

Article

An Algebraic Approach to Geodesics via Mobi Spaces

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Abstract

Mobi spaces were introduced by the authors as a possible algebraic axiomatization of spaces in which any two points are connected by a geodesic path. Previous work has focused primarily on the algebraic properties of these structures; here, we return to the original geometric motivation. We present new characterizations of mobi spaces inspired by the important class of smooth manifolds that arise as open subsets of Euclidean (n)-space endowed with a Riemannian metric. We show that such manifolds satisfy the axioms of a mobi space, thereby providing a broad family of natural geometric examples.

Keywords: mobi algebra; mobi space; geodesics; Riemannian manifold; Christoffel symbols; algebraic system

MSC: 53C22; 58E10; 08A99

1. Introduction

The concept of a mobi space [1–3] was introduced as an algebraic framework intended to capture the behaviour of spaces in which any two points can be connected by a geodesic path. The underlying idea is to encode the notion of a point lying between two others by means of an algebraic interpolation operation parametrized by elements of a mobi algebra, which is intended as an abstraction of the unit interval. In this way, the geometric intuition of moving along a path between two points can be expressed purely in algebraic terms.

Previous work mainly concentrated on the algebraic aspects of mobi algebras and mobi spaces, including their axiomatic properties and relationships with other algebraic structures [1–3]. However, the original motivation for introducing mobi spaces was geometric, namely, to provide an algebraic model for spaces endowed with geodesic paths.

The purpose of this paper is to further develop this geometric perspective. We present new characterizations of mobi spaces motivated by the behaviour of geodesics in smooth manifolds that are open subsets of Euclidean n -space equipped with a Riemannian metric. In this setting, geodesic segments naturally determine interpolation operations between pairs of points. We show, in Sections 3 and 4, that these examples satisfy the axioms of a mobi space, thereby providing a natural class of geometric models for the theory.

In Section 2, we recall the definition of a mobi algebra and the definition of a mobi space. Examples of mobi spaces, such as the 2-sphere, are presented here as working examples. Although they do not fully fall under the general result established in Sections 3 and 4 of this paper, they nevertheless serve as illustrative instances and have been studied in previous work.

These results reinforce the idea that mobi spaces provide a suitable algebraic setting in which geodesic structures can be studied without explicit reliance on metric or differential



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tools, and they contribute to the broader program of describing geometric phenomena through algebraic axiomatizations.

2. Mobility Algebra and Mobility Space

In this section, we recall the definitions of a mobi algebra [1] and of a mobi space [2,3] over a mobi algebra (also known as a mobility algebra). A mobi algebra was designed to model the unit interval $A = [0, 1]$, together with its affine structure given by the operation $p(a, b, c) = a + b(c - a)$, as well as the distinguished constants $0, \frac{1}{2}, 1 \in [0, 1]$. A mobi space, in turn, is intended to model a space X in which any two points $x, y \in X$ are canonically connected by a geodesic path, parametrized by elements t of the underlying mobi algebra and expressed as $q(x, t, y) \in X$.

Definition 1. A mobi algebra is a system $(A, p, 0, \frac{1}{2}, 1)$, where A is a set; p is a ternary operation on A ; and $0, \frac{1}{2}$ and 1 are distinguished elements of A , satisfying the following axioms:

- (A1) $p(1, \frac{1}{2}, 0) = \frac{1}{2}$;
- (A2) $p(0, a, 1) = a$;
- (A3) $p(a, b, a) = a$;
- (A4) $p(a, 0, b) = a$;
- (A5) $p(a, 1, b) = b$;
- (A6) $p(a, \frac{1}{2}, b_1) = p(a, \frac{1}{2}, b_2) \implies b_1 = b_2$;
- (A7) $p(a, p(c_1, c_2, c_3), b) = p(p(a, c_1, b), c_2, p(a, c_3, b))$;
- (A8) $p(p(a_1, c, b_1), \frac{1}{2}, p(a_2, c, b_2)) = p(p(a_1, \frac{1}{2}, a_2), c, p(b_1, \frac{1}{2}, b_2))$.

The mobi algebra $([0, 1], p, 0, \frac{1}{2}, 1)$, with $p(a, b, c) = a + b(c - a)$, is called the canonical mobi algebra. Any unitary ring R in which the element $2 = 1 + 1$ is invertible is also an example of a mobi algebra $(R, p, 0, 2^{-1}, 1)$, with $p(x, y, z) = x + y(z - x)$. Other mobi algebras can be found in [1]. An example of a mobi algebra in $\mathbb{R}^2$ is $(A, p, 0, \frac{1}{2}, 1)$, with

$$A = \left\{ (x, y) \in \mathbb{R}^2 : |y| \leq x \leq 1 - |y| \right\},$$

$$p(a, b, c) = (a_1 - b_1a_1 - b_2a_2 + b_1c_1 + b_2c_2, a_2 - b_1a_2 - b_2a_1 + b_1c_2 + b_2c_1)$$

and the three constants $\frac{1}{2} = (\frac{1}{2}, 0)$, $1 = (1, 0)$ and $0 = (0, 0)$.

A mobility space (or mobi space) over a mobi algebra is defined as a pair (X, q) satisfying the following conditions.

Definition 2. Let $(A, p, 0, \frac{1}{2}, 1)$ be a mobi algebra. An A -mobi space (X, q) consists of a set X and a map $q: X \times A \times X \rightarrow X$ such that the following hold:

- (M1) $q(x, 0, y) = x = q(y, 1, x)$;
- (M2) $q(x, \frac{1}{2}, y_1) = q(x, \frac{1}{2}, y_2) \implies y_1 = y_2$;
- (M3) $q(q(x, a, y), b, q(x, c, y)) = q(x, p(a, b, c), y)$.

Note that in [2], a mobi space was originally introduced with the following extra axiom:

$$q(x, a, x) = x. \quad (1)$$

Nevertheless, this property is deducible from the other axioms. Indeed, using (M1), (M3), (A3) and again (M1):

$$q(x, a, x) = q(q(x, 0, x), a, q(x, 0, x)) = q(x, p(0, a, 0), x) = q(x, 0, x) = x.$$

The canonical mobi space (X, q) over the canonical mobi algebra is obtained by considering $X = \mathbb{R}^n$ for some $n \in \mathbb{N}$, and $q(x, a, y) = (1 - a)x + ay$. The empty space is trivially an example of a mobi space over any mobi algebra. A space with a single point x and the operation $q(x, a, x) = x$ is another trivial example of a mobi space over any mobi algebra.

Note that every mobi algebra is a mobi space over itself. Consider, for example, $X = A = \{0, \frac{1}{2}, 1\}$ and $q(x, a, y) = p(x, a, y)$ defined as in the following tables:

$p(-, 0, -)$	0	$\frac{1}{2}$	1	$p(-, \frac{1}{2}, -)$	0	$\frac{1}{2}$	1	$p(-, 1, -)$	0	$\frac{1}{2}$	1
0	0	0	0	0	0	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	1
1	1	1	1	1	$\frac{1}{2}$	0	1	1	0	$\frac{1}{2}$	1

Then, (X, q) is a mobi space over the mobi algebra $(A, p, 0, \frac{1}{2}, 1)$.

Another intuitive mobi space is obtained by considering the euclidean 2-sphere

$$X = \{x \in \mathbb{R}^3 \mid \langle x, x \rangle = 1\},$$

where $\langle x, y \rangle$ is the usual euclidean scalar product, a map $v : X \rightarrow X$ such that

$$v(-x) = v(x) \text{ and } \langle x, v(x) \rangle = 0,$$

and the map $\theta : X \times X \rightarrow [0, \pi]$ defined by

$$\theta(x, y) = \arccos(\langle x, y \rangle).$$

With $q : X \times [0, 1] \times X \rightarrow X$ defined by

$$q(x, t, y) = \begin{cases} \frac{\sin[\theta(x, y)(1-t)]}{\sin[\theta(x, y)]}x + \frac{\sin[\theta(x, y)t]}{\sin[\theta(x, y)]}y & , \text{ if } \theta(x, y) \in]0, \pi[\\ \cos[\theta(x, y)t]x + \sin[\theta(x, y)t]v(x) & , \text{ if } \theta(x, y) \in \{0, \pi\} \end{cases},$$

(X, q) is a mobi space over the canonical mobi algebra. The second branch of q allows for choosing a path between antipodal points on the sphere [2].

As a last non-trivial example of a mobi space (X, q) over the canonical mobi algebra, consider $X = \mathbb{R} \times \mathbb{R}^+$ and

$$q((x_1, y_1), a, (x_2, y_2)) = \left(x_1 + \frac{2ay_1 + a^2(y_2 - y_1)}{y_1 + y_2}(x_2 - x_1), y_1 + a(y_2 - y_1) \right).$$

Readers can verify this example by straightforward substitution.

3. A Characterization of Mobi Spaces via Abstract Maps

We begin with some general results that can be used to construct mobi spaces using sets U, V, X and maps h, α, β , as presented in Proposition 1 below.

For an intuitive understanding of the underlying algebra, recall that the canonical mobi algebra is given by the real interval $A = [0, 1]$, together with the ternary operation $p(a, b, c) = a + b(c - a)$. Later, we will apply these general results to the particular case where U is an open subset of a vector space V , for example, $V = \mathbb{R}^n$, the map α is the identity, and β assigns to each pair of points in $X = U$ a vector $v \in V$ such that moving from x in the direction of v leads to y . In this situation, the map h is obtained as the (unique) solution of the corresponding initial value problem.

Proposition 1. Let $(A, p, 0, \frac{1}{2}, 1)$ be a mobi algebra and consider sets U, V, X and maps h, α, β such that

$$h: U \times A \times V \rightarrow X, \quad \alpha: X \rightarrow U, \quad \beta: X \times X \rightarrow V.$$

If the following three conditions hold:

(a) For every $x_1, x_2 \in X$, there exist $u \in U, v \in V$ and $a_1, a_2 \in A$ such that

$$\begin{cases} x_1 = h(u, a_1, v) \\ x_2 = h(u, a_2, v) \end{cases},$$

(b) For all $u \in U$ and $v, v' \in V$,

$$h(u, \frac{1}{2}, v) = h(u, \frac{1}{2}, v') \Rightarrow h(u, 1, v) = h(u, 1, v');$$

(c) For every $u \in U, v \in V$ and $a, b, c \in A$

$$h(u, p(a, b, c), v) = h(\alpha(h(u, a, v)), b, \beta(h(u, a, v), h(u, c, v))); \quad (2)$$

then the formula

$$q(x_1, a, x_2) = h(\alpha(x_1), a, \beta(x_1, x_2)) \quad (3)$$

defines an A -mobi space (X, q) .

Proof. First, in order to prove axiom (M1), we need to consider any two elements in X , say x_1 and x_2 . Under the hypothesis (a), we have the freedom to write $x_1 = h(u, a, v)$ and $x_2 = h(u, c, v)$, for some $u \in U, v \in V$ and $a, c \in A$. Now, for any $b \in A$, hypothesis (c) becomes

$$h(u, p(a, b, c), v) = h(\alpha(x_1), b, \beta(x_1, x_2))$$

which, by definition of q , is the same as

$$h(u, p(a, b, c), v) = q(x_1, b, x_2).$$

Consequently, $q(x_1, 0, x_2) = h(u, a, v) = x_1$ by (A4) and, by (A5), $q(x_1, 1, x_2) = h(u, c, v) = x_2$, which proves (M1). In order to prove (M2), let us suppose $x_1, x_2, x'_2 \in X$ are such that

$$q(x_1, \frac{1}{2}, x_2) = q(x_1, \frac{1}{2}, x'_2)$$

which is the same as

$$h(\alpha(x_1), \frac{1}{2}, \beta(x_1, x_2)) = h(\alpha(x_1), \frac{1}{2}, \beta(x_1, x'_2)).$$

From here, hypothesis (b) implies

$$h(\alpha(x_1), 1, \beta(x_1, x_2)) = h(\alpha(x_1), 1, \beta(x_1, x'_2))$$

which, by definition of q , is the same as

$$q(x_1, 1, x_2) = q(x_1, 1, x'_2)$$

and we make use of axiom (M1), already proved, to get the desired equality $x_2 = x'_2$. Finally, to prove (M3), let us take any $x_1, x_2 \in X$ and $a, b, c \in A$ and define $u = \alpha(x_1)$ and $v = \beta(x_1, x_2)$. Then, by definition of q , condition (2) is precisely axiom (M3). $\square$

A particular case arises when, in condition (a) of Proposition 1, we take $u = \alpha(x_1)$ and $v = \beta(x_1, x_2)$, as detailed below.

Corollary 1. Let $(A, p, 0, \frac{1}{2}, 1)$ be a mobi algebra and consider sets U, V, X and maps

$$h: U \times A \times V \rightarrow X, \quad \alpha: X \rightarrow U, \quad \beta: X \times X \rightarrow V.$$

Then, the Formula (3) defines an A -mobi space (X, q) if the following hold:

(a) For every $x_1, x_2 \in X$, there exist $a_1, a_2 \in A$ for which

$$\begin{cases} x_1 = h(\alpha(x_1), a_1, \beta(x_1, x_2)), \\ x_2 = h(\alpha(x_1), a_2, \beta(x_1, x_2)) \end{cases} \quad (4)$$

(b) $h(\alpha(x_1), \frac{1}{2}, \beta(x_1, x_2)) = h(\alpha(x_1), \frac{1}{2}, \beta(x_1, x'_2))$
 $\Rightarrow h(\alpha(x_1), 1, \beta(x_1, x_2)) = h(\alpha(x_1), 1, \beta(x_1, x'_2));$

(c) $h(\alpha(x_1), p(a, b, c), \beta(x_1, x_2)) =$
 $h(\alpha(h(\alpha(x_1), a, \beta(x_1, x_2))), b,$
 $\beta(h(\alpha(x_1), a, \beta(x_1, x_2)), h(\alpha(x_1), c, \beta(x_1, x_2)))).$

Let us now investigate to what extent the sufficient conditions in the previous result are also necessary. To express the relationships between elements of U and V arising from points in X , we introduce the following relations.

Definition 3. Let U, V , and X be sets, and let $\alpha: X \rightarrow U$ and $\beta: X \times X \rightarrow V$ be given maps. We define two relations as follows:

1. The relation $R \subseteq U \times V$ is defined by $(u, v) \in R$ if and only if there exists $x_1, x_2 \in X$ such that $u = \alpha(x_1)$ and $v = \beta(x_1, x_2)$.
2. The relation $S \subseteq U \times V \times V$ is defined by $(u, v, v') \in S$ if and only if $(u, v) \in R$ and $(u, v') \in R$.

We can now express necessary and sufficient conditions in order to have a mobi space out of Formula (3), repeated as Formula (5) below.

Theorem 1. Let $(A, p, 0, \frac{1}{2}, 1)$ be a mobi algebra and consider sets U, V, X and maps

$$h: U \times A \times V \rightarrow X, \quad \alpha: X \rightarrow U, \quad \beta: X \times X \rightarrow V.$$

Then, the formula

$$q(x_1, a, x_2) = h(\alpha(x_1), a, \beta(x_1, x_2)) \quad (5)$$

defines an A -mobi space (X, q) if and only if the following hold:

(a) For every $x_1, x_2 \in X$, there exist $(u, v) \in R$ and $a_1, a_2 \in A$ for which

$$\begin{cases} x_1 = h(u, a_1, v), \\ x_2 = h(u, a_2, v) \end{cases} \quad (6)$$

(b) For all $(u, v, v') \in S$, we have

$$h(u, \frac{1}{2}, v) = h(u, \frac{1}{2}, v') \Rightarrow h(u, 1, v) = h(u, 1, v');$$

(c) For every $(u, v) \in R$ and $a, b, c \in A$, the following condition holds:

$$h(u, p(a, b, c), v) = h(\alpha(h(u, a, v)), b, \beta(h(u, a, v), h(u, c, v))). \quad (7)$$

Proof. The proof of Proposition 1 may be replicated here to prove that (a), (b) and (c) turn (X, q) into a mobi space. Indeed, (a) of Theorem 1 implies (a) of Proposition 1 and, using

(b) and (c) in Proposition 1, the elements of U and V are always in relation. Therefore, it remains to prove that in a mobi space (X, q) , with q given by (5), (a), (b) and (c) are verified. Suppose then that (X, q) is a mobi space. Condition (a) is verified because, given any $x_1, x_2 \in X$, the choices $u = \alpha(x_1)$, $v = \beta(x_1, x_2)$, $a_1 = 0$ and $a_2 = 1$ imply

$$h(u, a_1, v) = h(\alpha(x_1), 0, \beta(x_1, x_2)) = q(x_1, 0, x_2) = x_1$$

and

$$h(u, a_2, v) = h(\alpha(x_1), 1, \beta(x_1, x_2)) = q(x_1, 1, x_2) = x_2.$$

To prove condition (b), consider that $h(u, \frac{1}{2}, v) = h(u, \frac{1}{2}, v')$, with $u = \alpha(x_1)$, $v = \beta(x_1, x_2)$ and $v' = \beta(x_1, x'_2)$ for some $x_1, x_2, x'_2 \in X$ existing because $(u, v, v') \in S$. This means, by (5), that

$$q(x_1, \frac{1}{2}, x_2) = q(x_1, \frac{1}{2}, x'_2).$$

Using (M2), we then get $x_2 = x'_2$ and, consequently, $v = v'$ and

$$h(u, 1, v) = h(u, 1, v').$$

Finally, the existence of $x_1, x_2 \in X$ with $u = \alpha(x_1)$ and $v = \beta(x_1, x_2)$ can be used again to show that condition (7) is true because it expresses precisely axiom (M3). $\square$

A simple example to illustrate Theorem 1 is obtained looking at the shortest paths between two points on the unit circle as follows. Consider $U = V = \mathbb{R}$, $A = [0, 1]$ as the unit interval mobi algebra with $p(a, b, c) = a + b(c - a)$, and $X = \{z \in \mathbb{C} \mid \text{abs}(z) = 1\}$. The map h is defined as $h(u, t, v) = e^{i\theta}$, where $\theta = u + tv$. The map α is defined as $\alpha(z) = \arg(z)$, with $\arg(z)$ meaning the phase angle of the complex number z considered in the interval $]-\pi, \pi]$, as usual, and the map β defined as

$$\beta(z_1, z_2) = \begin{cases} \arg(z_2/z_1) & \text{if } \arg(z_2/z_1) \neq \pi \\ \arg(z_2) - \arg(z_1) & \text{if } \arg(z_2/z_1) = \pi \end{cases}.$$

Note that in the case $\arg(z_2/z_1) = \pi$, a choice must be made since there are two shortest paths between the points, and only one may be selected. This example can be generalized from the unit circle to the group of non-zero complex numbers, as well as to the group of non-zero quaternions; however, this requires careful choices for the map β , and we postpone such a generalization to a future study where an appropriate framework will be developed.

As a second illustration of Theorem 1, consider $X = U = V = \mathbb{R}^2$; $A = \mathbb{R}$; and the maps h , α and β defined, for an arbitrary number $k \in \mathbb{R}$, as follows:

$$h\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, t, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = \begin{pmatrix} u_1 + v_1 t - k v_2^2 t^2 \\ u_2 + v_2 t \end{pmatrix} \quad (8)$$

$$\alpha\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (9)$$

$$\beta\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) = \begin{pmatrix} y_1 - x_1 + k(y_2 - x_2)^2 \\ y_2 - x_2 \end{pmatrix}. \quad (10)$$

Using $p(a, b, c) = a + b(c - a)$ for the mobi algebra $(\mathbb{R}, p, 0, \frac{1}{2}, 1)$, $(\mathbb{R}^2, q)$ is a mobi

space with

$$q\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, t, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) = \begin{pmatrix} x_1 + (y_1 - x_1)t + k(y_2 - x_2)^2 t(1-t) \\ x_2 + (y_2 - x_2)t \end{pmatrix}. \quad (11)$$

This example is inspired by the motion of a projectile and discussed in greater detail at the end of Section 4.

4. Geodesic Spaces as Mobi Spaces

In this section, we analyse spaces obtained as open subsets of Euclidean n -space in which a geodesic path between any two points is unique (or can be chosen uniquely), and we show that each such space gives rise to a mobi space. For the present discussion, we restrict our attention to open subsets of $\mathbb{R}^n$; however, the same principle extends to abstract Riemannian manifolds [4], which we plan to explore in future work. Our goal here is to illustrate that spaces with geodesics form a natural class of examples of mobi spaces.

Let $X \subseteq V$ be an open subset of an n -dimensional vector space V , for example, $V = \mathbb{R}^n$, and consider a map

$$g: X \times V \rightarrow V$$

satisfying

$$g(x, \lambda v) = \lambda^2 g(x, v), \quad (12)$$

for all $x \in X$, $v \in V$, and $\lambda \in \mathbb{R}$.

Suppose that for functions $x = x(t)$ and $v = v(t)$ defined on some interval $I \subseteq \mathbb{R}$ containing the unit interval $A = [0, 1]$, the initial value problem

$$\begin{cases} x' = v & , \quad x(0) = x_0 \in X \\ v' = g(x, v) & , \quad v(0) = v_0 \in V \end{cases} \quad (13)$$

has a unique solution, which we denote by

$$x(t) = h(x_0, t, v_0)$$

for every initial condition $(x_0, v_0) \in X \times V$.

Equivalently, for fixed $x \in X$ and $v \in V$, if $f: I \rightarrow X$ is a continuously differentiable map with continuous second derivative $f'': I \rightarrow V$ satisfying

$$f(0) = x, \quad f'(0) = v, \quad f''(t) = g(f(t), f'(t)) \quad \text{for all } t \in I, \quad (14)$$

then

$$f(t) = h(x, t, v). \quad (15)$$

For example, the constant function $f(t) = x$ satisfies (14) when $v = 0$, which immediately implies

$$h(x, t, 0) = x. \quad (16)$$

The classical conditions ensuring uniqueness of solutions to an initial value problem are well-known; see, for instance, [4].

Of course, the function h has also the following properties:

$$h(x, 0, v) = x \quad (17)$$

$$h'(x, 0, v) = v \quad (18)$$

$$h''(x, t, v) = g(h(x, t, v), h'(x, t, v)), \quad (19)$$

for every $t \in \mathbb{R}$, where $h'(x, t, v)$ stands for the first derivative of h with respect to its second argument and similarly $h''(x, t, v)$ stands for the second derivative of h with respect to its second argument.

The function h presented in (8) is an example of a solution of a problem of type (13) with

$$g\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = \begin{pmatrix} -2k v_2^2 \\ 0 \end{pmatrix}.$$

Lemma 1. Consider an initial value problem as in (13), with a unique solution h . Then, the following condition holds for every $x \in X$, $v \in V$, and $r, s, t \in \mathbb{R}$:

$$h(x, r + st, v) = h(h(x, r, v), t, s h'(x, r, v)). \quad (20)$$

Proof. Consider the map $f(t) = h(x, r + st, v)$ that satisfies conditions (14) for $x = h(x, r, v)$ and $v = s h'(x, r, v)$. Indeed,

$$f'(t) = s h'(x, r + st, v) \text{ and } f''(t) = s^2 h''(x, r + st, v)$$

so $f''(t) = s^2 g(h(x, r + st, v), h'(x, r + st, v))$, which may be rewritten using (12) as $f''(t) = g(f(t), f'(t))$. The desired equality then follows from the uniqueness of solutions with respect to initial conditions, as expressed in condition (15). $\square$

Lemma 2. Consider an initial value problem as in (13), with a unique solution h , and suppose the existence of a map $\beta: X \times X \rightarrow V$ such that

$$h(x, 1, \beta(x, y)) = y, \quad (21)$$

for every $x, y \in X$. Then, the following two conditions are equivalent:

- (a) $h(x, 1, v_1) = h(x, 1, v_2) \Rightarrow v_1 = v_2, \quad \forall x \in X, v_1, v_2 \in V$;
- (b) The map β with the property (21) is unique.

Moreover, we always have

$$\beta(x, y_1) = \beta(x, y_2) \Rightarrow y_1 = y_2.$$

Proof. Assuming condition (a), if there exists another map with the same property as β , say $\hat{\beta}: X \times X \rightarrow V$, then for every $x, y \in X$, we have

$$h(x, 1, \beta(x, y)) = y = h(x, 1, \hat{\beta}(x, y)).$$

Now, from (a), it immediately follows that $\beta = \hat{\beta}$. Conversely, let it be given any $x \in X$ and $v_1, v_2 \in V$ such that

$$h(x, 1, v_1) = h(x, 1, v_2)$$

and let us denote by $y \in X$, both $h(x, 1, v_1)$ and $h(x, 1, v_2)$. We now consider $\beta(x, y) \in V$ and observe that also $h(x, 1, \beta(x, y)) = y$, and hence, by the uniqueness of β , we obtain

$$\beta(x, y) = v_1 = v_2,$$

which concludes the first part of the proof. For the remaining part, we simply observe that if $\beta(x, y_1) = \beta(x, y_2)$, then $y_1 = h(x, 1, \beta(x, y_1)) = h(x, 1, \beta(x, y_2)) = y_2$. $\square$

Lemma 3. Consider an initial value problem as in (13), with a unique solution h . If there exists a unique $\beta: X \times X \rightarrow V$ such that for all $x, y \in X$,

$$h(x, 1, \beta(x, y)) = y$$

then we have

$$\beta(h(x, s, v), h(x, t, v)) = (t - s)h'(x, s, v) \quad (22)$$

for every $x \in X, v \in V$ and $t, s \in \mathbb{R}$.

Proof. On the one hand, the definition of β implies

$$h(h(x, s, v), 1, \beta(h(x, s, v), h(x, t, v))) = h(x, t, v).$$

On the other hand, a particular case of Lemma 1 reads

$$h(h(x, s, v), 1, (t - s)h'(x, s, v)) = h(x, t, v).$$

Therefore, the unicity of β proves (22). $\square$

Theorem 2. Consider an initial value problem as in (13) with a unique solution h , for which there is a unique $\beta: X \times X \rightarrow V$ such that (21) holds. Then, the structure (X, q) , with

$$q(x, t, y) = h(x, t, \beta(x, y))$$

is a mobi space over the mobi algebra $([0, 1], p, 0, \frac{1}{2}, 1)$, where

$$p(s, u, t) = s + u(t - s).$$

Proof. To prove this theorem, it is sufficient to show that the three hypotheses of Proposition 1 are satisfied. We may consider $U = X$ and $\alpha(x) = x$.

(a) For every $x_1, x_2 \in X$, there exist $x_1 \in U, \beta(x_1, x_2) \in V$ and $0, 1 \in A$ such that

$$\begin{cases} x_1 = h(x_1, 0, \beta(x_1, x_2)) \\ x_2 = h(x_1, 1, \beta(x_1, x_2)) \end{cases}.$$

(b) For all $u \in U$ and $v, v' \in V$,

$$h(u, \frac{1}{2}, v) = h(u, \frac{1}{2}, v') \Rightarrow h(u, 1, \frac{1}{2}v) = h(u, 1, \frac{1}{2}v') \Rightarrow v = v',$$

where we have used Lemma 2 and the property $h(u, s, v) = h(u, 1, sv)$ which is a particular case of Lemma 1 when (18) is also used. Consequently, $h(u, 1, v) = h(u, 1, v')$.

(c) For every $u \in U, v \in V$ and $a, b, c \in A$

$$\begin{aligned} h(u, p(a, b, c), v) &= h(u, a + (c - a)b, v) \\ &= h(h(u, a, v), b, (c - a)h'(u, a, v)) \\ &= h(h(u, a, v), b, \beta(h(u, a, v), h(u, c, v))), \end{aligned}$$

where Lemmas 1 and 3 were used.

$\square$

Corollary 2. Let $U \subseteq \mathbb{R}^n$ be an open subset of Euclidean space, considered as a Riemannian manifold, in which a unique geodesic connects any two points. Then, (U, q) is a mobi space, where $q(x, t, y)$ denotes the position at parameter $t \in [0, 1]$ along the geodesic from x to y .

Proof. If $\Gamma_{ij}^k: U \rightarrow \mathbb{R}$ are the Christoffel symbols (see [5], for instance) for the metric, then the function g in the initial value problem (13) is of the form

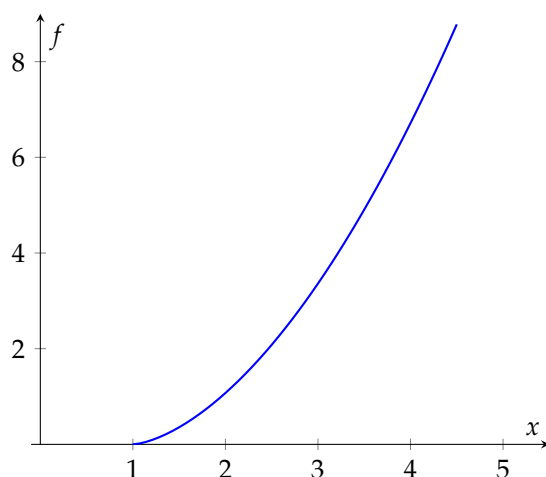
$$g_k(x, y) = - \sum_{i,j} y_i \Gamma_{ij}^k(x) y_j$$

and henceforth satisfies (12). The existence of unique geodesic paths between any two points implies the uniqueness of β in Theorem 2. $\square$

As a simple example of Corollary 2, consider a space X corresponding to the graphic of the real function of real variable f defined, for $x \geq 1$, by

$$f(x) = \frac{1}{2} \left(x \sqrt{x^2 - 1} - \ln(x + \sqrt{x^2 - 1}) \right).$$

The first derivative of this function is given by $f'(x) = \sqrt{x^2 - 1}$, and therefore, the function is strictly increasing and its graph is as follows:



The space $X = \{(x, y) \in \mathbb{R}^2 : x \in]1, +\infty[\wedge y = f(x)\}$ is parametrized by the one-dimensional open subset $U =]1, +\infty[$. Consequently, the square of an infinitesimal euclidean distance on X is

$$ds^2 = dx^2 + dy^2 = \left(1 + (f'(x))^2 \right) dx^2 = x^2 dx^2. \quad (23)$$

We can conclude from (23) that on the curve, the metric is given by

$$\gamma(x) = x^2 \quad (24)$$

and consequently that the Christoffel symbol Γ is

$$\Gamma(x) = \frac{1}{2} \gamma^{-1} \gamma' = \frac{1}{x}. \quad (25)$$

The equation for the geodesic $x'' + \Gamma(x) (x')^2 = 0$ is therefore the following:

$$x'' = -\frac{1}{x} (x')^2. \quad (26)$$

This differential equation is easily solved and, with arbitrary real parameters u and v , the solution can be written as

$$x(t) = \sqrt{u + tv}. \quad (27)$$

It is worth remembering that despite X being one-dimensional, not any displacement on X is considered a geodesic because these have a constant speed, which, in this case, is given by

$$\frac{ds}{dt} = x x' = \frac{v}{2}. \quad (28)$$

Using the notations of Proposition 1, with $U \cong X$, this example corresponds to the following sets and maps:

$$\begin{aligned} A &= [0, 1] \\ X &= \{(x, y) \in \mathbb{R}^2 : x > 1 \wedge y = f(x)\} \\ U &=]1, +\infty[\\ V &= \mathbb{R} \\ h &: U \times A \times V \rightarrow X \\ \alpha &: X \rightarrow U \\ \beta &: X \times X \rightarrow V \\ h(u, t, v) &= \left(\sqrt{u + tv}, f\left(\sqrt{u + tv}\right) \right), \text{ when } u + tv > 1 \\ \alpha(x, y) &= x^2 \\ \beta((x_1, y_1), (x_2, y_2)) &= x_2^2 - x_1^2 \\ q((x_1, y_1), t, (x_2, y_2)) &= \left(\sqrt{(1-t)x_1^2 + tx_2^2}, f\left(\sqrt{(1-t)x_1^2 + tx_2^2}\right) \right). \end{aligned}$$

Note that (U, q) is also a mobi space if we set $q(x_1, t, x_2) = \sqrt{(1-t)x_1^2 + tx_2^2}$.

As another example, consider the functions $x_1(t) \in \mathbb{R}$ and $x_2(t) \in \mathbb{R}$ verifying the following system of differential equations with initial conditions as presented in (13) for some real number k :

$$\begin{cases} x_1'' &= -2k x_1' x_2' - k^2 x_1 x_2'^2 \\ x_2'' &= 0 \end{cases}. \quad (29)$$

Then, we get the following expressions for h , β and q :

$$h\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, t, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = \begin{pmatrix} (x_1 + v_1 t + k x_1 v_2 t) e^{-k v_2 t} \\ x_2 + v_2 t \end{pmatrix} \quad (30)$$

$$\beta\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) = \begin{pmatrix} y_1 e^{k(y_2 - x_2)} - x_1 - k x_1 (y_2 - x_2) \\ y_2 - x_2 \end{pmatrix} \quad (31)$$

$$q\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, t, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) = \begin{pmatrix} x_1 e^{-k(y_2 - x_2)t} (1-t) + y_1 e^{k(y_2 - x_2)(1-t)} t \\ x_2 + (y_2 - x_2)t \end{pmatrix} \quad (32)$$

Our previous example is an application of the following corollary that shows that any second-order differential equation with unique solution for boundary conditions can produce a mobi space by introducing an extra dimension.

Corollary 3. Consider smooth functions $f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $x : \mathbb{R} \rightarrow \mathbb{R}^n$, and let $\dot{x}$ and $\ddot{x}$ be the first and second derivatives of $x = x(t)$ with respect to the variable $t \in \mathbb{R}$. If the problem

$$\begin{cases} \ddot{x} = f(x, t, \dot{x}) \\ x(t_1) = x_1 \\ x(t_2) = x_2 \end{cases} \quad (33)$$

has a unique solution for any $x_1, x_2 \in \mathbb{R}^n$ and $t_1, t_2 \in \mathbb{R}$, with $t_2 \neq t_1$, expressed as

$$x(t) = h((x_1, t_1), t, (x_2, t_2)),$$

then $(\mathbb{R}^{n+1}, q)$ is a mobi space over the mobi algebra $([0, 1], p, 0, \frac{1}{2}, 1)$, with

$$p(t_1, s, t_2) = t_1 + s(t_2 - t_1),$$

$$q((x_1, t_1), s, (x_2, t_2)) = (h((x_1, t_1), p(t_1, s, t_2), (x_2, t_2)), p(t_1, s, t_2)).$$

Proof. Considering a variable s such that $t = p(t_1, s, t_2) = t_1 + s(t_2 - t_1)$, and denoting by x' and x'' the first and second derivatives of $x = x(t) = x(t(s))$ with respect to s , and letting t' denote the derivative of $t = t(s)$ with respect to s , we get $x' = t'\dot{x}$ and $x'' = t'^2\ddot{x}$. With $\chi = (x, t) \in \mathbb{R}^{n+1}$ and $\omega = (x', t')$, we then have

$$\begin{cases} \chi' = \omega \\ \omega' = g(\chi, \omega) \end{cases}$$

where $g((x, t), (x', t')) = (t'^2 f(x, t, \frac{x'}{t'}), 0)$. It is clear that $g(\chi, \lambda\omega) = \lambda^2 g(\chi, \omega)$. The uniqueness of solutions for (33) implies that there exists an initial value problem (at the initial value $s = 0$), such as (13), equivalent to (33). Therefore, Theorem 2 can be applied and proves this Corollary. $\square$

As already observed, example (29) illustrates Corollary 3 with $f(x, t, \dot{x}) = -2k\dot{x} - k^2x$. The equation $\ddot{x} = f(x, t, \dot{x})$ is the differential equation of a critically damped harmonic oscillator. Note that if t is not considered as an extra dimension, the classical problem would be

$$\begin{cases} \ddot{x} = f(x, t, \dot{x}) \\ x(0) = x_1 \\ x(1) = x_2 \end{cases} \quad (34)$$

whose solution $x(x_1, t, x_2) = x_1(1-t)e^{-kt} + x_2te^{(1-t)k}$ does not satisfy all axioms of a mobi space, but solution (32) does. Note that in problem (29), $\chi = (x_1, x_2)$ and $g((x_1, x_2), (x'_1, x'_2)) = (-2kx'_1x'_2 - k^2x_1x_2^2, 0)$.

For another illustration of Corollary 3 and hence Theorem 2, consider the function $f(x, t, \dot{x}) = -2k$, where $k \in \mathbb{R}$ is a constant and $x = x(t) \in \mathbb{R}$. For $k > 0$, the equation $\ddot{x} = -2k$ is the equation of a projectile in one dimension. In this example, with $X = \mathbb{R}^2$ and $V = \mathbb{R}^2$, we obtain

$$\begin{aligned} g : X \times V &\rightarrow \mathbb{R}^2 \\ ((a_1, a_2), (w_1, w_2)) &\rightarrow (-2kw_2^2, 0). \end{aligned}$$

The solution of the following initial value problem involving the functions $\chi : \mathbb{R} \rightarrow X$ and $v : \mathbb{R} \rightarrow V$:

$$\begin{cases} \chi' = v & , \quad \chi(0) = (x_1, x_2) \in X \\ v' = g(\chi, v) & , \quad v(0) = (v_1, v_2) \in V \end{cases} \quad (35)$$

is

$$\chi(t) = h((x_1, x_2), t, (v_1, v_2)) = (x_1 + v_1 t - k v_2^2 t^2, x_2 + v_2 t).$$

It is easy to see that there is a unique $\beta : X \times X \rightarrow V$ such that

$$h((x_1, x_2), 1, \beta((x_1, x_2), (y_1, y_2))) = (y_1, y_2)$$

given by

$$\beta((x_1, x_2), (y_1, y_2)) = (y_1 - x_1 + k(y_2 - x_2)^2, y_2 - x_2).$$

We then obtain the ternary operation $q : X \times \mathbb{R} \times X \rightarrow X$ of the mobi space (X, q) , which is given by

$$\begin{aligned} q((x_1, x_2), t, (y_1, y_2)) &= h((x_1, x_2), t, \beta((x_1, x_2), (y_1, y_2))) \\ &= (x_1 + (y_1 - x_1)t + k(y_2 - x_2)^2(t - t^2), x_2 + (y_2 - x_2)t). \end{aligned}$$

This is exactly the example presented in Equation (11).

5. Conclusions

In Sections 3 and 4 of this paper, we have developed a framework for the generation of mobi spaces based on Riemannian metrics. The approach highlights how the underlying principles unify a range of examples, while also clarifying the limitations of the current formulation. As examples of these limitations are mobi spaces presented in Section 2. Those examples demonstrate the flexibility of the framework and provide concrete evidence of its broader applicability.

These considerations point naturally toward several directions for future research. Further investigation of the interplay between geometric structure and mobi space generation may lead to a deeper understanding of the framework and its potential applications.

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