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Normalized Solution for the Fractional Kirchhoff System with Hardy Nonlinearities

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Abstract

In this paper, we consider the existence of normalized solutions for the fractional Kirchhoff system with Hardy critical nonlinearities. The problem combines the challenges of nonlocal operators, singular critical effects, and a variational framework with fixed mass conditions. By employing the minimax theorem, analyzing the Palais–Smale sequence and restricting to radial functions, we overcome the loss of strong convergence at critical energy levels. Under L^2 -supercritical case, we prove the existence of positive radial normalized solutions, along with the positivity of the corresponding Lagrange multipliers.

Keywords: hardy nonlinearity; fractional kirchhoff system; normalized solutions; critical exponent

MSC: 35A15; 35A16; 35B09; 35B33; 35R11; 35J60

1. Introduction

In this paper, we focus on the following fractional Kirchhoff system with Hardy critical nonlinearities:

$$\begin{cases} (a + b \int_{\mathbb{R}^{3s}} |(-\Delta)^{\frac{s}{2}} u|^2 dx) (-\Delta)^s u + \lambda_1 u = \mu_1 |u|^{p-2} u + r_1 v \frac{|u|^{r_1-2} |v|^{r_2} u}{|x|^\alpha}, & \text{in } \mathbb{R}^{3s}, \\ (a + b \int_{\mathbb{R}^{3s}} |(-\Delta)^{\frac{s}{2}} v|^2 dx) (-\Delta)^s v + \lambda_2 v = \mu_2 |v|^{p-2} v + r_2 v \frac{|u|^{r_1} |v|^{r_2-2} v}{|x|^\alpha}, & \text{in } \mathbb{R}^{3s}, \end{cases} \quad (1)$$

having prescribed L^2 - norm:

$$\int_{\mathbb{R}^{3s}} u^2 dx = c^2, \quad \int_{\mathbb{R}^{3s}} v^2 dx = d^2, \quad (2)$$

where $s \in (0, 1)$, $\alpha \in (0, s)$, $a, b, c, d, \mu_1, \mu_2 > 0, v > 0$, the exponents p, r_1, r_2 satisfy

$$r_1 > 1, r_2 > 1, r_1 + r_2 = 2_{s,\alpha}^* := \frac{2(3s - \alpha)}{s}, \frac{14}{3} < p, q < 2_s^* := 6,$$

and $\lambda_1, \lambda_2 \in \mathbb{R}$ appear as unknown Lagrange multipliers. Actually, $2_s^* := 6$ represents the fractional Sobolev critical exponent, while $2_{s,\alpha}^* := \frac{2(3s - \alpha)}{s}$ corresponds to the fractional critical Hardy-Sobolev exponent. The fractional Laplacian operator $(-\Delta)^s$ is defined as follows:

$$(-\Delta)^s u(x) = C_{N,s} P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy, \quad x \in \mathbb{R}^N,$$



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where $u \in S(\mathbb{R}^N)$ and $S(\mathbb{R}^N)$ represents the Schwartz space of smooth functions decaying rapidly at infinity, P.V. indicates the principle value of the integral and $C_{N,s}$ represents a positive constant.

In recent years, numerous researches are focused on the normalized solutions of the systems with nonlinearities. Li and Zou [1] explored the following Schrödinger system:

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 |u|^{p-2} u + \beta r_1 |u|^{r_1-2} u |v|^{r_2}, & \text{in } \mathbb{R}^N, \\ -\Delta v + \lambda_2 v = \mu_2 |v|^{q-2} v + \beta r_2 |u|^{r_1} |v|^{r_2-2} v, & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = a_1^2, \quad \int_{\mathbb{R}^N} |v|^2 dx = a_2^2, \end{cases} \quad (3)$$

where $p, q, r_1 + r_2 < 2^*$. By studying the geometry properties of the Pohozaev manifold under some assumptions on parameters, they obtained a normalized positive ground state solution. Moreover, Bartsch, Jeanjean and Soave [2] investigated the system (3) with $p = 2^*$, when $N = 3, 4$ and $2 < r_1 + r_2 < 2 + \frac{4}{N}$, they got a normalized ground state solution with $0 < \beta < \beta_0$, and when $2 + \frac{4}{N} \leq r_1 + r_2 < 2^*$ they obtained the existence of a threshold $\beta_1 \geq 0$ such that when $\beta > \beta_1$, there exists a ground state normalized solution. Later, through Krasnoselskii genus approach, Bartsch and Soave [3] obtained the existence of an infinite number of solutions to (3) with $\mu_1 = \mu_2$ and $a_1 = a_2$. When the above system is equipped with Sobolev critical nonlinearities, i.e., $r_1 + r_2 = 2^*$, Zhang and Han [4] got a positive ground state solution under L^2 -subcritical case.

To extend the results mentioned above, a great deal of researchers are devoted to exploring the fractional Schrödinger systems. Zuo [5] examined the following coupled fractional Schrödinger systems:

$$\begin{cases} (-\Delta)^s u = \mu_1 u + |u|^{2_s^*-2} u + \eta_1 |u|^{p-2} u + \gamma \alpha |u|^{\alpha-2} u |v|^\beta & \text{in } \mathbb{R}^N, \\ (-\Delta)^s v = \mu_2 v + |v|^{2_s^*-2} v + \eta_2 |v|^{q-2} v + \gamma \beta |v|^{\beta-2} v |u|^\alpha & \text{in } \mathbb{R}^N, \\ \|u\|_{L^2}^2 = m_1^2 \text{ and } \|v\|_{L^2}^2 = m_2^2, \end{cases}$$

where $N > 2s$, p, q and $\alpha + \beta \in (2 + \frac{4s}{N}, 2_s^*]$. Through scaling transformation, concentration-compactness principle and classification discussion, they obtained a positive normalized solution with negative Lagrange multipliers. Additionally, they proved that no positive normalized solutions exist under the Sobolev-critical case. Moreover, Chen and Yang [6] explored the fractional Schrödinger system with Choquard terms, by applying minimax principle and describing the mountain pass geometry structure, they got the existence of the normalized positive ground state solution and the mountain pass type normalized solution under L^2 -subcritical case and L^2 -supercritical case respectively.

As for the study for Kirchhoff system has received considerable attention. Yang [7] considered the following Kirchhoff system with Sobolev nonlinearities:

$$\begin{cases} -(a_1 + b_1 \int_{\mathbb{R}^N} |\nabla u|^2 dx) \Delta u + \lambda_1 u = \mu_1 |u|^{p-2} u + r_1 \beta |u|^{r_1-2} |v|^{r_2} u & \text{in } \mathbb{R}^N, \\ -(a_2 + b_2 \int_{\mathbb{R}^N} |\nabla v|^2 dx) \Delta v + \lambda_2 v = \mu_2 |v|^{q-2} v + r_2 \beta |u|^{r_1} |v|^{r_2-2} v & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} u^2 dx = d_1^2, \quad \int_{\mathbb{R}^N} v^2 dx = d_2^2, \end{cases} \quad (4)$$

where $2 \leq N \leq 4$ and $2 + \frac{8}{N} < r_1 + r_2 < p, q < 2^*$. By minimax methods, the author obtained a ground state normalized solution under L^2 -supercritical case with the assumption that the coupling coefficient is big enough. When $a_1 = b_1 = a_2 = b_2 = 1$ and $r_1 + r_2 = 2^*$, by leveraging the Pohozaev manifold and minimizing the energy functional, Zhang and

Zhang [8] established a positive ground state solution to (4) under L^2 -subcritical case. Xu and Wang [9] explored the positive solutions through the constrained variational method.

For the fractional Kirchhoff system, Li et al. [10] investigated the following system:

$$\begin{cases} (1 + b \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx)(-\Delta)^s u + V_1(x)u + \lambda_1 u = \mu_1 |u|^{p_1-2} u + r_1 \beta |u|^{r_1-2} |v|^{r_2} u \text{ in } \mathbb{R}^N, \\ (1 + b \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v|^2 dx)(-\Delta)^s v + V_2(x)v + \lambda_2 v = \mu_2 |v|^{p_2-2} v + r_2 \beta |u|^{r_1} |v|^{r_2-2} v \text{ in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} u^2 dx = a_1^2, \int_{\mathbb{R}^N} v^2 dx = a_2^2, \end{cases}$$

where $2s < N < 4s$ and $2 + \frac{8s}{N} \leq p_1, p_2, r_1 + r_2 < 2s^*$. Through a constrained minimization problem and minimax theorem, they achieved the existence of non-negative solutions. When $N = 2, 3$ and $V_1(x) = V_2(x) = 0$, Kong and Chen [11] obtained the positive solutions under L^2 -subcritical case and L^2 -supercritical case by using minimax theorem and introducing some auxiliary minimization problems. For more results with regard to the Kirchhoff system, we can refer to [12–19].

Based on the investigation of the fractional Schrödinger systems, studies for the fractional Kirchhoff systems with nonlinearities are more challenging and interesting due to the existence of nonlocal terms $(\int |(-\Delta)^{\frac{s}{2}} u|^2 dx)(-\Delta)^s u$ and $(\int |(-\Delta)^{\frac{s}{2}} v|^2 dx)(-\Delta)^s v$. Thus, to extend the results above, we consider the normalized solutions for the fractional Kirchhoff system with Hardy critical terms (1) under L^2 -supercritical case.

Firstly, we introduce some notations for subsequent use. Throughout this paper, for any $1 \leq r < \infty, L^r(\mathbb{R}^{3s})$ stands for the Lebesgue space with the norm

$$\|u\|_r = \left(\int_{\mathbb{R}^{3s}} |u|^r dx \right)^{\frac{1}{r}}.$$

$H^s(\mathbb{R}^{3s})$ is the fractional Hilbert space:

$$H^s(\mathbb{R}^{3s}) := \left\{ u \in L^2(\mathbb{R}^{3s}) : (-\Delta)^{\frac{s}{2}} u \in L^2(\mathbb{R}^{3s}) \right\},$$

with its standard inner product and norm expressed respectively as

$$\langle u, v \rangle := \int_{\mathbb{R}^{3s}} \left((-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v + uv \right) dx, \quad \|u\|_{H^s}^2 = \langle u, u \rangle = \left\| (-\Delta)^{\frac{s}{2}} u \right\|_2^2 + \|u\|_2^2,$$

where

$$\left\| (-\Delta)^{\frac{s}{2}} u \right\|_2^2 = \int_{\mathbb{R}^{3s}} \int_{\mathbb{R}^{3s}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy.$$

Solutions to system (1) can be identified as critical points of energy functional $I(u, v) : H^s(\mathbb{R}^{3s}) \times H^s(\mathbb{R}^{3s}) \rightarrow \mathbb{R} :$

$$\begin{aligned} I(u, v) &:= \frac{a}{2} \left(\left\| (-\Delta)^{\frac{s}{2}} u \right\|_2^2 + \left\| (-\Delta)^{\frac{s}{2}} v \right\|_2^2 \right) + \frac{b}{4} \left(\left\| (-\Delta)^{\frac{s}{2}} u \right\|_2^4 + \left\| (-\Delta)^{\frac{s}{2}} v \right\|_2^4 \right) \\ &\quad - \frac{\mu_1}{p} \int_{\mathbb{R}^{3s}} |u|^p dx - \frac{\mu_2}{q} \int_{\mathbb{R}^{3s}} |v|^q dx - \nu \int_{\mathbb{R}^{3s}} \frac{|u|^{r_1} |v|^{r_2}}{|x|^\alpha} dx. \end{aligned}$$

One can readily verify that the energy functional is of class C^1 , and solution (u, v) serves as a normalized solution for (1) if $(u, v) \in \mathcal{T}(c, d)$, where

$$\mathcal{T}(c, d) := \left\{ (u, v) \in H^s(\mathbb{R}^{3s}) \times H^s(\mathbb{R}^{3s}) : \|u\|_2^2 = c^2, \|v\|_2^2 = d^2 \right\}.$$

We then present the definition of the homotopy-stable family and the minimax theorem, they can help to establish a Palais-Smale sequence later.

Definition 1 ([20], Definition 5.1). *Let X be a topological space and Y be a closed subset of X . We shall say that a class \mathcal{F} of compact subsets of X is a homotopy-stable family with extended boundary Y if for any set Z in \mathcal{F} and any $\gamma \in C([0, 1] \times X, X)$ satisfying $\gamma(t, x) = x$ for all $(t, x) \in (\{0\} \times X) \cup ([0, 1] \times Y)$ we have that $\gamma(\{1\} \times Z) \in \mathcal{F}$.*

Lemma 1 ([20], Theorem 3.2). *Let φ be a C^1 -functional on a complete connected C^1 -Finsler manifold X (without boundary) and consider a homotopy-stable family \mathcal{F} of compact subsets of X with a closed boundary B . Set*

$$c = c(\varphi, \mathcal{F}) = \inf_{A \in \mathcal{F}} \max_{x \in A} \varphi(x),$$

and suppose that

$$\sup_{x \in B} \varphi(x) < c.$$

Then, for any sequence of sets $\{A_n\}$ in \mathcal{F} such that

$$\lim_{n \rightarrow \infty} \sup_{x \in A_n} \varphi(x) = c,$$

there exists a sequence $\{x_n\}$ in X such that

- (1) $\lim_{n \rightarrow \infty} \varphi(x_n) = c;$
- (2) $\lim_{n \rightarrow \infty} \|d\varphi(x_n)\| = 0;$
- (3) $\lim_{n \rightarrow \infty} \text{dist}(x_n, A_n) = 0.$

Moreover, if $d\varphi$ is uniformly continuous, then x_n can be chosen to be in A_n for each n .

To accommodate the constraint $\mathcal{T}(c, d)$, we introduce a L^2 -norm preserving transformation $t \star u(x) := e^{\frac{N}{2}t} u(e^t x)$ and

$$(u, v) \in \mathcal{T}(c, d) \mapsto t \star (u, v) := (t \star u, t \star v) \in \mathcal{T}(c, d).$$

Then we have the fiber map:

$$\begin{aligned} \Phi_{(u,v)}(t) &= I(t \star (u, v)) \\ &= \frac{a}{2} e^{2st} \left(\|(-\Delta)^{\frac{s}{2}} u\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v\|_2^2 \right) + \frac{b}{4} e^{4st} \left(\|(-\Delta)^{\frac{s}{2}} u\|_2^4 + \|(-\Delta)^{\frac{s}{2}} v\|_2^4 \right) \\ &\quad - \frac{\mu_1}{p} e^{p s \gamma_{p,s} t} \int_{\mathbb{R}^{3s}} |u|^p dx - \frac{\mu_2}{q} e^{q s \gamma_{q,s} t} \int_{\mathbb{R}^{3s}} |v|^q dx - \nu e^{2^*_{s,\alpha} s t} \int_{\mathbb{R}^{3s}} \frac{|u|^{r_1} |v|^{r_2}}{|x|^\alpha} dx. \end{aligned} \tag{5}$$

We have the following result.

Theorem 1. *Let $N = 3s$, $\frac{14}{3} < p, q < 6$, $r_1, r_2 \geq 1$ and $r_1 + r_2 = 2^*_{s,\alpha}$. There exists a constant $\nu_1 > 0$, for all $\nu > \nu_1$, system (1) has a positive radial solution (u_0, v_0) with $\lambda_1, \lambda_2 > 0$.*

In this theorem, the main difficulty is to obtain the strong convergence of the Palais-Smale sequence. To overcome the difficulty, we establish the estimate of energy level σ which will be mentioned later. Moreover, we obtain the normalized solution through the minimax theorem and all the studies are investigated in the radial setting to ensure the critical sequence is strongly convergent in $L^p \times L^p$.

This paper is organized as follows. In Section 2, we give some preliminary results which will be used later. In Section 3, we give the proof of Theorem 1.

Throughout this work, $\|u\|_p$ denotes the L^p -norm of u , “ \rightarrow ” and “ \rightharpoonup ” stand for strong convergence and weak convergence in the corresponding space, u_n^- refers to the negative part of u_n . C is adopted to denote a generic constant, whose value may change from line to line.

2. Preliminaries

This section is devoted to establishing several auxiliary lemmas needed throughout the work.

Lemma 2 ([21] Fractional Gagliardo-Nirenberg inequality). *Let $p \in (\frac{14}{3}, 6)$. Then there exists a constant $C(N, p, s) > 0$ such that*

$$\|u\|_p^p \leq C(N, p, s) \|(-\Delta)^{\frac{s}{2}} u\|_2^{p\gamma_{p,s}} \|u\|_2^{p(1-\gamma_{p,s})}, \quad \forall u \in H^s(\mathbb{R}^{3s}), \tag{6}$$

where $\gamma_{p,s} := \frac{3(p-2)}{2p}$.

Then, we introduce the following fractional Sobolev embedding.

Lemma 3 ([22]). *There exists the best Hardy-Sobolev constant defined by*

$$\mathcal{S} := \inf_{u \in H^s(\mathbb{R}^{3s}) \setminus \{0\}} \frac{\|(-\Delta)^{\frac{s}{2}} u\|_2^2}{\left(\int_{\mathbb{R}^{3s}} \frac{|u|^{2^*_{s,\alpha}}}{|x|^\alpha} dx\right)^{\frac{2}{2^*_{s,\alpha}}}}, \tag{7}$$

and

$$\mathcal{S}_{r_1, r_2} := \inf_{u, v \in H^s(\mathbb{R}^{3s}) \setminus \{0\}} \frac{\|(-\Delta)^{\frac{s}{2}} u\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v\|_2^2}{\left(\int_{\mathbb{R}^{3s}} \frac{|u|^{r_1} |v|^{r_2}}{|x|^\alpha} dx\right)^{\frac{2}{2^*_{s,\alpha}}}}, \tag{8}$$

then we introduce the following Hardy-Sobolev inequality:

$$\int_{\mathbb{R}^{3s}} \frac{|u|^{r_1} |v|^{r_2}}{|x|^\alpha} \leq \mathcal{S}_{r_1, r_2}^{-\frac{2^*_{s,\alpha}}{2}} \left(\|(-\Delta)^{\frac{s}{2}} u\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v\|_2^2 \right)^{\frac{2^*_{s,\alpha}}{2}}. \tag{9}$$

Furthermore, we introduce the scalar equation:

$$\begin{cases} (a + b \int_{\mathbb{R}^{3s}} |(-\Delta)^{\frac{s}{2}} u|^2 dx) (-\Delta)^s u + \lambda u = \mu |u|^{p-2} u, & \text{in } \mathbb{R}^{3s}, \\ \int_{\mathbb{R}^{3s}} |u|^2 dx = c^2, \end{cases} \tag{10}$$

where $\frac{14}{3} < p < 6$, whose results are crucial to the proof of Theorem 1.

Normalized solutions to (10) correspond to the critical points of the energy functional presented below:

$$E_{\mu,p}(u) = \frac{a}{2} \int_{\mathbb{R}^{3s}} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \frac{b}{4} \left(\int_{\mathbb{R}^{3s}} |(-\Delta)^{\frac{s}{2}} u|^2 dx \right)^2 - \frac{\mu}{p} \int_{\mathbb{R}^{3s}} |u|^p dx, \tag{11}$$

under the constraint $S(c) = \{u \in H^s(\mathbb{R}^{3s}) : \|u\|_2^2 = c^2\}$. We introduce the least energy level

$$e_{\mu,p}(c) := \inf_{u \in S(c)} E_{\mu,p}(u), \tag{12}$$

there are following results.

Lemma 4 ([10]). Let $p \in (\frac{14}{3}, 6)$, $a, b, c > 0$ and $\mu > 0$. Set

$$e_{\mu,p}(c) = \inf_{\|u\|_2^2=c^2} E_{\mu,p}(u).$$

Then,

- (1) there exists a unique positive radial solution u_μ for (10);
- (2) $\sup_{t \in \mathbb{R}^{3s}} E_{\mu,p}(t \star u_\mu) = E_{\mu,p}(u_\mu)$;
- (3) $e_{\mu,p}(c) = \inf_{\|u\|_2^2=c^2} E_{\mu,p}(u) = E_{\mu,p}(u_\mu) > 0$;
- (4) the least energy $e_{\mu,p}(c)$ is strictly decreasing with respect to $c > 0$.

The subsequent Liouville-type result will allow us to obtain the strong convergence of a Palais-Smale sequence.

Lemma 5 ([23]). Suppose $u \in H^s(\mathbb{R}^{3s})$ is a solution of the following equation:

$$\begin{cases} (-\Delta)^s u \geq 0, & x \in \mathbb{R}^{3s}, \\ u \geq 0 \text{ and } u \in L^p(\mathbb{R}^{3s}), & p \in (0, 3], \end{cases}$$

then $u \equiv 0$.

3. Proof of the Theorem 3

In this section, we give the proof of Theorem 1. To begin with, we denote

$$H_r^s(\mathbb{R}^{3s}) := \{u \in H^s(\mathbb{R}^{3s}) : u(x) = u(|x|), \forall x \in \mathbb{R}^{3s}\},$$

and the embedding $H_r^s(\mathbb{R}^{3s}) \hookrightarrow L^p(\mathbb{R}^{3s})$ is compact for $2 < p < 6$. Then we set

$$\mathcal{T}_r(c, d) := \{(u, v) \in H_r^s(\mathbb{R}^{3s}) \times H_r^s(\mathbb{R}^{3s}) : \int_{\mathbb{R}^{3s}} |u|^2 dx = c^2 \text{ and } \int_{\mathbb{R}^{3s}} |v|^2 dx = d^2\}.$$

Applying the Palais’s symmetric criticality principle in [24] (Theorem 1.28), we obtain that the critical point of $I(u, v)$ in $\mathcal{T}_r(c, d)$ is equivalent to the critical point of $I(u, v)$ in $\mathcal{T}(c, d)$. Thus, we will investigate under the radial setting.

Before introducing the lemmas, we have the following conclusions. By direct calculations, for any $(u, v) \in \mathcal{T}(c, d)$,

$$\begin{aligned} & \lim_{t \rightarrow +\infty} \int_{\mathbb{R}^{3s}} \left(|(-\Delta)^{\frac{s}{2}}(t \star u)|^2 + |(-\Delta)^{\frac{s}{2}}(t \star v)|^2 \right) dx \\ &= \lim_{t \rightarrow +\infty} e^{2st} \left(\|(-\Delta)^{\frac{s}{2}} u\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v\|_2^2 \right) = +\infty, \end{aligned}$$

and

$$\begin{aligned} & \lim_{t \rightarrow -\infty} \int_{\mathbb{R}^{3s}} \left(|(-\Delta)^{\frac{s}{2}}(t \star u)|^2 + |(-\Delta)^{\frac{s}{2}}(t \star v)|^2 \right) dx \\ &= \lim_{t \rightarrow -\infty} e^{2st} \left(\|(-\Delta)^{\frac{s}{2}} u\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v\|_2^2 \right) = 0^+. \end{aligned}$$

Since

$$\begin{aligned} \Phi_{(u,v)}(t) &= I(t \star (u, v)) \\ &= \frac{a}{2} e^{2st} \left(\|(-\Delta)^{\frac{s}{2}} u\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v\|_2^2 \right) + \frac{b}{4} e^{4st} \left(\|(-\Delta)^{\frac{s}{2}} u\|_2^4 + \|(-\Delta)^{\frac{s}{2}} v\|_2^4 \right) \\ &\quad - \frac{\mu_1}{p} e^{ps\gamma_{p,s}t} \int_{\mathbb{R}^{3s}} |u|^p dx - \frac{\mu_2}{q} e^{qs\gamma_{q,s}t} \int_{\mathbb{R}^{3s}} |v|^q dx - \nu e^{2^*_s a st} \int_{\mathbb{R}^{3s}} \frac{|u|^{r_1} |v|^{r_2}}{|x|^\alpha} dx, \end{aligned}$$

and $p\gamma_{p,s}, q\gamma_{q,s}, 2_{s,\alpha}^* > 4$, we can conclude that

$$\lim_{t \rightarrow +\infty} \Phi_{(u,v)}(t) = -\infty, \tag{13}$$

and

$$\lim_{t \rightarrow -\infty} \Phi_{(u,v)}(t) = 0^+. \tag{14}$$

Lemma 6. *There exists a fixed constant $K > 0$ such that, for sufficiently small K ,*

$$\inf_{(u,v) \in D_1} I(u,v) > 0 \quad \text{and} \quad \sup_{(u,v) \in D_1} I(u,v) < \inf_{(u,v) \in D_2} I(u,v),$$

where

$$D_1 := \left\{ (u,v) \in \mathcal{T}_r(c,d) : \int_{\mathbb{R}^{3s}} \left(|(-\Delta)^{\frac{s}{2}} u|^2 + |(-\Delta)^{\frac{s}{2}} v|^2 \right) dx \leq K \right\},$$

and

$$D_2 := \left\{ (u,v) \in \mathcal{T}_r(c,d) : \int_{\mathbb{R}^{3s}} \left(|(-\Delta)^{\frac{s}{2}} u|^2 + |(-\Delta)^{\frac{s}{2}} v|^2 \right) dx = 2K \right\}.$$

Proof. In view of (6), for $(u,v) \in \mathcal{T}(c,d)$, we have

$$\|u\|_p^p \leq C(N,p,s) \|(-\Delta)^{\frac{s}{2}} u\|_2^{p\gamma_{p,s}} \|u\|_2^{p(1-\gamma_{p,s})}, \tag{15}$$

and

$$\|v\|_q^q \leq C(N,q,s) \|(-\Delta)^{\frac{s}{2}} v\|_2^{q\gamma_{q,s}} \|v\|_2^{q(1-\gamma_{q,s})}. \tag{16}$$

By (9), we have

$$\int_{\mathbb{R}^{3s}} \frac{|u|^{r_1} |v|^{r_2}}{|x|^\alpha} \leq S_{r_1,r_2}^{-\frac{2_{s,\alpha}^*}{2}} \left(\|(-\Delta)^{\frac{s}{2}} u\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v\|_2^2 \right)^{\frac{2_{s,\alpha}^*}{2}},$$

then for $(u,v) \in D_1$ we obtain that

$$\begin{aligned} I(u,v) &\geq \frac{a}{2} \left(\|(-\Delta)^{\frac{s}{2}} u\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v\|_2^2 \right) + \frac{b}{8} \left(\|(-\Delta)^{\frac{s}{2}} u\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v\|_2^2 \right)^2 \\ &\quad - \frac{\mu_1}{p} \int_{\mathbb{R}^{3s}} |u|^p dx - \frac{\mu_2}{q} \int_{\mathbb{R}^{3s}} |v|^q dx - v \int_{\mathbb{R}^{3s}} \frac{|u|^{r_1} |v|^{r_2}}{|x|^\alpha} dx \\ &\geq \frac{a}{2} \left(\|(-\Delta)^{\frac{s}{2}} u\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v\|_2^2 \right) + \frac{b}{8} \left(\|(-\Delta)^{\frac{s}{2}} u\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v\|_2^2 \right)^2 \\ &\quad - \frac{\mu_1}{p} C(N,p,s) c^{p(1-\gamma_{p,s})} \|(-\Delta)^{\frac{s}{2}} u\|_2^{p\gamma_{p,s}} - \frac{\mu_2}{q} C(N,q,s) d^{q(1-\gamma_{q,s})} \|(-\Delta)^{\frac{s}{2}} v\|_2^{q\gamma_{q,s}} \\ &\quad - S_{r_1,r_2}^{-\frac{2_{s,\alpha}^*}{2}} v \left(\|(-\Delta)^{\frac{s}{2}} u\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v\|_2^2 \right)^{\frac{2_{s,\alpha}^*}{2}}, \end{aligned}$$

since $\frac{14}{3} < p, q < 6$ and $2_{s,\alpha}^* > 4$, we can deduce that there exists sufficiently small $K > 0$ such that $\inf_{(u,v) \in D_1} I(u,v) > 0$.

For $\forall(u_1, v_1) \in D_2$ and $\forall(u_2, v_2) \in D_1$, there holds

$$\begin{aligned} I(u_1, v_1) - I(u_2, v_2) &\geq \frac{a}{2} \left(\|(-\Delta)^{\frac{s}{2}} u_1\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v_1\|_2^2 \right) + \frac{b}{8} \left(\|(-\Delta)^{\frac{s}{2}} u_1\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v_1\|_2^2 \right)^2 \\ &\quad - \frac{\mu_1}{p} C(N, p, s) \|(-\Delta)^{\frac{s}{2}} u\|_2^{p\gamma_{p,s}} - \frac{\mu_2}{q} C(N, q, s) \|(-\Delta)^{\frac{s}{2}} v\|_2^{q\gamma_{q,s}} \\ &\quad - \mathcal{S}_{r_1, r_2}^{-\frac{2^*s}{2}} v \left(\|(-\Delta)^{\frac{s}{2}} u\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v\|_2^2 \right)^{\frac{2^*s}{2}} \\ &\quad - \frac{a}{2} \left(\|(-\Delta)^{\frac{s}{2}} u_2\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v_2\|_2^2 \right) - \frac{b}{4} \left(\|(-\Delta)^{\frac{s}{2}} u_1\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v_1\|_2^2 \right)^2 \\ &\geq \frac{a}{2} K + \frac{b}{4} K^2 - \frac{\mu_1}{p} C(N, p, s) (2K)^{\frac{p\gamma_{p,s}}{2}} - \frac{\mu_2}{q} C(N, q, s) (2K)^{\frac{q\gamma_{q,s}}{2}} \\ &\quad - \mathcal{S}_{r_1, r_2}^{-\frac{2^*s}{2}} v (2K)^{\frac{2^*s}{2}}. \end{aligned}$$

Since $\frac{14}{3} < p, q < 6$ and $2^*_{s,\kappa} > 4$, we get $I(u_1, v_1) - I(u_2, v_2) > 0$ with small $K > 0$, that is $\sup_{(u,v) \in D_1} I(u, v) < \inf_{(u,v) \in D_2} I(u, v)$. \square

Denote

$$D_3 := \left\{ (u, v) \in \mathcal{T}_r(c, d) : \int_{\mathbb{R}^{3s}} \left(|(-\Delta)^{\frac{s}{2}} u|^2 + |(-\Delta)^{\frac{s}{2}} v|^2 \right) dx \geq 3K, I(u, v) \leq 0 \right\},$$

and

$$\Gamma := \{ \gamma \in C([0, 1], \mathcal{T}_r(c, d)) : \gamma(0) = (u_1, v_1), \gamma(1) = (u_2, v_2) \},$$

where $(u_1, v_1) \in D_1$ and $(u_2, v_2) \in D_3$.

According to (13), (14) and Lemma 6, we obtain that

$$\sigma := \inf_{\gamma \in \Gamma} \max_{\xi \in [0, 1]} I(\gamma(\xi)) > \max\{I(u_1, v_1), I(u_2, v_2)\} > 0.$$

Then we let

$$D_1^+ := \left\{ (u, v) \in D_1 : u, v \geq 0 \text{ a.e. in } \mathbb{R}^{3s} \right\},$$

and

$$D_3^+ := \left\{ (u, v) \in D_3 : u, v \geq 0 \text{ a.e. in } \mathbb{R}^{3s} \right\},$$

for any $(\bar{u}_1, \bar{v}_1) \in D_1^+$ and $(\bar{u}_2, \bar{v}_2) \in D_3^+$, we define

$$\Gamma_1 := \{ \gamma \in C([0, 1], \mathcal{T}_r(c, d)) : \gamma(0) = (\bar{u}_1, \bar{v}_1), \gamma(1) = (\bar{u}_2, \bar{v}_2) \},$$

in view of [25], D_1^+ and D_3^+ are connected by arcs and

$$\sigma := \inf_{\gamma \in \Gamma} \max_{\xi \in [0, 1]} I(\gamma(\xi)) = \inf_{\gamma \in \Gamma_1} \max_{\xi \in [0, 1]} I(\gamma(\xi)).$$

Nextly, we introduce some properties of the level σ .

Lemma 7. *There exists a constant $\nu_1 > 0$, such that for any $\nu > \nu_1$, we have*

$$0 < \sigma < \min\{e_{\mu_1, p}(c), e_{\mu_2, q}(d)\},$$

where

$$e_{\mu_1, p}(c) := \inf_{u \in S(c)} E_{\mu_1, p}(u) \text{ and } e_{\mu_2, q}(d) := \inf_{v \in S(d)} E_{\mu_2, q}(v).$$

Proof. Let u_{μ_1} denote the unique positive critical point to $E_{\mu_1,p}(u)$ and let v_{μ_2} denote the unique positive critical point to $E_{\mu_2,q}(v)$. According to (13) and (14) there exists $t_1 < 0$ and $t_2 > 0$ such that

$$(\bar{u}_1, \bar{v}_1) := (t_1 \star u_{\mu_1}, t_1 \star v_{\mu_2}) \in D_1^+$$

and

$$(\bar{u}_2, \bar{v}_2) := (t_2 \star u_{\mu_1}, t_2 \star v_{\mu_2}) \in D_3^+.$$

Let

$$\gamma_1(\xi) := ((1 - \xi)t_1 + \xi t_2) \star (u_{\mu_1}, v_{\mu_2}),$$

it is easy to see that

$$\gamma_1(0) := t_1 \star (u_{\mu_1}, v_{\mu_2}) \in D_1^+ \text{ and } \gamma_1(1) := t_2 \star (u_{\mu_1}, v_{\mu_2}) \in D_3^+,$$

which implies that $\gamma_1(\xi) \in \Gamma_1$, thus

$$\begin{aligned} \sigma &= \inf_{\gamma \in \Gamma_1} \max_{\xi \in [0,1]} I(\gamma(\xi)) \\ &\leq \max_{\xi \in [0,1]} I(\gamma_1(\xi)) \\ &= \max_{\xi \in [0,1]} I(((1 - \xi)t_1 + \xi t_2) \star (u_{\mu_1}, v_{\mu_2})) \\ &\leq \sup_{t \in \mathbb{R}} I((t \star u_{\mu_1}, t \star v_{\mu_2})). \end{aligned}$$

Then we only need to show that $\sup_{t \in \mathbb{R}} I((t \star u_{\mu_1}, t \star v_{\mu_2})) < \min\{e_{\mu_1,p}(c), e_{\mu_2,q}(d)\}$.

Through direct calculations, we have

$$E_{\mu_1,p}(t \star u_{\mu_1}) = \frac{a}{2} e^{2st} \int_{\mathbb{R}^{3s}} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \frac{b}{4} e^{4st} \left(\int_{\mathbb{R}^{3s}} |(-\Delta)^{\frac{s}{2}} u|^2 dx \right)^2 - \frac{\mu_1}{p} e^{p\gamma_{p,s}st} \int_{\mathbb{R}^{3s}} |u|^p dx,$$

and

$$E_{\mu_2,q}(t \star v_{\mu_2}) = \frac{a}{2} e^{2st} \int_{\mathbb{R}^{3s}} |(-\Delta)^{\frac{s}{2}} v|^2 dx + \frac{b}{4} e^{4st} \left(\int_{\mathbb{R}^{3s}} |(-\Delta)^{\frac{s}{2}} v|^2 dx \right)^2 - \frac{\mu_2}{q} e^{q\gamma_{q,s}st} \int_{\mathbb{R}^{3s}} |v|^q dx,$$

when $t \rightarrow -\infty$, $E_{\mu_1,p}(t \star u_{\mu_1}) \rightarrow 0^+$ and $E_{\mu_2,q}(t \star v_{\mu_2}) \rightarrow 0^+$. We can infer the existence of a small constant $t_0 < 0$ such that

$$\begin{aligned} \sup_{t \leq t_0} I((t \star u_{\mu_1}, t \star v_{\mu_2})) &< \sup_{t \leq t_0} \{E_{\mu_1,p}(t \star u_{\mu_1}) + E_{\mu_2,q}(t \star v_{\mu_2})\} \\ &< \min\{e_{\mu_1,p}(c), e_{\mu_2,q}(d)\}. \end{aligned}$$

When $t \geq t_0$, we have

$$\begin{aligned} \int_{\mathbb{R}^{3s}} \frac{|t \star u_{\mu_1}|^{r_1} |t \star v_{\mu_2}|^{r_2}}{|x|^\alpha} dx &= e^{2^*_{s,a}st} \int_{\mathbb{R}^{3s}} \frac{|u|^{r_1} |v|^{r_2}}{|x|^\alpha} dx \\ &\geq e^{2^*_{s,a}st_0} \int_{\mathbb{R}^{3s}} \frac{|u|^{r_1} |v|^{r_2}}{|x|^\alpha} dx. \end{aligned}$$

Then in view of Lemma 4, there exists a constant $\nu_1 > 0$, when $\nu > \nu_1$, it holds

$$\begin{aligned} \sup_{t \geq t_0} I((t \star u_{\mu_1}, t \star v_{\mu_2})) &\leq \sup_{t \geq t_0} \{E_{\mu_1,p}(t \star u_{\mu_1}) + E_{\mu_2,q}(t \star v_{\mu_2})\} - \nu e^{2^*_{s,a}st} \int_{\mathbb{R}^{3s}} \frac{|u|^{r_1}|v|^{r_2}}{|x|^\alpha} dx \\ &\leq \sup_{t \geq t_0} E_{\mu_1,p}(t \star u_{\mu_1}) + \sup_{t \geq t_0} E_{\mu_2,q}(t \star v_{\mu_2}) - \nu e^{2^*_{s,a}st_0} \int_{\mathbb{R}^{3s}} \frac{|u|^{r_1}|v|^{r_2}}{|x|^\alpha} dx \\ &\leq e_{\mu_1,p}(c) + e_{\mu_2,q}(d) - \nu e^{2^*_{s,a}st_0} \int_{\mathbb{R}^{3s}} \frac{|u|^{r_1}|v|^{r_2}}{|x|^\alpha} dx \\ &< \min\{e_{\mu_1,p}(c), e_{\mu_2,q}(d)\}. \end{aligned}$$

Thus, we obtain that $\sigma < \sup_{t \in \mathbb{R}} I((t \star u_{\mu_1}, t \star v_{\mu_2})) < \min\{e_{\mu_1,p}(c), e_{\mu_2,q}(d)\}$. This completes the proof. \square

Lemma 8. Let $\tilde{I}(t, u, v) := I(t \star (u, v))$ and

$$\tilde{\sigma} := \inf_{\gamma \in \tilde{\Gamma}} \max_{t \in [0,1]} I(\gamma(t)), \tag{17}$$

where

$$\tilde{\Gamma} := \{\tilde{\gamma} \in C([0, 1], \mathbb{R} \times \mathcal{T}_r(c, d)) : \tilde{\gamma}(0) = (0, u_1, v_1), \tilde{\gamma}(1) = (0, u_2, v_2)\},$$

then $\sigma = \tilde{\sigma}$.

Proof. For every $\gamma \in \Gamma$, we denote $\tilde{\gamma} := (0, \gamma) \in \tilde{\Gamma}$, which indicates $\tilde{\sigma} \leq \sigma$. On the other hand, for any $\tilde{\gamma} = (0, \gamma) = (0, \gamma_1, \gamma_2) \in \tilde{\Gamma}$, there holds

$$\tilde{I}(\tilde{\gamma}) = \tilde{I}(0, \gamma_1, \gamma_2) = I(\gamma_1, \gamma_2),$$

since $(\gamma_1, \gamma_2) \in \Gamma$, we can conclude that $\sigma \leq \tilde{\sigma}$. Thus we obtain that $\sigma = \tilde{\sigma}$. \square

Lemma 9. The functional $\tilde{I}(u, v)$ admits a Palais-Smale sequence $(u_n, v_n) \subset \mathcal{T}_r(c, d)$ at level σ with $u_n^-, v_n^- \rightarrow 0$ as $n \rightarrow \infty$.

Proof. In view of the Definition 1, it is obvious that $\mathcal{F} = \{Z = \gamma([0, 1]) : \gamma \in \tilde{\Gamma}\}$ is a homotopy stable family of compact subsets of $X = \mathbb{R} \times \mathcal{T}_r(c, d)$ with boundary $Y = (0, D_1^+) \cup (0, D_3^+)$. Since \tilde{I} is a C^1 -functional on the C^1 -Finsler manifold X , applying Lemma 1, we can deduce the existence of a Palais-Smale sequence (t_n, u_n, v_n) for $\tilde{I}(u, v)$ on $\mathbb{R} \times \mathcal{T}_r(c, d)$ at level $\tilde{\sigma}$ with the following properties:

- (1) $\lim_{n \rightarrow +\infty} \tilde{I}(t_n, u_n, v_n) = \tilde{\sigma} = \sigma$;
- (2) $\lim_{n \rightarrow +\infty} \partial_t \tilde{I}(t_n, u_n, v_n) = 0$;
- (3) $\lim_{n \rightarrow +\infty} |t_n| + \text{dist}((u_n, v_n), \gamma_n([0, 1])) = 0$.

According to the properties of Palais-Smale sequence, for all $u, v \in H_r^s(\mathbb{R}^{3s})$ with $\int_{\mathbb{R}^{3s}} u_n u = 0, \int_{\mathbb{R}^{3s}} v_n v = 0$, there holds

$$\langle \tilde{I}'(t_n, u_n, v_n), (t, u, v) \rangle = o(1)(|t| + \|u\|_{H^s} + \|v\|_{H^s}). \tag{18}$$

Moreover, (3) implies that $u_n^-, v_n^- \rightarrow 0$ a.e. in \mathbb{R}^{3s} . \square

Proof of Theorem 1. Step 1. Boundedness of $\{u_n, v_n\}$ in $H_r^s(\mathbb{R}^{3s}) \times H_r^s(\mathbb{R}^{3s})$.

Since $\lim_{n \rightarrow +\infty} \tilde{I}(t_n, u_n, v_n) = 0$, we have

$$\begin{aligned} \frac{\partial}{\partial t} \tilde{I}(t_n, u_n, v_n) &= \left\langle \tilde{I}'(t_n, u_n, v_n), (1, 0, 0) \right\rangle \\ &= ase^{2st} \left(\|(-\Delta)^{\frac{s}{2}} u\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v\|_2^2 \right) + bse^{4st} \left(\|(-\Delta)^{\frac{s}{2}} u\|_2^4 + \|(-\Delta)^{\frac{s}{2}} v\|_2^4 \right) \\ &\quad - \mu_1 s \gamma_{p,s} e^{ps\gamma_{p,s}t} \int_{\mathbb{R}^{3s}} |u|^p dx - \mu_2 s \gamma_{q,s} e^{qs\gamma_{q,s}t} \int_{\mathbb{R}^{3s}} |v|^q dx \\ &\quad - \nu 2_{s,\alpha}^* s e^{2_{s,\alpha}^* st} \int_{\mathbb{R}^{3s}} \frac{|u|^{r_1} |v|^{r_2}}{|x|^\alpha} dx \\ &= o_n(1). \end{aligned}$$

Combined with $\lim_{n \rightarrow +\infty} \tilde{I}(t_n, u_n, v_n) = \sigma$, we have

$$\begin{aligned} \sigma + o_n(1) &= \tilde{I}(t_n, u_n, v_n) - \frac{1}{4s} \left\langle \tilde{I}'(t_n, u_n, v_n), (1, 0, 0) \right\rangle \\ &= \frac{a}{4} e^{2st} \left(\|(-\Delta)^{\frac{s}{2}} u\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v\|_2^2 \right) + \left(\frac{\gamma_{p,s}}{4} - \frac{1}{p} \right) \mu_1 e^{ps\gamma_{p,s}t} \int_{\mathbb{R}^{3s}} |u|^p dx \\ &\quad + \left(\frac{\gamma_{q,s}}{4} - \frac{1}{q} \right) \mu_2 e^{qs\gamma_{q,s}t} \int_{\mathbb{R}^{3s}} |v|^q dx + \left(\frac{2_{s,\alpha}^*}{4} - 1 \right) \nu e^{2_{s,\alpha}^* st} \int_{\mathbb{R}^{3s}} \frac{|u|^{r_1} |v|^{r_2}}{|x|^\alpha} dx. \end{aligned}$$

Since $2 + \frac{8}{3} < p, q < 6$ and $2_{s,\alpha}^* > 4$, the coefficients of the terms above are positive, then there exists some $C > 0$,

$$\begin{aligned} \|(-\Delta)^{\frac{s}{2}} u\|_2^2 + \|(-\Delta)^{\frac{s}{2}} v\|_2^2 &\leq C, \quad \int_{\mathbb{R}^{3s}} |u|^p dx \leq C \\ \int_{\mathbb{R}^{3s}} |v|^q dx &\leq C, \quad \int_{\mathbb{R}^{3s}} \frac{|u|^{r_1} |v|^{r_2}}{|x|^\alpha} dx \leq C, \end{aligned}$$

thus $\{u_n, v_n\}$ is bounded in $H_r^s(\mathbb{R}^{3s}) \times H_r^s(\mathbb{R}^{3s})$.

Taking (t, u, v) in (18) to be $(0, u, v)$, combined with the boundedness of $\{u_n, v_n\}$ and $t_n \rightarrow 0$, we have

$$\begin{aligned} \langle I'(u_n, v_n), (u, v) \rangle &= \left\langle \tilde{I}'(t_n, u_n, v_n), (0, u, v) \right\rangle + O(|t_n|)(\|u\|_{H^s} + \|v\|_{H^s}) \\ &= o(1)(\|u\|_{H^s} + \|v\|_{H^s}), \end{aligned}$$

thus we obtain that $(u_n, v_n) \subset \mathcal{T}_r(c, d)$ is a Palais-Smale sequence for $I(u, v)$ at level σ . Similar to [6], for every $\phi, \psi \in H_r^s(\mathbb{R}^{3s})$, one can find a sequence $(\lambda_{1,n}, \lambda_{2,n})$ such that,

$$\begin{aligned} &a \left(\int_{\mathbb{R}^{3s}} (-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}} \phi dx + \int_{\mathbb{R}^{3s}} (-\Delta)^{\frac{s}{2}} v_n (-\Delta)^{\frac{s}{2}} \psi dx \right) \\ &+ b \left(\|(-\Delta)^{\frac{s}{2}} u_n\|_2^2 \int_{\mathbb{R}^{3s}} (-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}} \phi dx + \|(-\Delta)^{\frac{s}{2}} v_n\|_2^2 \int_{\mathbb{R}^{3s}} (-\Delta)^{\frac{s}{2}} v_n (-\Delta)^{\frac{s}{2}} \psi dx \right) \\ &+ \int_{\mathbb{R}^{3s}} (\lambda_{1,n} u_n \phi + \lambda_{2,n} v_n \psi) dx - \int_{\mathbb{R}^{3s}} \left(\mu_1 |u_n|^{p-2} u_n \phi + \mu_2 |v_n|^{q-2} v_n \psi \right) dx \\ &- \nu \left(\int_{\mathbb{R}^{3s}} r_1 \frac{|u_n|^{r_1-2} |v_n|^{r_2} u_n}{|x|^\alpha} \phi dx + \int_{\mathbb{R}^{3s}} r_2 \frac{|u_n|^{r_1} |v_n|^{r_2-2} v_n}{|x|^\alpha} \psi dx \right) \\ &= o(1), \end{aligned} \tag{19}$$

take $(\phi, \psi) = (u_n, 0)$ and $(0, v_n)$ respectively, there holds

$$\lambda_{1,n} c^2 = -a \|(-\Delta)^{\frac{s}{2}} u_n\|_2^2 - b \|(-\Delta)^{\frac{s}{2}} u_n\|_2^4 + \mu_1 \|u_n\|_p^p + \nu r_1 \int_{\mathbb{R}^{3s}} \frac{|u_n|^{r_1} |v_n|^{r_2}}{|x|^\alpha} dx + o(1), \tag{20}$$

and

$$\lambda_{2,n}d^2 = -a\|(-\Delta)^{\frac{s}{2}}v_n\|_2^2 - b\|(-\Delta)^{\frac{s}{2}}v_n\|_2^4 + \mu_2\|v_n\|_q^q + \nu r_2 \int_{\mathbb{R}^{3s}} \frac{|u_n|^{r_1}|v_n|^{r_2}}{|x|^\alpha} dx + o(1). \tag{21}$$

Since $\{u_n, v_n\}$ is bounded in $H_r^s(\mathbb{R}^{3s}) \times H_r^s(\mathbb{R}^{3s})$, we can deduce that $(\lambda_{1,n}, \lambda_{2,n})$ is bounded. Thus we conclude that there exists $u, v \in H^s(\mathbb{R}^N)$, $\lambda_1, \lambda_2 \in \mathbb{R}$ such that, after passing to a subsequence

$$\begin{aligned} (u_n, v_n) &\rightharpoonup (u_0, v_0) && \text{in } H_r^s(\mathbb{R}^{3s}) \times H_r^s(\mathbb{R}^{3s}), \\ (u_n, v_n) &\rightarrow (u_0, v_0) && \text{in } L^p(\mathbb{R}^{3s}) \times L^p(\mathbb{R}^{3s}), \text{ where } 2 < p < 2_s^*, \\ (u_n, v_n) &\rightarrow (u_0, v_0) && \text{in } \mathbb{R}^{3s}, \\ (\lambda_{1,n}, \lambda_{2,n}) &\rightarrow (\lambda_1, \lambda_2) && \text{in } \mathbb{R}^2. \end{aligned} \tag{22}$$

Step 2. $u_n \rightarrow u_0$ (resp. $v_n \rightarrow v_0$) in $H_r^s(\mathbb{R}^{3s})$ when $\lambda_1 > 0$ (resp. $\lambda_2 > 0$).

According to the boundedness of u_n in $H_r^s(\mathbb{R}^{3s})$, we suppose

$$\lim_{n \rightarrow \infty} \|(-\Delta)^{\frac{s}{2}}u_n\| = l \geq 0. \tag{23}$$

In view of (22), we have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^{3s}} \frac{|u_n|^{r_1}|v_n|^{r_2}}{|x|^\alpha} dx = \int_{\mathbb{R}^{3s}} \frac{|u_0|^{r_1}|v_0|^{r_2}}{|x|^\alpha} dx. \tag{24}$$

Taking $(\phi, 0)$ as test function in (19), when $n \rightarrow \infty$, we have

$$\begin{aligned} (a + bl) \int_{\mathbb{R}^{3s}} (-\Delta)^{\frac{s}{2}}u_0(-\Delta)^{\frac{s}{2}}\phi dx + \int_{\mathbb{R}^{3s}} \lambda_1 u_0 \phi \\ - \int_{\mathbb{R}^{3s}} \mu_1 |u_0|^{p-2}u_0 \phi - \nu \int_{\mathbb{R}^{3s}} r_1 \frac{|u_0|^{r_1-2}|v_0|^{r_2}u_0}{|x|^\alpha} \phi dx = 0, \end{aligned} \tag{25}$$

let $\phi = u_n - u_0$ in (25), we have

$$\begin{aligned} (a + bl) \int_{\mathbb{R}^{3s}} (-\Delta)^{\frac{s}{2}}u_0(-\Delta)^{\frac{s}{2}}(u_n - u_0) dx + \int_{\mathbb{R}^{3s}} \lambda_1 u_0(u_n - u_0) \\ = \int_{\mathbb{R}^{3s}} \mu_1 |u_0|^{p-2}u_0(u_n - u_0) + \nu \int_{\mathbb{R}^{3s}} r_1 \frac{|u_0|^{r_1-2}|v_0|^{r_2}u_0}{|x|^\alpha} (u_n - u_0) \\ = o(1). \end{aligned} \tag{26}$$

Using $(u_n - u_0, 0)$ as a test function in (19), we can easily conclude that

$$\begin{aligned} a \int_{\mathbb{R}^{3s}} (-\Delta)^{\frac{s}{2}}u_n(-\Delta)^{\frac{s}{2}}(u_n - u_0) dx + b\|(-\Delta)^{\frac{s}{2}}u_n\|_2^2 \int_{\mathbb{R}^{3s}} (-\Delta)^{\frac{s}{2}}u_n(-\Delta)^{\frac{s}{2}}(u_n - u_0) dx \\ + \int_{\mathbb{R}^{3s}} \lambda_{1,n}u_n(u_n - u_0) \\ = \int_{\mathbb{R}^{3s}} \mu_1 |u_n|^{p-2}u_n(u_n - u_0) + \nu \int_{\mathbb{R}^{3s}} r_1 \frac{|u_n|^{r_1-2}|v_n|^{r_2}u_n}{|x|^\alpha} (u_n - u_0) \\ = o(1). \end{aligned} \tag{27}$$

Since $\lim_{n \rightarrow \infty} \|(-\Delta)^{\frac{s}{2}}u_n\| = l \geq 0$ and $\lambda_{1,n} \rightarrow \lambda_1$, we can deduce from (26) and (27) that

$$(a + bl)\|(-\Delta)^{\frac{s}{2}}(u_n - u_0)\|_2^2 + \lambda_1\|u_n - u_0\|_2^2 = o(1),$$

when $\lambda_1 > 0$, it is easy to obtain that $u_n \rightarrow u_0$ in $H_r^s(\mathbb{R}^{3s})$

Step 3. The weak limit $u_0, v_0 \neq 0$ and $\lambda_1, \lambda_2 > 0$.

From (20), (21) and $I(u_n, v_n) = o(1)$ we get

$$\begin{aligned} \lambda_{1,n}c^2 + \lambda_{2,n}d^2 &= -a\left(\|(-\Delta)^{\frac{s}{2}}u_n\|_2^2 + \|(-\Delta)^{\frac{s}{2}}v_n\|_2^2\right) - b\left(\|(-\Delta)^{\frac{s}{2}}u_n\|_2^4 + \|(-\Delta)^{\frac{s}{2}}v_n\|_2^4\right) \\ &\quad + \mu_1\|u_n\|_p^p + \mu_2\|v_n\|_q^q + \nu(r_1 + r_2) \int_{\mathbb{R}^{3s}} \frac{|u_n|^{r_1}|v_n|^{r_2}}{|x|^\alpha} dx \\ &= -\mu_1\gamma_{p,s}\|u_n\|_p^p - \mu_2\gamma_{q,s}\|v_n\|_q^q - \nu 2_{s,\alpha}^* \int_{\mathbb{R}^{3s}} \frac{|u_n|^{r_1}|v_n|^{r_2}}{|x|^\alpha} dx \\ &\quad + \mu_1\|u_n\|_p^p + \mu_2\|v_n\|_q^q + \nu(r_1 + r_2) \int_{\mathbb{R}^{3s}} \frac{|u_n|^{r_1}|v_n|^{r_2}}{|x|^\alpha} dx + o(1) \\ &= \mu_1(1 - \gamma_{p,s})\|u_n\|_p^p + \mu_2(1 - \gamma_{q,s})\|v_n\|_q^q + o(1), \end{aligned}$$

since $\frac{14}{3} < p, q < 6$, there exists a positive constant C satisfying

$$\lambda_{1,n}c^2 + \lambda_{2,n}d^2 > C,$$

which indicates that at least one of λ_1 and λ_2 is a positive value.

Then we assume $\lambda_1 > 0$ and thus $u_n \rightarrow u_0$ in $H_r^s(\mathbb{R}^{3s})$. Suppose $\lambda_2 \leq 0$ and $\lim_{n \rightarrow \infty} \|(-\Delta)^{\frac{s}{2}}v_n\| = l_1 \geq 0$. Then u_0, v_0 satisfies the following equation:

$$\begin{cases} (a + b \int_{\mathbb{R}^{3s}} |(-\Delta)^{\frac{s}{2}}u_0|^2 dx)(-\Delta)^s u_0 + \lambda_1 u_0 = \mu_1 |u_0|^{p-2} u_0 + r_1 \nu \frac{|u_0|^{r_1-2} |v_0|^{r_2} u_0}{|x|^\alpha}, \\ (a + bl_1)(-\Delta)^s v_0 + \lambda_2 v_0 = \mu_2 |v_0|^{q-2} v_0 + r_2 \nu \frac{|u_0|^{r_1} |v_0|^{r_2-2} v_0}{|x|^\alpha}. \end{cases}$$

Since $u_0, v_0 \geq 0$ and $\lambda_2 \leq 0$, we have

$$(a + bl_1)(-\Delta)^s v_0 \geq -\lambda_2 v_0 \geq 0,$$

which implies that $(-\Delta)^s v_0 \geq 0$. According to Lemma 5, we obtain that $v_0 \equiv 0$, then u_0 satisfies

$$\begin{cases} (a + b \int_{\mathbb{R}^{3s}} |(-\Delta)^{\frac{s}{2}}u_0|^2 dx)(-\Delta)^s u_0 + \lambda_1 u_0 = \mu_1 |u_0|^{p-2} u_0, \\ \int_{\mathbb{R}^{3s}} u_n^2 dx = c^2, \end{cases}$$

and

$$\begin{aligned} \sigma &= \lim_{n \rightarrow \infty} I(u_n, v_n) \\ &= \lim_{n \rightarrow \infty} \frac{a}{2} \left(\|(-\Delta)^{\frac{s}{2}}u_n\|_2^2 + \|(-\Delta)^{\frac{s}{2}}v_n\|_2^2 \right) + \frac{b}{4} \left(\|(-\Delta)^{\frac{s}{2}}u_n\|_2^4 + \|(-\Delta)^{\frac{s}{2}}v_n\|_2^4 \right) \\ &\quad - \frac{\mu_1}{p} \int_{\mathbb{R}^{3s}} |u_n|^p dx \\ &= \frac{a}{2} \int_{\mathbb{R}^{3s}} |(-\Delta)^{\frac{s}{2}}u_0|^2 dx + \frac{b}{4} \left(\int_{\mathbb{R}^{3s}} |(-\Delta)^{\frac{s}{2}}u_0|^2 dx \right)^2 - \frac{\mu_1}{p} \int_{\mathbb{R}^{3s}} |u_0|^p dx \\ &= E_{\mu_1,p}(u_0) \\ &\geq e_{\mu_1,p}(c), \end{aligned}$$

which contradicts to Lemma 7. Therefore, $u_0, v_0 > 0$ and $\lambda_1, \lambda_2 > 0$. In view of the conclusion in Step 2, we can obtain that $(u_n, v_n) \rightarrow (u_0, v_0)$ in $H_r^s(\mathbb{R}^{3s}) \times H_r^s(\mathbb{R}^{3s})$.

Consequently, (u_0, v_0) is a positive radial solution to system (1) satisfying $\lambda_1, \lambda_2 > 0$. This completes the proof. \square

4. Conclusions

In this paper, we have investigated the existence of normalized solutions for a fractional Kirchhoff system with critical Hardy nonlinearities in three dimensions. By combining the minimax principle with radial rearrangement arguments, we overcame the lack of compactness induced by the nonlocal Kirchhoff operator and the Hardy singular term, and proved the existence of positive radial normalized solutions under a suitable condition on the coupling strength. From a mathematical perspective, our results extend the theory of normalized solutions for fractional Kirchhoff-type equation with Hardy critical nonlinearities, and provide a systematic approach to handling the loss of compactness in such nonlocal constrained problems.

Although the present work is mainly theoretical, the methods and results have potential implications in the study of fractional models arising in physics and engineering. In particular, fractional Kirchhoff equations have been widely used to describe anomalous diffusion, nonlocal elasticity, and the propagation of waves in complex media. While the classical integer-order Kirchhoff theory is well known for its applications to electrical circuits with nonlinear elements, our fractional generalization could offer a more accurate description of systems with memory effects or long-range interactions, which are common in complex electrical engineering problems with nonlinear components. We believe that the existence and qualitative properties of normalized solutions established here may serve as a theoretical basis for future studies on the dynamics and stability of such systems.

There remain several open problems that deserve further investigation. One may extend the present results to higher-dimensional cases and more general nonlinear structures. It is also valuable to study the multiplicity, symmetry and asymptotic behavior of normalized solutions. These problems are left for future work.

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