15) and (21) only along the curve (27), which is their intersection.

Table 1. The dynamics of $f(x)$ under condition (5) and represented by the matrix A, see (8) and (9).

Trace A	Discriminant A	Dynamics of $f(x)$
<0	>0	Theorem 1
>0	>0	Theorem 2
$=0$	>0	Theorem 3
$\in\mathbb{R}$	<0	Theorem 4
$\neq 0$	$=0$	Theorem 5

3. The approach of this paper is directly adapted to complex valued functions for $a, b, c, d, x \in \mathbb{C}$. It would be interesting to consider quaternion valued cases. This

is, of course, non-trivial due to a non-commutativity affecting the determinant and eigenvalue definitions. Also, for instance, functions

$$(ax+b)(cx+d)^{-1},$$

and

$$(cx+d)^{-1}(ax+b)$$

are different. Quaternion linear dynamical systems are studied in [22].

4. Extending these results to time scale cases like in [7] is also a promising investigation.
5. Results of this paper can be applied to periodic sequences of linear rational functions

$$f_i(x) = \frac{p_i x + q_i}{r_i x + s_i}, \quad i \in \mathbb{N}$$

with $p_{i+T} = p_i$ for a $1 < T \in \mathbb{N}$ and all $i \in \mathbb{N}$. An asymptotic case

$$\lim_{i\to\infty} p_i = a, \quad \lim_{i\to\infty} q_i = b, \quad \lim_{i\to\infty} r_i = c, \quad \lim_{i\to\infty} s_i = s$$

can be also investigated.

6. Results of this paper give explicit expansions of solutions of a perturbed function

$$f(x,\epsilon) = f(x) + \epsilon g(x),$$

i.e.,

$$x_n(\epsilon) = \sum_{k=0}^{j} \chi_k \epsilon^k + O(\epsilon^{j+1}),$$

where χ_k can be explicitly computed for any $j \in \mathbb{N}$, for a sufficiently smooth function $g(x)$ and for a small parameter ϵ.

7. We can scale coefficients (a,b,c,d) of $f(x)$ so that

$$a^2 + b^2 + c^2 + d^2 = 1.$$

Since $\det A \neq 0$, we have

$$a^2 + b^2 > 0, \quad c^2 + d^2 > 0,$$

and then, we can take

$$\begin{aligned} a &= \cos\phi\cos\psi, \\ b &= \cos\phi\sin\psi, \\ c &= \sin\phi\cos\zeta, \\ d &= \sin\phi\sin\zeta \end{aligned}$$

for $\phi \in \left(0, \frac{\pi}{2}\right)$, $\psi \equiv 2\pi \bmod$, $\zeta \equiv 2\pi \bmod$. We also know that coefficients (a_n, b_n, c_n, d_n) of $f^n(x)$ satisfy

$$a_n^2 + b_n^2 > 0, \quad c_n^2 + d_n^2 > 0$$

for all $n \in \mathbb{N}$. Thus, we also have

$$\begin{aligned} a_n &= \cos\phi_n\cos\psi_n, \\ b_n &= \cos\phi_n\sin\psi_n, \\ c_n &= \sin\phi_n\cos\zeta_n, \\ d_n &= \sin\phi_n\sin\zeta_n \end{aligned}$$

for $\phi_n \in \left(0, \frac{\pi}{2}\right)$, $\psi_n \equiv 2\pi \bmod$, $\zeta_n \equiv 2\pi \bmod$. Consequently, the sequence $\{(\phi_n, \psi_n, \zeta_n)\}_{n\in\mathbb{N}} \subset \left[0, \frac{\pi}{2}\right] \times [0, 2\pi]^2$ has an accumulation point. This implies that the omega limit set $\omega(x_0)$ for a generic iteration of $f(x)$ is nonempty. This justifies our main results, since dynamics of $f(x)$ are either with a global attractor, with periodic orbits, or with quasiperiodic orbits. In all of these cases, $\omega(x_0) \neq \emptyset$ for a generic iteration of $f(x)$.

6. Conclusions

1. We summarize our main results in Table 1.
2. When (5) fails, then Remark 1 is applied for understanding the solutions and dynamics of $f(x)$.

Funding: This work is partially supported by the Slovak Research and Development Agency under the contract No. APVV-23-0039, and the Slovak Grant Agency VEGA No.1/0084/23 and No.2/0062/24.

Data Availability Statement: No new data were created or analyzed in this study.

Acknowledgments: We thank the reviewers for fruitful comments.

Conflicts of Interest: The author declares no conflicts of interest.

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