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Article

New Results on Fixed Points and Coupled Fixed Points for α-Admissible Condensing Operators

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Abstract

This paper introduces a novel class of condensing mappings through the concept of α-admissibility. We prove several extensions of Darbo's fixed point theorem within this framework, including a generalized contractive condition and a coupled fixed point result. Our main results extend the classical condensing condition to a flexible inequality involving α-admissibility and an auxiliary function. An example is given to demonstrate the applicability of our theorems where traditional approaches fail. Also, by leveraging the properties of α-admissible condensing operators, we proceed to establish a coupled fixed point theorem that relaxes traditional compactness conditions. At the end of the article, we apply our main theorem to prove the existence of solutions for a nonlinear integral equation.

Keywords: fixed point; α-admissible map; condensing maps; Darbo's fixed point theorem; integral equations

MSC: 47H08; 47H10; 45G10

1. Introduction

The development of fixed point theory has been based on three results that established its modern foundations. The Brouwer's topological fixed point theorem [1], which guaranteed fixed points for continuous selfmappings in finite-dimensional Euclidean spaces. A decade later, Banach's celebrated contraction mapping principle [2] introduced the notion of contractive conditions in complete metric spaces by proving both the existence and uniqueness of fixed points together with a constructive iterative method. The theory achieved its mature form through Schauder's fundamental extension [3], which connected finite- and infinite-dimensional analysis via compactness arguments in Banach spaces. These three theorems serve as the foundational pillars of modern nonlinear functional analysis.

The introduction of measures of noncompactness (MNCs) by Kuratowski [4] in 1930 represented a turning point in the quantitative study of compactness. This concept allowed mathematicians to measure how far a bounded set is from being relatively compact. In 1955, Darbo [5] combined the ideas of Banach and Schauder by proving a fixed point theorem for condensing operators, i.e., operators that strictly decrease the measure of noncompactness. Darbo's result unified aspects of both Brouwer's and Banach's theorems; it reduces to the Schauder theorem when the contraction constant $\gamma = 0$ and generalizes the Banach

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contraction principle in the sense that every Banach contraction is a condensing map. The elegance of Darbo's result lies in its ability to handle situations where neither the Banach contraction principle (due to lack of contractivity) nor the Schauder theorem (due to lack of compactness) applies directly.

Over the decades, numerous generalizations of Darbo's fixed point theorem have appeared in the literature. These extensions typically modify either the condensing condition, the underlying space, or the class of admissible operators. For instance, Aghajani et al. [6] established extensions using various contractive conditions involving comparison functions and subadditive measures of noncompactness. Banaś et al. [7] provided comprehensive treatments of MNCs and their applications to integral and differential equations. Despite these advances, a common limitation remains: the classical condensing condition

$$\varrho(TX) \leq \gamma\varrho(X), \quad \gamma \in [0,1), \tag{1}$$

requires that the constant γ be strictly less than 1. In many practical applications, the natural estimate of the MNC leads to a constant $\gamma \geq 1$, rendering Darbo's theorem inapplicable. This observation has motivated the search for more flexible contractive conditions that can accommodate such situations.

In a different but related direction, Samet et al. [8] introduced the concept of α-admissible mappings in 2012. An operator $T : E \to E$ is called α-admissible with respect to a function $\alpha : E \times E \to \mathbb{R}_+$ if for all $x, y \in E$, $\alpha(x,y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$. This notion, originally formulated for metric spaces, provided a powerful framework for unifying and extending many existing contractive conditions. For example, by choosing α appropriately, one can recover standard Banach contractions, and various other classes of mappings. Subsequently, Aghajani et al. [9] adapted the concept of α-admissibility to the setting of noncompactness measures, where α is defined in the family of bounded subsets rather than in points. Their definition, which we adopt in the present work, requires that if $\alpha(X) \geq 1$ for some bounded set X, then $\alpha(\mathrm{Co}(TX)) \geq 1$. This adaptation has opened the door to combining the flexibility of α-admissibility with the quantitative power of MNC theory.

The present work is motivated by the following key observations. First, many nonlinear operators arising in applications, such as integral operators with non-Lipschitz kernels, do not satisfy the classical condensing condition (1) with $\gamma < 1$, but may satisfy a more general inequality involving an auxiliary function f and the α-admissibility factor. Second, the α-admissibility condition provides a mechanism to "turn off" the contractive requirement for subsets that are not sufficiently regular, thereby allowing the operator to behave poorly on certain subsets as long as those subsets are not encountered in the iterative construction. Third, in product spaces, which are essential for coupled fixed point problems, the interplay between the measure of noncompactness and admissibility conditions becomes more delicate and requires careful handling.

Specifically, in this work we replace the classical condensing condition (1) with the more general hypothesis

$$\alpha(X)\big[\varrho(TX) + f(\varrho(TX))\big] \leq \gamma\big[\varrho(X) + f(\varrho(X))\big], \quad \gamma \in [0,1), \tag{2}$$

where ϱ is a measure of noncompactness, $f : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, and $\alpha : \mathcal{M}_E \to \mathbb{R}_+$ is a function encoding admissibility. This extension allows us to handle operators where the Darbo constant γ in the classical sense would be $\geq$1, yet a fixed point still exists. The presence of f further generalizes the contraction by allowing a nonlinear comparison between $\varrho(TX)$ and $\varrho(X)$. When $\alpha(X) \equiv 1$ and $f \equiv 0$, our condition reduces to the classical Darbo condition.

Recent years have witnessed growing interest in generalizations of Darbo's theorem. Aghajani et al. [6] established extensions using various contractive conditions, including those involving comparison functions. The authors of [10–13] introduced new classes of condensing operators via α-admissibility and obtained fixed point results that generalize several known theorems in the literature. Our work unifies and extends these approaches by providing a comprehensive framework for α-admissible condensing mappings that simultaneously accounts for nonlinear corrections via f and product-space structures for coupled fixed points. This is the first time that a coupled fixed point theorem for α-admissible condensing operators has been established in the setting of an arbitrary MNC.

The main contributions of this paper are threefold. First, we prove a new fixed point theorem (Theorem 4) for α-admissible condensing operators satisfying the generalized condition (2). This theorem relaxes the classical condensing requirement and includes many existing results as special cases. Second, we provide a generalized version using comparison functions instead of a linear contraction, further extending the scope of applicability. Third, we establish a coupled fixed point theorem (Theorem 5) for mappings $F : C \times C \to C$ that are α-admissible in a product sense, yielding a pair (x, y) such that $F(x, y) = x$ and $F(y, x) = y$. Coupled fixed points have received considerable attention in recent years due to their application in systems of differential and integral equations, where the symmetry condition $F(x, y) = x$ and $F(y, x) = y$ often arises naturally.

To illustrate the practical utility of our theoretical advances, we apply our main theorem to prove the existence of solutions for a nonlinear Volterra-type integral equation of the form

$$x(t) = g(t) + \int_0^t K(t, s, x(s))\, ds, \quad t \in [0, 1],$$

where $x \in C([0, 1], \mathbb{R})$.

The paper is organized as follows. Section 2 recalls essential definitions and results, including the measure of noncompactness, Darbo's theorem, α-admissibility, and comparison functions. Section 3 contains the main results: first, the generalized fixed point theorem for α-admissible condensing operators (Theorem 4); second, a concrete example demonstrating the necessity and novelty of our approach; and third, the coupled fixed point theorem (Theorem 5), with a detailed proof. Section 4 applies Theorem 4 to a nonlinear Volterra-type integral equation. We verify the α-admissibility of the associated integral operator using equicontinuity, check the generalized contractive condition via the modulus of continuity, and prove the existence of a solution. A concrete example with $\varphi(t) = t/(1 + t)$ illustrates the power of our results over classical ones. Finally, we conclude with remarks on the limitations and potential extensions of our work, including the possibility of applying these methods to higher-dimensional systems and to fractional integral equations.

2. Preliminaries

Let E be a Banach space with $\mathcal{M}_E$ denoting the family of all bounded subsets of E and $\mathcal{N}_E$ the family of all relatively compact subsets of E. For $X \subset E$, we write $\overline{X}$ and $\mathrm{Co}(X)$ for its closure and closed convex hull, respectively.

Definition 1. *A function $\varrho : \mathcal{M}_E \to \mathbb{R}_+$ is called a measure of noncompactness (MNC) if it satisfies the following conditions:*

(i) $\ker \varrho = \{X \in \mathcal{M}_E : \varrho(X) = 0\} \neq \emptyset$ *and* $\ker \varrho \subset \mathcal{N}_E$*;*
(ii) $X \subset Y \Rightarrow \varrho(X) \leq \varrho(Y)$*;*
(iii) $\varrho(X) = \varrho(\overline{X}) = \varrho(\mathrm{Co}(X))$*;*
(iv) $\varrho(\delta X + (1 - \delta)Y) \leq \delta\varrho(X) + (1 - \delta)\varrho(Y)$ *for* $\delta \in [0, 1]$*;*

(v) *For all decreasing sequences $\{X_n\}$ of nonempty, closed, bounded subsets of E with $\lim_{n\to\infty} \varrho(X_n) = 0$, the intersection $\bigcap_{n=1}^{\infty} X_n$ is nonempty and compact (Cantor intersection property).*

The kernel $\ker \varrho$ is the collection of all bounded sets that are treated as "small" or "compact" by the MNC. Different choices of MNC yield different kernels. Two classical examples are the Kuratowski MNC and the Hausdorff (or ball) MNC. The Hausdorff MNC χ is defined by

$$\chi(X) = \inf\{\varepsilon > 0 : X \text{ can be covered by finitely many balls of radius } \varepsilon\}.$$

ϱ and χ satisfy the properties above, and they are equivalent in the sense that $\chi(X) \le \varrho(X) \le 2\chi(X)$ for all bounded X.

Definition 2. *A mapping $T : C \to E$ ($C \subset E$) is called condensing with respect to an MNC ϱ if for every bounded set $X \subset C$ with $\varrho(X) > 0$, we have $\varrho(TX) < \varrho(X)$. If there exists a constant $\gamma \in [0,1)$ such that $\varrho(TX) \le \gamma\varrho(X)$ for all $X \subset C$, then T is called contractive or a Darbo contraction.*

The classical fixed point theorems that motivate our work are stated below for completeness.

Theorem 1 (Brouwer [1])**.** *Every continuous selfmapping of a closed Euclidean ball in $\mathbb{R}^n$ has a fixed point.*

Theorem 2 (Schauder [3])**.** *Let C be a nonempty, closed, convex subset of a Banach space E. If $T : C \to C$ is continuous and $T(C)$ is relatively compact (i.e., T is compact), then T has a fixed point.*

Theorem 3 (Darbo [5])**.** *Let $C \subset E$ be nonempty, bounded, closed, and convex. Let $T : C \to C$ be a continuous mapping that is a contraction with respect to an MNC ϱ, i.e.,*

$$\varrho(TX) \le \gamma\varrho(X) \quad \textit{for all } X \subset C,$$

where $\gamma \in [0,1)$. Then, T has a fixed point. Moreover, the set of fixed points of T is compact.

The proof of Darbo's theorem proceeds by constructing a decreasing sequence of closed convex sets $C_{n+1} = \mathrm{Co}(TC_n)$ starting from $C_0 = C$. The contraction property leads to $\varrho(C_n) \to 0$, and the Cantor intersection property yields a nonempty compact invariant set to which Schauder's theorem applies.

The notion of MNC function extends naturally to α-admissibility settings as follows.

Definition 3 ([9])**.** *Let $\alpha : \mathcal{M}_E \to \mathbb{R}_+$ be a given function. An operator $T : E \to E$ is called α-admissible if for every bounded set $X \subset E$,*

$$\alpha(X) \ge 1 \quad \Longrightarrow \quad \alpha(\mathrm{Co}(TX)) \ge 1.$$

In other words, the property "$\alpha \ge 1$" is preserved when passing from X to the closed convex hull of its image under T.

The function α serves as a selector: it identifies "good" subsets for which the contractive condition will be imposed. For subsets with $\alpha(X) < 1$, no condensing condition is required. This flexibility is crucial when the operator has a complicated global behavior but exhibits nice properties on a well-chosen family of sets.

Example 1 ([9]). *Let $E = [0, \infty)$ and define $T : E \to E$ by $Tx = e^x$. Consider the function $\alpha : \mathcal{M}_E \to [0, \infty)$ given by $\alpha(X) = \mathrm{diam}(X)$, the diameter of X. For any bounded $X \subset [0, \infty)$, let $m = \inf X$ and $M = \sup X$. Then,*

$$\alpha(TX) = \mathrm{diam}(TX) = e^M - e^m = e^m(e^{M-m} - 1) \geq e^m(e^{\mathrm{diam}(X)} - 1).$$

If $\alpha(X) = \mathrm{diam}(X) \geq 1$, then $e^{\mathrm{diam}(X)} - 1 \geq e - 1 > 1$, so $\alpha(TX) > 1$. Since $\mathrm{Co}(TX)$ has the same diameter as TX, we obtain $\alpha(\mathrm{Co}(TX)) > 1 \geq 1$. Hence, T is α-admissible.

The above example is illustrative: it shows that even a rapidly expanding operator like the exponential can be α-admissible for a suitably chosen α. In our main results, α will typically depend on the MNC itself or on other regularity properties such as equicontinuity.

A function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a comparison function if it satisfies the following:

(i) ψ is increasing (i.e., $0 \leq t_1 < t_2$ implies $\psi(t_1) \leq \psi(t_2)$);
(ii) For every $t > 0$, the series $\sum_{n=1}^{\infty} \psi^n(t)$ converges, where ψ^n denotes the n-th iterate of ψ.

Comparison functions are a classical tool in fixed point theory for generalizing Banach's contraction principle. A typical example is $\psi(t) = \frac{t}{1+t}$, which satisfies $\psi(t) < t$ for $t > 0$ but has no global Lipschitz constant less than one. Another example is $\psi(t) = \lambda t$ with $\lambda \in (0, 1)$, which yields the standard Banach contraction. The convergence of $\sum \psi^n(t)$ implies, in particular, that $\psi^n(t) \to 0$ as $n \to \infty$ for each $t \geq 0$. Note that if ψ is a comparison function, then $\psi(t) < t$ for every $t > 0$, and $\psi(0) = 0$.

At the end of this section, we recall a useful observation regarding the behavior of MNCs under continuous mappings; If $T : E \to E$ is a continuous operator, then for every bounded set $X \subset E$, we have $\varrho(T(X)) \leq \varrho(X)$ whenever ϱ is the Hausdorff MNC or the Kuratowski MNC. For general MNCs, continuity of T alone does not guarantee such an inequality, but if T is also Lipschitz with constant L, then $\varrho(TX) \leq L\varrho(X)$ for many common MNCs. This explains why condensing conditions usually incorporate a factor γ that may depend on the Lipschitz constant of T when such a constant exists. In our main results, we do not require T to be Lipschitz; only continuity and the generalized condensing condition (2) are needed.

3. Main Results

We now proceed to the main results of the paper.

Theorem 4. *Let $C \subset E$ be nonempty, bounded, closed, and convex. Suppose $T : C \to C$ is a continuous α-admissible mapping such that the following holds:*

(i) There exists a closed convex $X_0 \subset C$ satisfying $TX_0 \subset X_0$ and $\alpha(X_0) \geq 1$;
(ii) For all $0 < a < b < 1$ and $X \subset C$, we have

$$a \leq \varrho(X) + f(\varrho(X)) \leq b \Rightarrow \alpha(X)\big[\varrho(TX) + f(\varrho(TX))\big] \leq \gamma\big[\varrho(X) + f(\varrho(X))\big], \quad (3)$$

where $\gamma \in [0, 1)$, $f : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, and ϱ is an MNC.

Then, T has a fixed point in C.

Proof. Define $\{X_n\}$ recursively by $X_{n+1} = \mathrm{Co}(TX_n)$. The inclusion $TX_0 \subset X_0$ implies

$$X_{n+1} \subset X_n \quad \text{and} \quad TX_n \subset X_n \quad \forall n \in \mathbb{N}. \quad (4)$$

The α-admissibility yields $\alpha(X_n) \geq 1$ for all n. If $\varrho(X_N) = 0$ for some N, then X_N is compact, and Schauder's theorem applies.

Assuming $\varrho(X_n) + f(\varrho(X_n)) > 0$ for all n, suppose for contradiction that for some n_0,

$$\varrho(X_{n_0+1}) + f(\varrho(X_{n_0+1})) > \varrho(X_{n_0}) + f(\varrho(X_{n_0})).$$

Then, condition (3) with $a = \varrho(X_{n_0}) + f(\varrho(X_{n_0}))$ and $b = \varrho(X_{n_0+1}) + f(\varrho(X_{n_0+1}))$ leads to $\gamma > 1$, a contradiction.

Thus, $\{\varrho(X_n) + f(\varrho(X_n))\}$ is non-increasing; let $c = \lim_{n\to\infty}(\varrho(X_n) + f(\varrho(X_n)))$. If $c > 0$, applying (3) yields $c \leq \gamma c$, implying $\gamma \geq 1$—again a contradiction. Hence, $c = 0$, and $\varrho(X_n) \to 0$.

By property (v) of Definition 1, $X_\infty = \bigcap_n X_n \neq \emptyset$ is compact and T-invariant. Schauder's theorem completes the proof. $\square$

Remark 1. *About the construction of successive approximations in the above proof with the sequence $X_{n+1} = \mathrm{Co}(TX_n)$, starting from X_0, this sequence converges to a compact set. Furthermore, the α-admissibility condition guarantees that beginning from an initial set X_0, the iterative sequence satisfies $\alpha(X_n) \geq 1$ for all n, and any fixed point can be obtained as a limit of a subsequence of approximations.*

Corollary 1. *Let $C \subset E$ be nonempty, bounded, closed, and convex. If $T : C \to C$ is a continuous α-admissible mapping such that*

(i) *There exists a closed convex $X_0 \subset C$ satisfying $TX_0 \subset X_0$ and $\alpha(X_0) \geq 1$;*
(ii) *For all $X \subset C$, we have*

$$\alpha(X)\big[\varrho(TX) + f(\varrho(TX))\big] \leq \gamma\big[\varrho(X) + f(\varrho(X))\big], \tag{5}$$

where $\gamma \in [0,1)$, $f : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, and ϱ is an MNC,

then, T has a fixed point in C.

Corollary 2 ([9])**.** *Let $C \subset E$ be nonempty, bounded, closed, and convex. Suppose $T : C \to C$ is a continuous α-admissible mapping satisfying*

$$\alpha(X)\big[\varrho(TX)\big] \leq \psi(\varrho(X)) \quad \text{for all } X \subset C,$$

where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function and ϱ is an MNC. Assume there exists a closed convex $X_0 \subset C$ such that $TX_0 \subset X_0$ and $\alpha(X_0) \geq 1$. Then, T has a fixed point in C.

Example 2. *Let $E = \mathbb{R}$, consider the mapping $T : [0,4] \to [0,4]$ defined by*

$$Tx = \begin{cases} \frac{x}{2}, & 0 \leq x \leq 2, \\ \frac{3}{2}x - 2, & 2 < x \leq 4 \end{cases}$$

and the function $\alpha : \mathcal{M}_{\mathbb{R}} \to \mathbb{R}_+$ by

$$\alpha(X) = \begin{cases} 1, & X \subset [0,2], \\ 0, & \text{otherwise}, \end{cases}$$

where ϱ is the measure of noncompactness of the diameter defined by

$$\varrho(X) = \mathrm{Diam}(X) = \sup\{|x-y| : x, y \in X\},$$

where $X \subset [0,4]$.

We have

$$\varrho(TX) = \frac{1}{2}\varrho(X)$$

if $X \subset [0,2]$ *and*

$$\varrho(TX) = \frac{3}{2}\varrho(X)$$

if $X \subset (2,4]$.

By taking $f(t) = t$*, we obtain*

$$\alpha(X)\big[\varrho(TX) + f(\varrho(TX))\big] \leq \frac{1}{2}\big[\varrho(X) + f(\varrho(X))\big].$$

Then, Theorem 4 yields the two fixed points $x = 4 \in (2,4]$ *and* $x = 0 \in [0,2]$.

Remark 2. *This example demonstrates that our result extends Darbo's theorem, particularly for operators where* $\varrho(TX) \leq \gamma\varrho(X)$ *with* $\gamma \geq 1$ *(in our example we have* $\varrho(TX) = \frac{3}{2}\varrho(X)$ *for* $X \subset (2,4]$*). In addition, the classical case is recovered when* $\alpha(X) \equiv 1$ *and* $f \equiv 0$.

We now study the existence of coupled fixed points.

Definition 4. *Let* E_1 *and* E_2 *be Banach spaces with respective MNCs* ϱ_1 *and* ϱ_2*. On the product space* $E_1 \times E_2$ *(equipped with, e.g., the norm* $\|(x,y)\| = \max\{\|x\|_{E_1}, \|y\|_{E_2}\}$*), we define the product MNC* $\widetilde{\varrho}$ *on bounded subsets* $\widetilde{X} \subset E_1 \times E_2$ *by*

$$\widetilde{\varrho}(\widetilde{X}) = \max\{\varrho_1(\pi_1(\widetilde{X})),\ \varrho_2(\pi_2(\widetilde{X}))\},$$

where π_i *are the canonical projections. For Cartesian products* $X \times Y$*, we have* $\widetilde{\varrho}(X \times Y) = \max\{\varrho_1(X), \varrho_2(Y)\}$.

It is easy to verify that $\widetilde{\varrho}$ satisfies all conditions of Definition 1 whenever ϱ_1 and ϱ_2 satisfy the conditions. This construction is essential for the coupled fixed point theorem in Section 3.

Theorem 5. *Let* $C \subset E$ *be nonempty, bounded, closed, and convex. Suppose* $F : C \times C \to C$ *is a continuous mapping such that the following holds:*

(i) *There exists* $\alpha : \mathcal{M}_{E\times E} \to \mathbb{R}_+$ *with* $\alpha(X \times Y) \geq 1$ *implies* $\alpha(\mathrm{Co}(F(X,Y)) \times \mathrm{Co}(F(Y,X))) \geq 1$;

(ii) *For all* $X, Y \subset C$,

$$\begin{aligned}\alpha(X \times Y)\big[\varrho(F(X,Y)) + f(\varrho(F(X,Y)))\big] \\ \leq \frac{\gamma}{2}\big[\varrho(X) + f(\varrho(X)) + \varrho(Y) + f(\varrho(Y))\big],\end{aligned}$$

where $\gamma \in [0,1)$*,* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *is continuous, and* ϱ *is an MNC on* E;

(iii) *There exist closed convex* X_0 *and* $Y_0 \subset C$ *such that* $F(X_0, Y_0) \subset X_0$*,* $F(Y_0, X_0) \subset Y_0$*, and* $\alpha(X_0 \times Y_0) \geq 1$.

Then, F *has a coupled fixed point, i.e., there exist* $x, y \in C$ *such that* $F(x,y) = x$ *and* $F(y,x) = y$.

Proof. Consider the product space $E \times E$ equipped with the norm $\|(x,y)\| = \max\{\|x\|, \|y\|\}$. Define $\widetilde{C} = C \times C$, which is bounded, closed, and convex in $E \times E$. Define an operator $\widetilde{T} : \widetilde{C} \to \widetilde{C}$ by

$$\widetilde{T}(x,y) = (F(x,y), F(y,x)).$$

For any $\widetilde{X} = X \times Y \subset \widetilde{C}$, define

$$\widetilde{\varrho}(\widetilde{X}) = \max\{\varrho(X), \varrho(Y)\}.$$

It can be verified that $\widetilde{\varrho}$ is an MNC on $E \times E$.

From condition (ii), we obtain

$$\begin{aligned}
&\alpha(X\times Y)\big[\widetilde{\varrho}(\widetilde{T}(X\times Y)) + f(\widetilde{\varrho}(\widetilde{T}(X\times Y)))\big] \\
&\quad= \alpha(X\times Y)\big[\max\{\varrho(F(X,Y)), \varrho(F(Y,X))\} + f(\max\{\varrho(F(X,Y)), \varrho(F(Y,X))\}\big] \\
&\quad\le \alpha(X\times Y)\big[\max\{\varrho(F(X,Y)) + f(\varrho(F(X,Y))), \varrho(F(Y,X)) + f(\varrho(F(Y,X)))\}\big] \\
&\quad\le \max\Big\{\frac{\gamma}{2}[\varrho(X) + f(\varrho(X)) + \varrho(Y) + f(\varrho(Y))], \frac{\gamma}{2}[\varrho(Y) + f(\varrho(Y)) + \varrho(X) + f(\varrho(X))]\Big\} \\
&\quad= \frac{\gamma}{2}[\varrho(X) + f(\varrho(X)) + \varrho(Y) + f(\varrho(Y))] \\
&\quad\le \gamma \max\{\varrho(X) + f(\varrho(X)), \varrho(Y) + f(\varrho(Y))\} \\
&\quad= \gamma[\widetilde{\varrho}(X\times Y) + f(\widetilde{\varrho}(X\times Y))].
\end{aligned}$$

Thus, $\widetilde{T}$ satisfies condition (5) on $\widetilde{C}$ with $\widetilde{X}_0 = X_0 \times Y_0$. By Theorem 4, $\widetilde{T}$ has a fixed point $(x,y) \in \widetilde{C}$, which gives $F(x,y) = x$ and $F(y,x) = y$. □

4. Application to Volterra-Type Integral Equations

In this section, we apply our main theorem (Theorem 4) to prove the existence of solutions for a nonlinear Volterra-type integral equation in the Banach space $E = C([0,1], \mathbb{R})$ of continuous functions equipped with the supremum norm

$$\|x\| = \sup_{t\in[0,1]} |x(t)|.$$

Consider the following integral equation:

$$x(t) = g(t) + \int_0^t K(t,s,x(s))\, ds, \quad t \in [0,1], \tag{6}$$

where $g \in C([0,1], \mathbb{R})$ and $K : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous.

We define the operator $T : C([0,1], \mathbb{R}) \to C([0,1], \mathbb{R})$ by

$$(Tx)(t) = g(t) + \int_0^t K(t,s,x(s))\, ds.$$

Let $C = \{x \in C([0,1], \mathbb{R}) : \|x\| \le R\}$ for some $R > 0$. We assume that

(A1) $\|g\| \le \frac{R}{2}$;

(A2) $|K(t,s,u)| \le \frac{R}{2} + L|u|$ for all $t,s \in [0,1]$, $u \in [-R,R]$, where $L \in \left[0, \frac{1}{2}\right)$;

(A3) $|K(t,s,u) - K(t,s,v)| \le \varphi(|u-v|)$ for all $t,s \in [0,1]$, $u,v \in [-R,R]$, where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, increasing, and $\varphi(t) < t$ for $t > 0$, and there exists $\gamma \in [0,1)$ such that $\varphi(t) \le \gamma t$ for all $t \in [0,2R]$.

Theorem 6. *Under assumptions (A1)–(A3), the integral Equation* (6) *has at least one solution* $x \in C([0,1], \mathbb{R})$.

Proof. First, we verify that T maps bounded sets to bounded sets. For $x \in C$ with $\|x\| \le R$, we have

$$\begin{aligned} |(Tx)(t)| &\le |g(t)| + \int_0^t |K(t,s,x(s))|\, ds \\ &\le \|g\| + \int_0^t \left(\frac{R}{2} + L|x(s)| \right) ds \\ &\le \frac{R}{2} + \frac{R}{2} + LR \\ &= R + LR \le R + \frac{R}{2} = \frac{3R}{2}. \end{aligned}$$

Thus, $T(C)$ is bounded.

Now, consider the set $C_0 = \overline{\text{Co}}(T(C))$, the closed convex hull of $T(C)$. Then, C_0 is nonempty, bounded, closed, and convex. Moreover, by construction, $T(C_0) \subset C_0$ since C_0 is convex and closed and contains $T(C)$.

We now apply Theorem 4. We use the measure of noncompactness ϱ defined on bounded subsets of $C([0,1], \mathbb{R})$ by the modulus of continuity

$$\varrho(X) = \lim_{\delta \to 0^+} \sup_{x \in X} \omega(x, \delta),$$

where

$$\omega(x, \delta) = \sup\{|x(t) - x(s)| : t, s \in [0,1], |t - s| \le \delta\}.$$

We note that the modulus of continuity is a standard MNC on $C([0,1])$ (see [7]). Define $\alpha : \mathcal{M}_{C([0,1],\mathbb{R})} \to \mathbb{R}_+$ by

$$\alpha(X) = \begin{cases} 1 & \text{if } X \text{ is equicontinuous,} \\ 0 & \text{otherwise.} \end{cases}$$

We need to show that T is α-admissible, i.e., if $\alpha(X) \ge 1$, then $\alpha(\text{Co}(TX)) \ge 1$.

Assume $\alpha(X) \ge 1$, so X is equicontinuous. We prove that TX is also equicontinuous. For any $x \in X$ and $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| \le \delta$, we have

$$\begin{aligned} |(Tx)(t_1) - (Tx)(t_2)| &\le |g(t_1) - g(t_2)| + \left| \int_0^{t_1} K(t_1, s, x(s))\, ds - \int_0^{t_2} K(t_2, s, x(s))\, ds \right| \\ &\le |g(t_1) - g(t_2)| + \left| \int_0^{t_1} [K(t_1, s, x(s)) - K(t_2, s, x(s))]\, ds \right| \\ &\quad + \left| \int_{t_2}^{t_1} K(t_2, s, x(s))\, ds \right|. \end{aligned}$$

Since g is continuous on $[0,1]$, it is uniformly continuous, so $|g(t_1) - g(t_2)| \to 0$ as $\delta \to 0$, uniformly in x.

The function K is uniformly continuous on the compact set $[0,1] \times [0,1] \times [-R, R]$, hence for any $\varepsilon > 0$, there exists $\eta > 0$ such that $|t_1 - t_2| < \eta$ implies $|K(t_1, s, u) - K(t_2, s, u)| < \varepsilon$ for all $s \in [0,1]$ and $u \in [-R, R]$.

Thus, the second term also tends to 0 uniformly. The third term is bounded by $|t_1 - t_2| \cdot \sup |K|$, which also tends to 0. Therefore, TX is equicontinuous. Consequently, $\text{Co}(TX)$ is also equicontinuous, so $\alpha(\text{Co}(TX)) \ge 1$. Thus, T is α-admissible.

Now, let $f(t) = t$. We verify condition (3) of Theorem 4. For any $X \subset C_0$ with $\alpha(X) \geq 1$, we have $\alpha(X) = 1$ and

$$\alpha(X)[\varrho(TX) + f(\varrho(TX))] = \varrho(TX) + \varrho(TX) = 2\varrho(TX).$$

We estimate $\varrho(TX)$. For any $x, y \in X$ and $t \in [0,1]$, using assumption (A3),

$$\begin{aligned}|(Tx)(t) - (Ty)(t)| &\leq \int_0^t |K(t,s,x(s)) - K(t,s,y(s))|\,ds \\ &\leq \int_0^t \varphi(|x(s) - y(s)|)\,ds \\ &\leq \int_0^1 \varphi(\|x - y\|)\,ds = \varphi(\|x - y\|).\end{aligned}$$

Taking the supremum over $t \in [0,1]$, we obtain $\|Tx - Ty\| \leq \varphi(\|x - y\|)$.

Now, for the modulus of continuity we have $\omega(Tx, \delta) = \omega(Ty, \delta)$ for all $x, y \in X$, so $\varrho(TX) = \varrho(\{Tx : x \in X\})$. Moreover, from the estimate above and the properties of the measure of noncompactness, we get

$$\varrho(TX) \leq \varphi(\varrho(X)).$$

Therefore,

$$\alpha(X)[\varrho(TX) + f(\varrho(TX))] = 2\varrho(TX) \leq 2\varphi(\varrho(X)).$$

By assumption (A3), there exists $\gamma \in [0,1)$ such that $\varphi(t) \leq \gamma t$ for all $t \in [0, 2R]$. Since $\varrho(X) \leq \mathrm{diam}(X) \leq \mathrm{diam}(C_0) \leq 2R$, we have

$$2\varphi(\varrho(X)) \leq 2\gamma\varrho(X) = \gamma[2\varrho(X)] = \gamma[\varrho(X) + f(\varrho(X))].$$

Thus, condition (3) is satisfied with the same γ.

Finally, we need a closed convex $X_0 \subset C_0$ such that $TX_0 \subset X_0$ and $\alpha(X_0) \geq 1$. Take $X_0 = C_0$. Since $C_0 = \overline{\mathrm{Co}}(T(C))$ and $T(C)$ is equicontinuous (as shown above), C_0 is also equicontinuous, so $\alpha(C_0) \geq 1$. Moreover, $T(C_0) \subset C_0$ by construction.

All conditions of Theorem 4 are satisfied. Therefore, T has a fixed point in C_0, which is a solution to the integral Equation (6). □

Remark 3. *Our theorem allows for more general situations where φ is not necessarily linear and where the α-admissibility condition provides additional flexibility. For instance, if*

$$\varphi(t) = \frac{t}{1+t},$$

which satisfies

$$\varphi(t) < t$$

for $t > 0$, but is not bounded above by γt for any fixed $\gamma < 1$ on $[0, \infty)$.

So, our theorem can still be applied by choosing R sufficiently small so that our assumptions (A1)–(A3) hold on $[0, 2R]$.

Remark 4. *As noted by Sidorov [14,15], solutions to nonlinear Volterra integral equations of the second kind may exist only locally and can blow up in finite time. Our assumptions (A1)–(A3) ensure global existence on $[0,1]$ by imposing a linear growth bound (A2) and a sublinear Lipschitz condition (A3). For cases where blow-up occurs, our fixed point approach would need to be applied on a smaller interval where the solution remains bounded.*

Remark 5. *In the classical Darbo fixed point theorem, one must verify the uniform condensing condition $\rho(TX) \leq \gamma\rho(X)$ for all bounded subsets $X \subset C$, which often fails when the nonlinearity K is not globally Lipschitz with constant < 1 (e.g., $\phi(t) = t/(1+t)$). In contrast, our α-admissible approach only requires this inequality on subsets with $\alpha(X) \geq 1$ (here, equicontinuous sets). The α-admissibility condition ensures that starting from an equicontinuous initial set X_0, the iterative sequence remains within the "good" family where the contraction holds.*

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