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Classification of Invariant 2-Conformal Vector Fields on 4D Non-Reductive Homogeneous Spaces

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Abstract

The notion of 2-conformal vector fields on pseudo-Riemannian manifolds, which arose naturally in the study of hyperbolic solitons, is introduced by Fasihi-Ramandi, De, and Shamkhali. In this paper, we study invariant 2-conformal vector fields on four-dimensional non-reductive pseudo-Riemannian homogeneous manifolds G/H. Consequently, the complete classification of such vector fields is achieved, together with the necessary and sufficient conditions for their existence. The results are then applied to Lorentzian and neutral signatures, where 2-conformal vector fields provide an effective criterion for detecting 2-conformal equivalences in geometries with limited algebraic symmetries.

Keywords: non-reductive homogeneous spaces; 2-conformal vector field; 2-Killing vector field; pseudo-Riemannian geometry

MSC: 53C30; 53C25; 37J37

1. Introduction

A connected pseudo-Riemannian manifold (M, g) is called homogeneous if its isometry group acts transitively on M. If (M, g) can be realized as a coset space $M = G/H$, where G is a Lie group acting transitively by isometries and H is the isotropy subgroup at a point, then (M, g) is said to be reductive. Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebra of G and H, respectively. For the reductive homogeneous space $M = G/H$, we have the following decomposition:

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}, \qquad \mathrm{Ad}(H)(\mathfrak{m}) \subseteq \mathfrak{m},$$

where $\mathfrak{m}$ is an $Ad(H)$-invariant subspace of $\mathfrak{g}$. A homogeneous Riemannian manifold is necessarily reductive. Furthermore, there do not exist any 2- and 3-dimensional non-reductive homogeneous pseudo-Riemannian manifolds [1].

In pseudo-Riemannian geometry, the four-dimensional case is of particular interest, since it admits a rich family of non-reductive homogeneous manifolds. Such spaces were classified by M. E. Fels and A. G. Renner in [1], comprising eight distinct geometric types: $\mathbf{A_1}, \mathbf{A_2}, \mathbf{A_3}, \mathbf{A_4}, \mathbf{A_5}, \mathbf{B_1}, \mathbf{B_2}, \mathbf{B_3}$.

Homogeneous spaces are characterized by their high symmetry and often serve as natural settings for the initial investigation of new geometric concepts and conjectures. In

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particular, the study of geometric vector fields on such spaces provides an effective framework for analyzing their geometric and physical properties. Among these geometric vector fields, Killing, 2-Killing, and conformal vector fields have long been of interest to mathematicians and physicists. In particular, Killing vector fields are the infinitesimal generators of isometries, that is, flows generated by Killing fields are isometries of their underlying manifolds, and conformal vector fields play a central role as a natural generalization of Killing vector fields [2].

While conformal vector fields do not, in general, preserve the metric tensor, they rescale it by a smooth scalar function. Precisely, for a smooth vector field X on a pseudo-Riemannian manifold (M, g), if

$$\mathcal{L}_X g = 2fg,$$

for a smooth function $f : M \to \mathbb{R}$, then X is called a conformal vector field. In the special case $f = 0$, the field X is *Killing* ; if f is constant, X is *homothetic*. The importance of conformal vector fields in geometry lies in their preservation of the metric's angular structure and their revelation of scale symmetries of spacetime. Geometrically, the existence of such fields reflects a deeper understanding of how metrics behave under local changes in size and angle.

In this paper, we classify invariant 2-conformal vector fields on non-reductive homogeneous four-spaces. These fields, first introduced by G. Fasihi-Ramandi, U. C. De, and F. Shamkhali [3] as follows.

Definition 1 ([3]). *A smooth vector field X on a pseudo-Riemannian manifold (M, g) is called a 2-conformal vector field if*

$$(\mathcal{L}_X \mathcal{L}_X) g = 2\sigma g, \tag{1}$$

for some smooth function $\sigma \in C^\infty(M)$.

Whenever $\sigma = 0$, the field defines a *2-Killing* vector field (see, e.g., [4–7]). Further, if σ is a non-zero constant, the vector field is called 2-homothetic.

From the definition of conformal vector fields, it is easy to verify that every conformal vector field is automatically a 2-conformal vector field. But the converse is not true in general. In fact, G. Fasihi-Ramandi, U. C. De, and F. Shamkhali [3] provided some examples of 2-conformal Killing vector fields on simply connected two-step nilpotent Lie groups of dimension five which are not conformal vector fields. Also, in [8] R. Bossly, G. Fasihi-Ramandi, and M. D. Siddiqi gave examples of 2-conformal vector fields in Sol space. Furthermore, we show in this article that there exist many 2-conformal vector fields on 4-dimensional non-reductive homogeneous spaces that are not conformal.

A hyperbolic Ricci soliton is a generalization of a Ricci soliton representing a self-similar solution to the hyperbolic Ricci flow. It is defined on a Riemannian manifold (M, g) by the existence of a potential vector field X and constants λ, μ satisfying the following equation:

$$Ric + \frac{1}{2}\mathcal{L}_X \mathcal{L}_X g + \lambda \mathcal{L}_X g = \mu g,$$

which Ric is the Ricci tensor of (M, g). Thus, 2-conformal vector fields arise naturally in the recent study of hyperbolic solitons (see, e.g., [3,8–11] for details).

Recently, self-similar solutions of the hyperbolic Yamabe flow, known as the hyperbolic Yamabe solitons, have been studied by Blaga [12]. A hyperbolic Yamabe soliton is a self-similar solution to the hyperbolic Yamabe flow. It is defined on a Riemannian manifold

(M, g) by the existence of a potential vector field X and constants λ, μ satisfying the following equation:

$$\frac{1}{2}\mathcal{L}_X\mathcal{L}_X g + \lambda\mathcal{L}_X g = (\mu - R)g,$$

where R stands for scalar curvature of (M, g). Considering the argument in [12], when $\lambda = 0$, we reach to a critical situation. This shows that the notion of 2-conformal vector fields are in close related to critical self-similar solutions to hyperbolic Yamabe solitons.

The study of 2-conformal vector fields on these non-reductive spaces is particularly important, as these spaces exhibit the least algebraic symmetry and the greatest geometric diversity. On these manifolds, the behaviour of the second Lie derivative of the metric tensor, $\mathcal{L}_X\mathcal{L}_X g$, along the group generating fields, provides a deeper understanding of how the metric deviates from isometries. The objective of investigating 2-conformal vector fields on four-dimensional non-reductive homogeneous spaces is the explicit description of β, the determination of the conditions for the existence of such fields, and the analysis of their connection to the underlying algebraic structure of $(\mathfrak{g}, \mathfrak{h})$. This investigation is especially significant in geometries with Lorentzian and neutral metrics, where angular variations in the metric are linked to the causal structure of space. Ultimately, such a study makes a significant contribution to the understanding of 2-conformal symmetries and equivalences in non-reductive spaces, providing a valuable theoretical tool for developing pseudo-Riemannian geometry in low dimensions.

2. Four-Dimensional Non-Reductive Homogeneous Spaces

Four-dimensional non-reductive homogeneous manifolds are classified according to their non-reductive Lie algebras. In what follows, we recall from [1,13–16] the classification of these spaces, as well as the pseudo-Riemannian metrics and the Lie derivative $\mathcal{L}_X g$ of the metric tensor along a vector field $X = X_i u_i \in \mathfrak{m}$.

2.1. Lorentzian Case

2.1.1. Case $(\mathbf{A_1})$

Let $\mathfrak{s}(2)$ be a two-dimensional solvable Lie algebra and $\mathfrak{g} = \mathfrak{a}_1$ be a five-dimensional decomposable Lie algebra of the form $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{s}(2)$. In addition, let $\{e_1, \ldots, e_5\}$ be a basis for $\mathfrak{a}_1$ such that the non-zero brackets are defined as:

$$[e_1, e_2] = -[e_1, e_3] = 2e_2, \ [e_2, e_3] = e_1, \ [e_4, e_5] = e_4,$$

and $\mathfrak{h} = \text{span}\{h_1 = e_3 + e_4\}$ is the isotropy subalgebra [16,17]. Therefore, we assume that

$$\mathfrak{m} = \text{span}\{u_1 = e_1, \ u_2 = e_2, u_3 = e_5, u_4 = e_3 - e_4\}$$

and the isotropy description for h_1 is as follows:

$$H_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -\frac{1}{2} & 0 \end{pmatrix}. \tag{2}$$

The metric g with respect to the basis $\{u_i\}$ is given by:

$$g = \begin{pmatrix} a & 0 & -\frac{a}{2} & 0 \\ 0 & b & c & a \\ -\frac{a}{2} & c & d & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \tag{3}$$

Whenever $a \neq 0$, this metric is non-degenerate.

For any vector field $X = X_i u_i \in \mathfrak{m}$, where $X_i \in \mathbb{R}$, using

$$(\mathcal{L}_X g)(e_i, e_j) = g(\nabla_{e_i} X, e_j) + g(e_i, \nabla_{e_j} X), \tag{4}$$

the Lie derivative of the metric g with respect to the vector field X is as follows:

$$\mathcal{L}_X g = \begin{pmatrix} 0 & 2bX_2 & 2cX_2 & aX_2 \\ 2bX_2 & -4bX_1 & -2cX_1 - aX_4 & -aX_1 + \frac{1}{2}aX_3 \\ 2cX_2 & -2cX_1 - aX_4 & 0 & \frac{a}{2}X_2 \\ aX_2 & -aX_1 + \frac{a}{2}X_3 & \frac{a}{2}X_2 & 0 \end{pmatrix}. \tag{5}$$

Since $a \neq 0$, the Formulas (3) and (5) show that in this case we have not any invariant conformal vector field.

Corresponding to the basis $\{u_1, u_2, u_3, u_4\}$ and using relations (4) and (5), we have:

$$\begin{aligned}
(\mathcal{L}_X \mathcal{L}_X g)_{11} &= -8b(X_2X_4 - X_2^2) \\
(\mathcal{L}_X \mathcal{L}_X g)_{22} &= 16bX_1^2 + 4bX_4X_2, \\
(\mathcal{L}_X \mathcal{L}_X g)_{33} &= -\frac{a}{2}X_2X_4, \\
(\mathcal{L}_X \mathcal{L}_X g)_{44} &= -4aX_1^2 + 2aX_3X_1 - 2aX_2^2, \\
(\mathcal{L}_X \mathcal{L}_X g)_{12} &= 8bX_1X_4 - 12bX_1X_2, \\
(\mathcal{L}_X \mathcal{L}_X g)_{13} &= 4cX_4X_1 - 4cX_1X_2 - \frac{5}{2}aX_2X_4 + 2aX_4^2, \\
(\mathcal{L}_X \mathcal{L}_X g)_{14} &= (4b - 2a)X_1X_2 + \frac{3}{2}aX_2X_3 + 2aX_1X_4 - aX_3X_4, \\
(\mathcal{L}_X \mathcal{L}_X g)_{23} &= 4cX_1^2 + \frac{5}{2}aX_1X_4 + 2cX_2X_4 - \frac{1}{4}aX_3X_4, \\
(\mathcal{L}_X \mathcal{L}_X g)_{24} &= (2a - 8b)X_1^2 + \frac{3}{2}aX_1X_3 + aX_2X_4 - 2bX_2^2 + \frac{1}{4}aX_3^2, \\
(\mathcal{L}_X \mathcal{L}_X g)_{34} &= -4cX_1^2 - 2aX_4X_1 - 2cX_2^2 + \frac{1}{4}aX_2X_3.
\end{aligned} \tag{6}$$

Consequently, using relations (1), (3) and (6), we obtain the following system for invariant 2-conformal vector fields:

$$\begin{aligned}
&(1)\ -8b(X_2X_4 - X_2^2) = a\beta, \\
&(2)\ 16bX_1^2 + 4bX_4X_2 = b\beta, \\
&(3)\ -\frac{a}{2}X_2X_4 = d\beta, \\
&(4)\ -4aX_1^2 + 2aX_3X_1 - 2aX_2^2 = 0, \\
&(5)\ 8bX_1X_4 - 12bX_1X_2 = 0, \\
&(6)\ 4cX_4X_1 - 4cX_1X_2 - \frac{5}{2}aX_2X_4 + 2aX_4^2 = -\frac{a}{2}\beta, \\
&(7)\ (4b - 2a)X_1X_2 + \frac{3}{2}aX_2X_3 + 2aX_1X_4 - aX_3X_4 = 0, \\
&(8)\ 4cX_1^2 + \frac{5}{2}aX_1X_4 + 2cX_2X_4 - \frac{1}{4}aX_3X_4 = c\beta, \\
&(9)\ (2a - 8b)X_1^2 + \frac{3}{2}aX_1X_3 + aX_2X_4 - 2bX_2^2 + \frac{1}{4}aX_3^2 = a\beta, \\
&(10)\ -4cX_1^2 - 2aX_4X_1 - 2cX_2^2 + \frac{1}{4}aX_2X_3 = 0.
\end{aligned} \tag{7}$$

According to the fifth equation, three cases arise:

$$X_1 = 0, \qquad 8X_4 - 12X_2 = 0, \qquad \text{or} \quad b = 0.$$

Whenever $X_1 = 0$ or $b = 0$, it follows after some algebraic manipulation that the system has only the trivial solution. If $X_1 = 0$, then by fourth equation, $X_2 = 0$. Now, the first equation shows $\beta = 0$. So, $\beta = 0$.

If $\beta = 0$, the right-hand side (RHS) of Equations (1)–(3), (6), and (8) becomes zero. Form Equation (9), we obtain $X_3 = 0$. Substituting these value in the system, the sixth equation guarantees that $X_4 = 0$. If $b = 0$, then by the first equation we have $\beta = 0$. From Equation (3), it follows that:

$$X_2 X_4 = 0.$$

Hence, either $X_2 = 0$ or $X_4 = 0$.

Case 1: $X_2 = 0$

Equation (4) simplifies to:

$$-4aX_1^2 + 2aX_3X_1 = 0 \Rightarrow X_3 = 2X_1.$$

Equation (6) becomes:

$$4cX_4X_1 + 2aX_4^2 = 0 \Rightarrow X_4(4cX_1 + 2aX_4) = 0.$$

If $X_4 = 0$, then from (9) and other equations, all variables are zero: $X_1 = 0$, $X_2 = 0$, $X_3 = 0$, $X_4 = 0$. If $X_4 \neq 0$, then:

$$X_4 = -\frac{2c}{a}X_1.$$

Substituting into (7) leads to $X_1 = 0$; hence all variables are zero.

Case 2: $X_4 = 0$

Equation (4) becomes:

$$-4aX_1^2 + 2aX_3X_1 - 2aX_2^2 = 0.$$

Equation (7) reduces to:

$$X_2\left(-2X_1 + \frac{3}{2}X_3\right) = 0,$$

implying either $X_2 = 0$ or $X_3 = \frac{4}{3}X_1$.

If $X_2 = 0$, all variables are zero as before. If $X_3 = \frac{4}{3}X_1$, substitution into Equation (4) yields:

$$-\frac{4}{3}aX_1^2 - 2aX_2^2 = 0,$$

which is impossible unless $X_1 = 0$. Therefore, all variables are zero. Hence, the only solution consistent with the system under the specified conditions is:

$$X_1 = X_2 = X_3 = X_4 = 0.$$

Therefore, we assume $X_1 \neq 0$ and $b \neq 0$. From the Equation (5), we obtain

$$X_4 = \tfrac{3}{2}X_2.$$

By substituting this into the remaining equations and eliminating β, the compatibility condition

$$3a^2 = 16bd$$

is derived. Using the fourth equation, we can write

$$X_3 = 2X_1 + \frac{X_2^2}{X_1}.$$

Then, from the first and third equations we obtain

$$\beta = -\frac{4b}{a}X_2^2.$$

By substituting these relations into the remaining equations, we arrive at the homogeneous equation

$$aX_3X_4 - bX_2^2 + cX_1X_4 = 0.$$

Assume $X_2 \neq 0$ (since if $X_2 = 0$, we get $X_1 = 0$, which contradicts the assumption). After algebraic simplification, we find the equation

$$3a + \frac{3}{2}ar^2 - br + \frac{3c}{2} = 0, \quad r = \frac{X_2}{X_1}.$$

Solving it gives

$$r = \frac{b \pm \sqrt{b^2 - 9a(2a+c)}}{3a}.$$

Thus, we obtain the following non-trivial solutions:

$$X_2 = rX_1, \quad X_3 = (2+r^2)X_1, \quad X_4 = \frac{3r}{2}X_1, \quad \beta = -\frac{4br^2}{a}X_1^2, \tag{8}$$

where $X_1 \in \mathbb{R}$. Consequently, we obtain the following.

Theorem 1. *Let (M, g) be a four-dimensional non-reductive homogeneous space of type $\mathfrak{a}_1$ associated with the basis $\{u_1, u_2, u_3, u_4\}$. Then, 2-conformal fields exist if and only if $3a^2 = 16bd$, and r must satisfy the above quadratic equation. Also, all non-trivial invariant 2-conformal vector fields are determined by (8).*

2.1.2. Case $(\mathbf{A_2})$

The five-dimensional one-parameter family of Lie algebras $A_{5,30}$ is given by $\mathfrak{g} = \mathfrak{a}_2$. Let $\{e_1, \ldots, e_5\}$ be a basis for $\mathfrak{a}_2$. Then for each $e \in \mathbb{R}$, the non-zero Lie brackets are:

$$\begin{aligned} &[e_1, e_5] = (e+1)e_1, \ [e_2, e_4] = e_1, && [e_2, e_5] = ee_2, \\ &[e_3, e_4] = e_2, && [e_3, e_5] = (e-1)e_3, \ [e_4, e_5] = e_4. \end{aligned}$$

The isotropy subalgebra is $\mathfrak{h} = \operatorname{span}\{h_1 = e_4\}$. Therefore, we consider

$$\mathfrak{m} = \operatorname{span}\{u_1 = e_1, \ u_2 = e_2, u_3 = e_3, u_4 = e_5\}, \tag{9}$$

and the isotropy description for h_1 is as follows:

$$H_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{10}$$

Consequently, the invariant metrics are obtained as:

$$g = \begin{pmatrix} 0 & 0 & -a & 0 \\ 0 & a & 0 & 0 \\ -a & 0 & b & c \\ 0 & 0 & c & d \end{pmatrix}, \tag{11}$$

and they are non-degenerate whenever $ad \neq 0$. For any vector field $X = X_i u_i \in \mathfrak{m}$, where $r = e + 1$ and $s = e - 1$, the Lie derivative with respect to $\{u_i\}$ is obtained as:

$$\mathcal{L}_X g = \begin{pmatrix} 0 & 0 & -2eaX_4 & asX_3 \\ 0 & 2eaX_4 & 0 & -eaX_2 \\ -2eaX_4 & 0 & 2bsX_4 & arX_1 - s(bX_3 - cX_4) \\ asX_3 & -eaX_2 & arX_1 - s(bX_3 - cX_4) & -2csX_3 \end{pmatrix}. \tag{12}$$

Corresponding to the basis $\{u_1, u_2, u_3, u_4\}$ and using relations (9), (10) and (12), we have:

$$\begin{aligned} &(\mathcal{L}_X\mathcal{L}_X g)_{11} = 0, \\ &(\mathcal{L}_X\mathcal{L}_X g)_{22} = 4ae^2X_4^2, \\ &(\mathcal{L}_X\mathcal{L}_X g)_{33} = -4bs^2X_4^2, \\ &(\mathcal{L}_X\mathcal{L}_X g)_{44} = -4arsX_1X_3 + 2e^2aX_2^2 + 2s^2bX_3^2 - 2cs^2X_3X_4, \\ &(\mathcal{L}_X\mathcal{L}_X g)_{12} = 0, \\ &(\mathcal{L}_X\mathcal{L}_X g)_{13} = -2reaX_4^2 - 2easX_4^2, \\ &(\mathcal{L}_X\mathcal{L}_X g)_{14} = rsaX_3X_4 + 2easX_3X_4, \\ &(\mathcal{L}_X\mathcal{L}_X g)_{23} = 0, \\ &(\mathcal{L}_X\mathcal{L}_X g)_{24} = -3ae^2X_2X_4, \\ &(\mathcal{L}_X\mathcal{L}_X g)_{34} = -(ars - 2ear)X_1X_4 - s^2bX_3X_4 + cs^2X_4^2. \end{aligned} \tag{13}$$

In conclusion, by using relations (1), (11), and (13), we obtain the following system of algebraic equations to find 2-conformal vector fields:

$$\begin{aligned} &(1)\ 4ae^2X_4^2 = a\beta, \\ &(2)\ 4bs^2X_4^2 = b\beta, \\ &(3)\ 4arsX_1X_3 - 2e^2aX_2^2 - 2s^2bX_3^2 + 2cs^2X_3X_4 = -d\beta, \\ &(4)\ 2reaX_4^2 + 2easX_4^2 = a\beta, \\ &(5)\ rsaX_3X_4 + 2easX_3X_4 = 0, \\ &(6)\ 3ae^2X_2X_4 = 0, \\ &(7)\ (ars - 2ear)X_1X_4 + s^2bX_3X_4 - cs^2X_4^2 = -c\beta. \end{aligned} \tag{14}$$

By solving the above system, we have the following:

Theorem 2. *Suppose (M, g) is a four-dimensional non-reductive homogeneous space of type $\mathfrak{a}_2$ associated with the basis $\{u_1, u_2, u_3, u_4\}$. In this case, invariant 2-conformal vector fields reduce to 2-Killing vector fields and are as follows:*

1. *If $\beta = 0$, $e \neq 0$ and $X_4 = 0$, then $X_2 \in \mathbb{R}$ and X_3 and X_1 must satisfy the equation:*

$$-4arsX_1X_3 + 2e^2aX_2^2 + 2s^2bX_3^2 = 0.$$

2. *If $\beta = e = 0$, then $X_4 = 0$, $X_3 = 0$, and $X_1, X_2 \in \mathbb{R}$.*
3. *If $\beta = e = 0$, then $X_4 = 0$, $X_1 = -\frac{b}{2a}X_3$, and $0 \neq X_3, X_2 \in \mathbb{R}$.*

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2.1.3. Case $(\mathbf{A_3})$

The Lie algebra $\mathfrak{g} = \mathfrak{a}_3$ is one of the 5-dimensional Lie algebras $A_{5,36}$ or $A_{5,37}$ in [18]. For $\mathfrak{a}_3$ the non-zero Lie brackets are as follows:

$$\begin{aligned}&[e_1, e_4] = 2e_1, \quad [e_2, e_3] = e_1, \ [e_2, e_4] = e_2,\\ &[e_2, e_5] = -\epsilon e_3, \ [e_3, e_4] = e_3, \ [e_3, e_5] = e_2,\end{aligned}$$

where $\epsilon = -1$ for $A_{5,36}$ and $\epsilon = 1$ for $A_{5,37}$. The isotropy subalgebra is given by $\mathfrak{h} = \text{span}\{h_1 = e_3\}$. Therefore, we consider

$$\mathfrak{m} = \text{span}\{u_1 = e_1, \ u_2 = e_2, u_3 = e_4, u_4 = e_5\}, \tag{15}$$

and the isotropy description for h_1 is as follows:

$$H_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{16}$$

Consequently, the invariant metrics are calculated as:

$$g = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & a & 0 & 0 \\ 0 & 0 & b & c \\ a & 0 & c & d \end{pmatrix}, \tag{17}$$

and they are non-degenerate whenever $ab \neq 0$.

For any vector field $X = X_i u_i \in \mathfrak{m}$, we calculate:

$$\mathcal{L}_X g = \begin{pmatrix} 0 & 0 & 0 & 2aX_3 \\ 0 & 2aX_3 & -aX_2 & 0 \\ 0 & -aX_2 & 0 & -2aX_1 \\ 2aX_3 & 0 & -2aX_1 & 0 \end{pmatrix}. \tag{18}$$

Corresponding to the basis $\{u_1, u_2, u_3, u_4\}$ and using relations (15) and (18), we have:

$$\begin{aligned}&(\mathcal{L}_X\mathcal{L}_X g)_{11} = 0,\\ &(\mathcal{L}_X\mathcal{L}_X g)_{22} = 4aX_3^2,\\ &(\mathcal{L}_X\mathcal{L}_X g)_{33} = 2aX_2^2,\\ &(\mathcal{L}_X\mathcal{L}_X g)_{44} = 0,\\ &(\mathcal{L}_X\mathcal{L}_X g)_{12} = 0,\\ &(\mathcal{L}_X\mathcal{L}_X g)_{13} = 0,\\ &(\mathcal{L}_X\mathcal{L}_X g)_{14} = 4aX_3^2,\\ &(\mathcal{L}_X\mathcal{L}_X g)_{23} = -3aX_2X_3,\\ &(\mathcal{L}_X\mathcal{L}_X g)_{24} = 0,\\ &(\mathcal{L}_X\mathcal{L}_X g)_{34} = -4aX_1X_3.\end{aligned} \tag{19}$$

Consequently, by using relations (1), (17), and (19), we obtain the following system of algebraic equations to find 2-conformal vector fields:

$$\begin{array}{l}(1)\ 4aX_3^2 = a\beta,\\ (2)\ 2aX_2^2 = b\beta,\\ (3)\ d\beta = 0,\\ (4)\ 4aX_3^2 = a\beta,\\ (5)\ 3aX_2X_3 = 0,\\ (6)\ 4aX_1X_3 = -c\beta.\end{array} \tag{20}$$

By solving the above system, we have the following.

Theorem 3. *Let (M, g) be a four-dimensional non-reductive homogeneous space of type $\mathfrak{a}_3$, corresponding to the basis $\{u_1, u_2, u_3, u_4\}$. In this case, invariant 2-conformal vector fields are 2-Killing vector fields, and we have*

$$X_2 = X_3 = 0, \quad X_1, X_4 \in \mathbb{R}.$$

2.1.4. Case $(\mathbf{A_4})$

The 6-dimensional Schrödinger Lie algebra $\mathfrak{sl}(2,\mathbb{R}) \ltimes \mathfrak{n}(3)$ is given by $\mathfrak{g} = \mathfrak{a}_4$, where $\mathfrak{n}(3)$ is the 3-dimensional Heisenberg algebra. Let $\{e_1, \ldots, e_6\}$ be a basis for $\mathfrak{a}_4$. The non-zero Lie brackets are:

$$\begin{array}{l}[e_1, e_2] = 2e_2,\ [e_1, e_3] = -2e_3,\ [e_2, e_3] = e_1,\ [e_1, e_4] = e_4,\\ [e_1, e_5] = -e_5, \quad [e_2, e_5] = e_4, \quad [e_3, e_4] = e_5,\ [e_4, e_5] = e_6.\end{array}$$

The isotropy subalgebra is given by $\mathfrak{h} = \text{span}\{h_1 = e_3 + e_6, h_2 = e_5\}$. In the next step, considering

$$\mathfrak{m} = \text{span}\{u_1 = e_1,\ u_2 = e_2, u_3 = e_3 - e_6, u_4 = e_4\}, \tag{21}$$

the isotropy descriptions for h_1 and h_2 are:

$$H_1 = \begin{pmatrix} 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}, \qquad H_2 = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{2}\\ 0 & -1 & 0 & 0\end{pmatrix}. \tag{22}$$

The invariant metrics are found as:

$$g = \begin{pmatrix} a & 0 & 0 & 0\\ 0 & b & a & 0\\ 0 & a & 0 & 0\\ 0 & 0 & 0 & \frac{a}{2}\end{pmatrix}, \tag{23}$$

and they are non-degenerate whenever $a \neq 0$. For any arbitrary vector field $X = X_i u_i \in \mathfrak{m}$, the Lie derivative is calculated as:

$$\mathcal{L}_{Xg} = \begin{pmatrix} 0 & 2bX_2 & aX_2 & \frac{a}{2}X_4\\ 2bX_2 & -4bX_1 & -aX_1 & 0\\ aX_2 & -aX_1 & 0 & 0\\ \frac{a}{2}X_4 & 0 & 0 & -aX_1\end{pmatrix}. \tag{24}$$

Corresponding to the basis $\{u_1, u_2, u_3, u_4\}$ and using relations (21) and (24), we have:

$$\begin{aligned}
(\mathcal{L}_X\mathcal{L}_X g)_{11} &= -2aX_3X_2 + aX_4^2 + 8bX_2^2,\\
(\mathcal{L}_X\mathcal{L}_X g)_{22} &= 16bX_1^2 + 4bX_3X_2,\\
(\mathcal{L}_X\mathcal{L}_X g)_{33} &= -2aX_2^2,\\
(\mathcal{L}_X\mathcal{L}_X g)_{44} &= 2aX_1^2,\\
(\mathcal{L}_X\mathcal{L}_X g)_{12} &= aX_1X_3 - 12bX_1X_2,\\
(\mathcal{L}_X\mathcal{L}_X g)_{13} &= -aX_1X_2,\\
(\mathcal{L}_X\mathcal{L}_X g)_{14} &= -\frac{3}{2}X_1X_4,\\
(\mathcal{L}_X\mathcal{L}_X g)_{23} &= aX_1^2 + aX_2X_3 - 2bX_2^2,\\
(\mathcal{L}_X\mathcal{L}_X g)_{24} &= \frac{a}{2}X_3X_4,\\
(\mathcal{L}_X\mathcal{L}_X g)_{34} &= -\frac{a}{2}X_2X_4.
\end{aligned} \tag{25}$$

As a result, by using relations (1), (23), and (25), to find 2-conformal vector fields, we obtain the following system of algebraic equations:

$$\begin{aligned}
&(1)\ aX_4^2 - 2aX_3X_2 + 8bX_2^2 = a\beta,\\
&(2)\ 16bX_1^2 + 4bX_3X_2 = b\beta,\\
&(3)\ 2aX_2^2 = 0,\\
&(4)\ 2X_1^2 = \frac{a}{2}\beta,\\
&(5)\ aX_1X_3 - 12bX_1X_2 = 0,\\
&(6)\ aX_1X_2 = 0,\\
&(7)\ \frac{3}{2}X_1X_4 = 0,\\
&(8)\ aX_1^2 + aX_2X_3 - 2bX_2^2 = a\beta,\\
&(9)\ \frac{a}{2}X_3X_4 = 0,\\
&(10)\ \frac{a}{2}X_2X_4 = 0.
\end{aligned} \tag{26}$$

By solving the above system, we obtain the following.

Theorem 4. *Let (M, g) be a non-reductive four-dimensional homogeneous space of type $\mathfrak{a}_4$ corresponding to the basis $\{u_1, u_2, u_3, u_4\}$. In this case, all invariant 2-conformal vector fields must be 2-Killing, leading to the condition:*

$$X_1 = X_2 = X_4 = 0, \quad X_3 \in \mathbb{R}.$$

2.1.5. Case $(\mathbf{A_5})$

The 7-dimensional Lie algebra $\mathfrak{sl}(2,\mathbb{R}) \ltimes A^1_{4,9}$ is given by $\mathfrak{g} = \mathfrak{a}_5$, where $A^1_{4,9}$ is the one defined in [18]. A basis for $\mathfrak{a}_5$ is $\{e_1, \ldots, e_7\}$, and the non-zero Lie brackets are:

$$\begin{aligned}
&[e_1, e_2] = 2e_2, \ [e_1, e_3] = -2e_3, \ [e_1, e_5] = -e_5, \ [e_1, e_6] = e_6,\\
&[e_2, e_3] = e_1, \quad [e_2, e_5] = e_6, \quad [e_3, e_6] = e_5, \quad [e_4, e_7] = 2e_4,\\
&[e_5, e_6] = e_4, \quad [e_5, e_7] = e_5, \quad [e_6, e_7] = e_6.
\end{aligned}$$

The isotropy subalgebra is given by $\mathfrak{h} = \operatorname{span}\{h_1 = e_1 + e_7, h_2 = e_3 - e_4, h_3 = e_5\}$. Thus, we set:

$$\mathfrak{m} = \operatorname{span}\{u_1 = e_1 - e_7,\ u_2 = e_2, u_3 = e_3 + e_4, u_4 = e_6\}.$$

For the isotropy description of h_1, h_2 and h_3, we have:

$$H_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; H_2 = \begin{pmatrix} 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; H_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

The invariant metrics are represented as follows:

$$g = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & \frac{a}{4} & 0 \\ 0 & \frac{a}{4} & 0 & 0 \\ 0 & 0 & 0 & \frac{a}{8} \end{pmatrix} \tag{27}$$

and they are non-degenerate whenever $a \neq 0$.

We have the Lie derivative of metric tensor along to any arbitrary vector field $X = X_i u_i \in \mathfrak{m}$ as follows:

$$\mathcal{L}_X g = \begin{pmatrix} 0 & \frac{a}{2}X_3 & 0 & \frac{a}{4}X_4 \\ \frac{a}{2}X_3 & 0 & -\frac{a}{2}X_1 & 0 \\ 0 & -\frac{a}{2}X_1 & 0 & 0 \\ \frac{a}{4}X_4 & 0 & 0 & -\frac{a}{2}X_1 \end{pmatrix}. \tag{28}$$

Corresponding to the basis $\{u_1, u_2, u_3, u_4\}$ and using relations Section 2.1.5 and (28), we have:

$$\begin{aligned} (\mathcal{L}_X\mathcal{L}_X g)_{11} &= 2aX_3X_2 + aX_4^2, \\ (\mathcal{L}_X\mathcal{L}_X g)_{22} &= \frac{a}{2}X_3^2, \\ (\mathcal{L}_X\mathcal{L}_X g)_{33} &= 0, \\ (\mathcal{L}_X\mathcal{L}_X g)_{44} &= 2aX_1^2, \\ (\mathcal{L}_X\mathcal{L}_X g)_{12} &= 0, \\ (\mathcal{L}_X\mathcal{L}_X g)_{13} &= -aX_1X_2, \\ (\mathcal{L}_X\mathcal{L}_X g)_{14} &= -\frac{3}{2}aX_1X_4, \\ (\mathcal{L}_X\mathcal{L}_X g)_{23} &= -\frac{a}{4}X_3X_2, \\ (\mathcal{L}_X\mathcal{L}_X g)_{24} &= -\frac{a}{8}X_3X_4, \\ (\mathcal{L}_X\mathcal{L}_X g)_{34} &= -\frac{a}{8}X_3X_2. \end{aligned} \tag{29}$$

As a result, by using the relations (1), (27), and (29), we obtain the following system of algebraic equations to find invariant 2-conformal fields.

$$\begin{aligned} &(1)\; 2aX_3X_2 + aX_4^2 = a\beta, \\ &(2)\; \frac{a}{2}X_3^2 = 0, \\ &(3)\; 2aX_1^2 = \frac{a}{8}\beta, \\ &(4)\; aX_1X_2 = 0, \\ &(5)\; \frac{3}{2}aX_1X_4 = 0, \\ &(6)\; \frac{a}{4}X_3X_2 = -\frac{a}{4}\beta \\ &(7)\; \frac{a}{8}X_3X_4 = 0, \\ &(8)\; \frac{a}{8}X_3X_2 = 0. \end{aligned} \tag{30}$$

It can be seen that the consistency condition for this system of equations is $\beta = 0$, and the following theorem is obtained.

Theorem 5. *Let (M, g) be a non-reductive four-dimensional homogeneous space of type $\mathfrak{a}_5$ corresponding to basis $\{u_1, u_2, u_3, u_4\}$. In this case, the invariant 2-conformal vector fields are as follows:*

(a) If $\beta \neq 0$, then the system has no solution.
(b) If $\beta = 0$, then $X_1 = X_3 = X_4 = 0$ and $X_2 \in \mathbb{R}$.

2.2. Sign (2, 2) Cases

Besides cases $\mathbf{A_1}, \mathbf{A_2}, \mathbf{A_3}$ when we accept the constant metrics with a neutral sign $(2, 2)$, the remaining Lie algebras are given in the following cases.

2.2.1. Case ($\mathbf{B_1}$)

The 5-dimensional Lie algebra $\mathfrak{sl}(2, \mathbb{R}) \ltimes \mathbb{R}^2$ is equal to $\mathfrak{g} = \mathfrak{b}_1$. The basis of $\mathfrak{b}_1$ is such that the non-zero brackets are as follows.

$$[e_1, e_2] = 2e_2, \ [e_1, e_3] = -2e_3, \ [e_2, e_3] = e_1, \ [e_1, e_4] = e_4,$$
$$[e_1, e_5] = -e_5, \quad [e_2, e_5] = e_4, \quad [e_3, e_4] = e_5,$$

and the isotropy subalgebra is $\mathfrak{h} = \text{span}\{h_1 = e_3\}$. Therefore, we assume

$$\mathfrak{m} = \text{span}\{u_1 = e_1, \ u_2 = e_2, \ u_3 = e_4, \ u_4 = e_5\},$$

and the isotropy description for h_1 is

$$H_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The invariant metrics are computed as follows:

$$g = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & b & c & a \\ a & c & d & 0 \\ 0 & a & 0 & 0 \end{pmatrix}. \tag{31}$$

So, they are non-degenerate whenever $a \neq 0$. For any vector field $X = X_i u_i \in \mathfrak{m}$, we have:

$$\mathcal{L}_X g = \begin{pmatrix} 2aX_3 & 2bX_2 + cX_3 & -aX_1 + 2cX_2 + dX_3 & aX_2 \\ 2bX_2 + cX_3 & -4bX_1 + 2cX_4 & -3cX_1 + dX_4 & -aX_1 - cX_2 \\ -aX_1 + 2cX_2 + dX_3 & -3cX_1 + dX_4 & -2dX_1 & -dX_2 \\ aX_2 & -aX_1 - cX_2 & -dX_2 & 0 \end{pmatrix}. \tag{32}$$

Corresponding to the basis $\{u_1, u_2, u_3, u_4\}$ and using relations Section 2.2.1 and (32), we have:

$$\begin{aligned}
(\mathcal{L}_X\mathcal{L}_X g)_{11} &= 8bX_2^2 + 8cX_2X_3 - 2aX_1X_3 + 2dX_3^2 - 2aX_2X_4, \\
(\mathcal{L}_X\mathcal{L}_X g)_{22} &= 16bX_1^2 - 14cX_1X_4 + 2dX_2X_4, \\
(\mathcal{L}_X\mathcal{L}_X g)_{33} &= 4dX_1^2, \\
(\mathcal{L}_X\mathcal{L}_X g)_{44} &= 2dX_2^2, \\
(\mathcal{L}_X\mathcal{L}_X g)_{12} &= -12bX_1X_2 + 7cX_2X_4 - 5cX_3X_1 + 2dX_3X_4, \\
(\mathcal{L}_X\mathcal{L}_X g)_{13} &= -8cX_1X_2 + 3dX_2X_4 - 3dX_3X_1 + aX_1^2, \\
(\mathcal{L}_X\mathcal{L}_X g)_{14} &= -4cX_2^2 - 2dX_2X_3, \\
(\mathcal{L}_X\mathcal{L}_X g)_{23} &= 9cX_1^2 - 5dX_1X_4, \\
(\mathcal{L}_X\mathcal{L}_X g)_{24} &= aX_1^2 + 4cX_1X_2 - 2dX_1X_4 - dX_2X_4, \\
(\mathcal{L}_X\mathcal{L}_X g)_{34} &= 2dX_1X_2.
\end{aligned} \tag{33}$$

In conclusion, using relations (1), (31) and (33), we obtain the following system of algebraic equations to find 2-conformal vector fields

$$\begin{aligned}
&(1)\ 8bX_2^2 + 8cX_2X_3 - 2aX_1X_3 + 2dX_3^2 - 2aX_2X_4 = 0,\\
&(2)\ 16bX_1^2 - 14cX_1X_4 + 2dX_2X_4 = b\beta,\\
&(3)\ 4dX_1^2 = d\beta,\\
&(4)\ 2dX_2^2 = 0,\\
&(5)\ -12bX_1X_2 + 7cX_2X_4 - 5cX_3X_1 + 2dX_3X_4 = 0,\\
&(6)\ -8cX_1X_2 + 3dX_2X_4 - 3dX_3X_1 + aX_1^2 = a\beta,\\
&(7)\ -4cX_2^2 - 2dX_2X_3 = 0,\\
&(8)\ 9cX_1^2 - 5dX_1X_4 = c\beta,\\
&(9)\ aX_1^2 + 4cX_1X_2 - 2dX_1X_4 - dX_2X_4 = a\beta,\\
&(10)\ 2dX_1X_2 = 0.
\end{aligned} \tag{34}$$

By solving the above system, we obtain the following.

Theorem 6. *Let (M, g) be a non-reductive four-dimensional homogeneous space of type $\mathfrak{b}_1$ corresponding to the basis $\{u_1, u_2, u_3, u_4\}$. In this case, the 2-conformal vector fields are as follows:*

(a) If $d = 0$, then invariant 2-conformal vector fields reduce to 2-Killing fields.

(b) If the consistency conditions $7c^2 = 6bd$ and $4bd = -7ac$ are satisfied, then

$$\begin{aligned}
&(a)\ X_2 = 0, \quad X_1 = \pm\frac{\sqrt{\beta}}{2}, \quad X_3 = 0,\\
&(b)\ X_2 = 0, \quad X_1 = \pm\frac{\sqrt{\beta}}{2}, \quad d \neq 0 \quad X_3 = \frac{aX_1}{d}.
\end{aligned}$$

In this case, X_4 can be determined in terms of X_1 and the structure constants of the algebra.

2.2.2. Case ($\mathbf{B_2}$)

The 6-dimensional Schrödinger–Lie algebras $\mathfrak{sl}(2,\mathbb{R}) \ltimes \mathfrak{n}(3)$ with $\mathfrak{g} = \mathfrak{b}_2$ and the isotropy subalgebra $\mathfrak{h} = \text{span}\{h_1 = e_3 - e_6, h_2 = e_5\}$ are equivalent. Therefore, we compute that

$$\mathfrak{m} = \text{span}\{u_1 = e_1,\ u_2 = e_2,\ u_3 = e_3 + e_6,\ u_4 = e_4\},$$

and the isotropy description for h_1, h_2 are

$$H_1 = \begin{pmatrix} 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -\frac{1}{2}\\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

The invariant metrics are described by

$$g = \begin{pmatrix} a & 0 & 0 & 0\\ 0 & b & a & 0\\ 0 & a & 0 & 0\\ 0 & 0 & 0 & -\frac{a}{2} \end{pmatrix}, \tag{35}$$

so they are non-degenerate whenever $a \neq 0$. We have the Lie derivative of the metric for any vector field $X = X_i u_i \in \mathfrak{m}$ as follows.

$$\mathcal{L}_X g = \begin{pmatrix} 0 & 2bX_2 & aX_2 & -\frac{a}{2}X_4 \\ 2bX_2 & -4bX_1 & -aX_1 & 0 \\ aX_2 & -aX_1 & 0 & 0 \\ -\frac{a}{2}X_4 & 0 & 0 & aX_1 \end{pmatrix}. \tag{36}$$

Corresponding to the basis $\{u_1, u_2, u_3, u_4\}$ and using relations Section 2.2.2 and (36), we have:

$$\begin{aligned} &(\mathcal{L}_X\mathcal{L}_X g)_{11} = -aX_4^2 + 8bX_2^2, \\ &(\mathcal{L}_X\mathcal{L}_X g)_{22} = -8bX_1X_2, \\ &(\mathcal{L}_X\mathcal{L}_X g)_{33} = 0, \\ &(\mathcal{L}_X\mathcal{L}_X g)_{44} = -2aX_1^2, \\ &(\mathcal{L}_X\mathcal{L}_X g)_{12} = 4bX_2^2, \\ &(\mathcal{L}_X\mathcal{L}_X g)_{13} = -2aX_1X_3, \\ &(\mathcal{L}_X\mathcal{L}_X g)_{14} = \frac{3}{2}aX_1X_4, \\ &(\mathcal{L}_X\mathcal{L}_X g)_{23} = -2aX_1X_2, \\ &(\mathcal{L}_X\mathcal{L}_X g)_{24} = 0, \\ &(\mathcal{L}_X\mathcal{L}_X g)_{34} = 0. \end{aligned} \tag{37}$$

Consequently, by using relations (1), (35), and (37), we obtain the following system of algebraic equations to find 2-conformal vector fields.

$$\begin{aligned} &(1)\ 8bX_2^2 - aX_4^2 = a\beta, \\ &(2)\ 8bX_1X_2 = -b\beta, \\ &(3)\ 2aX_1^2 = \frac{a}{2}\beta, \\ &(4)\ 4bX_2^2 = 0, \\ &(5)\ 2aX_1X_3 = 0, \\ &(6)\ \frac{3}{2}aX_1X_4 = 0, \\ &(7)\ 2aX_1X_2 = -a\beta. \end{aligned} \tag{38}$$

By solving the above system, we obtain the following theorem.

Theorem 7. *Suppose (M, g) is a four-dimensional non-reductive homogeneous space of type $\mathfrak{b}_2$ corresponding to the basis $\{u_1, u_2, u_3, u_4\}$. In this case, invariant 2-conformal vector fields are as follows:*

(1) If $X_2 = 0$ and $b \neq 0$, then $X_4 = X_1 = 0$, $\beta = 0$, and $X_3 \in \mathbb{R}$.
(2) If $X_2 \neq 0$ and $b = 0$, then $X_4 = X_1 = 0$, $\beta = 0$, and $X_3 \in \mathbb{R}$.

2.2.3. Case ($\mathbf{B_3}$)

The 6-dimensional Lie algebra $(\mathfrak{sl}(2, \mathbb{R}) \ltimes \mathbb{R}^2) \times \mathbb{R}$ is denoted by $\mathfrak{g} = \mathfrak{b}_3$. The basis of $\mathfrak{b}_3$ is $\{e_1, \ldots, e_6\}$, so the non-zero brackets are

$$\begin{aligned} &[e_5, e_2] = e_1, \quad [e_5, e_3] = -e_4, \quad [e_6, e_2] = -2e_6, \\ &[e_6, e_3] = -e_2, \quad [e_6, e_4] = e_1, \quad [e_1, e_2] = -e_1, \\ &[e_1, e_3] = e_4, \quad [e_2, e_3] = -2e_3, \quad [e_2, e_4] = -e_4, \end{aligned}$$

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and the isotropy subalgebra is given by $\mathfrak{h} = \text{span}\{h_1 = e_5, h_2 = e_6\}$. Therefore, we consider

$$\mathfrak{m} = \text{span}\{u_1 = e_1,\ u_2 = e_2, u_3 = e_3, u_4 = e_4\},$$

and the isotropy descriptions for h_1, h_2 are

$$H_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The invariant metrics are described as follows:

$$g = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \\ a & 0 & b & 0 \\ 0 & a & 0 & 0 \end{pmatrix}. \tag{39}$$

Thus, they are non-degenerate whenever $a \neq 0$. For any vector field $X = X_i u_i \in \mathfrak{m}$, we obtain:

$$\mathcal{L}_X g = \begin{pmatrix} 0 & -aX_3 & aX_2 & 0 \\ -aX_3 & -2aX_4 & -2bX_3 & aX_2 \\ aX_2 & -2bX_3 & 4bX_2 & 0 \\ 0 & aX_2 & 0 & 0 \end{pmatrix}. \tag{40}$$

Corresponding to the basis $\{u_1, u_2, u_3, u_4\}$ and using relations Section 2.2.3 and (40), we have:

$$\begin{aligned} &(\mathcal{L}_X\mathcal{L}_X g)_{11} = 0, \\ &(\mathcal{L}_X\mathcal{L}_X g)_{22} = -2aX_1X_3 + 8bX_3^2 - 2aX_2X_4, \\ &(\mathcal{L}_X\mathcal{L}_X g)_{33} = 16bX_2X_3, \\ &(\mathcal{L}_X\mathcal{L}_X g)_{44} = 0, \\ &(\mathcal{L}_X\mathcal{L}_X g)_{12} = aX_3X_2, \\ &(\mathcal{L}_X\mathcal{L}_X g)_{13} = -aX_2^2, \\ &(\mathcal{L}_X\mathcal{L}_X g)_{14} = 0, \\ &(\mathcal{L}_X\mathcal{L}_X g)_{23} = -4bX_2X_3, \\ &(\mathcal{L}_X\mathcal{L}_X g)_{24} = aX_2^2, \\ &(\mathcal{L}_X\mathcal{L}_X g)_{34} = 0. \end{aligned} \tag{41}$$

Consequently, by using relations (1), (39), and (41), we obtain the following system of algebraic equations to find the invariant 2-conformal fields:

$$\begin{aligned} &(1)\ 2aX_1X_3 - 8bX_3^2 + 2aX_2X_4 = 0, \\ &(2)\ 16bX_2X_3 = b\beta, \\ &(3)\ aX_3X_2 = 0, \\ &(4)\ aX_2^2 = -a\beta, \\ &(5)\ 4bX_2X_3 = 0, \\ &(6)\ aX_2^2 = a\beta. \end{aligned} \tag{42}$$

By solving the above system of equations, the following theorem is obtained.

Theorem 8. *Suppose (M, g) is a four-dimensional non-reductive homogeneous space of type $\mathfrak{b}_3$ corresponding to the basis $\{u_1, u_2, u_3, u_4\}$. In this case, invariant 2-conformal vector fields are as follows:*

(1) *If* $X_2 = 0$ *and* $X_3 \neq 0$, *then* $X_1 = \frac{4bX_3}{a}$, $\beta = 0$, *and* $X_4 \in \mathbb{R}$.

(2) *If* $X_2 \neq 0$ *and* $X_3 = 0$, *then the system has no solution.*

3. Conclusions

In pseudo-Riemannian geometry, dimension 4 is of particular interest, because there exist many non-reductive homogeneous 4-manifolds. These spaces were classified by M. E. Fels and A. G. Renner, comprising eight distinct geometric types $\mathbf{A_1}$, $\mathbf{A_2}$, $\mathbf{A_3}$, $\mathbf{A_4}$, $\mathbf{A_5}$, $\mathbf{B_1}$, $\mathbf{B_2}$, and $\mathbf{B_3}$. In this paper, we investigated and classified invariant 2-conformal vector fields on four-dimensional non-reductive homogeneous manifolds. The results are applied to Lorentzian and neutral signatures. Investigating the existence of such fields leads to solving systems of nonlinear equations, for which we completely determine the real solutions using analytical methods. In several cases, the 2-conformal and invariant vector fields reduce to 2-Killing vector fields.

Since 2-conformal vector fields have arisen naturally in the recent literature on hyperbolic solitons, it is an interesting geometric problem to classify hyperbolic solitons on non-reductive homogeneous spaces with 2-conformal vector fields as their potential vector fields.

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