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Complex Minkowski Spacetimes: Spin Matrices, Algebra and Quaternionic Structures

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Abstract

Complex metrics play a fundamental role in applied mathematical analysis and in the study of physical phenomena. In this work, two complex Minkowski spacetime metrics are examined in detail. The investigation centers on the spin matrices that generate these spacetimes, their corresponding algebraic properties, and the quaternionic structures that emerge from them. The key findings of this study include the identification of a novel set of spin matrices and the characterization of their symmetry, spectral, commutator, and anticommutator properties. Furthermore, a new generalized Lorentz algebra is derived, and a quaternionic mapping of the proposed spin matrices is performed, leading to the construction of new orthonormal quaternionic basis vectors.

Keywords: complex Minkowski spacetime; spin matrices; algebraic properties; generalized Lorentz algebra; quaternionic structures; orthonormal quaternionic basis vectors

MSC: 81R15; 22E43; 11R52; 51M15

1. Background

Complex metrics serve as essential frameworks in applied mathematical analysis and in the exploration of physical phenomena. Although not usually interpreted as literal physical spacetimes, studying metrics with complex components or more generally, complexified metrics has proven valuable in several concrete areas of physics. These include analytic continuation, path integrals, black hole thermodynamics and instantons, solution-generating methods in general relativity (GR) such as the Newman-Janis algorithm and engineered effective metrics in optics and analogue systems.

Building on earlier work involving complex metrics in quantum field theory and topology change from a real-time perspective, Witten (2022) [1] introduced a speculative proposal concerning potential restrictions on the allowed saddle points in the gravitational path integral. In that work, the author explored topological change in Lorentzian signature spacetimes and emphasized the important role of complex spacetime metrics in applications such as the Hartle–Hawking wavefunction of the universe [2]. Rotating black holes were also examined within this framework. A complementary contribution in this direction was given by Jonas et al. (2022) [3], who investigated the use of complex metrics in cosmology. Their work directly responds to the proposal in Witten (2022) [1], addressing whether complex metrics should be allowable in theoretical cosmological frameworks.

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They introduce a method to quickly determine the admissibility of minisuperspace metrics, finding that in well-understood cases the criterion yields reasonable results. For example, there appears to be no obstruction to calculating transition amplitudes between classical configurations of the universe. However, they caution that special care is required when applying the criterion to off-shell metrics with restricted functional freedom.

Another important application of complex metrics arises in optics. Castaldi et al. (2013) [4] extended the transformation-optics framework into a complex spatial coordinate domain to address electromagnetic metamaterials characterized by balanced loss and gain, specifically parity–time (PT) symmetric metamaterials. In their work, the authors applied general theory on complex-source-point radiation and anisotropic transmission resonances to demonstrate the potential of this approach for analytical modeling, systematic design, and for gaining physical insights into complex-coordinate wave objects (and resonant states). Another development in complex spacetime metrics was presented by Horsley et al. (2015) [5], who established a novel and general relation between the real and imaginary parts of a locally isotropic planar permittivity profile that guarantees zero reflection. Their approach leverages the properties of the permittivity profile at complex spatial coordinates to predict wave behavior when the spatial coordinate is real. An overview of transformation optics was provided in McCall et al., (2018) [6]. The authors in that work discuss complex coordinates and spacetime transformations in various contexts including photonics, lensing, metamaterials, wave guidance and antenna engineering.

Complex metrics are valuable tools for discovering solutions in GR. A well-known example is the Newman-Janis trick, which generates rotating black hole spacetimes (such as Kerr and Kerr–Newman) from static, spherically symmetric solutions. This approach provides a shortcut to the rotating case without directly resolving Einstein's field equations [7,8]. Although its generalizations to other spacetimes are not always effective, the Newman-Janis trick has inspired broader research into the role of complex transformations in GR and quantum gravity. A clear example was provided by Fazzini (2025) [9], where an effective Kerr metric was derived from an effective Schwarzschild metric inspired by loop quantum gravity using the Newman-Janis algorithm. By expressing the metric in generalized Painlevé–Gullstrand coordinates, the authors (i) offered a potential explanation of the resulting lower bound and (ii) analyzed the horizon structure and ergoregion (comparing key features with those of the classical Kerr spacetime). A related study by Ban et al. (2025) [10] examined two new spherically symmetric black hole models with covariance in the framework of effective quantum gravity. Using a modified Newman-Janis algorithm, the authors generated two rotating, quantum-corrected black hole solutions characterized by three parameters: mass, spin, and a quantum parameter. Their analysis focused on the properties of the black hole horizons and static limit surfaces, employing Event Horizon Telescope data to constrain the quantum parameter. Complex Einstein equations have also been explored in the fundamental work of Plebanski, (1975) [11]. In that work, the author outlined a general theory of certain types of complex spacetimes and provided examples of nontrivial solutions to all nondegenerate algebraic types. Similar works on complex approaches in GR are given in Plebanski and Demianski (1976) [12] and Plebanski and Robinson (1977) [13].

The standard Minkowski metric could be generated using the $\eta = \det(x_\mu \sigma^\mu)$ where σ^μ is the Pauli 4-vector with Pauli spin matrices as its components: $\sigma^\mu = (I_2, \vec{\sigma})$. Another mathematically consistent and generalized approach for recovering the usual four-dimensional spacetime of relativity (i.e., the Minkowski metric) is provided by twistor theory [14,15]. This framework reformulates physics in terms of twistors, which are complex geometric objects residing in a higher-dimensional complex space known as twistor space, $\mathbb{T}$. Since twistor space, $\mathbb{T}$ is inherently complex, mappings to conventional (Minkowski)

spacetime naturally yield complexified metrics. While these complex metrics may not directly correspond to physical spacetimes, they can serve as intermediate apparatuses—from which a physically meaningful Minkowski metric can be recovered by selecting an appropriate real slice [16]. This approach relaxes certain constraints in relativistic investigations. For example, complex Einstein manifolds (where solutions to Einstein's equations are not restricted to real metrics) are extensively studied within twistor theory, which provides systematic tools for their construction and classification [17,18]. In the Arnowitt–Deser–Misner (ADM) formalism, an effective method to represent GR in form closer to a gauge theory is to employ complex structures. This complex version of GR offers pathways towards quantization of GR—e.g., loop quantum gravity [19–21].

It is important to emphasize that generalizing the underlying metric structure naturally induces a corresponding generalization of the Lorentz algebra. One well-established route to such generalizations is through algebraic deformations. For instance, in the work of Ter-Kazarian (2024) [22], a deformed Lorentz symmetry arises within a master-space–induced supersymmetry framework, with applications to ultra-high-energy astrophysics. Similarly, in Carow-Watamura et al. (1991) [23], the authors constructed a quantum Lorentz group via q-deformation, employing both a q-deformed Minkowski space and an associated q-deformed Clifford algebra. Another notable direction involves quantum deformations derived from representation-theoretic structures. In Han (2025) [24], the quantum-deformed Lorentz algebra was formulated using irreducible representations of a quantum torus algebra at level N, emerging from the quantization of Chern–Simons theory with gauge group $SL(2,\mathbb{C})$. Broadly, these approaches illustrate that deformations; whether via q-parameters, noncommutative geometry, or quantum group constructions provide systematic mechanisms for extending the classical Lorentz algebra while preserving key structural features such as closure and representation theory. Related investigations into generalized and deformed Lorentz symmetries, and their implications for both algebraic structure and physical invariance principles can be found in the works of Hoff da Silva, (2025) [25], De Azcarraga et al. (1994) [26], Kovačević et al. (2012) [27], and Schmidke et al. (1991) [28].

In this work, two new complex spacetimes are derived from deformed metric structures, following the framework established by Ganesan (2024) [29]. The analysis focuses on the properties and behavior of the distinct sets of spin matrices employed in their construction. Section 2 provides the theoretical background underlying the derivation of these complex spacetimes, while Section 3 details their explicit construction from the proposed spin matrix sets. Section 4 introduces a generalized Lorentz algebra that emerges from these spacetimes, and Section 5 explores the quaternionic mapping of the novel spin matrices, leading to the formulation of new orthonormal quaternionic basis vectors. The paper concludes with a summary of the main findings, potential applications, and directions for future research.

2. Complex Spacetimes

In Ganesan (2024) [29], a deformed Hermitian spin basis was introduced to derive an analogue structure of Minkowski spacetime. This framework led to the formulation of the dual Minkowski metrics and an associated relativistic description tailored to material science applications—particularly for modeling spacetime-like structures in engineered metamaterials. Within this construction, the Lorentz factors emerge directly from the underlying Minkowski metric, reflecting the geometric origin of relativistic effects. Consequently, the introduction of dual Minkowski metrics implied the existence of corresponding dual Lorentz factors, each encoding distinct dynamical properties. These results suggest a pathway toward a deeper understanding of effective spacetime structures in metamaterial systems. In particular, negative-index metamaterials; where the permittivity and permeabil-

ity are typically complex-valued and may naturally accommodate such generalized metric structures. The Lorentz factors rigorously derived and presented in Ganesan (2024) [29] introduce a factor in the length contraction and time dilation in relativistic dynamics:

$$\gamma_X = \frac{2}{\sqrt{1-\frac{v^2}{c^2}}} \text{ and } \gamma_Y = \frac{1}{2\sqrt{1-\frac{v^2}{c^2}}}.$$

A factorized form of the relativistic time dilation in terms of two independent Lorentz factors, γ_X and γ_Y is:

$$(\Delta t)^2 = (\Delta t_0)^2 \gamma_X \gamma_Y.$$

This implies that the effective time dilation arises from two distinct geometric sectors, whose combined effect is multiplicative. To construct a complex representation, the complex time variables are introduced:

$$T = \tau + i\Delta t, \ \bar{T} = \tau - i\Delta t.$$

These variables satisfy the invariant relation, $T\bar{T} = \tau^2 + (\Delta t)^2$. Restricting to the initial condition $\tau = 0$, the relation, $T\bar{T} = (\Delta t)^2$ is obtained. A complex decomposition of the time dilation in terms of the dual Lorentz factors is defined:

$$i\Delta t = \frac{1}{\sqrt{2}}\gamma_X \Delta t_0 (i+1), \quad -i\Delta t = \frac{1}{\sqrt{2}}\gamma_Y \Delta t_0 (i-1).$$

Solving each expression for Δt, the following relations are obtained:

$$\Delta t = \frac{1}{\sqrt{2}}\gamma_X \Delta t_0 (1-i), \ \Delta t = -\frac{1}{\sqrt{2}}\gamma_Y \Delta t_0 (1+i). \tag{1}$$

Extracting constants and simplifying the equation gives:

$$(\Delta t)^2 = -\frac{1}{2}\gamma_X \gamma_Y (\Delta t_0)^2 (1-i)(1+i).$$

Using the identity $(1-i)(1+i) = 2$, this reduces to:

$$(\Delta t)^2 = -\gamma_X \gamma_Y (\Delta t_0)^2.$$

Taking the square root, the following expression is obtained:

$$\Delta t = i\sqrt{\gamma_X \gamma_Y}\, \Delta t_0.$$

Defining the effective Lorentz factor as $\gamma \equiv \sqrt{\gamma_X \gamma_Y}$, the complex time dilation relation becomes:

$$\Delta t = i\gamma \Delta t_0. \tag{2}$$

An analogous construction applies to spatial intervals. The complex length variables are defined as follows:

$$L = l + i\Delta L, \bar{L} = l - i\Delta L.$$

These variables satisfy: $L\bar{L} = l^2 + (\Delta L)^2$. At the initial condition $l = 0$, this reduces to:

$$L\bar{L} = (\Delta L)^2.$$

The complex decomposition is then introduced:

$$i\Delta L = \frac{1}{\sqrt{2}}\gamma_X^{-1} \Delta L_0 (i+1), -i\Delta L = \frac{1}{\sqrt{2}}\gamma_Y^{-1} \Delta L_0 (i-1).$$

Solving for ΔL, the following expressions are obtained:

$$\Delta L = \frac{1}{\sqrt{2}}\gamma_X^{-1}\Delta L_0(1-i), \Delta L = -\frac{1}{\sqrt{2}}\gamma_Y^{-1}\Delta L_0(1+i).$$

Multiplying the two expressions results in:

$$(\Delta L)^2\left[\frac{1}{\sqrt{2}}\gamma_X^{-1}\Delta L_0(1-i)\right]\left[-\frac{1}{\sqrt{2}}\gamma_Y^{-1}\Delta L_0(1+i)\right],$$

which simplifies to:

$$(\Delta L)^2 = -\frac{(\Delta L_0)^2}{\gamma_X\gamma_Y}.$$

Taking the square root then yields:

$$\Delta L = i\frac{\Delta L_0}{\sqrt{\gamma_X\gamma_Y}}.$$

Using the definition of γ, the final expression for the length contraction, ΔL is obtained:

$$\Delta L = i\gamma^{-1}\Delta L_0. \tag{3}$$

This formulation demonstrates that both time dilation and length contraction can be expressed through a unified complex structure governed by dual Lorentz factors. The effective relativistic scaling is controlled by the geometric mean $\gamma = \sqrt{\gamma_X\gamma_Y}$, while the imaginary unit encodes the Minkowski signature. The decomposition into $(1 \pm i)$ components suggest an underlying pair of orthogonal complex directions; whose combined contribution reproduces the standard relativistic invariants.

In view of these definitions, the matrix representations of the length contraction and the time dilation complex transformation are represented as follows:

$$\begin{bmatrix}\Delta L^x\\ \Delta L^y\\ \Delta L^z\\ \Delta t\end{bmatrix} = \overbrace{\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&i\end{bmatrix}}^{\eta_1}\begin{bmatrix}\gamma^{-1}\Delta L_0^x\\ \gamma^{-1}\Delta L_0^y\\ \gamma^{-1}\Delta L_0^z\\ \gamma\Delta t_0\end{bmatrix} \text{ and } \begin{bmatrix}\Delta L^x\\ \Delta L^y\\ \Delta L^z\\ \Delta t\end{bmatrix} = \overbrace{\begin{bmatrix}i&0&0&0\\0&i&0&0\\0&0&i&0\\0&0&0&1\end{bmatrix}}^{\eta_2}\begin{bmatrix}\gamma^{-1}\Delta L_0^x\\ \gamma^{-1}\Delta L_0^y\\ \gamma^{-1}\Delta L_0^z\\ \gamma\Delta t_0\end{bmatrix} \tag{4}$$

where timelike and spacelike skew-Hermitian Minkowski metrics: η_1 and η_2. The standard timelike and spacelike Minkowski metrics are then recovered by taking the square of the skew-Hermitian Minkowski metrics: $\eta = \eta_1^2$ and $\eta = \eta_2^2$ respectively. Using the metrics: η_1 and η_2, the spin matrices used for constructing the metrics could then be obtained.

3. Metric Construction & Algebraic Properties

In this section, the spin matrices used to construct the metrics η_1 and η_2 are derived, and their corresponding properties are examined. The use of the determinant to reconstruct a spacetime metric follows from the general correspondence between quadratic forms and 2×2 matrix representations. Given a linear map $\mathcal{X}(x^\mu) = \sum_\mu \rho_\mu x^\mu$, the determinant $\det(\mathcal{X})$ is necessarily a homogeneous polynomial of degree two in the coordinates x^μ, and therefore defines a quadratic form $Q(x) = g_{\mu\nu}x^\mu x^\nu$. This is a direct extension of the standard spinor–vector correspondence in relativistic theory, where a spacetime vector is encoded as a Hermitian matrix $X = x^\mu\sigma_\mu$, and $\det(X)$ reproduces the Minkowski interval. In the present construction, the replacement of the Hermitian basis by complex,

non-Hermitian matrices $\{\rho_\mu\}$ or $\{\tau_\mu\}$ leads naturally to generalized (complex-valued) metric structures.

More explicitly, the metric components are encoded algebraically through the identity $\det(A) = \frac{1}{2}[(\operatorname{tr} A)^2 - \operatorname{tr}(A^2)]$, which implies $g_{\mu\nu} = \frac{1}{2}[\operatorname{tr}(\rho_\mu)\operatorname{tr}(\rho_\nu) - \operatorname{tr}(\rho_\mu\rho_\nu)]$. Furthermore, the determinant is invariant under similarity transformations, $\det(S\mathcal{X}S^{-1}) = \det(\mathcal{X})$, establishing it as a natural candidate for a spacetime invariant under the associated symmetry group. Consequently, the determinant-based construction provides a systematic and algebraically controlled mechanism for generating spacetime metrics, including complex deformations, directly from an underlying spinorial basis. The construction of the metric η_1 from the spin matrices ρ_i is established in the following theorem.

Theorem 1. *Let the matrices $\{\rho_\mu\}_{\mu=0}^{3} \subset M_2(\mathbb{C})$ be defined by*

$$\rho_0 = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \rho_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \rho_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \rho_3 = \begin{pmatrix} \frac{1+i}{\sqrt{2}} & 0 \\ 0 & \frac{1-i}{\sqrt{2}} \end{pmatrix}.$$

Defining the matrix-valued spacetime map as $\mathcal{X}(t,x,y,z) := \rho_0\, ct + \rho_1\, x + \rho_2\, y + \rho_3\, z$, the determinant of $\mathcal{X}$ reproduces the quadratic form:

$$\eta_1 = \det(\mathcal{X}) = ic^2t^2 + x^2 + y^2 + z^2.$$

Proof. Substituting the explicit forms of ρ_μ, the matrix $\mathcal{X}$ is given by

$$\mathcal{X} = \begin{pmatrix} ct & 0 \\ 0 & ict \end{pmatrix} + \begin{pmatrix} 0 & -ix \\ -ix & 0 \end{pmatrix} + \begin{pmatrix} 0 & -y \\ y & 0 \end{pmatrix} + \begin{pmatrix} \frac{1+i}{\sqrt{2}}z & 0 \\ 0 & \frac{1-i}{\sqrt{2}}z \end{pmatrix}$$

$$\Rightarrow \mathcal{X} = \begin{pmatrix} ct + \frac{1+i}{\sqrt{2}}z & -(y+ix) \\ y - ix & ict + \frac{1-i}{\sqrt{2}}z \end{pmatrix}.$$

The determinant is computed via $\det(\mathcal{X}) = ad - bc$, where

$$a = ct + \frac{1+i}{\sqrt{2}}z, d = ict + \frac{1-i}{\sqrt{2}}z,$$

$$b = -(y+ix), c = y - ix.$$

The product ad is initially evaluated:

$$ad = \left(ct + \frac{1+i}{\sqrt{2}}z\right)\left(ict + \frac{1-i}{\sqrt{2}}z\right).$$

Expanding the expression yields:

$$ad = (ct)(ict) + \frac{ctz}{\sqrt{2}}(1-i) + \frac{ictz}{\sqrt{2}}(1+i) + \frac{(1+i)(1-i)}{2}z^2.$$

Using $(ct)(ict) = ic^2t^2$ and $(1+i)(1-i) = 2$ results in:

$$ad = ic^2t^2 + \frac{ctz}{\sqrt{2}}[(1-i) + i(1+i)] + z^2.$$

Since $i(1+i) = i-1$, the mixed terms cancel $(1-i)+(i-1) = 0$ resulting in:

$$ad = ic^2t^2 + z^2.$$

Next, the product bc is determined:

$$bc = (-(y+ix))(y-ix) = -(y^2+x^2).$$

Combining the results gives:

$$\det(\mathcal{X}) = ad - bc = (ic^2t^2 + z^2) - (-(x^2+y^2)),$$

which yields

$$\det(\mathcal{X}) = ic^2t^2 + x^2 + y^2 + z^2.$$

This establishes the result. □

The following symmetry then holds for the spin matrices described in Theorem 1:

$$-\rho_1^2 = -\rho_2^2 = \rho_0^4 = -\rho_3^4 = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \rho_3^2 = i\rho_0^2.$$

It is key to note that the cancellation of cross terms in Theorem 1 arises from the specific complex structure of ρ_3, ensuring that the determinant defines a quadratic invariant. This construction provides a spinorial realization of a complexified Minkowski metric, wherein the determinant plays the role of the spacetime interval. Table 1 provides an overview of the matrix properties of ρ_i:

Table 1. Overview of Properties of the Spin Matrices, ρ_i.

	ρ_0	ρ_1	ρ_2	ρ_3
Property	Unitary, hankel	Special unitary, skew hermitian, circulant, hankel	Special orthogonal, skew hermitian, toeplitz	Special unitary, hankel
Trace	$1+i$	0	0	$\sqrt{2}$
Determinant	i	1	1	1
Eigenvalues	$\lambda_1 = i$; $\lambda_2 = 1$	$\lambda_1 = i$; $\lambda_2 = -i$	$\lambda_1 = i$; $\lambda_2 = -i$	$\lambda_1 = \frac{1-i}{\sqrt{2}}$; $\lambda_2 = \frac{1+i}{\sqrt{2}}$
Eigenvectors	$v_1 = (0,1)$; $v_2 = (1,0)$	$v_1 = (-1,1)$; $v_2 = (1,1)$	$v_1 = (i,1)$; $v_2 = (-i,1)$	$v_1 = (0,1)$; $v_2 = (1,0)$

It is interesting to note that although the eigenvalues for the spin matrices, ρ_i are imaginary, the matrices, ρ_i are all unitary. An $SU(2)$ normalization can be performed on ρ_0 by removing its phase. This is done by introducing a scalar $\alpha^2 = e^{-\frac{i\pi}{2}} = -i$ where $\alpha^2 \det(\rho_0) = 1$ completes the $SU(2)$ normalization on ρ_0. The matrix $\rho_3 \in SO(2) \subset U(2)$, making it is unitary as well. It can be shown that the matrices ρ_1 and ρ_2 anticommute: $\{\rho_1, \rho_2\} = 0$ and $-\rho_1\rho_2 = i\rho_0^2$. The spin matrices commute, ρ_0 and ρ_3 where $[\rho_0, \rho_3] = 0$. The four matrices, ρ_1 with the identity matrix span the full 2×2 complex matrix algebra $M_2(\mathbb{C})$. Concretely, products of ρ_j close onto linear combinations of I_2, the diagonal matrix $\rho_0^2 = \mathrm{diag}(1,-1)$ and the off diagonal ρ_1 and ρ_2 matrices themselves. Therefore, the algebra

generated is four-dimensional over $\mathbb{C}$—i.e., all 2×2 matrices. The following theorem proves the construction of metric η_2:

Theorem 2: *Let* $\{I_2, \tau_i\}_{i=1}^3 \subset M_2(\mathbb{C})$ *be defined by*

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tau_1 = \begin{pmatrix} 0 & -i \\ 1 & 0 \end{pmatrix}, \tau_2 = \begin{pmatrix} 0 & -1 \\ i & 0 \end{pmatrix}, \tau_3 = \begin{pmatrix} \frac{1-i}{\sqrt{2}} & 0 \\ 0 & \frac{i-1}{\sqrt{2}} \end{pmatrix}.$$

The matrix-valued spacetime map is defined as: $\mathcal{Y}(t,x,y,z) := I_2\, ct + \tau_1\, x + \tau_2\, y + \tau_3\, z$. *Then* $\eta_2 = \det(\mathcal{Y}(t,x,y,z)) = c^2t^2 + ix^2 + iy^2 + iz^2$.

Proof. Substituting the explicit matrix representations, the following expression is obtained:

$$\begin{aligned} \mathcal{Y} &= \begin{pmatrix} ct & 0 \\ 0 & ct \end{pmatrix} + \begin{pmatrix} 0 & -ix \\ x & 0 \end{pmatrix} + \begin{pmatrix} 0 & -y \\ iy & 0 \end{pmatrix} + \begin{pmatrix} \frac{1-i}{\sqrt{2}}z & 0 \\ 0 & \frac{i-1}{\sqrt{2}}z \end{pmatrix} \\ &= \begin{pmatrix} ct + \frac{1-i}{\sqrt{2}}z & -(y+ix) \\ x+iy & ct + \frac{i-1}{\sqrt{2}}z \end{pmatrix}. \end{aligned}$$

The determinant is then: $\det(\mathcal{Y}) = ad - bc$, where,

$$a = ct + \frac{1-i}{\sqrt{2}}z, d = ct + \frac{i-1}{\sqrt{2}}z,$$

$$b = -(y+ix), c = x+iy.$$

Expanding the coefficient ad the following expression is obtained:

$$ad = \left(ct + \frac{1-i}{\sqrt{2}}z\right)\left(ct + \frac{i-1}{\sqrt{2}}z\right) = (ct)^2 + \frac{ctz}{\sqrt{2}}[(i-1) + (1-i)] + \frac{(1-i)(i-1)}{2}z^2.$$

The mixed terms cancel identically, and since $(1-i)(i-1) = 2i$, the relation above simplifies to:

$$ad = c^2t^2 + iz^2.$$

Next, the expression for bc is resolved:

$$bc = (-(y+ix))(x+iy) = -i\left(x^2 + y^2\right).$$

Therefore, $\det(\mathcal{Y}) = ad - bc = \left(c^2t^2 + iz^2\right) - \left[-i\left(x^2+y^2\right)\right] = c^2t^2 + i\left(x^2+y^2+z^2\right)$. This establishes the result. □

The symmetry, $\tau_1^2 = \tau_2^2 = \tau_3^2 = -iI_2$ holds for spin matrices described in Theorem 2. Table 2 provides an overview of the matrix properties of τ_i:

In Table 2 it can also be seen that unlike ρ_i, all the matrices, τ_i have the same trace and determinant. Each τ_i is unitary since $\tau_i^\dagger \tau_i = I_2$, but their determinants are all $\det(\tau_i) = i$, so they do not belong to $SU(2)$. To bring them into $SU(2)$, the phase can be removed by multiplying each τ_i by the scalar $\beta = e^{-\frac{i\pi}{4}} = \frac{1}{\sqrt{i}}$. This ensures that $\det(\beta\tau_i) = \beta^2\det(\tau_i) = 1$. Thus, the $SU(2)$ -normalized versions are precisely $\beta\tau_i$. It can be shown that when the

indices, $i \neq j$ the anticommutator $\{\tau_i,\ \tau_j\} = 0$. The anticommutator could also be described in a more compact form:

$$\{\tau_i,\ \tau_j\} = -2i\delta_{ij} I_2.$$

where δ_{ij} is the Dirac delta function. The commutators for the matrices τ_i close on the triple: $[\tau_1,\ \tau_2] = \sqrt{2}\,(i+1)\tau_3$; $[\tau_2,\ \tau_3] = \sqrt{2}\,(i+1)\tau_1$; $[\tau_3,\ \tau_1] = \sqrt{2}\,(i+1)\tau_2$. The commutators could be represented in the following compact form:

$$[\tau_i,\ \tau_j] = \sqrt{2}\,(i+1)\,\epsilon_{ijk}\tau_k.$$

where ϵ_{ijk} is the Levi-Civita symbol.

Table 2. Overview of Properties of the Spin Matrices, τ_i.

	τ_1	τ_2	τ_3
Property	Unitary, toeplitz	Unitary, toeplitz	Unitary, hankel
Trace	0	0	0
Determinant	i	i	i
Eigenvalues	$\lambda_1 = \frac{1-i}{\sqrt{2}}$; $\lambda_2 = \frac{i-1}{\sqrt{2}}$	$\lambda_1 = \frac{1-i}{\sqrt{2}}$; $\lambda_2 = \frac{i-1}{\sqrt{2}}$	$\lambda_1 = \frac{i-1}{\sqrt{2}}$; $\lambda_2 = \frac{1-i}{\sqrt{2}}$
Eigenvectors	$v_1 = \left(\frac{1-i}{\sqrt{2}}, 1\right)$; $v_2 = \left(\frac{i-1}{\sqrt{2}}, 1\right)$	$v_1 = \left(\frac{-1-i}{\sqrt{2}}, 1\right)$; $v_2 = \left(\frac{i+1}{\sqrt{2}}, 1\right)$	$v_1 = (0,1)$; $v_2 = (1,0)$

The matrices ρ_μ and τ_μ introduced in Theorems 1 and 2 can be naturally contextualized within the framework of complex Clifford algebras and 2×2 spinor representations. In the standard construction, spacetime vectors are encoded via the map $X = x^\mu \sigma_\mu$, where $\sigma_\mu = (I_2, \vec{\sigma})$ and $\vec{\sigma}$ are the Pauli matrices generating a representation of $SU(2)$. These matrices satisfy the Clifford relation $\{\sigma_\mu, \sigma_\nu\} = 2\eta_{\mu\nu} I_2$, with $\eta_{\mu\nu}$ the Minkowski metric. In the current case, the sets $\{\rho_\mu\}$ and $\{\tau_\mu\}$ do not satisfy a standard Clifford algebra with respect to a real signature—rather they define a complexified Clifford structure, in which the induced bilinear form arises through the determinant rather than an explicit anticommutation relation. Consequently, they should be viewed as elements of a representation of $Cl(1,3) \otimes \mathbb{C}$, but expressed in a non-canonical and non-Hermitian basis. More precisely, the matrices ρ_μ and τ_μ are not fundamentally new algebraic objects, but are related to the standard Pauli basis by similarity transformations in $GL(2,\mathbb{C})$, possibly combined with complex rescalings—i.e., there exists (in general non-unitary) matrices $S \in GL(2,\mathbb{C})$ such that $\rho_\mu = S\sigma_\mu S^{-1}$ (and similarly for τ_μ, up to phase factors). However, these transformations do not preserve hermiticity and therefore move the representation outside $SU(2)$ into the broader complex linear group. The novelty of the construction lies not in the algebra itself, but in the choice of complex, non-Hermitian basis, which effectively deforms the induced metric signature. In this sense, the resulting structures are best interpreted as alternative realizations of the same underlying spinor–Clifford framework, adapted to encode complexified or anisotropic spacetime geometries.

4. Generalized Lorentz Algebra

In this section, a generalized Lorentz algebra for the metrics, $\eta_1 = \text{diag}(i, 1, 1, 1)$ and $\eta_2 = \text{diag}(1, i, i, i)$ are explored. In the conventional Lorentz algebra, the generators, $M_{\mu\nu}$ satisfy the following commutation relation [30]:

$$[M^{\mu\nu},\ M^{\rho\sigma}] = i(M^{\mu\sigma}\eta^{\nu\rho} - M^{\nu\sigma}\eta^{\mu\rho} + M^{\nu\rho}\eta^{\mu\sigma} - M^{\mu\rho}\eta^{\nu\sigma}).$$

where $\mu, \rho, \nu, \sigma \in \{0, 1, 2, 3\}$ and the generators are then explicitly constructed and presented as 4×4 matrices using the following relation:

$$\left(M_{\alpha\beta}\right)_{\ \nu}^{\ \mu} = i\left(\delta_\alpha^\mu \eta_{\beta\nu} - \delta_\beta^\mu \eta_{\alpha\nu}\right).$$

where $\eta_{\beta\nu}$ is the complex spacetimes η_1 or η_2 with δ being the Kronecker delta. Including the imaginary factor i in the basis elements is a common convention—this induces the complex factor in the commutation relation as well. Then M^{0i} generates boosts while M^{ij} generates rotations. The boosts and rotations are defined as $K_i = M^{0i}$ and $J_i = \frac{1}{2}\epsilon_{ijk}M^{jk}$ respectively—where ϵ_{ijk} is the Levi-Civita symbol. The antisymmetric property for the Lorentz group holds where: $M^{\mu\nu} = -M^{\nu\mu}$ and $M^{\mu\mu} = 0$. The Lorentz algebra for the conventional Minkowski metric, $\eta = \text{diag}(-1, 1, 1, 1)$ is then represented by the following commutation relations:

$$\left[J_i, J_j\right] = i\epsilon_{ijk}\, J_k, \left[J_i, K_j\right] = i\epsilon_{ijk}\, K_k \text{ and } \left[K_i, K_j\right] = i\epsilon_{ijk}\, J_k.$$

Lemma 1 provides the commutator algebra for the complex metric $\eta_1 = \text{diag}(i, 1, 1, 1)$ in terms of the boost and rotation generators:

Lemma 1. *The commutator algebra for the complex metric* $\eta_1 = \text{diag}(i, 1, 1, 1)$ *in terms of the boost,* K *and rotation generators,* J *are:* $\left[J_i, J_j\right] = i\epsilon_{ijk}\, J_k$; $\left[J_i, K_j\right] = i\epsilon_{ijk}\, K_k$; $\left[K_i, K_j\right] = \epsilon_{ijk}\, J_k$.

Proof. In the case of the complex Minkowski metric presented in this work (η_1), the rotation-rotation commutator is:

$$\left[J_i, J_j\right] = [M^{jk},\ M^{ik}] = i(M^{jk}\eta^{ki} - M^{kk}\eta^{ji} + M^{ki}\eta^{jk} - M^{ji}\eta^{kk}).$$

Since the scenario, $i = j = 0$ the commutator vanishes, the term $\eta^{00} = i$ does not influence the commutator. Thus, the commutator for the complex metric η_1 is identical to that of the standard Minkowski metric: $\left[J_i, J_j\right] = i\epsilon_{ijk}\, J_k$. As for the rotation-boost commutator, $\left[J_i, K_j\right]$ the following scenarios exist for η^{00} to appear:

$$\begin{aligned}
&\Rightarrow j = 0\ :\ [J_i, K_0\,] = i(M^{00}\eta^{k0} - M^{k0}\eta^{00} + M^{k0}\eta^{00} - M^{00}\eta^{k0}).\\
&\Rightarrow k = 0\ :\ \left[J_i, K_j\right] = i\left(M^{jj}\eta^{00} - M^{0j}\eta^{j0} + M^{00}\eta^{jj} - M^{j0}\eta^{0j}\right).
\end{aligned}$$

In both cases it can be seen that η^{00} removes and does not remain in the relations. Hence, this commutator is also identical to the standard Minkowski metric: $\left[J_i, K_j\right] = i\epsilon_{ijk}\, K_k$. The commutator $\left[K_i, K_j\right]$ is given as follows:

$$\begin{aligned}
\left[K_i, K_j\right] = \left[M^{0i},\ M^{0j}\right] \quad &= i\left(M^{0j}\eta^{i0} - M^{ij}\eta^{00} + M^{i0}\eta^{0j} - M^{00}\eta^{ij}\right)\\
&= i\left(M^{0j}\eta^{i0} - M^{ij}\eta^{00} + M^{i0}\eta^{0j}\right).
\end{aligned}$$

At $i = 0$, $j = 0$ or $i = j$, the commutator vanishes. Therefore, $i \neq 0$ and $j \neq 0$, the $\eta^{0j} = \eta^{i0} = 0$ resulting in:

$$\left[K_i, K_j\right] = i(-M^{ij}\eta^{00}) = i(-M^{ij}i) = M^{ij}.$$

Since $M^{00} = 0$ and $M^{\mu\mu} = 0$ then the Levi-Civita symbol is employed to obtain the boost-boost commutator relation (which is different from the standard Lorentz algebra):

$$\left[K_i, K_j\right] = \epsilon_{ijk}\, J_k.$$

Thus, the commutator algebra is: $\left[J_i, J_j\right] = i\epsilon_{ijk}\, J_k$; $\left[J_i, K_j\right] = i\epsilon_{ijk}\, K_k$; $\left[K_i, K_j\right] = \epsilon_{ijk}\, J_k$. □

The commutator algebra for the complex metric $\eta_2 = \mathrm{diag}(1, i, i, i)$ with respect to the boost and rotation generators is given in Lemma 2:

Lemma 2. *The commutator algebra for the complex metric* $\eta_2 = \mathrm{diag}(1, i, i, i)$ *in terms of the boost, K and rotation generators, J is:* $\left[K_i, K_j\right] = -i\epsilon_{ijk}\, J_k$, $\left[J_i, K_j\right] = \epsilon_{ijk}K_k$ *and* $\left[J_i, J_j\right] = -\epsilon_{ijk}J_k$.

Proof. Similar to Lemma 1 and the standard Lorentz algebra the boost and rotation generators are defined as $K_i = M^{0i}$ and $J_i = \frac{1}{2}\epsilon_{ijk}M^{jk}$ respectively with ϵ_{ijk} being the Levi-Civita symbol. The boost commutator is then determined as follows:

$$\begin{aligned}\left[K_i, K_j\right] = [M^{0i}, M^{0j}] &= i(M^{0j}\eta^{i0} - M^{ij}\eta^{00} + M^{i0}\eta^{0j} - M^{00}\eta^{ij}) \\ &= i(-M^{ij}\eta^{00}) = i(-M^{ij}(1)) = -iM^{ij}. \\ \Rightarrow \left[K_i, K_j\right] &= -i\epsilon_{ijk}\, J_k.\end{aligned}$$

For the rotation-boost commutator:

$$\left[J_i, K_j\right] = \frac{1}{2}\epsilon_{iab}[M^{ab},\, M^{0j}] = \frac{1}{2}\epsilon_{iab}i(M^{aj}\eta^{b0} - M^{bj}\eta^{a0} + M^{b0}\eta^{aj} - M^{a0}\eta^{bj}).$$

where a and b are spatial indices, Since $\eta^{a0} = \eta^{b0} = 0$, it follows that:

$$\left[J_i, K_j\right] = \frac{1}{2}\epsilon_{iab}[M^{ab},\, M^{0j}] = \frac{1}{2}\epsilon_{iab}i(M^{aj}\eta^{b0} - M^{bj}\eta^{a0} + M^{b0}\eta^{aj} - M^{a0}\eta^{bj}).$$

The metric in terms of the Kronecker delta is represented by: $\eta^{aj} = i\delta^{aj}$. Since $M^{b0} = -M^{0b}$ $= -K_b$, then:

$$\begin{aligned}\left[J_i, K_j\right] &= \frac{1}{2}\epsilon_{iab}i\Big((-K_b)i\delta^{aj} - (-K_a)i\delta^{bj}\Big) = -\frac{1}{2}\epsilon_{iab}\Big(K_a\delta^{bj} - K_b\delta^{aj}\Big). \\ \Rightarrow \left[J_i, K_j\right] &= \epsilon_{ijk}K_k.\end{aligned}$$

The rotation-rotation commutator is then represented as follows:

$$\begin{aligned}\left[J_i, J_j\right] &= \frac{1}{4}\epsilon_{iab}\epsilon_{jcd}\Big[M^{ab},\, M^{cd}\Big] \\ &= \frac{1}{4}\epsilon_{iab}\epsilon_{jcd}i\Big(M^{ad}\eta^{bc} - M^{bd}\eta^{ac} + M^{bc}\eta^{ad} - M^{ac}\eta^{bd}\Big).\end{aligned}$$

Substituting the metric, $\eta^{mn} = i\delta^{mn}$ results in:

$$\left[J_i, J_j\right] = -\frac{1}{4}\epsilon_{iab}\epsilon_{jcd}\Big(M^{ad}\delta^{bc} - M^{bd}\delta^{ac} + M^{bc}\delta^{ad} - M^{ac}\delta^{bd}\Big).$$

Exercising the Levi-Civita contraction where $M^{ab} = \epsilon_{kab}\, J_k$, the commutator takes the following form:

$$[J_i, J_j] = -\frac{1}{4}\epsilon_{iab}\epsilon_{jcd}\Big(M^{ad}\delta^{bc} - M^{bd}\delta^{ac} + M^{bc}\delta^{ad} - M^{ac}\delta^{bd}\Big).$$
$$\Rightarrow [J_i, J_j] = -\epsilon_{ijk}J_k.$$

Therefore, the commutator algebra is: $[K_i, K_j] = -i\epsilon_{ijk}\ J_k$, $[J_i, K_j] = \epsilon_{ijk}K_k$ and $[J_i, J_j] = -\epsilon_{ijk}J_k$. □

The commutation relations derived in Lemmas 1 and 2 define Lie algebras over $\mathbb{C}$—as they satisfy bilinearity, antisymmetry and the Jacobi identity inherited from the general Lorentz commutator structure. Closure follows directly from the defining relation:

$$[M^{\mu\nu}, M^{\rho\sigma}] \ \propto \ M^{\alpha\beta},$$

which remains valid for any choice of non-degenerate metric $\eta_{\mu\nu}$ (including the complex metrics η_1 and η_2). Since the structure constants are modified only by factors of i, the resulting algebras remain closed within the span of the six generators $\{J_i, K_i\}$. In addition, since these algebras arise from a consistent deformation of the metric tensor, the Jacobi identity is automatically preserved; ensuring that the resulting structures are true Lie algebras. From a classification standpoint, these algebras are naturally interpreted as complexified forms of the Lorentz algebra $SO(1,3)$. Over $\mathbb{C}$, it is well known that

$$SO(1,3) \otimes \mathbb{C} \ \cong SO(4,\mathbb{C}) \cong SL(2,\mathbb{C}) \oplus SL(2,\mathbb{C}),$$

and different metric signatures correspond to different real forms embedded within this complex algebra. The modified commutation relations in Lemmas 1 and 2 can be considered as arising from non-standard real forms or complex basis choices of the same underlying complex Lie algebra. Specifically, the redistribution of factors of i between the rotation and boost sectors corresponds to a deformation of the Cartan decomposition, effectively interpolating between Lorentzian and Euclidean-type structures. Consequently, the algebras associated with η_1 and η_2 are not fundamentally new Lie algebras; but rather alternative realizations of $SO(1,3;\mathbb{C})$, expressed in a non-Hermitian basis adapted to the complexified metric structure. Since the matrices ρ_i and τ_i are isomorphic to Pauli matrices, these matrices also have a quaternion representation. These aspects of the spin matrices are explored in the next section.

5. Quaternionic & Rotational Properties

The spin matrices that construct metric η_1 could be mapped to the Pauli matrices, $\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, $\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ as follows:

$$\rho_0 = \frac{1+i}{2}I_2 + \frac{1-i}{2}\sigma_3\ ,\ \rho_1 = -i\sigma_3,\ \rho_2 = i\sigma_2,\ \rho_3 = \frac{\sigma_0 + i\sigma_3}{\sqrt{2}}.$$

It is known that quaternions have a direct isomorphism with Pauli matrices, σ_i—i.e., $\mathbf{i} \longmapsto -i\sigma_1 = -\sigma_2\sigma_3$, $\mathbf{j} \longmapsto -i\sigma_2 = -\sigma_3\sigma_1$, $\mathbf{k} \longmapsto -i\sigma_3 = -\sigma_1\sigma_2$ and $= 1 \longmapsto I_2$. A rigorous formulation of the correspondence between the spin matrices and their quaternionic representations proceeds by making explicit the algebra isomorphism between the biquaternions $\mathbb{H}_\mathbb{C}$ and the matrix algebra $\mathrm{Mat}(2,\mathbb{C})$, the latter generated by the Pauli matrices. The following map is then defined: $\Phi : \mathbb{H}_\mathbb{C} \to \mathrm{Mat}(2,\mathbb{C})$ by its action on the quaternion basis and extend linearly over $\mathbb{C}$. This map is an algebra isomorphism, since it preserves multiplication via the identities $\sigma_a\sigma_b = \delta_{ab}I_2 + i\epsilon_{abc}\sigma_c$, which reproduce the quaternion relations $ij = k$, $jk = i$, $ki = j$, and their antisymmetric counterparts. Consequently,

every biquaternion $q = a_0 + a_1 i + a_2 j + a_3 k$, with $a_\mu \in \mathbb{C}$, corresponds uniquely to a 2×2 complex matrix, $\Phi(q) = a_0 I_2 - i(a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3)$ establishing $\mathbb{H}_\mathbb{C} \cong \mathrm{Mat}(2, \mathbb{C})$ as associative *-algebras.

The spin matrices, ρ_i share this isomorphism. To simplify the analysis, the following notation is used for the quaternion representation: $\rho_i \longmapsto P_i$. The quaternion representation of the spin matrices, ρ_i is then as follows:

$$\begin{aligned}
&\rho_0 = \frac{1+i}{2} I_2 + \frac{1-i}{2}\sigma_3 = \frac{1+i}{2} + \frac{i(\sigma_3 - i\sigma_3)}{2i} \longmapsto \tfrac{1+i}{2} + \frac{-\mathbf{k} + i\mathbf{k}}{2i}.\\
&\Rightarrow \rho_0 \longmapsto \tfrac{1+i}{2} + \frac{(i-1)\mathbf{k}}{2i} = \frac{1+i}{2} + \frac{(1+i)\mathbf{k}}{2} = \frac{1+i}{2}(1+\mathbf{k}).\\
&\rho_1 = -i\sigma_3 \longmapsto \mathbf{k},\\
&\rho_2 = i\sigma_2 \longmapsto -\mathbf{j},\\
&\rho_3 = \frac{I_2 + i\sigma_3}{\sqrt{2}} \longmapsto \frac{1-\mathbf{k}}{\sqrt{2}}
\end{aligned}$$

where ρ_0 is a biquaternion and ρ_i with $i \in [1, 3]$ are quaternions. To simplify the analysis, the following notation is used for the quaternion representation of ρ_i ($\rho_i \longmapsto P_i$). It is easy to check that all the quaternion representations in this system $\{P_0, P_1, P_2, P_3\}$ are unit quaternions where $\|P_i\| = |P_i|^2 = 1$ for $i \in [0, 3]$. Lemma 3 proves the orthogonal property for the quaternion system:

Lemma 3. *(Orthogonal Property): All quaternion pairs in the basis* $\{P_0, P_1, P_2, P_3\} = \left\{\frac{1+i}{2}(1+\mathbf{k}), \mathbf{k}, -\mathbf{j}, \frac{1-\mathbf{k}}{\sqrt{2}}\right\}$ *are orthogonal except the pairs* $\{P_0, P_1\}$ *and* $\{P_1, P_3\}$.

Proof. To check the pairwise orthogonality of the quaternion basis, the Hermitian inner product is used: $\langle u, v\rangle = \sum_i \overline{u_i} v_i$. In quaternion form, the following Hermitian inner product vanishes

$$\langle P_0, P_2\rangle = \langle P_0, P_3\rangle = \langle P_1, P_2\rangle = \langle P_2, P_3\rangle = 0.$$

Thus, all the quaternion pairs are orthogonal with the exception of $\langle P_0, P_1\rangle$ and $\langle P_1, P_3\rangle$ since:

$$\langle P_0, P_1\rangle = \frac{\overline{1+i}}{2}\cdot 0 + \frac{\overline{1+i}}{2}\cdot 1 = \frac{\overline{1+i}}{2}\ ;\ \langle P_1, P_3\rangle = 1 \cdot \left(-\frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}}\,.$$

□

Although some quaternion pairs are nonorthogonal (as shown in Lemma 3), there exists a true orthonormal triple within this system: $\{P_0, P_2, P_3\} = \left\{\frac{1+i}{2}(1+\mathbf{k}), -\mathbf{j}, \frac{1-\mathbf{k}}{\sqrt{2}}\right\}$. Similarly, the spin matrices, τ_i that construct the metric η_2 is mapped to the Pauli matrices, σ_i as follows:

$$\tau_1 = \frac{1-i}{2}(\sigma_1 + \sigma_2);\ \tau_2 = \frac{1-i}{2}(\sigma_2 - \sigma_1); \tau_3 = \frac{1-i}{\sqrt{2}}\sigma_3$$

Since the spin matrices, τ_i could be mapped to Pauli matrices, they are then also isomorphic to quaternions. This is verified below, where the following notation is used for the quaternion representation of τ_i ($\tau_i \longmapsto T_i$):

$$\tau_1 = \frac{1-i}{2}(\sigma_1 + \sigma_2) = \frac{\sigma_1 - i\sigma_1 + \sigma_2 - i\sigma_2}{2} = \frac{i\sigma_1 - i(i\sigma_1) + i\sigma_2 - i(i\sigma_2)}{2i},$$

$$\tau_1 \longmapsto T_1 = \frac{-\mathbf{i} + i(\mathbf{i}) - \mathbf{j} + i(\mathbf{j})}{2i} = \frac{i-1}{2i}\mathbf{i} + \frac{i-1}{2i}\mathbf{j} = \frac{i+1}{2}(\mathbf{i}+\mathbf{j}).$$

$$\tau_2 = \frac{1-i}{2}(\sigma_2 - \sigma_1) = \frac{\sigma_2 - i\sigma_2 - \sigma_1 + i\sigma_1}{2} = \frac{i\sigma_2 - i(i\sigma_2) - i\sigma_1 + i(i\sigma_1)}{2i},$$

$$\tau_2 \longmapsto T_2 = \frac{-\mathbf{j} + i(\mathbf{j}) + \mathbf{i} - i(\mathbf{i})}{2i} = \frac{i-1}{2i}\mathbf{j} - \frac{i-1}{2i}\mathbf{i} = \frac{i-1}{2}(\mathbf{i}+\mathbf{j}).$$

$$\tau_3 = \frac{1-i}{\sqrt{2}}\sigma_3 = \frac{\sigma_3 - i\sigma_3}{\sqrt{2}} = \frac{i\sigma_3 - i(i\sigma_3)}{i\sqrt{2}},$$

$$\tau_3 \longmapsto T_3 = \frac{-\mathbf{k} + i(\mathbf{k})}{i\sqrt{2}} = \frac{1-i}{i\sqrt{2}}\mathbf{k} = \frac{-i-1}{\sqrt{2}}\mathbf{k}.$$

where $T_1,\ T_2$ and T_3 are all biquaternions. It is readily verified that all the quaternion representations in this system $\{T_1,\ T_2, T_3\} = \left\{\frac{i+1}{2}(\mathbf{i}+\mathbf{j}), \frac{i-1}{2}(\mathbf{i}+\mathbf{j}), \frac{-i-1}{\sqrt{2}}\mathbf{k}\right\}$ are unit quaternions. Lemma 4 proves the orthogonal property of the basis, $\{T_1,\ T_2, T_3\}$.

Lemma 4. *(Orthogonal Property): The quaternion pairs, $\{T_2, T_3\}$ and $\{T_1, T_3\}$ in the basis $\left\{\frac{i+1}{2}(\mathbf{i}+\mathbf{j}), \frac{i-1}{2}(\mathbf{i}+\mathbf{j}), \frac{-i-1}{\sqrt{2}}\mathbf{k}\right\}$ are orthogonal. The pair $\{T_1, T_2\}$ are not orthogonal but linearly dependent.*

Proof. The pairwise orthogonality of the quaternion basis is checked using the Hermitian inner product: $\langle u, v\rangle = \sum_i \overline{u_i} v_i$.

In quaternion form, the following Hermitian inner products vanish: $\langle T_2, T_3\rangle = \langle T_1, T_3\rangle = 0$ except:

$$\langle T_1, T_2\rangle = \frac{\overline{i+1}}{2}\cdot\frac{i-1}{2} + \frac{\overline{i+1}}{2}\cdot\frac{i-1}{2} = \frac{i}{2} + \frac{i}{2} = i.$$

Thus, all the quaternion pairs are orthogonal with the exception of $\langle T_1, T_2\rangle$. □

If quaternion vectors, T_1 and T_2 are simply non-orthogonal but still independent, it would still be possible to construct an orthogonal basis including all three vectors using the Gram–Schmidt process. However, since $T_2 = iT_1$, the set is linearly dependent from the start and therefore cannot form a basis (orthogonal or otherwise).

Using the Gram–Schmidt process two orthogonal quaternions $\{T_1, T_3\}$ and $\{T_2, T_3\}$ it is possible to construct an orthonormal triple. Beginning with $\{T_1, T_3\}$, the following third quaternion vector is considered: $v = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$—where its quaternion representation is, $T'_2 = \frac{\mathbf{i}-\mathbf{j}}{\sqrt{2}}$. Using the Hermitian inner product, it can be verified that: $\langle T_1, T'_2\rangle = \langle T_3, T'_2\rangle = 0$. The quaternion vector $|T'_2|^2 = 1$, thus T'_2 is a unit quaternion as well. Therefore, the quaternion vectors $\{T_1, T'_2,\ T_3\}$ form an orthonormal basis set. Similarly for the basis, $\{T_2, T_3\}$ the same third quaternion vector is considered: $v = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right) \rightarrow T'_1 = \frac{\mathbf{i}-\mathbf{j}}{\sqrt{2}}$. It can be seen that quaternion vectors $\{T'_1,\ T_2,\ T_3\}$ are orthogonal $\langle T_2, T'_1\rangle = \langle T_3, T'_1\rangle = 0$. Thus, three orthonormal quaternion basis sets are obtained:

- $\mathcal{P} = \{P_0, P_2, P_3\} = \left\{\frac{1+i}{2}(1+\mathbf{k}), -\mathbf{j}, \frac{1-\mathbf{k}}{\sqrt{2}}\right\}$.
- $\mathcal{T}_1 = \{T'_1, T_2, T_3\} = \left\{\frac{\mathbf{i}-\mathbf{j}}{\sqrt{2}}, \frac{i-1}{2}(\mathbf{i}+\mathbf{j}), \frac{-i-1}{\sqrt{2}}\mathbf{k}\right\}$.
- $\mathcal{T}_2 = \{T_1, T'_2, T_3\} = \left\{\frac{i+1}{2}(\mathbf{i}+\mathbf{j}), \frac{\mathbf{i}-\mathbf{j}}{\sqrt{2}}, \frac{-i-1}{\sqrt{2}}\mathbf{k}\right\}$.

To further formalize the orthogonality structure underlying the quaternion representations $\{P_i\}$ and $\{T_i\}$, these elements are regarded not as vectors in the real quaternion algebra $\mathbb{H}$, but in its complexification $\mathbb{H}_{\mathbb{C}} \cong \mathrm{Mat}(2, \mathbb{C})$—i.e., the same algebraic setting as the Pauli matrices. This identification endows the space with a natural complex vector space structure; requiring the use of a Hermitian (rather than bilinear) inner product. Accordingly, the following definition is employed:

$$\langle u, v\rangle := \sum_{\alpha=0}^{3} \overline{u}_\alpha v_\alpha,$$

where $u = \sum u_\alpha e_\alpha$, $v = \sum v_\alpha e_\alpha$, and $e_\alpha \in \{1, i, j, k\}$ with the involution given by the combined action of complex conjugation and quaternionic conjugation. Under the matrix isomorphism, this coincides with the Hilbert–Schmidt inner product $\langle U, V\rangle = \mathrm{Tr}(U^\dagger V)$; ensuring conjugate symmetry, linearity and positive-definiteness. This structure is uniquely compatible with the complex scalar field and the unitary action of spin operators, making it the appropriate notion of inner product for analyzing orthogonality in this setting.

In this Hermitian framework, orthogonality is defined by the vanishing of $\langle u, v\rangle = 0$ —which corresponds to trace orthogonality in $\mathrm{Mat}(2, \mathbb{C})$. The observed properties of the systems $\{P_i\}$ and $\{T_i\}$ follow directly: the partial orthogonality of the P-basis reflects that it is not fully orthogonal under the induced inner product; while the relation $T_2 = iT_1$ demonstrates linear dependence over $\mathbb{C}$, precluding orthogonality. The validity of the Gram–Schmidt process in this context follows from the completeness and positive-definite nature of the Hermitian inner product, allowing the construction of orthonormal bases from linearly independent subsets. Moreover, multiplication by unit biquaternions preserves the inner product, i.e., $\langle uq, vq\rangle = \langle u, v\rangle$ for $|q| = 1$, establishing compatibility between the inner product and quaternionic algebra. Consequently, the orthonormal bases derived from $\{P_i\}$ and $\{T_i\}$ can be interpreted as unitary frames in a complex Hilbert space representation of the quaternion algebra, with symmetry group $U(2)$ (or $SU(2)$ upon normalization).

Quaternions are geometrically interesting objects as they physically represent three-dimensional rotation. For instance, the standard unit quaternion: $q = \cos\left(\frac{\theta}{2}\right) + \mathbf{u}\sin\left(\frac{\theta}{2}\right)$ with $\mathbf{u} = \mathbf{i}, \mathbf{j}, \mathbf{k}$. Consider the orthonormal basis set: $\mathcal{P} = \{P_0, P_2, P_3\} = \left\{\frac{1+i}{2}(1+\mathbf{k}), -\mathbf{j}, \frac{1-\mathbf{k}}{\sqrt{2}}\right\}$. For the case of: since $(1+i) = \sqrt{2}e^{\frac{i\pi}{4}}$ a multiplication by a factor of $e^{-\frac{i\pi}{4}}$ results in the following relation:

$$e^{-\frac{i\pi}{4}} P_0 = \frac{\sqrt{2}}{1+i}\frac{1+i}{2}(1+\mathbf{k}) = \frac{1}{\sqrt{2}}(1+\mathbf{k}) = \cos\left(\frac{\theta}{2}\right) + \mathbf{k}\sin\left(\frac{\theta}{2}\right).$$

Therefore, it can be seen that up to a complex phase, $e^{-\frac{i\pi}{4}}$ the quaternion basis element P_0 corresponds to a rotation $\theta = 90^\circ$ about the axis $\mathbf{k}$. For P_2:

$$P_2 = -\mathbf{j} = \cos\left(\frac{\theta}{2}\right) + \mathbf{j}\sin\left(\frac{\theta}{2}\right),$$

where $\sin\left(\frac{\theta}{2}\right) = -1$ and $\cos\left(\frac{\theta}{2}\right) = 0 \Longrightarrow \theta = 180^\circ$ about the $-\mathbf{j}$ axis. Similarly for the case of P_3:

$$P_3 = \frac{1-\mathbf{k}}{\sqrt{2}} = \cos\left(\frac{\theta}{2}\right) - \mathbf{k}\sin\left(\frac{\theta}{2}\right),$$

where the quaternion basis element P_3 corresponds to a rotation $\theta = 90^\circ$ about the $-\mathbf{k}$ axis. It can be seen that the orthonormal basis set: $\{P_0, P_2, P_3\}$ corresponds to two dimensional rotations about the $\mathbf{j}$ and $\mathbf{k}$ axes. Their respective 3×3 rotational matrices in $SO(3)$ are:

$$R_0^{\mathcal{P}} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \; R_2^{\mathcal{P}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}; \; R_3^{\mathcal{P}} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The orthonormal basis $\mathcal{T}_1 = \{T'_1, T_2, T_3\}$ is represented as $\left\{\frac{\mathbf{i}-\mathbf{j}}{\sqrt{2}}, \frac{i-1}{2}(\mathbf{i}+\mathbf{j}), \frac{-i-1}{\sqrt{2}}\mathbf{k}\right\}$. For T'_1, there is no complex phase removal required and $\frac{\mathbf{i}-\mathbf{j}}{\sqrt{2}} = \cos\left(\frac{\theta}{2}\right) + \frac{\mathbf{i}-\mathbf{j}}{\sqrt{2}}\sin\left(\frac{\theta}{2}\right)$ which corresponds to $\theta = 180^\circ$ about the $\frac{\mathbf{i}-\mathbf{j}}{\sqrt{2}}$. For the axis T_2, the complex phase can be removed as follows since $\sqrt{2}e^{i\frac{3\pi}{4}} = i - 1 \Longrightarrow e^{-\frac{i3\pi}{4}} T_2 = \frac{\sqrt{2}}{i-1}\frac{i-1}{2}(\mathbf{i}+\mathbf{j}) = \frac{(\mathbf{i}+\mathbf{j})}{\sqrt{2}} = \cos\left(\frac{\theta}{2}\right) + \frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}\sin\left(\frac{\theta}{2}\right)$.

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Thus, $\theta = 180^\circ$ about the $\frac{\mathbf{i}-\mathbf{j}}{\sqrt{2}}$ axis. For T_3, the complex phase is removed. Since $\sqrt{2}e^{-i\frac{3\pi}{4}} = -i-1$, it follows that, $e^{\frac{i3\pi}{4}}T_3 = \frac{\sqrt{2}}{-i-1}\frac{-i-1}{\sqrt{2}}\mathbf{k} = \mathbf{k}$. Thus, $e^{\frac{i3\pi}{4}}T_3 = \cos\left(\frac{\theta}{2}\right) + \mathbf{k}\sin\left(\frac{\theta}{2}\right) \Longrightarrow \theta = 180^\circ$ about the $\mathbf{k}$ axis. The rotational matrices in $SO(3)$ are as follows:

$$R_0^{\mathcal{P}} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \; R_2^{\mathcal{P}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}; \; R_3^{\mathcal{P}} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For the orthonormal basis, $\mathcal{T}_2 = \{T_1, T'_2, \; T_3\} = \left\{\frac{i+1}{2}(\mathbf{i}+\mathbf{j}), \frac{\mathbf{i}-\mathbf{j}}{\sqrt{2}}, \frac{-i-1}{\sqrt{2}}\mathbf{k}\right\}$, the complex phase of T_1 can be removed. Since $e^{\frac{i\pi}{4}} = \frac{i+1}{\sqrt{2}}$, then $e^{-\frac{i\pi}{4}}T_1 = \frac{\sqrt{2}}{i+1}\frac{i+1}{2}(\mathbf{i}+\mathbf{j}) = \frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}$. It follows that: $\frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}} = \cos\left(\frac{\theta}{2}\right) + \frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}\sin\left(\frac{\theta}{2}\right) \Rightarrow \theta = 180^\circ$ about the $\frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}$ axis. As for T'_2: $\frac{\mathbf{i}-\mathbf{j}}{\sqrt{2}} = \cos\left(\frac{\theta}{2}\right) + \frac{\mathbf{i}-\mathbf{j}}{\sqrt{2}}\sin\left(\frac{\theta}{2}\right)$ which corresponds to $\theta = 180^\circ$ about the $\frac{\mathbf{i}-\mathbf{j}}{\sqrt{2}}$. Finally, for the T_3 the phase needs to be removed (similar to T_1). Since $e^{-\frac{i3\pi}{4}} = \frac{-i-1}{\sqrt{2}}$ then $e^{\frac{i3\pi}{4}}T_3 = \frac{\sqrt{2}}{-i-1}\frac{-i-1}{\sqrt{2}}\mathbf{k} = \mathbf{k}$. This corresponds to: $\mathbf{k} = \cos\left(\frac{\theta}{2}\right) + \mathbf{k}\sin\left(\frac{\theta}{2}\right)$ where $\theta = 180^\circ$ about the $\mathbf{k}$ axis. Their corresponding rotational matrices in $SO(3)$ are:

$$R_0^{\mathcal{P}} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \; R_2^{\mathcal{P}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}; \; R_3^{\mathcal{P}} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

6. Main Findings, Potential Applications & Future Research Directions

In this work, two complex spacetime metrics, η_1 and η_2, are obtained using the dual Lorentz factors proposed by Ganesan (2024) [29]. The analysis focuses on the spin matrices that construct these spacetimes, their associated algebraic properties, and the quaternionic structures that arise from them. The key findings of this study are as follows:

- A novel set of spin matrices, denoted τ_i and ρ_i, is obtained to construct the complex metrics η_1 and η_2. Their associated symmetry, spectral, commutator, and anticommutator properties are subsequently examined.
- For these complex metrics, a new generalized Lorentz algebra is obtained and is given as follows:

$$\begin{aligned} &\text{Metric } \eta_1\text{: } \left[J_i, J_j\right] = i\epsilon_{ijk}\, J_k; \left[J_i, K_j\right] = i\epsilon_{ijk}\, K_k; \left[K_i, K_j\right] = \epsilon_{ijk}\, J_k. \\ &\text{Metric } \eta_2\text{: } \left[K_i, K_j\right] = -i\epsilon_{ijk}\, J_k; \left[J_i, K_j\right] = \epsilon_{ijk} K_k; \left[J_i, J_j\right] = -\epsilon_{ijk} J_k. \end{aligned}$$

- The examination of the quaternionic mapping of these novel spin matrices, new orthonormal basis vectors are obtained: $\mathcal{P}, \mathcal{T}_1$ and $\mathcal{T}_2$. Their respective 3×3 rotational matrices in $SO(3)$ are also derived:

$$\left\{R_0^{\mathcal{P}} \; R_2^{\mathcal{P}} \; R_3^{\mathcal{P}}\right\}, \left\{R_1^{\mathcal{T}_1}, \; R_2^{\mathcal{T}_1}, R_3^{\mathcal{T}_1}\right\} \text{ and } \left\{R_1^{\mathcal{T}_2}, \; R_2^{\mathcal{T}_2}, R_3^{\mathcal{T}_2}\right\}.$$

Complex spacetimes have been known to have key applications in explorations in physics and applied mathematics. These spacetimes may serve as metrics resulting from analytic continuation techniques within quantum field theory (QFT). A familiar example is thermal QFT, where a Wick rotation $t \to i\tau$ improves the convergence of path integrals [31]. Temporal Wick rotations have also been applied to suppress the large fluctuations characteristic of chaotic tunneling, as illustrated in the kicked rotor model (quantum standard map), where the map period is extended into the complex domain [32]. By contrast, η_2 involves imaginary spatial dimensions, making it relevant for calculations employing double Wick rotations. One example is the analysis of the double Wick–rotated Bañados-Teitelboim–

Zanelli (BTZ) black hole in Euclidean signature, where Dai et al. (2025) [33] computed the total spacetime energy, thermodynamic quantities, and holographic two-point functions. Double Wick rotations also play a role in certain string-theory—based models [34]. A further application of η_2 lies in the analytic continuation between Anti–de Sitter (AdS) and de Sitter (dS) spacetimes, achieved by flipping the sign of the cosmological constant. This approach has been explored in, for example, Akhmedov and Sadofyev (2012) [35], Iacobacci et al. (2023) [36], and Sleight and Taronna (2021) [37]. Orthonormal quaternion basis sets play a significant role in the formulation of quaternionic quantum mechanics [38]. Such basis sets are frequently employed to construct mathematical frameworks that generalize complex Hilbert spaces to quaternionic ones. This approach is exemplified in Gantner (2018) [39], who established an equivalence between any quaternionic quantum system and the quaternionification of a corresponding complex quantum system. Similarly, De Leo and Scolarici (2000) [40] demonstrate how orthogonal quaternionic vectors serve as a bridge between quaternionic and complex geometries, highlighting their importance in extending conventional quantum mechanics into the quaternionic domain. Collectively, these studies underscore the foundational role of orthonormal quaternion bases in linking quaternionic formulations to the familiar complex quantum framework.

A promising direction for future research is to explore the implications of the proposed spin matrices within the biquaternionic algebra $\mathbb{H}_{\mathbb{C}} \cong \mathrm{Mat}(2, \mathbb{C})$, where the correspondence with the Pauli matrices is made explicit through the algebra isomorphism. In particular, the standard lattice spin models could be generalized by replacing the conventional Pauli generators with the matrices ρ_μ or τ_i, thereby inducing a modified local spin algebra that encodes complexified rotational structure. Within the framework of the Heisenberg model, this substitution leads to Hamiltonians of the form:

$$H = -\sum_{i,j} J_{ab}\, S_i^{(a)} S_j^{(b)},$$

where $S^{(a)} \in \{\rho_\mu\}$ or $\{\tau_i\}$, and the coupling tensor J_{ab} may inherit anisotropies or complex structure from the underlying representation. This naturally extends the model beyond standard $SU(2)$-symmetric interactions to a broader class of unitary (or complexified) spin systems. Such a generalization opens the possibility of probing qualitatively new physical regimes within statistical mechanics. In particular, the modified commutation relations and inner product structure could alter the spectra of excitations, influence correlation functions, and shift the location or nature of critical points. One may investigate whether the presence of biquaternionic degrees of freedom induces novel magnetic ordering patterns, complex order parameters, or nontrivial phase structure, especially in low-dimensional systems where quantum fluctuations are enhanced. Furthermore, the interplay between the Hermitian inner product (ensuring unitary evolution) and the complexified algebra may provide a natural setting for studying generalized symmetry breaking, crossover phenomena, or extensions of universality classes. These directions suggest that the proposed spin matrices may serve as a foundation for constructing enriched spin models with potentially observable consequences in quantum many-body systems.

Another fruitful avenue of investigation involves the construction of corresponding Dirac matrices derived from these spin structures. Their algebraic representations, symmetry properties, and potential physical interpretations could reveal novel insights into relativistic quantum systems. The generalized Lorentz algebra formulated in this work also presents a fertile ground for further exploration. In particular, examining its isomorphisms with other mathematical groups, such as the special linear group, spin group, symplectic group, and projective special linear group could help elucidate its deeper geometric and algebraic significance. Moreover, identifying physical theories (or extensions thereof) that

remain invariant under the generalized Lorentz symmetry such as Maxwell's equations, the Dirac equation, or even aspects of the Standard Model would provide a valuable link between abstract algebraic generalizations and physical law. Extending this framework to the context of quantum gravity could also shed light on potential symmetries at the Planck scale. Finally, a natural setting for applying the quaternionic basis introduced in this work is quaternionic quantum mechanics. Investigating how a change of basis, using the proposed quaternionic structures, influences the formulation and interpretation of quaternionic quantum systems constitutes an intriguing and potentially impactful direction for future study.

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