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Topological Information Systems in a Complete Co-Residuated Lattice

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Abstract

In this paper, we introduce topological information systems between objects with a distance and attributes with a distance based on a complete co-residuated lattice. We can obtain two distances and Alexandrov topologies for objects and attributes from an information system. Moreover, we investigate the relations between residuated frames and residuated connections on Alexandrov topologies induced by distances instead of fuzzy equalities. We define a fuzzy concept lattice based on Alexandrov topologies and show that it is a complete lattice. We study its properties as a topological viewpoint and provide examples. Moreover, we introduce *R*-*R* embedding maps and *R*-*R* frame embedding maps.

Keywords: complete co-residuated lattice; Alexandrov topologies; residuated frames; residuated connections; fuzzy concept lattices; *R*-*R* (frame) embedding maps

MSC: 03G27; 06D50; 08A72; 54F05; 94A16

1. Introduction

Residuated connections were introduced by Blyth and Janovitz [1] as a pair of maps from a partially ordered set to a partially ordered set. In modal logics [2–5], the operators of necessity and possibility form this connection. Moreover, rough sets form a residuated connection [6]. From this view point, Wille [7] introduced a formal concept analysis, which is a mathematical framework used to analyze relationships between objects and their attributes. Rodabaugh [8] interpreted the image and preimage of fuzzy sets as a residuated connection. Kim [9] extended the image and preimage of filter bases. Various maps between algebraic structures form residuated connections [4,10–13].

As extensions of Boolean algebras, Heyting algebras and left continuous t-norms, a residuated lattice is an algebraic structure with a residuated connection. It appears in areas like order theory, and many valued areas of logic and theoretical computer science [4,8–13].

Georgescu and Popescu [10,14] introduced residuated connection on a complete residuated lattice with fuzzy partial orders. Moreover, they proposed a fuzzy concept, which is an extension of Wille' concepts [7]. Using various fuzzy concepts, information systems and decision rules were investigated in complete residuated lattices [5,13,15–20].

On the other hand, dualities between algebras (Boolean algebras, Heyting algebras, and BL-algebras) and relational semantics (classical logic, intuitionistic logic, and basic fuzzy logic) are useful information systems, and data analysis and decision rules [2–5,21]. As a duality, the relations between residuated connections as algebras and

Check for updates

Academic Editor: Hsien-Chung Wu

Received: 31 March 2026
Revised: 21 April 2026
Accepted: 27 April 2026
Published: 3 May 2026

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residuated frames as relational semantics were investigated in a complete residuated lattice [11,21].

Zheng and Wang [22] introduced a complete co-residuated lattice as the generalization of t-conorm. Qiao and Hu [23] investigated a $(\odot, \&)$-generalized fuzzy rough set for operations $\odot$ and $\&$ on complete residuated lattices and complete co-residuated lattices, but it does not behave for residuated connections. Using distances instead of fuzzy partial orders, Oh and Kim [24] introduced various fuzzy concepts on a co-residuated lattice. This concept plays an important role in changing the existing method of classifying information using the equivalence relation to the method of using distance.

In this paper, we define a topological information system (Definition 5) with two distances for objects and attributes based on a complete co-residuated lattice. Here, two distances induce two Alexandrov topologies.

In Section 3, we introduce the notions of residuated connections and residuated frames as a duality. From a residuated frame as relational semantics, in Theorem 2, we have a residuated connection based on two Alexandrov topologies. In Theorem 3, from an information system, we can obtain a topological information system with two distances for objects and attributes satisfying a residuated frame. Moreover, we obtain a residuated connection in Theorem 2.

In Theorem 5, from a residuated frame as relational semantics, we show that a fuzzy concept lattice based on two Alexandrov topologies is a complete lattice as an algebra. By Remark 3, it is an extension of classical fuzzy concepts in [25]. From these results, big data with an information system can also be variously classified into small groups with a viewpoint of Alexandrov topologies induced by distances.

In Section 4, we investigate the relations *R*-*R* embedding maps and *R*-*R* frame embedding maps as structure-preserving maps in Definitions 7 and 8.

2. Preliminaries

In this section, we recall definitions and properties of complete co-residuated lattices and a distance.

Definition 1 ([12,13,24,26]). *An algebra $(L, \wedge, \vee, \oplus, \bot, \top)$ is called a complete co-residuated lattice if it satisfies the following three conditions:*

(C1) *$L = (L, \vee, \wedge, \bot, \top)$ is a complete lattice where $\bot$ is the bottom element and $\top$ is the top element.*

(C2) *$x = x \oplus \bot$, $x \oplus y = y \oplus x$ and $x \oplus (y \oplus z) = (x \oplus y) \oplus z$ for all $x, y, z \in L$.*

(C3) *$(\bigwedge_{i\in\Gamma} x_i) \oplus y = \bigwedge_{i\in\Gamma}(x_i \oplus y)$ for all $x_i, y \in L$.*

Let $(L, \wedge, \vee, \oplus, \bot, \top)$ be a complete co-residuated lattice. For all $x, y \in L$, define

$$x \ominus y = \bigwedge\{z \in L \mid y \oplus z \geq x\}.$$

Then $(x \oplus y) \geq z$ if and only if $x \geq (z \ominus y)$.

For all $\alpha \in L$ and $A \in L^X$, we denote $(A \ominus \alpha_X)$, $(\alpha_X \oplus A)$ and $\alpha_X \in L^X$ by $(A \ominus \alpha_X)(x) = A(x) \ominus \alpha$, $(\alpha_X \oplus A)(x) = \alpha \oplus A(x)$ and $\alpha_X(x) = \alpha$, respectively.

Let $x^* = \top \ominus x$. The condition $(x^*)^* = x$ for all $x \in L$ is called a *double negative law*.

Lemma 1 ([12,13,24,26]). *Let $(L, \wedge, \vee, \oplus, \ominus, \bot, \top)$ be a complete co-residuated lattice with a double negative law. For all $x, y, z, x_i, y_i \in L$, the following holds.*

(1) *If $y \leq z$, then $x \oplus y \leq x \oplus z$, $y \ominus x \leq z \ominus x$ and $x \ominus z \leq x \ominus y$.*

(2) *$(\bigvee_{i\in\Gamma} x_i) \ominus y = \bigvee_{i\in\Gamma}(x_i \ominus y)$ and $x \ominus (\bigwedge_{i\in\Gamma} y_i) = \bigvee_{i\in\Gamma}(x \ominus y_i)$.*

https://doi.org/10.3390/axioms15050336

(3) $(\bigwedge_{i\in\Gamma} x_i) \ominus y \leq \bigwedge_{i\in\Gamma}(x_i \ominus y)$.
(4) $x \ominus (\bigvee_{i\in\Gamma} y_i) \leq \bigwedge_{i\in\Gamma}(x \ominus y_i)$.
(5) $x \ominus x = \bot$, $x \ominus \bot = x$ *and* $\bot \ominus x = \bot$. *Moreover,* $x \ominus y = \bot$ *if and only if* $x \leq y$.
(6) $y \oplus (x \ominus y) \geq x$, $y \geq x \ominus (x \ominus y)$ *and* $(x \ominus y) \oplus (y \ominus z) \geq x \ominus z$.
(7) $x \ominus (y \oplus z) = (x \ominus y) \ominus z = (x \ominus z) \ominus y$.
(8) $(x \oplus y)^* = x^* \ominus y = y^* \ominus x$, $(x \ominus y)^* = x^* \oplus y$ *and* $x \ominus y = y^* \ominus x^*$.

Definition 2 ([12,13]). *Let* $(L, \wedge, \vee, \oplus, \ominus, \bot, \top)$ *be a complete co-residuated lattice. Let* X *be a set. A function* $d_X : X \times X \to L$ *is called a distance if it satisfies the following two conditions:*

(M1) $d_X(x, x) = \bot$ *for all* $x \in X$.
(M2) $d_X(x, y) \oplus d_X(y, z) \geq d_X(x, z)$, *for all* $x, y, z \in X$.
(M3) If $d_X(x, y) = d_X(y, x) = \bot$, *then* $x = y$.

The pair (X, d_X) *is called a distance space.*

Let $d_X : X \times X \to L$ be a distance. Define $d_X^{-1} : X \times X \to L$ by $d_X^{-1}(x, y) = d_X(y, x)$ for all $x, y \in X$. It is easy to see that d_X^{-1} is a distance.

In this paper, we assume that $(L, \wedge, \vee, \oplus, \ominus, \bot, \top)$ is a complete co-residuated lattice with a double negative law.

3. Fuzzy Concept Lattices Based on Alexandrov Topologies

In this section, we investigate the relations between residuated connections and residuated frames. Moreover, we show that fuzzy concept lattices based on Alexandrov topologies are complete lattices.

Definition 3 ([12,13]). *(1) Let* $\tau \subseteq L^X$. τ *is called an Alexandrov topology on* X *if it satisfies the following two conditions:*

(A1) If $A_i \in \tau$ *for all* $i \in I$, *then* $\bigvee_{i\in I} A_i, \bigwedge_{i\in I} A_i \in \tau$.
(A2) If $A \in \tau$ *and* $\alpha \in L$, *then* $A \ominus \alpha_X, A \oplus \alpha_X \in \tau$.

The pair (X, τ) *is called an Alexandrov topological space on* X.

Lemma 2. *Let* $\tau_X \subset L^X$. *Define* $d_{\tau_X} : \tau_X \times \tau_X \to L$ *as* $d_{\tau_X}(A, B) = \bigvee_{x\in X}(B(x) \ominus A(x))$. *Then* (τ_X, d_{τ_X}) *is a distance space.*

Proof. (1) (M1) $d_{\tau_X}(A, A) = \bigvee_{x\in X}(A(x) \ominus A(x)) = \bot$ for all $A \in \tau_X$,

(M2) By Lemma 1 (6), $d_{\tau_X}(A, B) \oplus d_{\tau_X}(B, C) = \bigvee_{x\in X}(B(x) \ominus A(x)) \oplus \bigvee_{x\in X}(C(x) \ominus B(x)) \geq \bigvee_{x\in X}((B(x) \ominus A(x)) \oplus (C(x) \ominus A(x))) \geq d_{\tau_X}(A, C)$, for all $A, B, C \in \tau_X$,

(M3) If $d_{\tau_X}(A, B) = d_{\tau_X}(B, A) = \bot$, By Lemma 1 (5), $A = B$. Hence (τ_X, d_{τ_X}) is a distance space. □

Theorem 1. *Let* (X, d_X) *be a distance space. Define*

$$\tau_{d_X} = \{A \in L^X \mid A(x) \oplus d_X(x, z) \geq A(z)\}.$$

Then the following holds.

(1) τ_{d_X} *is an Alexandrov topology on* X *such that*

$$\begin{aligned} \tau_{d_X} &= \{\textstyle\bigwedge_{x\in X}(A(x) \oplus d_X(x, -)) \mid A \in L^X\} \\ &= \{\textstyle\bigvee_{x\in X}(A(x) \ominus d_X(-, x)) \mid A \in L^X\}. \end{aligned}$$

(2) $A \in \tau_{d_X^{-1}}$ *iff* $A^* \in \tau_{d_X}$. *For each* $(d_X)_x = d_X(x, -)$ *and* $(d_X)^z = d_X(-, z)$, $(d_X)_x \in \tau_{d_X}$, $(d_X)^z \in \tau_{d_X^{-1}}$ *and* $((d_X)^z)^* \in \tau_{d_X}$.

Proof. (1) (A1) If $A_i \in \tau_{d_X}$ for all $i \in I$, then

$$\begin{aligned}&(\bigwedge_{i\in I} A_i) \oplus d_X(x,y) = \bigwedge_{i\in I}(A_i \oplus d_X(x,y))\\ &\geq \bigwedge_{i\in I} A_i,\\ &(\bigvee_{i\in I} A_i) \oplus d_X(x,y) \geq \bigvee_{i\in I}(A_i \oplus d_X(x,y))\\ &\geq \bigvee_{i\in I} A_i, \text{ (by Lemma 1 (1))}\end{aligned}$$

then $\bigwedge_{i\in I} A_i, \bigvee_{i\in I} A_i \in \tau_{d_X}$.

(A2) If $A \in \tau_{d_X}$ and $\alpha \in L$, then $\alpha \oplus (A(x) \ominus \alpha) \oplus d_X(x,y) \geq A(x) \oplus d_X(x,y) \geq A(y)$ implies $(A(x) \ominus \alpha) \oplus d_X(x,y) \geq A(y) \ominus \alpha$. So, $A \ominus \alpha_X \in \tau_{d_X}$. Easily, $\alpha_X \oplus A \in \tau_{d_X}$. Hence τ_{d_X} is an Alexandrov topology on X.

Put $\tau_1 = \{\bigwedge_{x\in X}(A(x) \oplus d_X(x,-)) \mid A \in L^X\}$. Let $A \in \tau_{d_X}$. Then $A = \bigwedge_{x\in X}(A(x) \oplus d_X(x,-)) \in \tau_1$.

Let $\bigwedge_{x\in X}(A(x) \oplus d_X(x,-)) \in \tau_1$. Then $\bigwedge_{x\in X}(A(x) \oplus d_X(x,y)) \oplus d_X(y,z) \geq \bigwedge_{x\in X}(A(x) \oplus d_X(x,z))$. Thus $\bigwedge_{x\in X}(A(x) \oplus d_X(x,-)) \in \tau_{d_X}$. Hence $\tau_1 = \tau_{d_X}$

Put $\tau_2 = \{\bigvee_{x\in X}(A(x) \ominus d_X(-,x)) \mid A \in L^X\}$. Let $A \in \tau_{d_X}$. Then $A(y) \geq \bigvee_{x\in X}(A(x) \ominus d_X(y,x)) \geq A(y)$. Thus $A \in \tau_2$.

Let $\bigvee_{x\in X}(A(x) \ominus d_X(-,x)) \in \tau_2$. Since $(A(x) \ominus d_X(y,x)) \oplus d_X(y,z) \oplus d_X(z,x) \geq (A(x) \ominus d_X(y,x)) \oplus d_X(y,x) \geq A(x)$, $(A(x) \ominus d_X(y,x)) \oplus d_X(y,z) \geq A(x) \ominus d_X(z,x)$. It implies $\bigvee_{x\in X}(A(x) \ominus d_X(y,x)) \oplus d_X(y,z) \geq \bigvee_{x\in X}(A(x) \ominus d_X(z,x))$. Thus $\bigvee_{x\in X}(A(x) \ominus d_X(-,x)) \in \tau_{d_X}$. Hence $\tau_2 = \tau_{d_X}$

(2) By Lemma 1 (8), $A(x) \oplus d_X^{-1}(x,y) \geq A(y)$ iff $A(x) \oplus d_X(y,x) \geq A(y)$ iff $A^*(x) \ominus d_X(y,x) \leq A^*(y)$ iff $A^*(y) \oplus d_X(y,x) \geq A^*(x)$ iff $A^* \in \tau_{d_X}$. Other cases are easily proved. □

Remark 1. *(1) Let $(X, \perp_{\triangle_X})$ be a distance space where $\perp_{\triangle_X}(x,x) = \perp$ and $\perp_{\triangle_X}(x,y) = \top$ for $x \neq y \in X$. For each $A \in L^X$, $A(y) = \bigwedge_{x\in X}(A(x) \oplus \perp_{\triangle_X}(x,y)) \in \tau_{\perp_{\triangle_X}}$. Thus $\tau_{\perp_{\triangle_X}} = L^X$ and $d_{\tau_{\perp_{\triangle_X}}} = d_{L^X} : L^X \times L^X \to L$ as $d_{L^X}(A,B) = \bigvee_{x\in X}(B(x) \ominus A(x))$.*

(2) Let $(X, \perp_{X\times X})$ be a distance space where $\perp_{X\times X}(x,y) = \perp$ for each $x, y \in X$. Since $A(y) = \bigwedge_{x\in X}(A(x) \oplus \perp_{X\times X}(x,y)) = \alpha$ Thus $\tau_{\perp_{X\times X}} = \{\alpha_X \in L^X \mid \alpha \in L\}$ and $d_{\tau_{\perp_{X\times X}}} : \tau_{\perp_{X\times X}} \times \tau_{\perp_{X\times X}} \to L$ as $d_{\tau_{\perp_{X\times X}}}(\alpha_X, \beta_X) = \beta \ominus \alpha$.

Definition 4. *Let (X, d_X) and (Y, d_Y) be distance spaces and $f : X \to Y$ and $g : Y \to X$ maps.*

(1) (d_X, f, g, d_Y) is called a residuated connection if for all $x \in X, y \in Y$, $d_Y(f(x), y) = d_X(x, g(y))$.

(2) f is an isotone map if $d_Y(f(x_1), f(x_2)) \leq d_X(x_1, x_2)$ for all $x_1, x_2 \in X$.

(3) f is an embedding map if $d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$ for all $x_1, x_2 \in X$.

For $R_1 \in L^{X\times Y}, R_2 \in L^{Y\times Z}$, we define

$$\begin{aligned}&R_1 \circ R_2(x,z) = \bigwedge_y (R_1(x,y) \oplus R_2(y,z)),\\ &R_1^{-1}(y,x) = R_1(x,y).\end{aligned}$$

Definition 5. *Let (X, d_X) and (Y, d_Y) be distance spaces and $R \in L^{X\times Y}$.*

(1) A structure (d_X, R, d_Y) is called a residuated frame if $d_X \circ R \circ d_Y = R$.

(2) If X is a set of objects and Y is a set of attributes, a triple (X, Y, R) is called an information system. Moreover, $((X, d_X), (Y, d_Y), R)$ is called a topological information system.

Lemma 3. *Let (X, d_X) and (Y, d_Y) be distance spaces. For $R \in L^{X\times Y}$, we have the following properties:*

(1) $d_X \circ R \circ d_Y = R$ iff $d_X \circ R = R$ and $R \circ d_Y = R$.

(2) *If* $d_X \circ R = R$, *then* $R^y \in \tau_{d_X^{-1}}$ *for each* $y \in Y$ *where* $R^y(x) = R(x,y)$.

(3) *If* $R \circ d_Y = R$, *then* $R_x \in \tau_{d_Y}$ *for each* $x \in X$ *where* $R_x(y) = R(x,y)$.

Proof. (1) Since $d_X \circ R \circ d_Y = R$, $R(x,y) \leq d_X \circ R \circ d_Y(x,y) = \bigwedge_{y_1 \in Y}(d_X \circ R)(x,y_1) \oplus d_Y(y_1,y) \leq (d_X \circ R)(x,y) \oplus d_Y(y,y) = (d_X \circ R)(x,y) \leq R(x,y)$. Thus $d_X \circ R = R$. Similarly, $R \circ d_Y = R$. Conversely, it easily proved.

(2) Since $d_X(x,z) \oplus R(z,y) \geq (d_X \circ R)(x,y) = R(x,y)$, $R^y \in \tau_{d_X^{-1}}$.

(3) Since $R(x,y) \oplus d_Y(y,w)) \geq R(x,w)$, $R_x \in \tau_{d_Y}$. □

Theorem 2. *Let* (d_X, R, d_Y) *be a residuated frame. Let* τ_{d_X} *and* τ_{d_Y} *be Alexandrov topologies. Then* $(d_{\tau_{d_X}}, F, G, d_{\tau_{d_Y}})$ *is a residuated connection where*

$$F(A)(y) = \bigwedge_{x \in X}(R(x,y) \oplus A(x)),$$
$$G(B)(x) = \bigvee_{y \in Y}(B(y) \ominus R(x,y)).$$

Moreover, for each $x, z \in X$ *and* $y, w \in Y$,

$$F((d_X)_x)(y) = R(x,y), G(((d_Y)^y)^*)(x) = R^*(x,y),$$
$$F(R^*(-,w))(y) = \bigwedge_{x \in X}(R(x,y) \oplus R^*(x,w)),$$
$$G(R(z,-))(x) = \bigvee_{y \in Y}(R(z,y) \ominus R(x,y)).$$

Proof. Since $R \circ d_Y = R$ and $d_X \circ R = R$ from Lemma 3 (1), $F(A) \in \tau_{d_Y}$ and $G(B) \in \tau_{d_X}$ from:

$$\begin{aligned}
&F(A)(y) \oplus d_Y(y,w)\\
&= (\bigwedge_{x \in X}(R(x,y) \oplus A(x))) \oplus d_Y(y,w)\\
&\geq \bigwedge_{x \in X}(R(x,w) \oplus A(x)) = F(A)(w),\\
&G(B)(x) \oplus d_X(x,z) \oplus R(z,y)\\
&\geq \bigvee_{w \in Y}(B(w) \ominus R(x,w)) \oplus R(x,y)\\
&\geq (B(y) \ominus R(x,y)) \oplus R(x,y) \geq B(y),\\
&G(B)(x) \oplus d_X(x,z) \geq B(y) \ominus R(z,y).
\end{aligned}$$

Moreover, $A \in \tau_{d_X}, B \in \tau_{d_Y}$, by Lemma 1 (7),

$$\begin{aligned}
&d_{\tau_{d_X}}(A, G(B))\\
&= \bigvee_{x \in X}(G(B)(x) \ominus A(x))\\
&= \bigvee_{x \in X}\Big(\bigvee_{y \in Y}(B(y) \ominus R(x,y)) \ominus A(x)\Big)\\
&= \bigvee_{x \in X}\bigvee_{y \in Y}\Big(B(y) \ominus (R(x,y) \oplus A(x))\Big)\\
&= \bigvee_{y \in Y}\Big(B(y) \ominus \bigwedge_{x \in X}(R(x,y) \oplus A(x))\Big)\\
&= \bigvee_{y \in Y}\Big(B(y) \ominus F(A)(y)\Big) = d_{\tau_{d_Y}}(F(A), B).
\end{aligned}$$

Moreover, since $d_X \circ R \circ d_Y = R$, $d_X \circ R = R$ and $R \circ d_Y = R$. Thus,

$$\begin{aligned}
&F((d_X)_x)(y) = \bigwedge_{z \in X}(R(z,y) \oplus (d_X)_x(z))\\
&= \bigwedge_{z \in X}(R(z,y) \oplus d_X(x,z))\\
&= (d_X \circ R)(x,y) = R(x,y),\\
&G(((d_Y)^y)^*)(x) = \bigvee_{w \in Y}(((d_Y)^y)^*(w) \ominus R(x,w))\\
&= (\bigwedge_{w \in Y}(d_Y(w,y) \oplus R(x,w)))^*\\
&= (R \circ d_Y)^*(x,y) = R^*(x,y).
\end{aligned}$$

Since $d_X \circ R = R$ and $R \circ d_Y = R$, $R^*(-,w) \in \tau_{d_X}$ and $R(z,-) \in \tau_{d_Y}$ for each $z \in X$ and $w \in Y$. Thus $F(R^*(-,w))(y) = \bigwedge_{x \in X}(R(x,y) \oplus R^*(x,w)), G(R(z,-))(x) = \bigvee_{y \in Y}(R(z,y) \ominus R(x,y))$. □

Remark 2. *(1) Since* $(\perp_{\triangle_X}, d_X, \perp_{\triangle_X})$ *is a residuated frame where* d_X *is a distance and* $\tau_{\perp_{\triangle_X}} = L^X$ *from Remark 1 (1), then* (d_{L^X}, F, G, d_{L^X}) *is a residuated connection defined as*

$$F(A)(y) = \bigwedge_{x\in X}(A(x) \oplus d_X(x,y)),$$
$$G(B)(x) = \bigvee_{y\in X}(B(y) \ominus d_X(x,y)).$$

(2) Let $(X, \perp_{\triangle_X})$ *and* $(Y, \perp_{\triangle_Y})$ *be distance spaces with* $\tau_{\perp_{\triangle_X}} = L^X$ *and* $\tau_{\perp_{\triangle_Y}} = L^Y$. *Let* $R \in L^{X\times Y}$ *be given. Since* $\perp_{\triangle_X} \circ R \circ \perp_{\triangle_Y} = R$, $(\perp_{\triangle_X}, R, \perp_{\triangle_Y})$ *is a residuated frame. Hence* (d_{L^X}, F, G, d_{L^Y}) *is a residuated connection defined as*

$$F(A)(y) = \bigwedge_{x\in X}(A(x) \oplus R(x,y)),$$
$$G(B)(x) = \bigvee_{y\in Y}(B(y) \ominus R(x,y)).$$

Moreover, let $f : X \to Y$ *be a function and* $R(x,y) = (f(x), y)$. *Then*

$$F(A)(y) = \begin{cases} \bigwedge_{x\in f^{-1}(\{y\})} A(x), & \text{if } y \in f(X), \\ \top, & \text{if } y \notin f(X), \end{cases}$$

and $G(B)(x) = B(f(x))$ *for each* $x \in X$.

In following theorem, we can obtain a topological information system, a residuated frame and a residuated connection from an information system.

Theorem 3. *Let* (X, Y, R) *be an information system. Define* $d_X : X \times X \to L$ *and* $d_Y : Y \times Y \to L$ *as*

$$d_X(x,z) = \bigvee_{y\in Y}(R(x,y) \ominus R(z,y)),$$
$$d_Y(y,w) = \bigvee_{x\in X}(R(x,w) \ominus R(x,y)).$$

Then the following holds.

(1) A triple $((X, d_X), (Y, d_Y), R)$ *is a topological information system. Moreover,* (d_X, R, d_Y) *is a residuated frame.*

(2) Then $(d_{\tau_{d_X}}, F, G, d_{\tau_{d_Y}})$ *is a residuated connection defined as*

$$F(A)(y) = \bigwedge_{x\in X}(R(x,y) \oplus A(x)),$$
$$G(B)(x) = \bigvee_{y\in Y}(B(y) \ominus R(x,y)).$$

Moreover, $F(R^*(-,w))(y) = d_Y^*(y,w), G(R(z,-))(x) = d_X(z,x)$.

Proof. (1) For each $x, y \in X$,

$$\begin{aligned}
&(d_X \circ R)(x,y) = \textstyle\bigwedge_{z\in X}(d_X(x,z) \oplus R(z,y)) \\
&= \textstyle\bigwedge_{z\in X}(\bigvee_{w\in Y}(R(x,w) \ominus R(z,w)) \oplus R(z,y)) \\
&\geq \textstyle\bigwedge_{z\in X}((R(x,y) \ominus R(z,y)) \oplus R(z,y)) \\
&\geq R(x,y), \\
&(d_X \circ R)(x,y) = \textstyle\bigwedge_{z\in X}(d_X(x,z) \oplus R(z,y)) \\
&= \textstyle\bigwedge_{z\in X}(\bigvee_{w\in Y}(R(x,w) \ominus R(z,w)) \oplus R(z,y)) \\
&\leq \textstyle\bigvee_{w\in Y}(R(x,w) \ominus R(x,w)) \oplus R(x,y) \\
&= R(x,y),
\end{aligned}$$

$$
\begin{aligned}
&(R \circ d_Y)(x,y) = \bigwedge_{w\in Y}(R(x,w) \oplus d_Y(w,y))\\
&= \bigwedge_{w\in Y}(R(x,w) \oplus \bigvee_{z\in X}(R(z,y) \ominus R(z,w)))\\
&\geq \bigwedge_{w\in Y}(R(x,w) \oplus (R(x,y) \ominus R(x,w)))\\
&\geq R(x,y),\\
&(R \circ d_Y)(x,y) = \bigwedge_{w\in Y}(R(x,w) \oplus d_Y(w,y))\\
&= \bigwedge_{w\in Y}(R(x,w) \oplus \bigvee_{z\in X}(R(z,y) \ominus R(z,w)))\\
&\leq R(x,y) \oplus \bigvee_{z\in X}(R(z,y) \ominus R(z,y))\\
&= R(x,y).
\end{aligned}
$$

Thus $R \circ d_Y = R$ and $d_X \circ R = R$, by Lemma 3 (1), and (d_X, R, d_Y) is a residuated frame.

(2) By Theorem 2 and (1), $(d_{\tau_{d_X}}, F, G, d_{\tau_{d_Y}})$ is a residuated connection where

$$
\begin{aligned}
&F(A)(y) = \bigwedge_{x\in X}(R(x,y) \oplus A(x)),\\
&G(B)(x) = \bigvee_{y\in Y}(B(y) \ominus R(x,y)).
\end{aligned}
$$

For each $x, z \in X$ and $y, w \in Y$,

$$
\begin{aligned}
&F(R^*(-,w))(y) = \bigwedge_{x\in X}(R(x,y) \oplus R^*(x,w))\\
&= (\bigvee_{x\in X}(R(x,w) \ominus R(x,y)))^* = d_Y^*(y,w),\\
&G(R(z,-))(x) = \bigvee_{y\in Y}(R(z,y) \ominus R(x,y))\\
&= d_X(z,x).
\end{aligned}
$$

□

Definition 6. *Let $F : \tau_{d_X} \to \tau_{d_Y}$ and $G : \tau_{d_Y} \to \tau_{d_X}$ be two maps with a residuated connection $(d_{\tau_{d_X}}, F, G, d_{\tau_{d_Y}})$. A family $\mathcal{F}(d_{\tau_{d_X}}, F, G, d_{\tau_{d_Y}})$ is called a fuzzy concept lattice based on τ_{d_X} and τ_{d_Y} where*

$$
\begin{aligned}
&\mathcal{F}(d_{\tau_{d_X}}, F, G, d_{\tau_{d_Y}})\\
&= \{(A,B) \in \tau_{d_X} \times \tau_{d_Y} \mid F(A) = B, G(B) = A\},\\
&(A_1,B_1) \leq (A_2,B_2) \textit{ iff } A_1 \leq A_2 (\textit{or } B_1 \leq B_2).
\end{aligned}
$$

Theorem 4. *Let $F : \tau_{d_X} \to \tau_{d_Y}$ and $G : \tau_{d_Y} \to \tau_{d_X}$ be two maps. The following statements are equivalent.*

(1) $(d_{\tau_{d_X}}, F, G, d_{\tau_{d_Y}})$ *is a residuated connection.*

(2) $F(\bigwedge_{i\in\Gamma} A_i) = \bigwedge_{i\in\Gamma} F(A_i)$, $F(A \oplus \alpha) = F(A) \oplus \alpha$ *and* $G(\bigvee_{i\in\Gamma} B_i) = \bigvee_{i\in\Gamma} G(B_i)$, $G(B \ominus \alpha) = G(B) \ominus \alpha$. *Moreover, for each $x \in X, y \in Y$,*

$$
\begin{aligned}
&F(A)(y) = \bigwedge_{x\in X}(A(x) \oplus F(d_X(x,-))(y)),\\
&G(B)(x) = \bigvee_{y\in Y}(F(d_X(x,-))(y) \ominus B(y)),\\
&F^*(d_X(x,-))(y) = G(d_Y^*(-,y))(x).
\end{aligned}
$$

(3) For each $A, A_1, A_2 \in \tau_{d_X}, B, B_1, B_2 \in \tau_{d_Y}$,

$$
\begin{aligned}
&G(F(A)) \leq A,\ B \leq F(G(B)),\\
&d_{\tau_{d_X}}(A_1,A_2) \geq d_{\tau_{d_Y}}(F(A_1),F(A_2)),\\
&d_{\tau_{d_Y}}(B_1,B_2) \geq d_{\tau_{d_X}}(G(B_1),G(B_2)),\\
&F(G(F(A))) = F(A), G(F(G(B))) = G(B).
\end{aligned}
$$

Proof. (1) ⇒ (2) since

$$
\begin{aligned}
&d_{\tau_{d_Y}}(F(\bigwedge_{i\in\Gamma} A_i), B) = d_{\tau_{d_X}}(\bigwedge_{i\in\Gamma} A_i, G(B))\\
&= \bigvee_{x\in X}(G(B)(x) \ominus \bigwedge_{i\in\Gamma} A_i(x))\\
&= \bigvee_{i\in\Gamma}(\bigvee_{x\in X}(G(B)(x) \ominus A_i(x)))\\
&\quad ((\text{by Lemma 1 (2)}))\\
&= \bigvee_{i\in\Gamma} d_{\tau_{d_X}}(A_i, G(B)) = \bigvee_{i\in\Gamma} d_{\tau_{d_Y}}(F(A_i), B)\\
&= d_{\tau_{d_Y}}(\bigwedge_{i\in\Gamma} F(A_i), B),
\end{aligned}
$$

we have $F(\bigwedge_{i\in\Gamma} A_i) = \bigwedge_{i\in\Gamma} F(A_i)$. Since

$$
\begin{aligned}
&d_{\tau_{d_Y}}(F(A \oplus \alpha), B) = d_{\tau_{d_X}}(A \oplus \alpha, G(B))\\
&= \bigvee_{x\in X}(G(B)(x) \ominus (A(x) \oplus \alpha))\\
&\quad (\text{by Lemma 1 (2,7)})\\
&= \bigvee_{x\in X}(G(B)(x) \ominus A(x)) \ominus \alpha\\
&= d_{\tau_{d_X}}(F(A), B) \ominus \alpha\\
&= \bigvee_{y\in Y}(B(y) \ominus F(A)(y)) \ominus \alpha\\
&= \bigvee_{y\in Y}(B(y) \ominus (F(A)(y) \oplus \alpha))\\
&= d_{\tau_{d_Y}}(F(A) \oplus \alpha, B),
\end{aligned}
$$

we have $F(A \oplus \alpha) = F(A) \oplus \alpha$ for all $\alpha \in L$.

Since

$$
\begin{aligned}
&d_{\tau_{d_X}}(A, G(\bigvee_{i\in\Gamma} B_i)) = d_{\tau_{d_Y}}(F(A), \bigvee_{i\in\Gamma} B_i)\\
&= \bigvee_{i\in\Gamma} d_{\tau_{d_X}}(A, G(B_i)) = d_{\tau_{d_X}}(A, \bigvee_{i\in\Gamma} G(B_i)),
\end{aligned}
$$

we have $G(\bigvee_{i\in\Gamma} B_i) = \bigvee_{i\in\Gamma} G(B_i)$. Note that

$$
\begin{aligned}
&d_{\tau_{d_X}}(A, G(B \ominus \alpha)) = d_{\tau_{d_Y}}(F(A), B \ominus \alpha)\\
&= \bigvee_{y\in Y}((B(y) \ominus \alpha) \ominus F(A)(y))\\
&= \bigvee_{y\in Y}((B(y) \ominus F(A)(y)) \ominus \alpha)\\
&= d_{\tau_{d_Y}}(F(A), B) \ominus \alpha\\
&= d_{\tau_{d_X}}(A, G(B)) \ominus \alpha\\
&= \bigvee_{x\in X}(G(B)(x) \ominus A(x)) \ominus \alpha\\
&= \bigvee_{x\in X}(G(B)(x) \ominus \alpha) \ominus A(x))\\
&= d_{\tau_{d_X}}(A, G(B) \ominus \alpha).
\end{aligned}
$$

Then $G(B \ominus \alpha) = G(B) \ominus \alpha$ for all $\alpha \in L$. Moreover, for $A \in \tau_{d_X}$, since $A = \bigwedge_{z\in X}(A(z) \oplus d_X(z, -))$,

$$
\begin{aligned}
&F(A)(y) = F(\bigwedge_{z\in X}(A(z) \oplus d_X(z, -)))\\
&= \bigwedge_{z\in X}(A(z) \oplus F(d_X(z, -))(y))
\end{aligned}
$$

and

$$
\begin{aligned}
&d_{\tau_{d_Y}}(F(d_X(x, -)), B) = d_{\tau_{d_X}}(d_X(x, -), G(B))\\
&= \bigvee_{z\in X}\Big(G(B)(z) \ominus d_X(x, z)\Big) = G(B)(x),\\
&G(B)(x) = \bigvee_{y\in Y}(B(y) \ominus F(d_X(x, -))(y)).
\end{aligned}
$$

and

$$
\begin{aligned}
&F^*(d_X(x, -)) = d_{\tau_{d_Y}}(F(d_X(x, -)), d_Y^*(-, y))\\
&= d_{\tau_{d_X}}(d_X(x, -), G(d_Y^*(-, y)))\\
&= \bigvee_{z\in X}(G(d_Y^*(-, y))(z) \ominus d_X(x, z))\\
&= G(d_Y^*(-, y))(x).
\end{aligned}
$$

(2) $\Rightarrow$ (1). Put $R(x, y) = F(d_X(x, -))(y)$. Then

$$F(A)(y) = \bigwedge_{x\in X}(R(x,y) \oplus A(x))$$
$$G(B)(x) = \bigvee_{y\in Y}(B(y) \ominus R(x,y)).$$

Moreover, (d_X, R, d_Y) is a residuated frame from:

$$\begin{aligned}
&R(x,y) = F(d_X(x,-))(y) = \bigwedge_{z\in X}(d_X(x,-)(z) \oplus R(z,y))\\
&= (d_X \circ R)(x,y)\\
&R(x,y) = F^*(d_X(x,-))(y)\\
&= \bigwedge_{z\in X}(R(z,y) \oplus d_X(x,z)) = (d_X \circ R)(x,y),\\
&R^*(x,y) = F^*(d_X(x,-))(y) = G(d_Y^*(-,y))(x)\\
&= \bigvee_{w\in Y}(d_Y^*(-,y)(w) \ominus R(x,w))\\
&= (\bigwedge_{w\in Y}(R(x,w) \oplus d_Y(w,y)))^* = (R \circ d_Y(x,y))^*.
\end{aligned}$$

From Theorem 2, $(d_{\tau_{d_X}}, F, G, d_{\tau_{d_Y}})$ is a residuated connection.

(1) $\Rightarrow$ (3). For each $A \in \tau_{d_X}, B \in \tau_{d_Y}$,

$$\begin{aligned}
&\bot = d_{\tau_{d_Y}}(F(A), F(A)) = d_{\tau_{d_X}}(A, G(F(A)))\\
&= \bigvee_{x\in X}(G(F(A)(x) \ominus A(x)).
\end{aligned}$$

By Lemma 1 (5), $G(F(A)) \leq A$. Moreover, $\bot = d_{\tau_{d_X}}(G(B), G(B)) = d_{\tau_{d_Y}}(F(G(B)), B)$. Thus $B \leq F(G(B))$.

For each $A_1, A_2 \in \tau_{d_X}, B_1, B_2 \in \tau_{d_Y}$,

$$\begin{aligned}
&d_{\tau_{d_Y}}(F(A_1), F(A_2)) = d_{\tau_{d_X}}(A_1, G(F(A_2)))\\
&\leq d_{\tau_{d_X}}(A_1, A_2) \oplus d_{\tau_{d_X}}(A_2, G(F(A_2)))\\
&= d_{\tau_{d_X}}(A_1, A_2),\\
&d_{\tau_{d_X}}(G(B_1), G(B_2)) = d_{\tau_{d_Y}}(F(G(B_1)), B_2)\\
&\leq d_{\tau_{d_Y}}(F(G(B_1)), B_1) \oplus d_{\tau_{d_Y}}(B_1, B_2)\\
&= d_{\tau_{d_Y}}(B_1, B_2).
\end{aligned}$$

Since $G(F(A)) \leq A$, $F(G(F(A))) \leq F(A)$. Moreover, $F(A) \leq F(G(F(A)))$. Thus $F(A) = F(G(F(A)))$. Similarly, $G(F(G(B))) = G(B)$.

(3) $\Rightarrow$ (1). For each $A \in \tau_{d_X}$ and $B \in \tau_{d_Y}$,

$$\begin{aligned}
&d_{\tau_{d_Y}}(F(A), B) \geq d_{\tau_{d_X}}(G(F(A)), G(B))\\
&\oplus d_{\tau_{d_X}}(A, G(F(A))) \geq d_{\tau_{d_X}}(A, G(B)).
\end{aligned}$$

$$\begin{aligned}
&d_{\tau_{d_X}}(A, G(B)) \geq d_{\tau_{d_Y}}(F(A), F(G(B)))\\
&\oplus d_{\tau_{d_Y}}(F(G(B)), B) \geq d_{\tau_{d_Y}}(F(A), B).
\end{aligned}$$

Hence $d_{\tau_{d_Y}}(F(A), B) = d_{\tau_{d_X}}(A, G(B))$. □

Theorem 5. *Let $\mathcal{F}(d_{\tau_{d_X}}, F, G, d_{\tau_{d_Y}})$ be a fuzzy concept lattice. Then the following properties hold.*

(1) For each $A \in \tau_{d_X}$ and $B \in \tau_{d_Y}$,

$$\begin{aligned}
&\mathcal{F}(d_{\tau_{d_X}}, F, G, d_{\tau_{d_Y}})\\
&= \{(A, F(A)) \in \tau_{d_X} \times \tau_{d_Y} \mid G(F(A)) = A\}\\
&= \{(G(B), B) \in \tau_{d_X} \times \tau_{d_Y} \mid F(G(B)) = B\}.
\end{aligned}$$

(2) For each $(A_1, B_1), (A_2, B_2) \in \mathcal{F}(d_{\tau_{d_X}}, F, G, d_{\tau_{d_Y}})$,

$$A_1 \leq A_2 \text{ iff } B_1 \leq B_2.$$

(3) *For each* $(A_1, B_1), (A_2, B_2), (A_i, B_i)_{i\in I} \in \mathcal{F}(d_{\tau_{d_X}}, F, G, d_{\tau_{d_Y}})$, *define*

$$(A_1, B_1) \leq (A_2, B_2) \text{ iff } A_1 \leq A_2 (\text{or } B_1 \leq B_2),$$

$$\begin{aligned} \bigvee_{i\in I}(A_i, B_i) &= (\bigvee_{i\in I} A_i, F(G(\bigvee_{i\in I} B_i))) \\ &= (\bigvee_{i\in I} A_i, F(\bigvee_{i\in I} A_i)), \end{aligned}$$

$$\begin{aligned} \bigwedge_{i\in I}(A_i, B_i) &= (G(F(\bigwedge_{i\in I} A_i)), \bigwedge_{i\in I} B_i) \\ &= (G(\bigwedge_{i\in I} B_i), \bigwedge_{i\in I} B_i). \end{aligned}$$

Then $\mathcal{F}(d_{\tau_{d_X}}, F, G, d_{\tau_{d_Y}})$ *forms a complete lattice.*

Proof. (1) Let $\mathcal{B} = \{(A, F(A)) \in \tau_{d_X} \times \tau_{d_Y} \mid G(F(A)) = A\}$. Let $(A, B) \in \mathcal{F}(d_{\tau_{d_X}}, F, G, d_{\tau_{d_Y}})$. Then $G(F(A)) = G(B) = A, B = F(A)$. Thus $(A, F(A)) \in \mathcal{B}$.

Let $(A, F(A)) \in \mathcal{B}$. Since $G(F(A)) = A, F(A) = F(A)$, $(A, F(A)) \in \mathcal{F}(d_{\tau_{d_X}}, F, G, d_{\tau_{d_Y}})$.

Let $\mathcal{C} = \{(G(B), B) \in \tau_{d_X} \times \tau_{d_Y} \mid F(G(B)) = B\}$. Let $(A, B) \in \mathcal{F}(\tau_{d_X}, F, G, \tau_{d_Y})$. Then $F(G(B)) = F(A) = B, A = G(B)$. Thus $(G(B), B) \in \mathcal{C}$.

Let $(G(B), B) \in \mathcal{C}$. Since $F(G(B)) = B, G(B) = G(B)$, $(G(B), B) \in \mathcal{F}(d_{\tau_{d_X}}, F, G, d_{\tau_{d_Y}})$.

(2) For each $(A_1, B_1), (A_2, B_2) \in \mathcal{F}(d_{\tau_{d_X}}, F, G, d_{\tau_{d_Y}})$,

Let $A_1 \leq A_2$. By Theorem 4 (3), $d_{\tau_{d_X}}(A_1, A_2) \geq d_{\tau_{d_Y}}(F(A_1), F(A_2))$, Then $B_1 = F(A_1) \leq F(A_2) = B_2$. Let $B_1 \leq B_2$. Then $A_1 = G(B_1) \leq G(B_2) = A_2$.

(3) For each $i \in I$, $(A_i, B_i) \in \mathcal{F}(d_{\tau_{d_X}}, F, G, d_{\tau_{d_Y}})$, by Theorem 4,

$$\begin{aligned} &G(F(G(\bigvee_{i\in I} B_i))) = G(\bigvee_{i\in I} B_i) \\ &= \bigvee_{i\in I} G(B_i) = \bigvee_{i\in I} A_i, \\ &F(G(\bigvee_{i\in I} B_i)) = F(\bigvee_{i\in I} G(B_i)) = F(\bigvee_{i\in I} A_i). \end{aligned}$$

Hence $\bigvee_{i\in I}(A_i, B_i) \in \mathcal{F}(d_{\tau_{d_X}}, F, G, d_{\tau_{d_Y}})$.

Since $A_i \leq \bigvee_{i\in I} A_i$, $B_i = F(A_i) \leq F(\bigvee_{i\in I} A_i) = F(G(\bigvee_{i\in I} B_i))$. Thus $(A_i, B_i) \leq \bigvee_{i\in I}(A_i, B_i)$.

If $(A_i, B_i) \leq (A, B)$ for each $i \in I$, then $\bigvee_{i\in I} A_i \leq A$ and $F(\bigvee_{i\in I} A_i) \leq F(A) = B$. Hence $\bigvee_{i\in I}(A_i, B_i) = (\bigvee_{i\in I} A_i, F(\bigvee_{i\in I} A_i)) \leq (A, B)$. Thus $\bigvee_{i\in I}(A_i, B_i)$ is the supremum of (A_i, B_i) for each $i \in I$

For each $i \in I$, $(A_i, B_i) \in \mathcal{F}(\tau_{d_X}, F, G, \tau_{d_Y})$,

$$\begin{aligned} &F(G(F(\bigwedge_{i\in I} A_i))) = F(\bigwedge_{i\in I} A_i) \\ &= \bigwedge_{i\in I} F(A_i) = \bigwedge_{i\in I} B_i, \\ &G(F(\bigwedge_{i\in I} A_i)) = G(\bigwedge_{i\in I} F(A_i)) = G(\bigwedge_{i\in I} B_i). \end{aligned}$$

Hence $\bigwedge_{i\in I}(A_i, B_i) \in \mathcal{F}(d_{\tau_{d_X}}, F, G, d_{\tau_{d_Y}})$. Moreover, since $\bigwedge_{i\in I} B_i \leq B_i, F(G(\bigwedge_{i\in I} B_i)) \leq \bigwedge_{i\in I} F(G(B_i)) = \bigwedge_{i\in I} A_i \leq B_i$. Thus, $\bigwedge_{i\in I}(A_i, B_i) \leq (A_i, B_i)$ for each $i \in I$.

If $(A, B) \leq (A_i, B_i)$ for each $i \in I$, then $B \leq \bigwedge_{i\in I} B_i$ and $A = G(B) \leq G(\bigwedge_{i\in I} B_i)$. Hence $(A, B) \leq \bigwedge_{i\in I}(A_i, B_i) = (G(\bigwedge_{i\in I} B_i), \bigwedge_{i\in I} B_i)$. Thus $\bigwedge_{i\in I}(A_i, B_i)$ is the infimum of (A_i, B_i) for each $i \in I$. □

Remark 3. *Let* $(X, \perp_{\triangle_X})$ *and* $(Y, \perp_{\triangle_Y})$ *be distance spaces with* $\tau_{\perp_{\triangle_X}} = L^X$ *and* $\tau_{\perp_{\triangle_Y}} = L^Y$. *Let* $F : L^X \to L^Y$ *and* $G : L^Y \to L^X$ *be two maps with a residuated connection* $(d_{\tau_{\perp_{\triangle_X}}}, F, G, d_{\tau_{\perp_{\triangle_Y}}})$. *By Theorem 5,* $\mathcal{F}(d_{\tau_{\perp_{\triangle_X}}}, F, G, d_{\tau_{\perp_{\triangle_Y}}})$ *forms a complete lattice as a sense in [17].*

Example 1. *Let* $X = \{x_1, x_2, x_3\}$ *be a set of objects* x_i *and* $Y = \{y_1, y_2, y_3, y_4\}$ *be a set of attributes* y_j. *Define a binary operation* $\oplus$ *on* $L = [0, 1]$ *by*

$$x \oplus y = \min\{1, x + y\},\ x \ominus y = \max\{x - y, 0\}.$$

Then $(L = [0,1], \oplus, \ominus, 0, 1)$ *is a complete coresiduated lattice.*

Let (X, Y, R) *be an information system with* $R \in [0,1]^{X \times Y}$ *as follows:*

$$\begin{pmatrix} R & y_1 & y_2 & y_3 & y_4 \\ x_1 & 0.1 & 0.4 & 0.5 & 0.2 \\ x_2 & 0.3 & 0.6 & 0.4 & 0.5 \\ x_3 & 0.5 & 0.5 & 0.3 & 0.6 \end{pmatrix}$$

(1) Let (X, d_X) *and* (Y, d_Y) *be distance spaces as follows:*

$$d_X = \begin{pmatrix} 0 & 0.4 & 0.5 \\ 0.6 & 0 & 0.3 \\ 0.4 & 0.2 & 0 \end{pmatrix} d_Y = \begin{pmatrix} 0 & 0.3 & 0.5 & 0.2 \\ 0.5 & 0 & 0.2 & 0.5 \\ 0.3 & 0.2 & 0 & 0.5 \\ 0.6 & 0.3 & 0.5 & 0 \end{pmatrix}$$

Then $\tau_{d_X} = \{\bigwedge_{x \in X}(A(x) \oplus d_X(x, -)) \mid A \in L^X\}$ *and* $\tau_{d_Y} = \{\bigwedge_{y \in Y}(B(y) \oplus d_Y(y, -)) \mid B \in L^Y\}$ *are Alexandrov topologies on X and Y, respectively.*

Since $d_X \circ R \circ d_Y = R$, *is a residuated frame, by Theorem 2,* $(d_{\tau_{d_X}}, F, G, d_{\tau_{d_Y}})$ *is a residuated connection where*

$$F(A)(y) = \bigwedge_{x \in X}(R(x,y) \oplus A(x)),$$
$$G(B)(x) = \bigvee_{y \in X}(B(y) \ominus R(x,y)).$$

Moreover, $F((d_X)_x)(y) = R(x,y), G(((d_Y)^y)^*)(x) = R^*(x,y)$, $F((R^*(-, y_2))(y_4) = \bigwedge_{x \in X}(R(x,y_4) \oplus R^*(x,y_2)) = 0.8, G(R(x_3, -))(x_1) = \bigvee_{y \in Y}(R(x_3,y) \ominus R(x_1,y)) = 0.4$.

For $A_1 = (0.5, 0.1, 0.7) \notin \tau_{d_X}$, $0.4 = A_1(x_2) \oplus d_X(x_2, x_3) \not\geq A_1(x_3) = 0.7$. *Then* $A_1 \notin \tau_{d_X}$. *But,* $A = (0.5, 0.1, 0.4) = \bigwedge_{x \in X}(A_1(x) \oplus d_X(x, -)) \in \tau_{d_X}$ *and* $F(A) = (0.4, 0.7, 0.5, 0.6) \in \tau_{d_Y}$.

Then a fuzzy concept lattice $\mathcal{F}(\tau_{d_X}, F, G, \tau_{d_Y})$ *is defined as*

$$\mathcal{F}(d_{\tau_{d_X}}, F, G, d_{\tau_{d_Y}})$$
$$= \{(A, B) \in \tau_{d_X} \times \tau_{d_X} \mid F(A) = B, G(B) = A\},$$
$$(A_1, B_1) \leq (A_2, B_2) \text{ iff } A_1 \leq A_2 (\text{or } B_1 \leq B_2).$$

Since $A \neq G(F(A)) = (0.4, 0.1, 0.2)$,

$$(A, F(A)) \notin \mathcal{F}(d_{\tau_{d_X}}, F, G, d_{\tau_{d_Y}}),$$
$$(G(F(A)), F(A)) \in \mathcal{F}(d_{\tau_{d_X}}, F, G, d_{\tau_{d_Y}}),$$

$$((d_X)_{x_1}, F((d_X)_{x_1}))$$
$$= ((0, 0.6, 0.4), (0.1, 0.4, 0.5, 0.2)) \notin \mathcal{F}(d_{\tau_{d_X}}, F, G, d_{\tau_{d_Y}}),$$
$$(G(F((d_X)_{x_1})), F((d_X)_{x_1}))$$
$$= ((0, 0.1, 0.2), (0.1, 0.4, 0.5, 0.2)) \in \mathcal{F}(d_{\tau_{d_X}}, F, G, d_{\tau_{d_Y}}),$$

$$((d_X)_{x_2}, F((d_X)_{x_2}))$$
$$= ((0.4, 0, 0.2), (0.3, 0.6, 0.4, 0.5)) \notin \mathcal{F}(d_{\tau_{d_X}}, F, G, d_{\tau_{d_Y}}),$$
$$(G(F((d_X)_{x_2})), F((d_X)_{x_2}))$$
$$= ((0.3, 0, 0.1), (0.3, 0.6, 0.4, 0.5)) \in \mathcal{F}(d_{\tau_{d_X}}, F, G, d_{\tau_{d_Y}}),$$

$$((d_X)_{x_3}, F((d_X)_{x_3}))$$
$$= ((0.5, 0.3, 0), (0.5, 0.5, 0.3, 0.6)) \notin \mathcal{F}(d_{\tau_{d_X}}, F, G, d_{\tau_{d_Y}}),$$
$$(G(F((d_X)_{x_3})), F((d_X)_{x_3}))$$
$$= ((0.4, 0.2, 0), (0.5, 0.5, 0.3, 0.6)) \in \mathcal{F}(d_{\tau_{d_X}}, F, G, d_{\tau_{d_Y}}),$$

$$(G(F((d_X)_{x_1})) \vee G(F((d_X)_{x_2})), F((d_X)_{x_1}) \vee F((d_X)_{x_2}))$$
$$= ((0.3, 0.1, 0.2), (0.3, 0.6, 0.5, 0.5)) \notin \mathcal{F}(d_{\tau_{d_X}}, F, G, d_{\tau_{d_Y}}),$$
$$(G(F((d_X)_{x_1})) \vee G(F((d_X)_{x_2})), F(G(F((d_X)_{x_1})) \vee G(F((d_X)_{x_2}))))$$
$$= ((0.3, 0.1, 0.2), (0.4, 0.7, 0.5, 0.5)) \in \mathcal{F}(d_{\tau_{d_X}}, F, G, d_{\tau_{d_Y}}),$$

$$(G(F((d_X)_{x_1})) \wedge G(F((d_X)_{x_3})), F((d_X)_{x_1}) \wedge F((d_X)_{x_3}))$$
$$= ((0, 0.1, 0), (0.1, 0.4, 0.3, 0.2)) \notin \mathcal{F}(d_{\tau_{d_X}}, F, G, d_{\tau_{d_Y}}),$$
$$(G(F((d_X)_{x_1}) \wedge F((d_X)_{x_3})), F((d_X)_{x_1}) \wedge F((d_X)_{x_3}))$$
$$= ((0, 0, 0), (0.1, 0.4, 0.3, 0.2)) \in \mathcal{F}(\tau_{d_X}, F, G, \tau_{d_Y}).$$

(2) For an information system (X, Y, R), we can obtain two distances $d^1_X : X \times X \to L$ and $d^1_Y : Y \times Y \to L$ as

$$d^1_X(x, z) = \bigvee_{y \in Y}(R(x, y) \ominus R(z, y)),$$
$$d^1_Y(y, w) = \bigvee_{x \in X}(R(x, y) \ominus R(x, w)).$$

as follows:

$$d^1_X = \begin{pmatrix} 0 & 0.1 & 0.2 \\ 0.3 & 0 & 0.1 \\ 0.4 & 0.2 & 0 \end{pmatrix} d^1_Y = \begin{pmatrix} 0 & 0.3 & 0.4 & 0.2 \\ 0 & 0 & 0.1 & 0.1 \\ 0.2 & 0.2 & 0 & 0.3 \\ 0 & 0.2 & 0.3 & 0 \end{pmatrix}$$

Then $\mathcal{F}(d_{\tau_{d^1_X}}, F, G, d_{\tau_{d^1_Y}})$ is a fuzzy concept lattice where F, G in (1) and

$$\mathcal{F}(d_{\tau_{d^1_X}}, F, G, d_{\tau_{d^1_Y}})$$
$$= \{(A, B) \in \tau_{d^1_X} \times \tau_{d^1_Y} \mid F(A) = B, G(B) = A\},$$
$$(A_1, B_1) \leq (A_2, B_2) \textit{ iff } A_1 \leq A_2 (\textit{or } B_1 \leq B_2).$$

For $A_1 = (0.5, 0.1, 0.7) \notin \tau_{d^1_X}$, $C = (0.4, 0.1, 0.2) = \bigwedge_{x \in X}(A_1(x) \oplus d^1_X(x, -)) \in \tau_{d^1_X}$ and $F(C) = (0.4, 0.7, 0.5, 0.6) \in \tau_{d^1_Y}$.

Since $C = G(F(C)) = (0.4, 0.1, 0.2)$,

$$(C, F(C)) \in \mathcal{F}(d_{\tau_{d^1_X}}, F, G, d_{\tau_{d^1_Y}}),$$

For $B_1 = (0.3, 0.6, 0.1) \notin \tau_{d^1_X}$, $D = (0.3, 0.3, 0.1) = \bigwedge_{x \in X}(B_1(x) \oplus d^1_X(x, -)) \in \tau_{d^1_X}$ and $F(D) = (0.4, 0.6, 0.4, 0.5) \in \tau_{d^1_Y}$.

Since $D \neq G(F(D)) = (0.3, 0.1, 0.1)$,

$$(D, F(D)) \notin \mathcal{F}(d_{\tau_{d_X}}, F, G, d_{\tau_{d_Y}}),$$
$$(G(F(D)), F(D)) \in \mathcal{F}(d_{\tau_{d_X}}, F, G, d_{\tau_{d_Y}}),$$

$$((d^1_X)_{x_1}, F((d^1_X)_{x_1}))$$
$$= ((0, 0.1, 0.2), (0.1, 0.4, 0.5, 0.2)) \in \mathcal{F}(d_{\tau_{d^1_X}}, F, G, d_{\tau_{d^1_Y}}),$$

$$((d^1_X)_{x_2}, F((d^1_X)_{x_2}))$$
$$= ((0.3, 0, 0.1), (0.3, 0.6, 0.4, 0.5)) \in \mathcal{F}(d_{\tau_{d^1_X}}, F, G, d_{\tau_{d^1_Y}}),$$

$$(d^1_X)_{x_3}, F((d^1_X)_{x_3}))$$
$$= ((0.4, 0.2, 0), (0.5, 0.5, 0.3, 0.6)) \in \mathcal{F}(d_{\tau_{d^1_X}}, F, G, d_{\tau_{d^1_Y}}),$$

$$
\begin{aligned}
&((d^1_X)_{x_1} \vee (d^1_X)_{x_2}, F((d^1_X)_{x_1}) \vee F((d^1_X)_{x_2})) \\
&= ((0.3, 0.1, 0.2), (0.3, 0.6, 0.5, 0.5)) \notin \mathcal{F}(d_{\tau_{d^1_X}}, F, G, d_{\tau_{d^1_Y}}), \\
&((d^1_X)_{x_1} \vee (d^1_X)_{x_2}, F((d^1_X)_{x_1} \vee (d^1_X)_{x_2})) \\
&= ((0.3, 0.1, 0.2), (0.4, 0.7, 0.5, 0.5)) \in \mathcal{F}(d_{\tau_{d^1_X}}, F, G, d_{\tau_{d^1_Y}}),
\end{aligned}
$$

$$
\begin{aligned}
&((d^1_X)_{x_1} \wedge (d^1_X)_{x_3}, F((d^1_X)_{x_1}) \wedge F((d^1_X)_{x_3})) \\
&= ((0, 0.1, 0), (0.1, 0.4, 0.3, 0.2)) \notin \mathcal{F}(d_{\tau_{d^1_X}}, F, G, d_{\tau_{d^1_Y}}), \\
&(G(F((d^1_X)_{x_1}) \wedge F((d^1_X)_{x_3})), F((d^1_X)_{x_1}) \wedge F((d^1_X)_{x_3})) \\
&= ((0, 0, 0), (0.1, 0.4, 0.3, 0.2)) \in \mathcal{F}(d_{\tau_{d^1_X}}, F, G, d_{\tau_{d^1_Y}}).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&F(R^*(-, y_1))(y_3) = F((0.9, 0.7, 0.5))(y_3) \\
&= (d^1_Y)^*(y_3, y_1) = 0.8, \\
&G(R(x_2, -))(x_3) = G((0.3, 0.6, 0.4, 0.5))(x_3) \\
&= d^1_X(x_2, x_3) = 0.1.
\end{aligned}
$$

(3) Let $(X, \perp_{\triangle_X})$ and $(Y, \perp_{\triangle_Y})$ be distance spaces with $\tau_{\perp_{\triangle_X}} = L^X$ and $\tau_{\perp_{\triangle_Y}} = L^Y$. Define $f : X \to Y$ be a function as

$$f(x_1) = f(x_2) = y_1,\ f(x_3) = y_2.$$

By Remark 2 (2),

$$F(A)(y) = \begin{cases} \bigwedge_{x \in f^{-1}(\{y\})} A(x), & \text{if } y \in f(X), \\ \top, & \text{if } y \notin f(X), \end{cases}$$

and $G(B)(x) = B(f(x))$ for each $x \in X$.

For $A_1 = (0.5, 0.1, 0.7) \in \tau_{\perp_{\triangle_X}} = L^X$,

$$
\begin{aligned}
&F(A_1) = (0.1, 1, 0.7, 1) \in \tau_{\perp_{\triangle_Y}} = L^Y, \\
&G(F(A_1)) = (0.1, 0.1, 0.7) \in L^X, \\
&(A_1, F(A_1)) \notin \mathcal{F}(d_{L^X}, F, G, d_{L^X}), \\
&(G(F(A_1)), F(A_1)) \in \mathcal{F}(d_{L^X}, F, G, d_{L^X}).
\end{aligned}
$$

4. Various Embedding Maps Based on Alexandrov Topologies

In this section, we investigate the relations between *R*-*R* embedding maps and *R*-*R* frame embedding maps.

Theorem 6. *Let (X, d_X) be a distance space, and τ_{d_X} and $\tau_{d_{\tau_{d_X}}} = \{\alpha \in L^{\tau_{d_X}} \mid \alpha(A) \oplus d_{\tau_{d_X}}(A, B) \geq \alpha(B)\}$ be Alexandrov topologies. Define a map $k_X : X \to \tau_{d_{\tau_{d_X}}}$ as $k_X(x)(A) = \hat{x}(A) = A(x)$. Then $k_X : (X, d_X) \to (\tau_{d_{\tau_{d_X}}}, d_{\tau_{d_{\tau_{d_X}}}})$ is an embedding map.*

Proof. We show $k_X(x) = \hat{x} \in \tau_{d_{\tau_{d_X}}}$ from

$$
\begin{aligned}
&\hat{x}(A) \oplus d_{\tau_{d_X}}(A, B) \\
&= \hat{x}(A) \oplus \bigvee_{y \in X}(B(y) \ominus A(y)) \\
&\geq A(x) \oplus (B(x) \ominus A(x)) \geq B(x) = \hat{x}(B),
\end{aligned}
$$

If $k_X(x)(A) = k_X(y)(A)$ for each $A \in \tau_{d_X}$, then $k_X(x)((d_X)_x) = k_X(y)((d_X)_x) = d_X(x, y) = \perp$ for $(d_X)_x \in \tau_{d_X}$, and $k_X(x)((d_X)_y) = k_X(y)((d_X)_y) = d_X(y, x) = \perp$ for $(d_X)_y \in \tau_{d_X}$. Thus $x = y$. Hence k_X is injective.

For each $A \in \tau_{d_X}$, since $A(x) \oplus d_X(x, y) \geq A(y)$, we have

$$d_X(x,y) \geq \bigvee_{A\in\tau_{d_X}}(A(y)\ominus A(x))$$
$$= \bigvee_{A\in\tau_{d_X}}(\hat{y}(A)\ominus \hat{x}(A)) = d_{\tau_{d_{\tau_{d_X}}}}(\hat{x},\hat{y}).$$

Put $(d_X)_z(x) = d_X(z,x)$. Since $(d_X)_z(x)\oplus d_X(x,y) \geq (d_X)_z(y)$, then $(d_X)_z \in \tau_{d_X}$ for all $z \in X$. Hence

$$d_{\tau_{d_{\tau_{d_X}}}}(\hat{x},\hat{y}) = \bigvee_{A\in\tau_{d_X}}(A(y)\ominus A(x))$$
$$\geq \bigvee_{(d_X)_z\in\tau_{d_X}}((d_X)_z(y)\ominus (d_X)_z(x))$$
$$= \bigvee_{z\in X}(d_X(z,y)\ominus d_X(z,x)) = d_X(x,y).$$

Thus, $d_{\tau_{d_{\tau_{d_X}}}}(\hat{x},\hat{y}) = d_X(x,y)$. □

Definition 7. *Let (d_X,R,d_Y) and $(d_Z,\tilde{R},d_W)$ be residuated frames. Let $k_X : (X,d_X)\to(Z,d_Z)$ and $k_Y : (Y,d_Y)\to(W,d_W)$ be maps. An injective function $k = (k_X,k_Y) : (d_X,R,d_Y)\to(d_Z,\tilde{R},d_W)$ is called R-R frame embedding if*

$$d_X(x,z) = d_Z(k_X(x),k_X(z)),$$
$$d_Y(y,w) = d_W(k_Y(y),k_Y(w)),$$
$$R(x,y) = \tilde{R}(k_X(x),k_Y(y)).$$

Lemma 4. *Let (X,d_X) and (Y,d_Y) be distance spaces, and $f : X\to Y$ and $g : Y\to X$ be two maps. Then the following holds.*

(1) If (d_X,f,g,d_Y) is a residuated connection and a define map $R : X\times Y\to L$ as

$$R(x,y) = d_X(x,g(y)) = d_Y(f(x),y).$$

Then (d_X,R,d_Y) is a residuated frame.

(2) If g is isotone and $R_2(x,y) = d_X(x,g(y))$, then $d_X\circ R_2\circ d_Y = R_2$.

(3) If f is isotone and $R_1(x,y) = d_Y(f(x),y)$, then $d_X\circ R_1\circ d_Y = R_1$.

Proof. (1) For each $x,x_1\in X, y,y_1\in Y$,

$$d_X(x,x_1)\oplus R(x_1,y_1)\oplus d_Y(y_1,y)$$
$$= d_X(x,x_1)\oplus d_X(x_1,g(y_1))\oplus d_Y(y_1,y)$$
$$\geq d_X(x,g(y_1))\oplus d_X(y_1,y)$$
$$= d_Y(f(x),y_1)\oplus d_Y(y_1,y)$$
$$\geq d_Y(f(x),y) = R(x,y).$$

Hence $d_X\circ R\circ d_Y \geq R$. Easily, $d_X\circ R\circ d_Y \leq R$.

(2) For each $x,x_1\in X, y,y_1\in Y$,

$$d_X(x,x_1)\oplus R_2(x_1,y_1)\oplus d_Y(y_1,y)$$
$$= d_X(x,x_1)\oplus d_X(x_1,g(y_1))\oplus d_Y(y_1,y)$$
$$\geq d_X(x,x_1)\oplus d_X(x_1,g(y_1))\oplus d_X(g(y_1),g(y))$$
$$\geq d_X(x,x_1)\oplus d_X(x_1,g(y))$$
$$\geq d_X(x,g(y)) = R_2(x,y).$$

Hence $d_X\circ R_2\circ d_Y \geq R_2$. Easily, $d_X\circ R_2\circ d_Y \leq R_2$.

(3) is similarly proved as in (2). □

Theorem 7. *Let (X,d_X) and (Y,d_Y) be distance spaces, τ_{d_X} and τ_{d_Y} be Alexandrov topologies and $f : X\to Y$, $g : Y\to X$ be maps. Then the following statements (1), (2) and (3) are equivalent.*

(1) (d_X,f,g,d_Y) is a residuated connection, that is, $d_Y(f(x),y) = d_X(x,g(y))$ for all $x,y\in X$.

(2) There exists a residuated connection $(d_{\tau_X},F,G,d_{\tau_Y})$ such that

$$\begin{aligned} F(A)(y) &= \bigwedge_{z\in X}(d_Y(f(z),y)\oplus A(z)) \\ &= \bigwedge_{z\in X}(d_Y(z,g(y))\oplus A(z)), \\ G(B)(x) &= \bigvee_{y\in Y}(B(y)\ominus d_Y(f(x),y)) \\ &= \bigvee_{w\in Y}(B(w)\ominus d_X(x,g(w))) \end{aligned}$$

with isotone maps $f: X\to Y$, $g: Y\to X$.

(3) *There exists a relation* $R: \tau_{d_X}\times\tau_{d_Y}\to L$ *as*

$$\begin{aligned} R(A,B) &= \bigvee_{x\in X}(B(f(x))\ominus A(x)) \\ &= \bigvee_{y\in Y}(B(y)\ominus A(g(y))) \end{aligned}$$

with isotone maps $f: X\to Y$, $g: Y\to X$ *such that* $(d_{\tau_{d_X}}, R, d_{\tau_{d_Y}})$ *is a residuated frame. Moreover,*

$$\begin{aligned} &R(A,B) = d_{\tau_{d_X}}(A, f^{\leftarrow}(B)) = d_{\tau_{d_Y}}(g^{\leftarrow}(A), B) \\ &= d_{\tau_{d_Y}}(F(A), B) = d_{\tau_{d_X}}(A, G(B)) \end{aligned}$$

where $f^{\leftarrow}(B)(x) = B(f(x)), g^{\leftarrow}(A)(y) = A(g(y))$, *for each* $x\in X, y\in Y$, F *and* G *in (2).*

Proof. (1) $\Rightarrow$ (2). Put $R_1(x,y) = d_Y(f(x),y) = d_X(x,g(y))$ for all $x,y\in X$. By Lemma 4 (1), $d_X\circ R_1\circ d_Y = R_1$. By Theorem 2, there exists a residuated connection $(d_{\tau_X}, F, G, d_{\tau_Y})$, where

$$\begin{aligned} F(A)(y) &= \bigwedge_{z\in X}(d_Y(f(z),y)\oplus A(z)) \\ &= \bigwedge_{z\in X}(d_Y(z,g(y))\oplus A(z)), \\ G(B)(x) &= \bigvee_{y\in Y}(B(y)\ominus d_Y(f(x),y)) \\ &= \bigvee_{w\in Y}(B(w)\ominus d_X(x,g(w))). \end{aligned}$$

Since $\bot = d_Y(f(z),f(z)) = d_X(z,g(f(z)))$ and $\bot = d_X(g(y),g(y)) = d_Y(f(g(y)),y)$,

$$\begin{aligned} &d_Y(f(x),f(z)) = d_X(x,g(f(z))) \\ &\le d_X(x,z)\oplus d_X(z,g(f(z))) = d_X(x,z), \\ &d_X(g(y),g(w)) = d_Y(f(g(y)),w) \\ &\le d_Y(f(g(y)),y)\oplus d_Y(y,w) = d_X(y,w). \end{aligned}$$

Hence $f: X\to Y$, $g: Y\to X$ are isotone maps.

(2) $\Rightarrow$ (1). By Lemma 3 (1), since $\bigwedge_{z\in X}(d_X(z,g(w))\oplus(d_X)_x(z)) = d_X\circ R_1(x,w) = R_1(x,w) = d_X(x,g(w))$ and $\bigwedge_{w\in Y}(d_Y(w,y)\oplus d_X(x,g(w))) = R_1\circ d_Y(x,y) = R_1(x,y) = d_X(x,g(y))$, for $(d_X)_x\in\tau_{d_X}$ and $((d_Y)^y)^*\in\tau_{d_Y}$,

$$\begin{aligned} &d_{\tau_Y}(F((d_X)_x),((d_Y)^y)^*) = \bigvee_{w\in Y}(((d_Y)^y)^*(w) \\ &\ominus\bigwedge_{z\in X}(d_X(z,g(w))\oplus(d_X)_x(z))) \\ &= \bigvee_{w\in Y}(((d_Y)^y)^*(w)\ominus d_X(x,g(w))) \\ &\text{(by Lemma 1 (8))} \\ &= (\bigwedge_{w\in Y}(d_Y(w,y)\oplus d_X(x,g(w))))^* \\ &= d_X^*(x,g(y)). \end{aligned}$$

Hence $d_{\tau_Y}(F((d_X)_x),((d_Y)^y)^*) = d_X^*(x,g(y))$ for each $x\in X$ and $y\in Y$.

On the other hand, since $\bigwedge_{w\in Y}(d_Y(w,y)\oplus d_Y(f(x),w)) = R_1\circ d_Y(x,y) = R_1(x,y) = d_Y(f(x),y)$ for $(d_X)_x\in\tau_{d_X}$ and $((d_Y)^y)^*\in\tau_{d_Y}$,

$$
\begin{aligned}
&d_{\tau_Y}((d_X)_x, G(((d_Y)^y)^*))\\
&= \bigvee_{z\in X}(\bigvee_{w\in Y}(((d_Y)^y)^*(w) \ominus d_Y(f(z),w)))\\
&\ominus (d_X)_x(z))\\
&\text{(by Lemma 1 (8))}\\
&= \bigvee_{z\in X}((\bigwedge_{w\in Y}(d_Y(w,y) \oplus d_Y(f(z),w)))^*\\
&\ominus d_X(x,z))\\
&= \bigvee_{z\in X}(d_Y(f(z),y)^* \ominus d_X(x,z))\\
&= (\bigwedge_{z\in X}(d_Y(f(z),y) \oplus d_X(x,z)))^*\\
&= d_Y^*(f(x),y).
\end{aligned}
$$

Hence $d_{\tau_Y}((d_X)_x, G(((d_Y)^y)^*)) = d_Y^*(f(x),y)$ for each $x \in X$ and $y \in Y$. Thus, $d_X^*(x,g(y)) = d_{\tau_X}(F((d_X)_x), ((d_Y)^y)^*) = d_{\tau_Y}((d_X)_x, G(((d_Y)^y)^*)) = d_Y^*(f(x),y)$ implies $d_Y(f(x),y) = d_X(x,g(y))$ for each $x \in X$ and $y \in Y$.

(1) $\Rightarrow$ (3). From the proof of (1) $\Rightarrow$ (2), f and g are isotone maps.

Put $S(A,B) = \bigvee_{y\in Y}(B(y) \ominus A(g(y)))$ for $A \in \tau_{d_X}$ and $B \in \tau_{d_Y}$. Since $B(f(g(y))) = B(f(g(y))) \oplus d_Y(f(g(y)),y) \geq B(y), d_Y(f(g(y)),y) = \bot, A(x) = A(x) \oplus d_X(x,g(f(x))) \geq A(g(f(x)))$ and $d_X(x,g(f(x))) = \bot$, we have $R(A,B) = S(A,B)$ from:

$$
\begin{aligned}
&R(A,B) = \bigvee_{x\in X}(B(f(x)) \ominus A(x))\\
&\geq \bigvee_{y\in Y}(B(f(g(y))) \ominus A(g(y)))\\
&= \bigvee_{y\in Y}((B(f(g(y))) \oplus d_Y(f(g(y)),y)) \ominus A(g(y)))\\
&\geq \bigvee_{y\in Y}(B(y) \ominus A(g(y))) = S(A,B),
\end{aligned}
$$

$$
\begin{aligned}
&S(A,B) = \bigvee_{y\in Y}(B(y) \ominus A(g(y)))\\
&\geq \bigvee_{x\in X}(B(f(x)) \ominus A(g(f(x))))\\
&\geq \bigvee_{y\in X}(B(f(x)) \ominus (A(x) \oplus d_X(x,g(f(x)))))\\
&= \bigvee_{y\in X}(B(f(x)) \ominus A(x)) = R(A,B).
\end{aligned}
$$

For $A \in \tau_{d_X}$ and $B^* \in \tau_{d_Y^{-1}}$, $A = \bigwedge_{z\in X}(A(z) \oplus d_X(z,-))$ and $B^* = \bigwedge_{y\in Y}(B^*(y) \oplus d_Y(-,y))$ implies $B = \bigvee_{y\in Y}(d_Y^*(-,y) \ominus B^*(y))$. Thus,

$$
\begin{aligned}
&R(A,B) = \bigvee_{x\in X}(B(f(x)) \ominus A(x))\\
&= \bigvee_{x\in X}(\bigvee_{y\in Y}(d_Y^*(f(x),y) \ominus B^*(y))\\
&\ominus \bigwedge_{z\in X}(A(z) \oplus d_X(z,x)))\\
&= \bigvee_{y\in Y}\bigvee_{z\in X}(\bigvee_{x\in X}(d_Y^*(f(x),y) \ominus d_X(z,x)))\\
&\ominus (A(z) \oplus B^*(y)))\\
&= \bigvee_{y\in Y}\bigvee_{z\in X}((\bigwedge_{x\in X}(d_Y(f(x),y) \oplus d_X(z,x))^*\\
&\ominus (A(z) \oplus B^*(y)))\\
&((d_X \circ R_1)(z,y) = R_1(z,x) = d_Y(f(z),y))\\
&= \bigvee_{y\in Y}\bigvee_{z\in X}(d_Y^*(f(z),y)) \ominus (A(z) \oplus B^*(y)))\\
&= \bigvee_{y\in Y}(\bigvee_{z\in X}(d_Y^*(f(z),y) \ominus A(z)) \ominus B^*(y)))\\
&= \bigvee_{y\in Y}(B(y) \ominus \bigwedge_{z\in X}(d_Y(f(z),y) \oplus A(z)))\\
&= d_{\tau_{d_Y}}(F(A),B)\\
&= \bigvee_{z\in X}(\bigvee_{y\in Y}(d_Y^*(f(z),y) \ominus B^*(y)) \ominus A(z))\\
&= \bigvee_{z\in X}(\bigvee_{y\in Y}(B(y) \ominus d_Y(f(z),y)) \ominus A(z))\\
&= d_{\tau_{d_X}}(A, G(B)).
\end{aligned}
$$

(3) $\Rightarrow$ (1). Since $\bigwedge_{z\in X}(d_Y(f(z),y) \oplus d_X(x,z)) = d_X \circ R_1(x,y) = R_1(x,y) = d_Y(f(x),y)$, for $(d_X)_x \in \tau_{d_X}$ and $((d_Y)^y)^* \in \tau_{d_Y}$,

$$\begin{aligned}&R((d_X)_x,((d_Y)^y)^*)\\&=\bigvee_{z\in X}(((d_Y)^y)^*(f(z))\ominus(d_X)_x(z))\\&=(\bigwedge_{z\in X}(d_Y(f(z),y)\oplus d_X(x,z))^*=d_Y^*(f(x),y).\end{aligned}$$

Since $\bigwedge_{z\in X}(d_Y(z,y)\oplus d_X(x,g(z)))=(R_1\circ d_Y)(x,y)=R_1(x,y)=d_X(x,g(y))$, for $(d_X)_x\in\tau_{d_X}$ and $((d_Y)^y)^*\in\tau_{d_Y}$,

$$\begin{aligned}&R((d_X)_x,((d_Y)^y)^*)\\&=\bigvee_{z\in X}(((d_Y)^y)^*(z)\ominus(d_X)_x(g(z)))\\&=(\bigwedge_{z\in X}(d_Y(z,y)\oplus d_X(x,g(z))))^*\qquad=d_X(x,g(y))^*\end{aligned}$$

Hence, for $(d_X)_x\in\tau_{d_X}$ and $((d_Y)^y)^*\in\tau_{d_Y}$,

$$R((d_X)_x,((d_Y)^y)^*)=d_Y^*(f(x),y)=d_X^*(x,g(y)).$$

Thus $d_Y(f(x),y)=d_X(x,g(y))$ for all $x,y\in X$. □

Example 2. *Define a binary operation $\oplus$ on $[0,1]$ by*

$$x\oplus y=\min\{x+y,1\},\ x\ominus y=\max\{x-y,0\}.$$

(1) Let $d_{[0,1]}:[0,1]\times[0,1]\to[0,1]$ be a map as follows:

$$d_{[0,1]}(x,y)\ =\begin{cases}0, & \text{if } x\geq y,\\ 1, & \text{if } x<y.\end{cases}$$

By Lemma 1 (6), $d_{[0,1]}:[0,1]\times[0,1]\to[0,1]$ is a distance. Then

$$\begin{aligned}&\tau_{d_{[0,1]}}\\&=\{A\in[0,1]^{[0,1]}\mid\bigwedge_{x\in X}(A(x)\oplus d_{[0,1]}(x,-))=A\}\\&=\{A\in[0,1]^{[0,1]}\mid\bigwedge_{x\geq y}A(x)=A(y),\forall y\in Y\}\\&=\{A\in[0,1]^{[0,1]}\mid A(x)\geq A(y),\forall x\geq y\}.\end{aligned}$$

For $a\in[0,1]$, define $f_a,g_a:[0,1]\to[0,1]$ as

$$f_a(x)=x\oplus a,\ \ g_a(x)=x\ominus a.$$

Since $f_a(x)=x\oplus a\geq y$ iff $x\geq g_a(y)=y\ominus a$, $d_{[0,1]}(f_a(x),y)=d_{[0,1]}(x,g_a(y))$; i.e., $(d_{[0,1]},f_a,g_a,d_{[0,1]})$ is a residuated connection. There exists a residuated connection $(d_{\tau_{d_{[0,1]}}},F,G,d_{\tau_{d_{[0,1]}}})$ such that

$$\begin{aligned}&F(A)(y)=\bigwedge_{f_a(z)\geq y}A(z)\\&=\bigwedge_{z\geq g_a(y)}A(z)=A(y\ominus a),\\&G(B)(x)=\bigvee_{y\leq f_a(x)}B(y)\\&=\bigvee_{x\geq g_a(y)}B(y)=B(f_a(x))=B(x\oplus a).\end{aligned}$$

$$\begin{aligned}&d_{\tau_{d_{[0,1]}}}(F(A),B)=\bigvee_{y\in[0,1]}(B(y)\ominus F(A)(y))\\&=\bigvee_{y\in[0,1]}(B(y)\ominus A(y\ominus a))\\&d_{\tau_{d_{[0,1]}}}(A,G(B))=\bigvee_{x\in[0,1]}(G(B)(x)\ominus A(x))\\&=\bigvee_{x\in[0,1]}(B(x\oplus a)\ominus A(x))\\&R(A,B)=\bigvee_{x\in[0,1]}(B(x\oplus a)\ominus A(x))\\&=\bigvee_{y\in[0,1]}(B(y)\ominus A(y\ominus a))\\&=d_{\tau_{d_{[0,1]}}}(A,f_a^{\leftarrow}(B))=d_{\tau_{d_{[0,1]}}}(g_a^{\leftarrow}(A),B)\\&=d_{\tau_{d_{[0,1]}}}(F(A),B)=d_{\tau_{d_{[0,1]}}}(A,G(B)).\end{aligned}$$

For $A(x) = 0.6, B(x) = x$ *for each* $x \in [0,1]$, *then* $A, B \in \tau_{d_{[0,1]}}$. *Put* $f_{0.3}(x) = x \oplus 0.3$, $g_{0.3}(x) = x \ominus 0.3$. *Since* $F(A)(y) = A(y \ominus 0.3) = g^{\leftarrow}_{0.3}(A)(y), G(B)(x) = B(x \oplus 0.3) = f^{\leftarrow}_{0.3}(B)(x), F(A)(y) = 0.6$,

$$G(B)(x) = \begin{cases} x + 0.3, & \text{if } 0 \le x \le 0.7, \\ 1, & \text{if } x > 0.7, \end{cases}$$

$$R(A,B) = d_{\tau_{d_{[0,1]}}}(A, f^{\leftarrow}_{0.3}(B)) = d_{\tau_{d_{[0,1]}}}(g^{\leftarrow}_{0.3}(A), B)$$
$$= d_{\tau_{d_{[0,1]}}}(F(A), B) = d_{\tau_{d_{[0,1]}}}(A, G(B)) = 0.4.$$

(2) Let $d : [0,1] \times [0,1] \to [0,1]$ *be a distance as* $d(x,y) = x \ominus y$. *Then* $\tau_d = \{A \in [0,1]^{[0,1]} \mid \bigwedge_{x \in X}(A(x) \oplus d(x,-)) = A\}$.

Define $g : [0,1] \to [0,1]$ *as* $g(x) = \frac{1}{2}x$. *Then* $d(g(x), g(y)) = \frac{1}{2}x \ominus \frac{1}{2}y \le d(x,y) = x \ominus y$; *i.e., g is an isotone map. Put* $R_2(x,y) = d(x, g(y))$. *By Lemma 4 (2),* (d, R_2, d) *is a residuated frame. Then* $(d_{\tau_d}, F_1, G_1, d_{\tau_d})$ *is a residuated connection where*

$$F_1(A)(y) = \bigwedge_{x \in X}(A(x) \oplus (x \ominus \tfrac{1}{2}y)),$$
$$G_1(B)(x) = \bigvee_{y \in Y}(B(y) \ominus (x \ominus \tfrac{1}{2}y)).$$

Theorem 8. *Let* (d_X, R, d_Y) *be a residuated frame,* τ_{d_X} *and* τ_{d_Y} *be Alexandrov topologies on X and Y and* $\tau_{d_{\tau_{d_X}}}$ *and* $\tau_{d_{\tau_{d_Y}}}$ *be Alexandrov topologies on* τ_{d_X} *and* τ_{d_Y}, *respectively. Define a map* $k = (k_X, k_Y) : X \times Y \to \tau_{d_{\tau_{d_X}}} \times \tau_{d_{\tau_{d_Y}}}$ *as* $k(x,y)(A \times B) = (\hat{x}(A), \hat{y}(B)) = (A(x), B(y))$. *Then* $k : (d_X, R, d_Y) \to (d_{\tau_{d_{\tau_{d_X}}}}, \hat{R}, d_{\tau_{d_{\tau_{d_Y}}}})$ *is an R-R frame embedding map where*

$$F(A)(y) = \bigwedge_{x \in X}(R(x,y) \oplus A(x)),$$
$$G(B)(x) = \bigvee_{y \in Y}(B(y) \ominus R(x,y)),$$
$$\hat{R}(\alpha, \beta) = \bigvee_{A \in \tau_{d_X}}(\beta(F(A)) \ominus \alpha(A)),$$
$$= \bigvee_{B \in \tau_{d_Y}}(\beta(B) \ominus \alpha(G(B))).$$

Proof. By Theorem 6, $k_X : (X, d_X) \to (\tau_{d_{\tau_{d_X}}}, d_{\tau_{d_{\tau_{d_X}}}})$ and $k_Y : (Y, d_Y) \to (\tau_{d_{\tau_{d_Y}}}, d_{\tau_{d_{\tau_{d_Y}}}})$ are embedding maps. Hence $d_X(x,z) = d_{\tau_{d_{\tau_{d_X}}}}(\hat{x}, \hat{z})$ and $d_Y(y,w) = d_{\tau_{d_{\tau_{d_Y}}}}(\hat{y}, \hat{w})$. By Theorem 2, $(d_{\tau_{d_X}}, F, G, d_{\tau_{d_Y}})$ is a residuated connection where

$$F(A)(y) = \bigwedge_{x \in X}(R(x,y) \oplus A(x)),$$
$$G(B)(x) = \bigvee_{y \in X}(B(y) \ominus R(x,y)).$$

By Theorem 7 (3), $(d_{\tau_{d_{\tau_{d_X}}}}, \hat{R}, d_{\tau_{d_{\tau_{d_X}}}})$ is a residuated frame where

$$\hat{R}(\alpha, \beta) = \bigvee_{A \in \tau_{d_X}}(\beta(F(A)) \ominus \alpha(A))$$
$$= \bigvee_{B \in \tau_{d_X}}(\beta(B) \ominus \alpha(G(B))).$$

For each $\hat{x} \in \tau_{d_{\tau_{d_X}}}, \hat{y} \in \tau_{d_{\tau_{d_Y}}}$,

$$\hat{R}(\hat{x}, \hat{y}) = \bigvee_{A \in \tau_{d_X}}(\hat{y}(F(A)) \ominus \hat{x}(A))$$
$$= \bigvee_{A \in \tau_{d_X}}(F(A)(y) \ominus A(x))$$
$$= \bigvee_{A \in \tau_{d_X}}\Big(\bigwedge_{z \in X}(R(z,y) \oplus A(z)) \ominus A(x)\Big)$$
$$\le \bigvee_{A \in \tau_{d_X}}\Big((R(x,y) \oplus A(x)) \ominus A(x)\Big)$$
$$\le R(x,y).$$

https://doi.org/10.3390/axioms15050336

Put $(d_X)_x(z) = d_X(x,z)$. Then $(d_X)_x \in \tau_{d_X}$. Since $d_X \circ R(x,y) = \bigwedge_{z\in X}(d_X(x,z) \oplus R(z,y)) = R(x,y)$,

$$\begin{aligned}\hat{R}(\hat{x},\hat{y}) &= \bigvee_{A\in\tau_{d_X}} \left(\bigwedge_{z\in X}(R(z,y)\oplus A(z)) \ominus A(x)\right)\\ &\geq \left(\bigwedge_{z\in X}(R(z,y)\oplus (d_X)_x(z)) \ominus (d_X)_x(x)\right)\\ &= R(x,y).\end{aligned}$$

Thus $\hat{R}(\hat{x},\hat{y}) = R(x,y)$ for each $x,y \in X$. □

Example 3. *Let* $(X = \{x_1,x_2,x_3\}, d_X)$ *be a distance space with*

$$d_X = \begin{pmatrix} 0 & 0.4 & 0.5\\ 0.4 & 0 & 0.3\\ 0.5 & 0.3 & 0\end{pmatrix}.$$

Define a binary operation $\oplus$ *on* $[0,1]$ *by*

$$x\oplus y = \min\{x+y,1\},\ x\ominus y = \max\{x-y,0\}.$$

Put

$$R = \begin{pmatrix} 0.3 & 0.6 & 0.7\\ 0.4 & 0.2 & 0.5\\ 0.7 & 0.5 & 0.2\end{pmatrix}.$$

Since (d_X,R,d_X) *is residuated frame, by Theorem 2,* $(d_{\tau_{d_X}},F,G,d_{\tau_{d_X}})$ *is a residuated connection where*

$$\begin{aligned}F(A)(y) &= \bigwedge_{x\in X}(R(x,y)\oplus A(x)),\\ G(B)(x) &= \bigvee_{y\in X}(B(y)\ominus R(x,y)).\end{aligned}$$

For $(d_X)_{x_1} = (0,0.4,0.5)$, $((d_X)^{x_2})^* = (0.6,1,0.7) \in \tau_{d_X}$,

$$\begin{aligned}&F((d_X)_{x_1}) = \bigwedge_{x\in X}(R(x,-)\oplus (d_X)_{x_1}(x))\\ &= R(x_1,-) = (0.3,0.6,0.7),\\ &G(((d_X)^{x_2})^*) = \bigvee_{y\in X}(((d_X)^{x_2})^*(y)\ominus R(-,y))\\ &= R^*(-,x_2) = (0.4,0.8,0.5),\\ &d_{\tau_{d_X}}(F((d_X)_{x_1}),((d_X)^{x_2})^*)\\ &= d_{\tau_{d_X}}((d_X)_{x_1},G(((d_X)^{x_2})^*)) = 0.4.\end{aligned}$$

By Theorem 7 (3), $(d_{\tau_{d_{\tau_{d_X}}}},\hat{R},d_{\tau_{d_{\tau_{d_X}}}})$ *is a residuated frame where*

$$\begin{aligned}\hat{R}(\alpha,\beta) &= \bigvee_{A\in\tau_{d_X}}(\beta(F(A))\ominus\alpha(A))\\ &= \bigvee_{B\in\tau_{d_X}}(\beta(B)\ominus\alpha(G(B))).\end{aligned}$$

For $\overline{0.4} \in \tau_{d_{\tau_{d_X}}}$ *defined as* $\overline{0.4}(A) = A$ *for each* $A \in \tau_{d_X}$,

$$\begin{aligned}&\hat{R}(\overline{0.4},\hat{x_1}) = \bigvee_{A\in\tau_{d_X}}(\hat{x_1}(F(A))\ominus\overline{0.4}(A))\\ &= \bigvee_{A\in\tau_{d_X}}(F(A)(x_1)\ominus 0.4))\\ &= \bigvee_{A\in\tau_{d_X}}(\bigwedge_{x\in X}(R(x,x_1)\oplus A(x))\ominus 0.4))\\ &= 1\ominus 0.4 = 0.6.\end{aligned}$$

A map $k = (k_X,k_X) : X\times X \to \tau_{d_{\tau_{d_X}}} \times \tau_{d_{\tau_{d_X}}}$ *as* $k(x,y)(A\times B) = (\hat{x}(A),\hat{y}(B))$ *is an embedding map. But* $k_X(x) \neq \overline{0.4}$ *for all* $x \in X$. *Thus* k_X *is not surjective.*

Moreover, $k : (d_X, R, d_X) \to (d_{\tau_{d_{\tau_{d_X}}}}, \hat{R}, d_{\tau_{d_{\tau_{d_X}}}})$ *is an R-R frame embedding map with* $d_X(x,y) = d_{\tau_{d_{\tau_{d_X}}}}(\hat{x},\hat{y})$, $\hat{R}(\hat{x}_i,\hat{x}_j) = R(x_i,x_j)$ *for each* $x_i, x_j \in X$.

Definition 8. *Let* (d_X, f, g, d_Y) *and* $(d_Z, \tilde{f}, \tilde{g}, d_W)$ *be residuated connections. An injective function* $k : (d_X, f, g, d_Y) \to (d_Z, \tilde{f}, \tilde{g}, d_W)$ *is called an R-R embedding if for each* $x, z \in X$ *and* $y, w \in Y$,

$$\begin{aligned} &d_X(x,z) = d_Z(k(x),k(z)),\\ &d_Y(y,w) = d_W(k(y),k(w)),\\ &d_Y(f(x),y) = d_W(\tilde{f}(k(x)),k(y))\\ &= d_X(x,g(y)) = d_Z(k(x),\tilde{g}(k(y))). \end{aligned}$$

Theorem 9. *Let* (d_X, f, g, d_Y) *be a residuated connection,* τ_{d_X} *and* τ_{d_Y} *be Alexandrov topologies on* X *and* Y *and* $\tau_{d_{\tau_{d_X}}}$ *and* $\tau_{d_{\tau_{d_Y}}}$ *be Alexandrov topologies on* τ_{d_X} *and* τ_{d_Y}, *respectively. There exists a residuated connection* $(d_{\tau_{d_{\tau_{d_X}}}}, \Theta, \Phi, d_{\tau_{d_{\tau_{d_Y}}}})$ *defined as*

$$\begin{aligned} &R(A,B) = \bigvee_{x\in X}(B(f(x)) \ominus A(x))\\ &= \bigvee_{y\in X}(B(y) \ominus A(g(y))),\\ &\Theta(\alpha)(B) = \bigwedge_{A\in\tau_{d_X}}(\alpha(A) \oplus R(A,B)),\\ &\Phi(\beta)(A) = \bigvee_{B\in\tau_{d_Y}}(\beta(B) \ominus R(A,B)). \end{aligned}$$

Define a map $k = (k_X, k_Y) : X \times Y \to \tau_{d_{\tau_{d_X}}} \times \tau_{d_{\tau_{d_X}}}$ *as* $k(x,y)(A \times B) = (\hat{x}(A), \hat{y}(B)) = (A(x), B(y))$. *Then* $k : (d_X, f, g, d_Y) \to (d_{\tau_{d_{\tau_{d_X}}}}, \Theta, \Phi, d_{\tau_{d_{\tau_{d_Y}}}})$ *is an R-R embedding map such that*

$$\begin{aligned} &\Theta(\hat{x})(B) = \hat{f(x)}(B) = B(f(x)),\\ &\Theta(\hat{x})(((d_Y)^y)^*) = d^*_{\tau_{d_{\tau_{d_X}}}}(\hat{x}, \hat{g(y)}) = d^*_X(x, g(y)),\\ &\Phi(\hat{y})(A) = \hat{g(y)}(A) = A(g(y)),\\ &\Phi(\hat{y})((d_X)_x) = d_{\tau_{d_{\tau_{d_Y}}}}(\hat{f(x)}, \hat{y}) = d_Y(f(x), y). \end{aligned}$$

Proof. By Theorem 6, since $k_X : (X, d_X) \to (\tau_{d_{\tau_{d_X}}}, d_{\tau_{d_{\tau_{d_X}}}})$ and $k_Y : (Y, d_Y) \to (\tau_{d_{\tau_{d_Y}}}, d_{\tau_{d_{\tau_{d_Y}}}})$ are embedding maps,

$$\begin{aligned} &d_X(x,z) = d_{\tau_{d_{\tau_{d_X}}}}(k_X(x), k_X(z)) = d_{\tau_{d_{\tau_{d_X}}}}(\hat{x},\hat{z})\\ &d_Y(y,w) = d_{\tau_{d_{\tau_{d_Y}}}}(k_Y(y), k_Y(w)) = d_{\tau_{d_{\tau_{d_Y}}}}(\hat{y},\hat{w}). \end{aligned}$$

By Theorem 7 (3), $(d_{\tau_{d_X}}, R, d_{\tau_{d_Y}})$ is a residuated frame where

$$\begin{aligned} R(A,B) \quad &= \bigvee_{x\in X}(B(f(x)) \ominus A(x))\\ &= \bigvee_{y\in Y}(B(y) \ominus A(g(y))). \end{aligned}$$

By Theorem 7, $(d_{\tau_{d_{\tau_{d_X}}}}, \Theta, \Phi, d_{\tau_{d_{\tau_{d_Y}}}})$ is a residuated connection where

$$\begin{aligned} &\Theta(\alpha)(B) = \bigwedge_{A\in\tau_{d_X}}(R(A,B) \oplus \alpha(A))\\ &= \bigwedge_{A\in\tau_{d_X}}\Big(\bigvee_{x\in X}(B(f(x)) \ominus A(x)) \oplus \alpha(A)\Big),\\ &\Phi(\beta)(A) = \bigvee_{B\in\tau_{d_Y}}(\beta(B) \ominus R(A,B))\\ &= \bigvee_{B\in\tau_{d_Y}}\Big(\beta(B) \ominus \bigvee_{y\in Y}(B(y) \ominus A(g(y)))\Big). \end{aligned}$$

Since $B \in \tau_{d_Y}$ and f is isotone from Theorem 7 (2), then

$$\begin{aligned}&B(f(x)) \oplus d_Y(f(x), f(y)) \geq B(f(y))\\ &\Rightarrow B(f(x)) \oplus d_X(x, y)\\ &\geq B(f(x)) \oplus d_Y(f(x), f(y)) \geq B(f(y)).\end{aligned}$$

Thus $f^{\leftarrow}(B) \in \tau_{d_X}$. Put $A = f^{\leftarrow}(B)$. Then

$$\begin{aligned}&\Theta(\hat{x})(B)\\ &= \bigwedge_{A \in \tau_{d_X}} \left(\hat{x}(A) \oplus \bigvee_{z \in X}(B(f(z)) \ominus A(z))\right)\\ &\leq \hat{x}(f^{\leftarrow}(B)) \oplus \bigvee_{z \in X}(B(f(z)) \ominus f^{\leftarrow}(B)(z))\\ &= B(f(x)).\end{aligned}$$

Moreover,

$$\begin{aligned}&\Theta(\hat{x})(B) = \bigwedge_{A \in \tau_{d_X}}(\hat{x}(A) \oplus R(A, B))\\ &= \bigwedge_{A \in \tau_{d_X}} \left(\hat{x}(A) \oplus \bigvee_{z \in X}(B(f(z)) \ominus A(z))\right)\\ &\geq \bigwedge_{A \in \tau_{d_X}} \left(A(x) \oplus (B(f(x)) \ominus A(x))\right)\\ &\geq B(f(x)).\end{aligned}$$

Hence $\Theta(\hat{x})(B) = \hat{f(x)}(B) = B(f(x))$.

Since $f^{\leftarrow}(B) \in \tau_{d_X}$, $\bigvee_{z \in X}(B(f(z)) \ominus (d_X)_x(z)) = B(f(x))$. For $(d_X)_x \in \tau_{d_X}$,

$$\begin{aligned}&\Phi(\hat{y})((d_X)_x)\\ &= \bigvee_{B \in \tau_{d_Y}} \left(\hat{y}(B) \ominus \bigvee_{z \in X}(B(f(z)) \ominus (d_X)_x(z))\right)\\ &= \bigvee_{B \in \tau_{d_Y}} \left(B(y) \ominus B(f(x))\right)\\ &= d_{\tau_{d_{\tau_{d_Y}}}}(\hat{f(x)}, \hat{y}) = d_Y(f(x), y) \text{ (by Theorem 6).}\end{aligned}$$

Since $A \in \tau_{d_X}$ and g is isotone from Theorem 7 (2),

$$\begin{aligned}&A(g(y)) \oplus d_Y(y, w) \geq A(g(y)) \oplus d_X(g(y), g(w))\\ &\geq A(g(w)).\end{aligned}$$

Thus $g^{\leftarrow}(A) \in \tau_{d_Y}$. Put $B = g^{\leftarrow}(A)$.

$$\begin{aligned}&\Phi(\hat{y})(A) = \bigvee_{B \in \tau_{d_Y}}(\hat{y}(B) \ominus R(A, B))\\ &= \bigvee_{B \in \tau_{d_Y}} \left(\hat{y}(B) \ominus \bigvee_{y \in Y}(B(y) \ominus A(g(y)))\right)\\ &\geq \hat{y}(g^{\leftarrow}(A)) \ominus \bigvee_{y \in Y}(g^{\leftarrow}(A)(y) \ominus A(g(y)))\\ &= A(g(y)).\end{aligned}$$

$$\begin{aligned}&\Phi(\hat{y})(A) = \bigvee_{B \in \tau_{d_Y}}(\hat{y}(B) \ominus R(A, B))\\ &= \bigvee_{B \in \tau_{d_Y}} \left(\hat{y}(B) \ominus \bigvee_{y \in Y}(B(y) \ominus A(g(y)))\right)\\ &\leq \bigvee_{B \in \tau_{d_Y}} \left(B(y) \ominus (B(y) \ominus A(g(y)))\right)\\ &\leq A(g(y)).\end{aligned}$$

Hence $\Phi(\hat{y})(A) = \hat{g(y)}(A) = A(g(y))$.

For each $x \in X$ and $((d_Y)^y)^* \in \tau_{d_Y}$,

$$\begin{aligned}
&\Theta(\hat{x})(((d_Y)^y)^*)\\
&= \bigwedge_{A\in\tau_{d_X}} \left(\hat{x}(A) \oplus \bigvee_{w\in Y}(((d_Y)^y)^*(w) \ominus A(g(w)))\right)\\
&\geq \bigwedge_{A\in\tau_{d_X}} \left(\hat{x}(A) \oplus (d_Y(y,y)^* \ominus A(g(y)))\right)\\
&= \left(\bigvee_{A\in\tau_{d_X}} (g\hat{(}y)(A) \ominus \hat{x}(A))\right)^* \text{ (by Lemma 1 (8))}\\
&= d^*_{\tau_{d_{\tau_{d_X}}}}(\hat{x}, g\hat{(}y)) = d^*_X(x, g(y)) = R^*_2(x,y).
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\Theta(\hat{x})(((d_Y)^y)^*)\\
&= \bigwedge_{A\in\tau_{d_X}} \left(\hat{x}(A) \oplus \bigvee_{w\in Y}(((d_Y)^y)^*(w) \ominus A(g(w)))\right)\\
&\leq \hat{x}((d_X)_x) \oplus \bigvee_{w\in Y}(((d_Y)^y)^*(w) \ominus (d_X)_x(g(w)))\Big)\\
&= d_X(x,x) \oplus \left(\bigwedge_{w\in Y}(d_Y(w,y) \oplus d_X(x,g(w)))\right)^*\\
&= \left(\bigwedge_{w\in Y}(d_Y(w,y) \oplus R_2(x,w))\right)^*\\
&= R_2(x,y)^* = d^*_X(x,g(y)) \text{ (by Lemma 3 (1)).}
\end{aligned}$$

Hence $\Theta(\hat{x})(((d_Y)^y)^*) = d^*_{\tau_{d_{\tau_{d_X}}}}(\hat{x}, g\hat{(}y)) = R_2(x,y)^* = d^*_X(x,g(y))$. Thus,

$$\begin{aligned}
&\Theta(\hat{x})(B) = f\hat{(}x)(B) = B(f(x)),\\
&\Theta(\hat{x})(((d_Y)^y)^*) = d^*_{\tau_{d_{\tau_{d_X}}}}(\hat{x}, g\hat{(}y)) = d^*_X(x,g(y)),\\
&\Phi(\hat{y})(A) = g\hat{(}y)(A) = A(g(y)),\\
&\Phi(\hat{y})((d_X)_x) = d_{\tau_{d_{\tau_{d_Y}}}}(f\hat{(}x), \hat{y}) = d_Y(f(x),y).
\end{aligned}$$

Moreover,

$$\begin{aligned}
&d_{\tau_{d_{\tau_{d_Y}}}}(\Theta(\hat{x}), \hat{y}) = d_{\tau_{d_{\tau_{d_Y}}}}(f\hat{(}x), \hat{y}) = d_Y(f(x),y)\\
&= d_X(x,g(y)) = d_{\tau_{d_{\tau_{d_X}}}}(\hat{x}, g\hat{(}y)) = d_{\tau_{d_{\tau_{d_X}}}}(\hat{x}, \Phi(\hat{y})).
\end{aligned}$$

Hence $k : (d_X, f, g, d_Y) \to (d_{\tau_{d_{\tau_{d_X}}}}, \Theta, \Phi, d_{\tau_{d_{\tau_{d_Y}}}})$ is an R-R embedding map. □

Example 4. *Let $X = \{x_1, x_2, x_3\}$ be a set and $f : X \to X$ a function as $f(x_1) = x_2$, $f(x_2) = x_1, f(x_3) = x_3$ and $f = f^{-1}$. Define a binary operation $\oplus$ on $L = [0,1]$ by*

$$x \oplus y = \min\{1, x+y\},\ x \ominus y = \max\{x-y, 0\}.$$

Let (X, d_X) be a distance space as follows:

$$d_X = \begin{pmatrix} 0 & 0.5 & 0.4\\ 0.5 & 0 & 0.4\\ 0.3 & 0.3 & 0 \end{pmatrix}$$

Since $d_X(f(x),y) = d_X(x,f(y))$ for all $x,y \in X$, (d_X, f, f, d_X) is a residuated connection. By Theorem 7, (d_X, R_1, d_X) is a residuated frame where $R_1(x,y) = d_X(f(x),y) = d_X(x,f(y))$ for all $x,y \in X$ such that

$$R_1 = \begin{pmatrix} 0.5 & 0 & 0.4\\ 0 & 0.5 & 0.4\\ 0.3 & 0.3 & 0 \end{pmatrix}$$

By Theorem 7 (2), there exists a residuated connection $(d_{\tau_{d_X}}, F, G, d_{\tau_{d_X}})$ where

$$\begin{aligned} F(A)(y) &= \bigwedge_{z\in X}(d_X(f(z),y)\oplus A(z)) \\ &= \bigwedge_{z\in X}(d_X(z,f(y))\oplus A(z)) \\ G(B)(x) &= \bigvee_{y\in X}(B(y)\ominus d_X(f(x),y)) \\ &= \bigvee_{w\in X}(B(w)\ominus d_X(x,f(w))). \end{aligned}$$

By Theorem 7 (3), $(d_{\tau_{d_X}}, R, d_{\tau_{d_X}})$ *is a residuated frame where* $R : \tau_{d_X}\times\tau_{d_X}\to L$ *as*

$$\begin{aligned} R(A,B) &= \bigvee_{x\in X}(B(f(x))\ominus A(x)) \\ &= \bigvee_{y\in X}(B(y)\ominus A(f(y))). \end{aligned}$$

For $A_1 = (0.5, 0.2, 0.8)$, $0.6 = A_1(x_2)\oplus d_X(x_2,x_3) \not\geq A_1(x_3) = 0.8$, $A_1 \notin \tau_{d_X}$. *Then* $A = \bigwedge_{z\in X}(d_X(z,y)\oplus A_1(z)) = (0.5, 0.2, 0.6) \in \tau_{d_X}$.

For $B_1 = (0.2, 0.7, 0.8)$, $0.6 = B_1(x_1)\oplus d_X(x_1,x_3) \not\geq B_1(x_3) = 0.8$, $B_1 \notin \tau_{d_X}$. *Then* $B = \bigwedge_{z\in X}(d_X(z,y)\oplus B_1(z)) = (0.2, 0.7, 0.6) \in \tau_{d_X}$. *For* $A = (0.5, 0.2, 0.6) \in \tau_{d_X}$ *and* $B = (0.2, 0.7, 0.6) \in \tau_{d_X}$,

$$\begin{aligned} &F(A) = \textstyle\bigwedge_{z\in X}(d_X(f(z),-)\oplus A(z)) = (0.2, 0.5, 0.6) \\ &G(B) = \textstyle\bigvee_{y\in X}(B(y)\ominus d_X(f(-),y)) = (0.7, 0.2, 0.6) \\ &d_{\tau_{d_X}}(F(A),B) = \textstyle\bigvee_{w\in Y}(B(w)\ominus F(A)(w)) \\ &d_{\tau_{d_X}}(A,G(B)) = \textstyle\bigvee_{x\in X}(G(B)(x)\ominus A(x)) \\ &R(A,B) = d_{\tau_{d_X}}(F(A),B) = d_{\tau_{d_X}}(A,G(B)) = 0.2. \end{aligned}$$

By Theorem 5,

$\{(A,F(A)),(B,F(B))\} \subset \mathcal{F}(d_{\tau_{d_X}}, F, G, d_{\tau_{d_X}})$. *Moreover,*

$$\begin{aligned} &F((d_X)_x)(y) = \textstyle\bigwedge_{z\in X}(d_X(f(z),y)\oplus d_X(x,z)) \\ &= d_X(f(x),y), \\ &R((d_X)_x,((d_X)^y)^*) = d_X^*(x,f(y)) = d_X^*(f(x),y) \\ &= G(((d_X)^y)^*) = d_{\tau_{d_X}}(F((d_X)_x),((d_X)^y)^*) \\ &= d_{\tau_{d_X}}((d_X)_x, G(((d_X)^y)^*)). \end{aligned}$$

By Theorem 9, $(d_{\tau_{d_{\tau_{d_X}}}}, \Theta, \Phi, d_{\tau_{d_{\tau_{d_X}}}})$ *is a residuated connection where*

$$\begin{aligned} &\Theta(\alpha)(B) = \textstyle\bigwedge_{A\in\tau_{d_X}}(\alpha(A)\oplus R(A,B)), \\ &\Phi(\beta)(A) = \textstyle\bigvee_{B\in\tau_{d_X}}(\beta(B)\ominus R(A,B)). \end{aligned}$$

Since (d_X, f, f, d_X) *is a residuated connection, by Theorem 9,* $\Theta(\hat{x})(A) = \hat{f(x)}(A) = A(f(x))$ *and* $\Phi(\hat{y})(A) = \hat{f(y)}(A) = A(f(y))$ *for each* $A \in \tau_{d_X}$. *Thus,* $\Theta(\hat{x}) = \Phi(\hat{x})$ *for each* $x \in X$. *For* $A = (0.5, 0.2, 0.6) \in \tau_{d_X}$ *and* $B = (0.2, 0.7, 0.6) \in \tau_{d_X}$,

$$\begin{aligned} &\Theta(\hat{x_1})(A) = \hat{f(x_1)}(A) = A(f(x_1)) = 0.2, \\ &\Theta(\hat{x_1})(B) = \hat{f(x_1)}(B) = B(f(x_1)) = 0.7. \end{aligned}$$

Hence $k : (d_X, f, f, d_X) \to (d_{\tau_{d_{\tau_{d_X}}}}, \Theta, \Phi, d_{\tau_{d_{\tau_{d_X}}}})$ *is an R-R embedding map. Moreover,*

$$\begin{aligned} &d_{\tau_{d_{\tau_{d_X}}}}(\Theta(\hat{x_2}),\hat{x_3}) = d_{\tau_{d_{\tau_{d_X}}}}(\hat{f(x_2)},\hat{x_3}) \\ &= d_X(f(x_2),x_3) = 0.4 = d_X(x_2,f(x_3)) \\ &= d_{\tau_{d_{\tau_{d_X}}}}(\hat{x_2},\hat{f(x_3)}) = d_{\tau_{d_{\tau_{d_X}}}}(\hat{x_2},\Phi(\hat{x_3})). \end{aligned}$$

https://doi.org/10.3390/axioms15050336

5. Conclusions

In this paper, we have introduced that residuated frames for fuzzy logics induce residuated connections on Alexandrov topologies. Moreover, we have studied that fuzzy residuated connections with distances induce fuzzy concept lattices based on Alexandrov topologies. From an information system, we can define two distances for objects and attributes satisfying a residuated frame. Moreover, we obtain a residuated connection and a fuzzy concept lattice. It has shown that big data with an information system can also be variously classified into small groups with a viewpoint of Alexandrov topologies induced by distances.

In the future, by using the concepts of various connections and frames, we may study various fuzzy concepts, information systems and decision rules on algebraic structures.

Author Contributions: Conceptualization, Y.C.K.; methodology, Y.C.K. and Y.-H.K.; validation, Y.C.K. and Y.-H.K.; formal analysis, Y.C.K. and Y.-H.K.; writing, original draft, Y.-H.K.; review and editing, Y.C.K.; project administration, Y.-H.K.; funding acquisition, Y.-H.K. All authors have read and agreed to the published version of the manuscript.

Funding: The present Research has been conducted by the Research Grant of Kwangwoon University in 2025.

Data Availability Statement: No new data were created or analyzed in this study.

Conflicts of Interest: The authors declare no conflicts of interest.

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