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On the Asymptotic Properties of Functions of Finite Order, Analytic in a Disk, Represented by Series in Systems of Functions

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Abstract

In terms of order, lower order, type, lower type and two-member power asymptotic for the function $A(z) = \sum\limits_{n=1}^{\infty} a_n f(\lambda_n z)$, analytic in a finite disk, where f is an entire transcendental function and (λ_n) is sequence of positive numbers increasing to $+\infty$, the relationship between the growth of $\mathfrak{M}(r, A) = \sum\limits_{n=1}^{\infty} |a_n| M_f(r\lambda_n)$, where $M_f(r) = \max\{|f(z)| : |z| = r\}$, and the behavior of the coefficients a_n, is studied.

Keywords: series in system of functions; order; type; two-member power asymptotic

MSC: 30B50; 30D20

1. Introduction

The study of asymptotic properties of analytic functions represented by series in systems of functions has deep roots, beginning with classical works such as those by Beuermann [1], Kamthan [2], and Reddy [3], as well as the structural developments of Zelisko [4]. Important advances were made through the theory of regularly converging series explored by Sheremeta [5–7] and later expanded in the analyses of maximal terms by Gal' and Sheremeta [8]. The foundations of series expansions in suitable function systems were also shaped by contributions of Saurer [9] and the extensive series of results by Vinnitskii [10–13]. Further refinements involving Borel-type relations and positive functional series were introduced by Skaskiv and collaborators [14–16], with additional developments by Bodnarchuk, Skaskiv, and Trusevych [17]. Recent progress on two-member asymptotics and inequalities analogous to those of Lockhart and Straus is reflected in the works of Holovata and Sheremeta [18], while classical insights into the relation between maximal terms and maximum modulus can be traced back to Lockhart and Straus [19] and later to Filevych [20] and Skaskiv [21]. Altogether, these references form a coherent theoretical background for the present investigation. The purpose of this study is to establish explicit formulas describing the order, lower order, type, lower type, and two-member asymptotics of functions defined by such series in system of functions, expressed directly in terms of their coefficients.

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Let

$$f(z) = \sum_{k=0}^{\infty} f_k z^k \tag{1}$$

be an entire transcendental function,

$$M_f(r) = \max\{|f(z)| : |z| = r\},$$

(λ_n) be a sequence of positive numbers increasing to $+\infty$ and

$$A(z) = \sum_{n=1}^{\infty} a_n f(\lambda_n z) \tag{2}$$

be a series in the system $f(\lambda_n z)$. Let $R[A]$ be the radius of regular convergence of series (2), i.e.,

$$\mathfrak{M}(r, A) = \sum_{n=1}^{\infty} |a_n| M_f(r\lambda_n) < +\infty \tag{3}$$

holds for $r < R[A]$ and does not hold for $r > R[A]$. If (3) holds for all $r \geq 0$ then we put $R[A] = +\infty$.

Denote $\Gamma_f(r) = \dfrac{d \ln M_f(r)}{d \ln r}$ (at points where the derivative does not exist, the notation $\dfrac{d \ln M_f(r)}{d \ln r}$ means right-hand derivative). Since the function $\ln M_f(r)$ is logarithmically convex, we have $\Gamma_f(r) \nearrow +\infty$ as $r \to +\infty$.

Remark 1. *Let $M_f(r) = \max_{|z|=r} |f(z)|$, where f is an entire function, and consider the function $\varphi(r) = \log M_f(r)$, for $r > 0$. The derivative in the variable r exists almost everywhere. The function has such known properties: $M_f(r)$ is a continuous and non-decreasing function and $\log M_f(r)$ is a convex function as a function of $\log r$ (Hadamard three-circle theorem). So, $\varphi(r)$ has one-sided derivatives everywhere, but the ordinary (two-sided) derivative may not exist at points where the "dominating" term changes in power series of f. If $f(z) = \sum_{k=0}^{\infty} a_k z^k$, then the following estimate is often valid: $M_f(r) \approx \max_k |a_k| r^k$. Then for the logarithm, one has $\log M_f(r) = \max_k (\log |a_k| + k \log r)$. It is the maximum of the family of smooth functions. At points where the maximum is attained by two different indices k, a break occurs. Let us present a simple example where only the right-hand derivative exists. Put $f(z) = z + z^2$. For $r > 0$, one has $M_f(r) = \max_{|z|=r} |z + z^2| = r + r^2$. Then $\varphi(r) = \log(r + r^2)$. We have such asymptotic estimates:*

for $r < 1$: $M_f(r) \sim r$, $\varphi(r) \sim \log r$;

for $r > 1$: $M(r) \sim r^2$, $\varphi(r) \sim 2 \log r$. At the point $r = 1$, the left derivative is $\frac{d}{dr} \log r \Big|_{r=1} = 1$, and the right derivative is $\frac{d}{dr} 2 \log r \Big|_{r=1} = 2$. Hence, $\varphi'(1-0) \neq \varphi'(1+0)$. Then the ordinary derivative does not exist, but the right derivative exists.

It is known [5] that if $\Gamma_f(cr) \asymp \Gamma_f(r)$ as $r \to +\infty$ for each $c \in (0, +\infty)$ and $\ln n = o(\Gamma_f(\lambda_n))$ as $n \to \infty$ then $R[A] = \varliminf\limits_{n \to \infty} \dfrac{1}{\lambda_n} M_f^{-1}\left(\dfrac{1}{|a_n|}\right)$. If $0 < R[A] < +\infty$ then quantities

$$\varrho[A] = \varlimsup_{r \uparrow R[A]} \frac{\ln \ln \mathfrak{M}(r, A)}{-\ln (R[A] - r)},$$

$$\lambda[A] = \varliminf_{r \uparrow R[A]} \frac{\ln \ln \mathfrak{M}(r, A)}{-\ln (R[A] - r)}$$

are called the order and lower order of the function A, respectively. If $0 < \varrho[A] < +\infty$, then

$$T[A] = \varlimsup_{r\uparrow R[A]} (R[A]-r)^{\varrho[A]} \ln \mathfrak{M}(r,A),$$

$$t[A] = \varliminf_{r\uparrow R[A]} (R[A]-r)^{\varrho[A]} \ln \mathfrak{M}(r,A)$$

are called type and lower type of the function A, respectively.

The purpose of this article is to find formulas for calculating $\varrho[A]$, $\lambda[A]$, $T[A]$ and $t[A]$ by the coefficients a_n. In addition, the two-member power asymptotic of $\ln \mathfrak{M}(r,A)$ will be studied.

2. Formulas for the Order and Lower Order

Let

$$\mu(r,A) = \max\{|a_n|M_f(r\lambda_n) : n \geq 1\}$$

be the maximal term of series (3) and

$$\nu(r,A) = \max\{n \geq 1 : |a_n|M_f(r\lambda_n) = \mu(r,A)\}$$

be its central index. We remark that if $R[A] < +\infty$ then the function $\mu(r,A)$ can be bounded on $[0,R[A])$. But if

$$\varlimsup_{n\to\infty} |a_n|M_f(R[A]\lambda_n) = +\infty,$$

then $\mu(r,A) \to +\infty$ as $r \to R[A]$. In what follows, we will assume that this condition is satisfied.

Let us recall that a (measurable or smooth) function $f : \mathbb{R}^n \to \mathbb{C}$ is called slowly increasing, if there exist constants $C > 0$ and $N \geq 0$ such that $|f(x)| \leq C(1+|x|)^N$ for all $x \in \mathbb{R}^n$. In other words, a slowly increasing function grows at most polynomially at infinity. No restriction is imposed near the origin; the condition concerns only the behavior as $|x| \to \infty$. The same concept can be expressed in Big-O formulation $f(x) = O(|x|^N)$ as $|x| \to \infty$ for some finite N or by the logarithmic growth $\log|f(x)| = O(\log|x|)$.

Clearly, $\mu(r,A) \leq \mathfrak{M}(r,A)$. On the other hand, the following statement is true.

Lemma 1. *Let $\Gamma_f(x) = x + O(l(x))$ as $x \to +\infty$, where l is a slowly increasing function, and $\ln\ln n(t) = o(\ln t)$ as $t \to +\infty$, where $n(t) = \sum_{\lambda_n \leq t} 1$ is the counting function of the sequence (λ_n). Then for every $q \in (0,1)$, $\varepsilon \in (0,1)$ and all $r \in [r_0(\varepsilon), R[A])$ one has*

$$\frac{\mathfrak{M}(r,A)}{\mu(r+q(R[A]-r),A)} \leq 2\left(\frac{4p}{q(R[A]-r)}\right)^{\varepsilon/(1-\varepsilon)} \exp\left\{\left(\frac{4p}{q(R[A]-r)}\right)^{\varepsilon/(1-\varepsilon)}\right\} + K, \quad (4)$$

where $p > 1$ and K is a constant independent of r.

Proof. For $q \in (0,1)$ we have

$$\mathfrak{M}(r,A) = \sum_{n=1}^{\infty} |a_n|M_f((r+q(R[A]-r))\lambda_n)\frac{M_f(r\lambda_n)}{M_f((r+q(R[A]-r))\lambda_n)} \leq$$

$$\leq \mu(r+q(R[A]-r),A)\sum_{n=1}^{\infty}\frac{M_f(r\lambda_n)}{M_f((r+q(R[A]-r))\lambda_n)}. \quad (5)$$

On the other hand,

$$\frac{M_f((r+q(R[A]-r)\lambda_n)}{M_f(r\lambda_n)} = \exp\{\ln\, M_f((r+q(R[A]-r))\lambda_n) - \ln\, M_f(r\lambda_n)\} =$$
$$= \exp\left\{\int\limits_{r\lambda_n}^{(r+q(R[A]-r))\lambda_n} \frac{d}{d\ln\, t}\ln\, M_f(t) d\ln\, t\right\} = \exp\left\{\int\limits_{r\lambda_n}^{(r+q(R[A]-r))\lambda_n} \Gamma_f(t) d\ln\, t\right\}.$$

Since $\Gamma_f(r) \nearrow +\infty$ as $r \to +\infty$, one has $\Gamma_f(t) \geq \Gamma_f(r\lambda_n)$ for $t \in [r\lambda_n, (r+q(R[A]-r))\lambda_n]$. Then

$$\begin{aligned}\frac{M_f((r+q(R[A]-r)\lambda_n)}{M_f(r\lambda_n)} &\geq \exp\left\{\Gamma_f(r\lambda_n)\int\limits_{r\lambda_n}^{(r+q(R[A]-r))\lambda_n} d\ln\, t\right\} = \\ &= \exp\left\{\Gamma_f(r\lambda_n)\ln\,\tfrac{r+q(R[A]-r)}{r}\right\} = \\ &= \exp\left\{\Gamma_f(r\lambda_n)\ln\left(1+\tfrac{q(R[A]-r)}{r}\right)\right\} \geq \exp\left\{\Gamma_f(r\lambda_n)\tfrac{q(R[A]-r)}{2r}\right\}\end{aligned} \tag{6}$$

provided $r \geq R[A]/2$. Above in the last inequality we used that $\ln(1+x) \geq \frac{x}{2}$ for $x \in (0,1]$. Since $q \in (0,1)$, one has $\ln\left(1+\frac{q(R[A]-r)}{r}\right) \geq \frac{q(R[A]-r)}{2r}$, if $\frac{q(R[A]-r)}{r} \leq \frac{(R[A]-r)}{r} \in (0,1]$. From $\frac{(R[A]-r)}{r} \leq 1$ we deduce $R[A] \leq 2r$.

From (5) and (6) we get

$$\frac{\mathfrak{M}(r,A)}{\mu(r+q(R[A]-r),A)} \leq \sum_{n=1}^{\infty}\exp\left\{-\Gamma_f(r\lambda_n)\frac{q(R[A]-r)}{2r}\right\}, \quad r \geq R[A]/2, \tag{7}$$

and, thus, we need an estimate for the series on the right-hand side of (7).

Since $\Gamma_f(x) = x + O(l(x))$ as $x \to +\infty$, where l is a slowly increasing function, we have

$$\sum_{n=1}^{\infty}\exp\left\{-\Gamma_f(r\lambda_n)\frac{q(R[A]-r)}{2r}\right\} = \sum_{n=1}^{\infty}\exp\left\{-(1+o(1))\frac{q(R[A]-r)}{2}\lambda_n\right\} \leq$$

$$\leq \sum_{n=1}^{\infty}\exp\left\{-\frac{q(R[A]-r)}{2p}\lambda_n\right\} + K_1 \leq \int\limits_0^{\infty}\exp\left\{-\frac{q(R[A]-r)}{2p}t\right\}dn(t) + K_1, \tag{8}$$

for $p > 1$. Here and below, K_j denotes positive constants independent of r.

The condition $\ln\,\ln\, n(t) = o(\ln\, t)$ as $t \to +\infty$ implies $\ln\, n(t) \leq t^{\varepsilon}$ for every $\varepsilon \in (0,1)$ and all $t \geq t_0 = t_0(\varepsilon)$. We put

$$T(r) = \left(\frac{4p}{q(R[A]-r)}\right)^{1/(1-\varepsilon)}.$$

Then

$$\int\limits_0^{\infty}\exp\left\{-\frac{q(R[A]-r)}{2p}t\right\}dn(t) \leq \frac{q(R[A]-r)}{2p}\int\limits_0^{\infty} n(t)\exp\left\{-\frac{q(R[A]-r)}{2p}t\right\}dt \leq$$

$$\leq \frac{q(R[A]-r)}{2p}\left(\int\limits_0^{t_0} + \int\limits_{t_0}^{T(r)} + \int\limits_{T(r)}^{\infty}\right)\exp\left\{-\frac{q(R[A]-r)}{2p}t + \ln\, n(t)\right\}dt. \tag{9}$$

On the other hand,

$$\begin{aligned}
&\tfrac{q(R[A]-r)}{2p}\int\limits_{T(r)}^{\infty}\exp\Big\{-t\Big(\tfrac{q(R[A]-r)}{2p}-\tfrac{1}{t^{1-\varepsilon}}\Big)\Big\}dt\le\\
&\le\tfrac{q(R[A]-r)}{2p}\int\limits_{T(r)}^{\infty}\exp\Big\{-t\Big(\tfrac{q(R[A]-r)}{2p}-\tfrac{1}{T(r)^{1-\varepsilon}}\Big)\Big\}dt=\\
&=\tfrac{q(R[A]-r)}{2p}\int\limits_{T(r)}^{\infty}\exp\Big\{-t\Big(\tfrac{q(R[A]-r)}{4p}\Big)\Big\}dt=\\
&=\tfrac{q(R[A]-r)}{2p}\cdot\tfrac{-4p}{q(R[A]-r)}\lim_{B\to+\infty}\exp\Big\{-t\Big(\tfrac{q(R[A]-r)}{4p}\Big)\Big\}\Big|_{T(r)}^{B}=\\
&=2\exp\Big\{-T(r)\Big(\tfrac{q(R[A]-r)}{4p}\Big)\Big\}=2\exp\Big\{-\Big(\frac{4p}{q(R[A]-r)}\Big)^{1/(1-\varepsilon)}\Big(\tfrac{q(R[A]-r)}{4p}\Big)\Big\}=\\
&=2\exp\Big\{-\Big(\tfrac{q(R[A]-r)}{4p}\Big)^{-\varepsilon/(1-\varepsilon)}\Big\}\le 2
\end{aligned}\tag{10}$$

because $\frac{q(R[A]-r)}{4p}\ge 0$ and $\left(\frac{q(R[A]-r)}{4p}\right)^{-\varepsilon/(1-\varepsilon)}\ge 0$. Moreover,

$$\begin{aligned}
&\tfrac{q(R[A]-r)}{2p}\int\limits_{t_0}^{T(r)}\exp\Big\{-\tfrac{q(R[A]-r)}{2p}t+t^{\varepsilon}\Big\}dt\le\\
&\le\tfrac{q(R[A]-r)}{2p}\int\limits_{0}^{T(r)}\exp\{t^{\varepsilon}\}dt\le\tfrac{q(R[A]-r)}{2p}T(r)\exp\{T(r)^{\varepsilon}\}=\\
&=2\Big(\frac{4p}{q(R[A]-r)}\Big)^{\varepsilon/(1-\varepsilon)}\exp\Big\{\Big(\frac{4p}{q(R[A]-r)}\Big)^{\varepsilon/(1-\varepsilon)}\Big\}
\end{aligned}\tag{11}$$

and, similarly,

$$\begin{aligned}
&\frac{q(R[A]-r)}{2p}\int\limits_{0}^{t_0}\exp\Big\{-\frac{q(R[A]-r)}{2p}t+t^{\varepsilon}\Big\}dt\le\frac{q(R[A]-r)}{2p}t_0\exp\{t_0^{\varepsilon}\}\le\\
&\le\frac{R[A]}{2}t_0\exp\{t_0^{\varepsilon}\}=K_2<+\infty,
\end{aligned}\tag{12}$$

because $q\in(0,1)$, $r\in[r_0(\varepsilon),R[A])$, $p>1$ (see above lines). So, we put $r=0$, $q=1$, $p=1$. Recall that $t_0(\varepsilon)$ is chosen such that $\ln n(t)\le t^{\varepsilon}$ for every $\varepsilon\in(0,1)$ and all $t\ge t_0=t_0(\varepsilon)$. Hence, the expression $\frac{R[A]}{2}t_0\exp\{t_0^{\varepsilon}\}=K_2$ is finite. Estimates (8)–(12) imply (4). Lemma 1 is proved. □

Using Lemma 1, we prove the following theorem.

Theorem 1. *If $\Gamma_f(x)=x+O(l(x))$ as $x\to+\infty$, where l is a slowly increasing function, and $\ln\ln n(t)=o(\ln t)$ as $t\to+\infty$, then*

$$\varrho[A]=\varrho[\mu]:=\varlimsup_{r\uparrow R[A]}\frac{\ln\ln\mu(r,A))}{-\ln(R[A]-r)},\quad \lambda[A]=\lambda[\mu]:=\varliminf_{r\uparrow R[A]}\frac{\ln\ln\mu(r,A))}{-\ln(R[A]-r)}.$$

Proof. Since $\mu(r,A)\le\mathfrak{M}(r,A)$, we have $\varrho[\mu]\le\varrho[A]$ and $\lambda[\mu]\le\lambda[A]$. On the other hand, from (4) we obtain

$$\ln\mathfrak{M}(r,A)\le\ln\mu(r+q(R[A]-r),A)+(1+o(1))\Big(\frac{4p}{q(R[A]-r)}\Big)^{\varepsilon/(1-\varepsilon)},\quad r\uparrow R[A],$$

and, thus,

$$\ln\ln\mathfrak{M}(r,A)\le\ln\ln\mu(r+q(R[A]-r),A)+\frac{\varepsilon}{1-\varepsilon}\ln\frac{4p}{q(R[A]-r)}+o(1)+\ln 2,\quad r\uparrow R[A].$$

Therefore,

$$\frac{\ln\ln\mathfrak{M}(r,A)}{\ln(1/(R[A]-r)}\le\frac{\ln\ln\mu(r+q(R[A]-r),A)}{\ln(1/(R[A]-r-q(R[A]-r)))}\frac{\ln(1/((1-q)(R[A]-r)))}{\ln(1/(R[A]-r))}+$$

$$+\frac{\varepsilon}{1-\varepsilon}\frac{\ln(1/(R[A]-r))+O(1)}{\ln(1/(R[A]-r)}+o(1),\quad r\uparrow R[A],$$

whence it follows that $\varrho[A]\le\varrho[\mu]+\varepsilon/(1-\varepsilon)$ and $\lambda[A]\le\lambda[\mu]+\varepsilon/(1-\varepsilon)$. In view of the arbitrariness of ε we get $\varrho[A]\le\varrho[\mu]$ and $\lambda[A]\le\lambda[\mu]$, i.e., $\varrho[A]=\varrho[\mu]$ and $\lambda[A]=\lambda[\mu]$. Theorem 1 is proved. □

We now obtain formulas for the finding $\varrho[\mu]$ and $\lambda[\mu]$ by the coefficients a_n. At first, we remark that the condition $\Gamma_f(x)=x+O(l(x))$ as $x\to+\infty$ implies $\ln M_f(x)=x+O(L(x))$ as $x\to+\infty$, where $L(x)=\int_1^x l(t)d\ln t$ is also a slowly increasing function.

Let $\varrho[\mu]<+\infty$. Then $\ln\mu(r,A)\le(1/(R[A]-r))^\varrho$ for every $\varrho>\varrho[\mu]$ and all $r\in[r_0(\varrho),R[A])$. Therefore,

$$\ln|a_n|\le\ln\mu(r,A)-\ln M_f(r\lambda_n)\le\frac{1}{(R[A]-r)^\varrho}-r\lambda_n+K_3L(\lambda_n)$$

for all $n\ge1$ and $r\in[r_0(\varrho),R[A])$. We choose $r=r_n:=R[A]-(\varrho/\lambda_n)^{1/(\varrho+1)}$. Then $r_n\ge r_0(\varrho)$ for $n\ge n_0(\varrho)$ and

$$\ln|a_n|\le\left(\frac{\lambda_n}{\varrho}\right)^{\varrho/(\varrho+1)}-R[A]\lambda_n+\varrho\left(\frac{\lambda_n}{\varrho}\right)^{\varrho/(\varrho+1)}+K_3L(\lambda_n),$$

i.e.,

$$\ln(|a_n|e^{R[A]\lambda_n})\le(\varrho+1)\left(\frac{\lambda_n}{\varrho}\right)^{\varrho/(\varrho+1)}(1+o(1)),\quad n\to\infty,$$

and

$$\ln\ln(|a_n|e^{R[A]\lambda_n})\le(1+o(1))\frac{\varrho}{\varrho+1}\ln\lambda_n,\quad n\to\infty,$$

whence in view of the arbitrariness of ϱ we get

$$\alpha[A]:=\varlimsup_{n\to\infty}\frac{\ln\ln(|a_n|e^{R[A]\lambda_n})}{\ln\lambda_n}\le\frac{\varrho[\mu]}{\varrho[\mu]+1}<1.\tag{13}$$

On the contrary, if $\alpha[A]<1$ then $\ln(|a_n|e^{R[A]\lambda_n})\le\lambda_n^\alpha$ for every $\alpha\in(\alpha[A],1)$ and all $n\ge n_0=n_0(\alpha)$. Therefore,

$$\ln\mu(r,A)\le K_4+\max_{n\ge n_0}\{\ln|a_n|+\ln M_f(r\lambda_n)\}\le$$

$$\le\max_{n\ge n_0}\{\lambda_n^\alpha-R[A]\lambda_n+r\lambda_n+K_5L(r\lambda_n)\}+K_4\le$$

$$\le\max_{x\ge0}\{px^\alpha-(R[A]-r)x\}+K_4=\max_{x\ge0}\{-x((R[A]-r)-px^{\alpha-1})\}+K_4$$

for some $p>1$. Put $x(r)=\left(\frac{p}{R[A]-r}\right)^{1/(1-\alpha)}$. Then $(R[A]-r)-px^{\alpha-1}=0$ and the maximum point of the function $px^\alpha-(R[A]-r)x$ does not exceed $x(r)$. Therefore,

$$\ln\mu(r,A)\le px(r)^\alpha+K_4=p\left(\frac{p}{R[A]-r}\right)^{\alpha/(1-\alpha)}+K_4,$$

whence in view of the arbitrariness of α it follows that $\varrho[\mu] \le \alpha[A]/(1-\alpha[A])$. Thus, the following statement is true.

Proposition 1. *If $\Gamma_f(x) = x + O(l(x))$ as $x \to +\infty$, where l is a slowly increasing function, then $\varrho[\mu] = \alpha[A]/(1-\alpha[A])$.*

To find the formula for $\lambda[\mu]$ we need a result from [8]. Suppose that the function Γ_f is increasing to $+\infty$. Then $\dfrac{M_f(r\lambda_{n+1})}{M_f(r\lambda_n)} \uparrow +\infty$ as $r \uparrow R[A]$. Therefore, if all $a_n \neq 0$ then the equation $\dfrac{M_f(r\lambda_{n+1})}{M_f(r\lambda_n)} = \dfrac{|a_n|}{|a_{n+1}|}$ has unique solution $r = \kappa_n$, i.e., $\dfrac{|a_n|}{|a_{n+1}|} = \dfrac{M_f(\kappa_n\lambda_{n+1})}{M_f(\kappa_n\lambda_n)}$. The following statement is proved in [8] for $R[A] = +\infty$, but from the proof given in [8] it is clear that it is also true for $0 < R[A] < +\infty$.

Lemma 2. *If $\kappa_n \nearrow R[A]$ as $n \to \infty$ then $\mu(\kappa_n, A) = |a_n| M_f(\kappa_n\lambda_n)$. If $\kappa_n \uparrow R[A]$ as $n \to \infty$ then $\mu(r, A) = |a_n| M_f(r\lambda_n)$ and $\nu(r, A) = n$ for all $r \in [\kappa_{n-1}, \kappa_n)$ and all $n \ge 1$.*

Now we suppose that

$$\beta[A] := \varliminf_{n\to\infty} \frac{\ln \ln \left(|a_n| e^{R[A]\lambda_n}\right)}{\ln \lambda_n} < 1$$

Then for every $\beta \in (\beta[A], 1)$ there exists a sequence $(n_k) \uparrow \infty$ such that $\ln \left(|a_{n_k}| e^{R[A]\lambda_{n_k}}\right) \le \lambda_{n_k}^{\beta}$. If $\kappa_n \nearrow R[A]$ as $n \to +\infty$, then by Lemma 2 we have $\mu(\kappa_{n_k}, A) = |a_{n_k}| M_f(\kappa_{n_k}\lambda_{n_k})$. Let $m \in \{n_j\}$. Then

$$\ln \mu(\kappa_m, A) \le \lambda_m^{\beta} - R[A]\lambda_m + \ln M_f(\kappa_m\lambda_m) \le \ln \mu^*(\kappa_m),$$

where

$$\ln \mu^*(r) = \max_{n\ge 1}\{\lambda_n^{\beta} - R[A]\lambda_n + \ln M_f(r\lambda_n)\}$$

and as in the proof of Proposition 1 we get

$$\ln \mu(\kappa_m, A) \le \ln \mu^*(\kappa_m, A) \le p(p/(R[A] - \kappa_m))^{\beta/(1-\beta)},$$

whence in view of the arbitrariness of β it follows that

$$\lambda[\mu] = \varliminf_{r\uparrow R[A]} \frac{\ln \ln \mu(r, A))}{-\ln (R[A] - r)} \le \varliminf_{m\to\infty} \frac{\ln \ln \mu(\kappa_m, A))}{-\ln (R[A] - \kappa_m)} \le \frac{\beta[A]}{1-\beta[A]} \tag{14}$$

On the other hand, if $\beta[A] > 0$ then $\ln \left(|a_n| e^{R[A]\lambda_n}\right) \ge \lambda_n^{\beta}$ for every $\beta \in (0, \beta[A])$ and all $n \ge n_0 = n_0(\beta)$. Therefore, if we choose $r_n = R[A] - \lambda_n^{\beta-1}/2$ then

$$\ln \mu(r_n, A) \ge \ln |a_n| + \ln M_f(r_n\lambda_n) \ge \lambda_n^{\beta} - R[A]\lambda_n + r_n\lambda_n + O(L(r_n\lambda_n)) =$$

$$= \lambda_n^{\beta}/2 + O(L(\lambda_n)) = (1 + o(1))\lambda_n^{\beta}/2, \quad n \to \infty,$$

i.e.,

$$\ln \ln \mu(r_n, A) \ge (1 + o(1))\beta \ln \lambda_n, \quad n \to \infty.$$

Similarly,

$$\ln \ln \mu(r_{n+1}, A) \ge (1 + o(1))\beta \ln \lambda_{n+1} = (1 + o(1))\beta \ln \lambda_n, \quad n \to \infty.$$

if $\ln \lambda_{n+1} = (1+o(1)) \ln \lambda_n$ as $n \to \infty$. Since $\lambda_n = (1/(2(R[A]-r_n)))^{1/(1-\beta)}$, from hence for $r_n \le r \le r_{n+1}$ we get

$$\ln \ln \mu(r,A) \ge \ln \ln \mu(r_n,A) \ge (1+o(1)) \frac{\beta}{1-\beta} \ln \frac{1}{R[A]-r}, \quad r \uparrow R[A],$$

i.e., in view of the arbitrariness of β we obtain the inequality $\lambda[\mu] \ge \dfrac{\beta[A]}{1-\beta[A]}$, that is obvious if $\beta[A] = 0$. Thus, the following statement is true.

Proposition 2. *Let $\Gamma_f(x) \uparrow +\infty$ and $\Gamma_f(x) = x + O(l(x))$ as $x \to +\infty$, where l is a slowly increasing function. If $\ln \lambda_{n+1} = (1+o(1)) \ln \lambda_n$ and $\kappa_n \nearrow R[A]$ as $n \to \infty$ then $\lambda[\mu] = \beta[A]/(1-\beta[A])$.*

Combining Propositions 1 and 2 with Theorem 1, we arrive at the following theorem.

Theorem 2. *Let $\Gamma_f(x) = x + O(l(x))$ as $x \to +\infty$, where l is a slowly increasing function, and $\ln \ln n(t) = o(\ln t)$ as $t \to +\infty$. Then $\varrho[\mu] = \alpha[A]/(1-\alpha[A])$ and if, moreover, $\Gamma_f(x) \uparrow +\infty$ as $x \to +\infty$, $\ln \lambda_{n+1} = (1+o(1)) \ln \lambda_n$ and $\kappa_n \nearrow [A]$ as $n \to \infty$ then $\lambda[\mu] = \beta[A]/(1-\beta[A])$.*

3. Formulas for the Type and Lower Type

Let's start with the following analogue of Theorem 1.

Theorem 3. *If $\Gamma_f(x) = x + O(l(x))$ as $x \to +\infty$, where l is a slowly increasing function, and $\ln n(t) = o(t^{\varrho[A]/(\varrho[A]+1)})$ as $t \to +\infty$ then*

$$T[A] = T[\mu] := \varlimsup_{r \uparrow R[A]} (R[A]-r)^{\varrho[A]} \ln \mu(r,A)),$$

$$\tau[A] = \tau[\mu] := \varliminf_{r \uparrow R[A]} (R[A]-r)^{\varrho[A]} \ln \mu(r,A)).$$

Proof. Since $\ln n(t) \le q t^{\varrho[A]/(\varrho[A]+1)}$ for every $q \in (0,1)$ and all $t \ge t_0 = t_0(q)$, choosing $T(r) = \left(\dfrac{4p}{R[A]-r}\right)^{\varrho[A]+1}$, from (7) and (8) as above we get

$$\frac{\mathfrak{M}(r,A)}{\mu(r+q(R[A]-r),A)} \le$$

$$\le \frac{q(R[A]-r)}{2p} \left(\int_0^{t_0} + \int_{t_0}^{T(r)} + \int_{T(r)}^{\infty} \right) \exp\left\{ -\frac{q(R[A]-r)}{2p} t + \ln n(t) \right\} dt + K_5.$$

As in the proof of Theorem 1

$$\frac{q(R[A]-r)}{2p} \int_{T(r)}^{\infty} \exp\left\{ -\frac{q(R[A]-r)}{2p} t + \ln n(t) \right\} dt \le$$

$$\le \frac{q(R[A]-r)}{2p} \int_{T(r)}^{\infty} \exp\left\{ -\frac{q(R[A]-r)}{2p} t + q t^{\varrho[A]/(\varrho[A]+1)} \right\} dt =$$

$$= \frac{q(R[A]-r)}{2p} \int_{T(r)}^{\infty} \exp\left\{ -qt \left(\frac{R[A]-r}{2p} - \frac{1}{t^{1/(\varrho[A]+1)}} \right) \right\} dt \le$$

$$\leq \frac{q(R[A]-r)}{2p}\int\limits_{T(r)}^{\infty}\exp\left\{-qt\left(\frac{R[A]-r}{2p}-\frac{1}{T(r)^{1/(\varrho[A]+1)}}\right)\right\}dt=$$

$$=\frac{q(R[A]-r)}{2p}\int\limits_{T(r)}^{\infty}\exp\left\{-t\left(\frac{q(R[A]-r)}{4p}\right)\right\}dt\leq 2$$

and

$$\frac{q(R[A]-r)}{2p}\int\limits_{t_0}^{T(r)}\exp\left\{-\frac{q(R[A]-r)}{2p}t+\ln\, n(t)\right\}dt\leq$$

$$\leq\frac{q(R[A]-r)}{2p}T(r)\exp\{\ln\, n(T(r))\}=$$

$$=\frac{q(4p)^{\varrho[A]+1}}{2p}\frac{1}{(R[A]-r)^{\varrho[A]}}\exp\left\{q\left(\frac{4p}{R[A]-r}\right)^{\varrho[A]}\right\}$$

and

$$\frac{q(R[A]-r)}{2p}\int\limits_{0}^{t_0}\exp\left\{-\frac{q(R[A]-r)}{2p}t+\ln\, n(t)\right\}dt\leq K_6.$$

Thus, we get

$$\ln\,\mathfrak{M}(r,A)\leq\ln\,\mu(r+q(R[A]-r),A)+q\left(\frac{4p}{R[A]-r}\right)^{\varrho[A]}+O\left(\ln\,\frac{1}{R[A]-r}\right),$$

i.e.,

$$(R[A]-r)^{\varrho[A]}\ln\,\mathfrak{M}(r,A)\leq$$

$$\leq(R[A]-r-q(R[A]-r))^{\varrho[A]}\ln\,\mu(r+q(R[A]-r),A))(1-q)^{-\varrho[A]}+4pq+o(1)$$

as $r\uparrow R[A]$, whence $T[A]\leq\dfrac{T[\mu]}{(1-q)^{\varrho[A]}}+4pq$ and $\tau[A]\leq\dfrac{\tau[\mu]}{(1-q)^{\varrho[A]}}+4pq$. In view of the arbitrariness of q we get $T[A]\leq T[\mu]$ and $\tau[A]\leq\tau[\mu]$, which, together with the obvious inequalities $T[A]\geq T[\mu]$ and $\tau[A]\geq\tau[\mu]$, completes the proof of Theorem 3. □

Now we obtain formulas for finding $T[\mu]$ and $\tau[\mu]$ by the coefficients a_n.

Let $T[\mu]<+\infty$. Then $\ln\,\mu(r,A)\leq T/(R[A]-r))^{\varrho[A]}$ for every $T>T[\mu]$ and all $r\in[r_0(T),R[A])$. Therefore, as above,

$$\ln\,|a_n|\leq\frac{T}{(R[A]-r)^{\varrho[A]}}-r\lambda_n+K_7L(\lambda_n)$$

for all $n\geq 1$ and $r\in[r_0(T),R[A])$. Now we choose $r=r_n:=R[A]-\left(\frac{T\varrho[A]}{\lambda_n}\right)^{1/(\varrho[A]+1)}$. Then $r_n\geq r_0(T)$ for $n\geq n_0(T)$ and

$$\ln\,|a_n|\leq T\left(\frac{\lambda_n}{T\varrho[A]}\right)^{\varrho[A]/(\varrho[A]+1)}-R[A]\lambda_n+T\varrho\left(\frac{\lambda_n}{T\varrho[A]}\right)^{\varrho[A]/(\varrho[A]+1)}+K_7L(\lambda_n),$$

i.e.,

$$\ln\left(|a_n|e^{R[A]\lambda_n}\right)\leq(1+o(1))T(\varrho[A]+1)\left(\frac{\lambda_n}{T\varrho[A]}\right)^{\varrho[A]/(\varrho[A]+1)},\quad n\to\infty,$$

whence in view of the arbitrariness of T we get

$$\gamma[A]:=\varlimsup_{n\to\infty}\lambda_n^{-\varrho[A]/(\varrho[A]+1)}\ln\left(|a_n|e^{R[A]\lambda_n}\right)\leq(\varrho[A]+1)\varrho[A]^{-\varrho[A]/(\varrho[A]+1)}T[\mu]^{1/(\varrho[A]+1)},$$

i.e.,

$$T[\mu] \geq \frac{\varrho[A]^{\varrho[A]}}{(\varrho[A]+1)^{\varrho[A]+1}}\gamma[A]^{\varrho[A]+1}$$

On the contrary, if $\gamma[A] < +\infty$ then $\ln(|a_n|e^{R[A]\lambda_n}) \leq \gamma\lambda_n^{\varrho[A]/(\varrho[A]+1)}$ for every $\gamma > \gamma[A]$ and all $n \geq n_0 = n_0(\gamma)$. Therefore, if $\gamma^* > \gamma$ then as above we get

$$\ln \mu(r,A) \leq \max_{x\geq 0}\{(1+o(1))\gamma x^{\varrho[A]/(\varrho[A]+1)} - (R[A]-r)x\} + K_8 \leq$$

$$\leq \max_{x\geq 0}\{\gamma^* x^{\varrho[A]/(\varrho[A]+1)} - (R[A]-r)x\} + K_8 =$$

$$= \frac{\varrho[A]^{\varrho[A]}}{(\varrho[A]+1)^{\varrho[A]+1}}(\gamma^*)^{\varrho[A]+1}\left(\frac{1}{R[A]-r}\right)^{\varrho[A]} + K_8,$$

whence in view of the arbitrariness of γ^* and γ it follows that

$$T[\mu] \leq \frac{\varrho[A]^{\varrho[A]}}{(\varrho[A]+1)^{\varrho[A]+1}}\gamma[A]^{\varrho[A]+1}.$$

Thus, the following statement is true.

Proposition 3. *If $\Gamma_f(x) = x + O(l(x))$ as $x \to +\infty$, where l is a slowly increasing function, then*

$$T[\mu] = \frac{\varrho[A]^{\varrho[A]}}{(\varrho[A]+1)^{\varrho[A]+1}}\gamma[A]^{\varrho[A]+1}.$$

Now let $\kappa_n \uparrow R[A]$ as $n \to +\infty$ and

$$\delta[A] := \varliminf_{n\to\infty} \lambda_n^{-\varrho[A]/(\varrho[A]+1)} \ln(|a_n|e^{R[A]\lambda_n}) < +\infty.$$

Then for every $\delta > \delta[A]$ there exists a sequence $(n_k) \uparrow \infty$ such that

$$\ln(|a_{n_k}|e^{R[A]\lambda_{n_k}}) \leq \delta\lambda_{n_k}^{\varrho[A]/(\varrho[A]+1)}$$

and by Lemma 2 for $m \in \{n_j\}$ as above we have

$$\ln \mu(\kappa_m, A) \leq \delta\lambda_m^{\varrho[A]/(\varrho[A]+1)} - R[A]\lambda_m + \ln M_f(\kappa_m\lambda_m) \leq \ln \mu^*(\kappa_m),$$

where

$$\ln \mu^*(r) = \max_{n\geq 1}(\delta\lambda_n^{\varrho[A]/(\varrho[A]+1)} - R[A]\lambda_n + \ln M_f(r\lambda_n))$$

and as in the proof of Proposition 2 we get

$$\ln \mu(\kappa_m, A) \leq \ln \mu^*(\kappa_m, A) \leq \frac{\varrho[A]^{\varrho[A]}}{(\varrho[A]+1)^{\varrho[A]+1}}\delta^{\varrho[A]+1}\left(\frac{1}{R[A]-\kappa_m}\right)^{\varrho[A]},$$

whence in view of the arbitrariness of δ it follows that

$$\tau[\mu] := \varliminf_{r\uparrow R[A]} (R[A]-r)^{\varrho[A]} \ln \mu(r,A) \leq \varliminf_{m\to\infty} (R[A]-\kappa_m)^{\varrho[A]} \ln \mu(\kappa_m, A) \leq$$

$$\leq \frac{\varrho[A]^{\varrho[A]}}{(\varrho[A]+1)^{\varrho[A]+1}}\delta[A]^{\varrho[A]+1}.$$

On the other hand, if $\delta[A] > 0$ then $\ln(|a_n|e^{R[A]\lambda_n}) \geq \delta\lambda_n^{\varrho[A]/(\varrho[A]+1)}$ for every $\delta \in (0, \delta[A])$ and all $n \geq n_0 = n_0(\delta)$. Therefore, if we choose $r_n = R[A] - \varepsilon\delta\lambda_n^{-1/(\varrho[A]+1)}$, $0 < \varepsilon < 1$, then

$$\ln \mu(r_n, A) \geq \ln |a_n| + \ln M_f(r_n\lambda_n) \geq \delta\lambda_n^{\varrho[A]/(\varrho[A]+1)} - R[A]\lambda_n + r_n\lambda_n + O(L(r_n\lambda_n)) =$$

$$= \delta(1-\varepsilon)\lambda_n^{\varrho[A]/(\varrho[A]+1)} + O(L(\lambda_n)) = (1+o(1))\delta(1-\varepsilon)\lambda_n^{\varrho[A]/(\varrho[A]+1)}, \quad n \to \infty.$$

Similarly, one has as $n \to \infty$

$$\ln \mu(r_{n+1}, A) \geq (1+o(1))\delta(1-\varepsilon)\lambda_{n+1}^{\varrho[A]/(\varrho[A]+1)} = (1+o(1))\delta(1-\varepsilon)\lambda_n^{\varrho[A]/(\varrho[A]+1)},$$

if $\lambda_{n+1} = (1+o(1))\lambda_n$ as $n \to \infty$. Since $\lambda_n = (\varepsilon\delta/(R[A]-r_n))^{\varrho[A]+1}$, from hence for $r_n \leq r \leq r_{n+1}$ we get

$$\ln \mu(r, A) \geq \ln \mu(r_n, A) \geq (1+o(1))(1-\varepsilon)\varepsilon^{\varrho[A]}\delta^{\varrho[A]+1}\left(\frac{1}{R[A]-r}\right)^{\varrho[A]} =$$

$$= (1+o(1))\frac{\varrho[A]^{\varrho[A]}}{(\varrho[A]+1)^{\varrho[A]+1}}\delta^{\varrho[A]+1}\left(\frac{1}{R[A]-r}\right)^{\varrho[A]}, \quad r \uparrow R[A],$$

if we choose $\varepsilon = \varrho[A]/(\varrho[A]+1)$. Hence, in view of the arbitrariness of δ, we obtain the inequality

$$\tau[\mu] \geq \frac{\varrho[A]^{\varrho[A]}}{(\varrho[A]+1)^{\varrho[A]+1}}\delta[A]^{\varrho[A]+1},$$

that is obvious if $\delta[A] = 0$.

Thus, the following statement is true.

Proposition 4. *Let $\Gamma_f(x) \uparrow +\infty$ and $\Gamma_f(x) = x + O(l(x))$ as $x \to +\infty$, where l is a slowly increasing function. If $\lambda_{n+1} = (1+o(1))\lambda_n$ and $\kappa_n \uparrow R[A]$ as $n \to \infty$ then*

$$\tau[\mu] = \frac{\varrho[A]^{\varrho[A]}}{(\varrho[A]+1)^{\varrho[A]+1}}\delta[A]^{\varrho[A]+1}.$$

Combining Propositions 3 and 4 with Theorem 3, we arrive at the following theorem.

Theorem 4. *Let $\Gamma_f(x) = x + O(l(x))$ as $x \to +\infty$, where l is a slowly increasing function, and $\ln n(t) = o(t^{\varrho[A]/(\varrho[A]+1)})$ as $t \to +\infty$. Then*

$$T[\mu] = \varrho[A]^{\varrho[A]}(\varrho[A]+1)^{-(\varrho[A]+1)}\gamma[A]^{\varrho[A]+1}$$

and if, moreover, $\Gamma_f(x) \uparrow +\infty$ as $x \to +\infty$, $\lambda_{n+1} = (1+o(1))\lambda_n$ and $\kappa_n \uparrow R[A]$ as $n \to \infty$ then

$$\tau[\mu] = \varrho[A]^{\varrho[A]}(\varrho[A]+1)^{-(\varrho[A]+1)}\delta[A]^{\varrho[A]+1}.$$

4. Two-Member Asymptotic

The following result is proved in [3].

Proposition 5. *Let $0 < R[A] < +\infty$, $0 < \varrho_1 < \varrho[A]$ and $T_1 \neq 0$. If $\Gamma_f(r) \asymp r$ and $\ln n(t) = O(\ln t)$ as $t \to +\infty$ then*

$$\ln \mathfrak{M}(r, A) \leq \frac{T[A]}{(R[A]-r)^{\varrho[A]}} + \frac{(1+o(1))T_1}{(R[A]-r)^{\varrho_1}}, \quad r \uparrow R[A],$$

if and only if

$$\ln \mu(r,A) \le \frac{T[A]}{(R[A]-r)^{\varrho[A]}} + \frac{(1+o(1))T_1}{(R[A]-r)^{\varrho_1}}, \quad r \uparrow R[A]. \tag{15}$$

Here we will study under what conditions on the coefficients a_n inequality (15) is true.

Theorem 5. *Let $\Gamma_f(x) = x + O(l(x))$ as $x \to +\infty$, where l is a slowly increasing function. Then* (15) *holds if and only if*

$$\ln\left(|a_n|e^{R[A]\lambda_n}\right) \le T[A](\varrho[A]+1)\left(\frac{\lambda_n}{T[A]\varrho[A]}\right)^{\frac{\varrho[A]}{\varrho[A]+1}} +$$

$$+(1+o(1))T_1\left(\frac{\lambda_n}{T[A]\varrho[A]}\right)^{\frac{\varrho_1}{\varrho[A]+1}}, \quad n \to \infty. \tag{16}$$

Proof. If (15) holds then

$$\ln|a_n| \le \ln \mu(r,A) - \ln M_f(r\lambda_n) \le \frac{T[A]}{(R[A]-r)^{\varrho[A]}} + \frac{(1+o(1))T_1}{(R[A]-r)^{\varrho_1}} - r\lambda_n + O(L(\lambda_n))$$

If we choose

$$r = r_n := R[A] - \left(\frac{T[A]\varrho[A]}{\lambda_n - \lambda_n^\eta}\right)^{\frac{1}{\varrho[A]+1}}, \quad 0 < \eta < \frac{1}{\varrho[A]+1},$$

then we get

$$\ln|a_n| \le T[A]\frac{(\lambda_n - \lambda_n^\eta)^{\frac{\varrho[A]}{\varrho[A]+1}}}{(T[A]\varrho[A])^{\frac{\varrho[A]}{\varrho[A]+1}}} + (1+o(1))T_1\left(\frac{\lambda_n}{T[A]\varrho[A]}\right)^{\frac{\varrho_1}{\varrho[A]+1}} -$$

$$-R[A]\lambda_n + \lambda_n\left(\frac{T[A]\varrho[A]}{\lambda_n - \lambda_n^\eta}\right)^{\frac{1}{\varrho[A]+1}}, \quad n \to \infty. \tag{17}$$

Since as $n \to \infty$ one has

$$\left(\frac{1}{\lambda_n - \lambda_n^\eta}\right)^{\frac{1}{\varrho[A]+1}} - \left(\frac{1}{\lambda_n}\right)^{\frac{1}{\varrho[A]+1}} = \frac{\lambda_n^{\frac{1}{\varrho[A]+1}} - (\lambda_n - \lambda_n^\eta)^{\frac{1}{\varrho[A]+1}}}{\lambda_n^{\frac{1}{\varrho[A]+1}}(\lambda_n - \lambda_n^\eta)^{\frac{1}{\varrho[A]+1}}} = \frac{1+o(1)}{\varrho[A]+1}\lambda_n^{\eta - \frac{\varrho[A]+2}{\varrho[A]+1}},$$

from (17) it follows that

$$\ln\left(|a_n|e^{R[A]\lambda_n}\right) \le T[A]\frac{\lambda_n^{\frac{\varrho[A]}{\varrho[A]+1}}}{(T[A]\varrho[A])^{\frac{\varrho[A]}{\varrho[A]+1}}} + (1+o(1))T_1\left(\frac{\lambda_n}{T[A]\varrho[A]}\right)^{\frac{\varrho_1}{\varrho[A]+1}} +$$

$$+(T[A]\varrho[A])^{\frac{1}{\varrho[A]+1}}\lambda_n^{\frac{\varrho[A]}{\varrho[A]+1}} + (1+o(1))(T[A]\varrho[A])^{1/(\varrho[A]+1)}\lambda_n^{\eta - 1/(\varrho[A]+1)} =$$

$$= T[A]^{\frac{1}{\varrho[A]+1}}(\varrho[A]+1)\left(\frac{\lambda_n}{\varrho[A]}\right)^{\frac{\varrho[A]}{\varrho[A]+1}} + (1+o(1))T_1\left(\frac{\lambda_n}{T[A]\varrho[A]}\right)^{\frac{\varrho_1}{\varrho[A]+1}}, \quad n \to \infty,$$

i.e., (16) holds.

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On the contrary, if (16) holds then

$$\ln \mu(r,A) \le \max_{n\ge 1}\left\{ T[A]^{\frac{1}{\varrho[A]+1}}(\varrho[A]+1)\left(\frac{\lambda_n}{\varrho[A]}\right)^{\frac{\varrho[A]}{\varrho[A]+1}} + \right.$$

$$\left. +(1+o(1))T_1\left(\frac{\lambda_n}{T[A]\varrho[A]}\right)^{\frac{\varrho_1}{\varrho[A]+1}} - R[A]\lambda_n + r\lambda_n \right\} + O(1) \le \max_{x\ge 0}\Phi(x) + \text{const},$$

where

$$\Phi(x) = \frac{T[A]^{\frac{1}{\varrho[A]+1}}(\varrho[A]+1)}{\varrho[A]^{\frac{\varrho[A]}{\varrho[A]+1}}}x^{\frac{\varrho[A]}{\varrho[A]+1}} + \frac{(1+o(1))T_1}{(T[A]\varrho[A])^{\frac{\varrho_1}{\varrho[A]+1}}}x^{\frac{\varrho_1}{\varrho[A]+1}} - (R[A]-r)x.$$

Calculating the first derivative of the function Φ in the variable x and equating it to zero, it can be shown that the function Φ reaches its maximum at the point

$$x(r) = \frac{T[A]\varrho[A]}{(R[A]-r)^{\varrho[A]+1}}\left(1 + \frac{(1+o(1))T_1\varrho_1}{T[A]\varrho[A]}(R[A]-r)^{\varrho[A]-\varrho_1}\right) = \frac{(1+o(1))T[A]\varrho[A]}{(R[A]-r)^{\varrho[A]+1}}$$

as $r \uparrow R[A]$ and

$$x(r)^{\frac{\varrho[A]}{\varrho[A]+1}} = \left(\frac{T[A]\varrho[A]}{(R[A]-r)^{\varrho[A]+1}}\right)^{\frac{\varrho[A]}{\varrho[A]+1}}\left(1 + \frac{(1+o(1))T_1\varrho_1}{T[A]\varrho[A]}(R[A]-r)^{\varrho[A]-\varrho_1}\right)^{\frac{\varrho[A]}{\varrho[A]+1}} =$$

$$= \frac{(T[A]\varrho[A])^{\frac{\varrho[A]}{\varrho[A]+1}}}{(R[A]-r)^{\varrho[A]}} + (1+o(1))(T[A]\varrho[A])^{-\frac{1}{\varrho[A]+1}}\frac{\varrho[A]}{\varrho[A]+1}\frac{T_1\varrho_1}{(R[A]-r)^{\varrho_1}}, \quad r\uparrow R[A].$$

Therefore,

$$\frac{T[A]^{\frac{1}{\varrho[A]+1}}(\varrho[A]+1)}{\varrho[A]^{\frac{\varrho[A]}{\varrho[A]+1}}}x^{\frac{\varrho[A]}{\varrho[A]+1}} = \frac{T[A](\varrho[A]+1)}{(R[A]-r)^{\varrho[A]}} + \frac{(1+o(1))T_1\varrho_1}{(R[A]-r)^{\varrho_1}}, \quad r\uparrow R[A],$$

$$\frac{T_1}{(T[A]\varrho[A])^{\frac{\varrho_1}{\varrho[A]+1}}}x(r)^{\frac{\varrho_1}{\varrho[A]+1}} = \frac{(1+o(1))T_1}{(R[A]-r)^{\varrho_1}}, \quad r\uparrow R[A],$$

and

$$-(R[A]-r)x(r) = -\frac{T[A]\varrho[A]}{(R[A]-r)^{\varrho[A]}}\left(1 + (1+o(1))\frac{T_1\varrho_1}{T[A]\varrho[A]}(R[A]-r)^{\varrho[A]-\varrho_1}\right) =$$

$$= -\frac{T[A]\varrho[A]}{(R[A]-r)^{\varrho[A]}} - \frac{(1+o(1))T_1\varrho_1}{(R[A]-r)^{\varrho_1}}, \quad r\uparrow R[A].$$

Hence, it follows that

$$\Phi(x(r)) = \frac{T[A]}{(R[A]-r)^{\varrho[A]}} + \frac{(1+o(1))T_1}{(R[A]-r)^{\varrho_1}}, \quad r\uparrow R[A],$$

that is (15) holds. Theorem 5 is proved. □

In conclusion, we will find the conditions under which

$$\ln \mu(r,A) = \frac{T[A]}{(R[A]-r)^{\varrho[A]}} + \frac{(1+o(1))T_1}{(R[A]-r)^{\varrho_1}}, \quad r\uparrow R[A]. \tag{18}$$

The following theorem is true.

Theorem 6. *Let $\Gamma_f(x) \uparrow +\infty$ and $\Gamma_f(x) = x + O(l(x))$ as $x \to +\infty$, where l is a slowly increasing function. If $\kappa_n \nearrow +R[A]$, $\lambda_n(R[A]-\kappa_n)^{\varrho_1} = O(1)$ and $\lambda_{n+1} - \lambda_n = o(\lambda_n^{(\varrho_1+1)/(\varrho[A]+1)})$ as $n \to \infty$, then (18) holds if and only if*

$$\ln\left(|a_n|e^{R[A]\lambda_n}\right) = T[A](\varrho[A]+1)\left(\frac{\lambda_n}{T[A]\varrho[A]}\right)^{\frac{\varrho[A]}{\varrho[A]+1}} +$$

$$+(1+o(1))T_1\left(\frac{\lambda_n}{T[A]\varrho[A]}\right)^{\frac{\varrho_1}{\varrho[A]+1}}, \quad n \to \infty. \tag{19}$$

Proof. Clearly, (18) implies (15) and by Theorem 5 (16) holds. From (18) we also get

$$\ln \mu(r,A) \geq \frac{T[A]}{(R[A]-r)^{\varrho}} + \frac{(1+o(1))T_1}{(R[A]-r)^{\varrho_1}}, \quad r \uparrow R[A].$$

By Lemma 2, $\ln \mu(\kappa_n, A) = \ln |a_n| + \ln M_f(\kappa_n\lambda_n)$ for all $n \geq 1$, that is in view of the condition $\lambda_n(R[A]-\kappa_n)^{\varrho_1} = O(1)$ as $n \to \infty$ we have

$$\ln |a_n| = \ln \mu(\kappa_n, A) - \ln M_f(\kappa_n\lambda_n) \geq$$

$$\geq \frac{T[A]}{(R[A]-\kappa_n)^{\varrho[A]}} + \frac{(1+o(1))T_1}{(R[A]-\kappa_n)^{\varrho_1}} - \kappa_n\lambda_n + O(L(\lambda_n))$$

$$\geq \frac{T[A]}{(R[A]-\kappa_n)^{\varrho[A]}} + \frac{(1+o(1))T_1}{(R[A]-\kappa_n)^{\varrho_1}} - \kappa_n\lambda_n \geq \min_{x\geq 0}\Phi_*(x),$$

where $\Phi_*(x) = \dfrac{T[A]}{(R[A]-x)^{\varrho[A]}} + \dfrac{(1+o(1))T_1}{(R[A]-x)^{\varrho_1}} - x\lambda_n$. Calculating the first derivative of the function Φ in the variable x and equating it to zero, it can be shown that the function Φ_* reaches its minimum at the point

$$x_n = R - \left(\frac{T[A]\varrho[A]}{\lambda_n}\right)^{\frac{1}{\varrho[A]+1}} - \frac{(1+o(1))T_1\varrho_1}{T[A]\varrho[A](\varrho[A]+1)}\left(\frac{T[A]\varrho[A]}{\lambda_n}\right)^{1-\frac{\varrho_1}{\varrho+1}}, \quad n \to \infty,$$

whence

$$\frac{1}{R[A]-x_n} = \left(\frac{\lambda_n}{T[A]\varrho[A]}\right)^{\frac{1}{\varrho[A]+1}}\left(1 - \frac{(1+o(1))T_1\varrho_1}{T[A]\varrho[A](\varrho[A]+1)}\left(\frac{\lambda_n}{T[A]\varrho[A]}\right)^{\frac{\varrho_1-\varrho[A]}{\varrho[A]+1}}\right), \quad n \to \infty.$$

Therefore,

$$\frac{T[A]}{(R[A]-x_n)^{\varrho[A]}} =$$

$$= T[A]\left(\frac{\lambda_n}{T[A]\varrho[A]}\right)^{\frac{\varrho[A]}{\varrho[A]+1}}\left(1 - \frac{(1+o(1))T_1\varrho_1}{T[A]\varrho[A](\varrho[A]+1)}\left(\frac{\lambda_n}{T[A]\varrho[A]}\right)^{\frac{\varrho_1-\varrho[A]}{\varrho[A]+1}}\right)^{\varrho[A]} =$$

$$= T[A]\left(\frac{\lambda_n}{T[A]\varrho[A]}\right)^{\frac{\varrho[A]}{\varrho[A]+1}}\left(1 - \frac{(1+o(1))T_1\varrho_1}{T[A](\varrho[A]+1)}\left(\frac{\lambda_n}{T[A]\varrho[A]}\right)^{\frac{\varrho_1-\varrho[A]}{\varrho[A]+1}}\right) =$$

$$= T[A]\left(\frac{\lambda_n}{T[A]\varrho[A]}\right)^{\frac{\varrho[A]}{\varrho[A]+1}} - \frac{(1+o(1))T_1\varrho_1}{\varrho[A]+1}\left(\frac{\lambda_n}{T[A]\varrho[A]}\right)^{\frac{\varrho_1}{\varrho[A]+1}}, \quad n \to \infty,$$

and

$$\frac{(1+o(1))T_1}{(R[A]-x)^{\varrho_1}} = (1+o(1))T_1\left(\frac{\lambda_n}{T[A]\varrho[A]}\right)^{\frac{\varrho_1}{\varrho[A]+1}}, \quad n\to\infty,$$

and

$$-x_n\lambda_n = -\left(R-\left(\frac{T[A]\varrho[A]}{\lambda_n}\right)^{\frac{1}{\varrho[A]+1}} - \frac{(1+o(1))T_1\varrho_1}{T[A]\varrho[A](\varrho[A]+1)}\left(\frac{T[A]\varrho[A]}{\lambda_n}\right)^{1-\frac{\varrho_1}{\varrho[A]+1}}\right)\lambda_n =$$

$$= -R\lambda_n + T[A]\varrho[A]\left(\frac{\lambda_n}{T[A]\varrho[A]}\right)^{\frac{\varrho[A]}{\varrho[A]+1}} + \frac{(1+o(1))T_1\varrho_1}{\varrho[A]+1}\left(\frac{\lambda_n}{T[A]\varrho[A]}\right)^{\frac{\varrho_1}{\varrho[A]+1}}, \quad n\to\infty.$$

Thus,

$$\Phi_*(x_n) = -R\lambda_n + T[A](\varrho[A]+1)\left(\frac{\lambda_n}{T[A]\varrho[A]}\right)^{\frac{\varrho[A]}{\varrho[A]+1}} + (1+o(1))T_1\left(\frac{\lambda_n}{T[A]\varrho[A]}\right)^{\frac{\varrho_1}{\varrho[A]+1}},$$

and, therefore,

$$\ln\left(|a_n|e^{R[A]\lambda_n}\right) \geq T[A](\varrho[A]+1)\left(\frac{\lambda_n}{T[A]\varrho[A]}\right)^{\frac{\varrho[A]}{\varrho[A]+1}} + (1+o(1))T_1\left(\frac{\lambda_n}{T[A]\varrho[A]}\right)^{\frac{\varrho_1}{\varrho[A]+1}} \tag{20}$$

as $n\to\infty$, that is in view of (16) we get (19).

On the other hand, (19) implies (20), that is

$$\ln\,\mu(r,A) \geq T[A](\varrho[A]+1)\left(\frac{\lambda_n}{T[A]\varrho[A]}\right)^{\frac{\varrho[A]}{\varrho[A]+1}} + (1+o(1))T_1\left(\frac{\lambda_n}{T[A]\varrho[A]}\right)^{\frac{\varrho_1}{\varrho[A]+1}} -$$

$$-(R-r)\lambda_n + O(L(\lambda_n)), \quad n\to\infty.$$

We put $r_n = R-\left(\frac{T[A]\varrho[A]}{\lambda_n}\right)^{1/(\varrho[A]+1)}$. From the condition

$$\lambda_{n+1}-\lambda_n = o(\lambda_n^{(\varrho_1+1)/(\varrho[A]+1)}) \text{ as } n\to\infty$$

it follows that

$$\frac{1}{(R-r_{n+1})^{\varrho[A]}} - \frac{1}{(R-r_n)^{\varrho[A]}} = o\left(\frac{1}{(R-r_n)^{\varrho_1}}\right)$$

and

$$\frac{1}{(R-r_{n+1})^{\varrho_1}} - \frac{1}{(R-r_n)^{\varrho_1}} = o\left(\frac{1}{(R-r_n)^{\varrho_1}}\right)$$

as $n\to\infty$. Therefore, for $r_n\leq r\leq r_{n+1}$ we have

$$\ln\,\mu(r,A) \geq \ln\,\mu(r_n,A) \geq \frac{T[A](\varrho[A]+1)}{(R-r_n)^{\varrho[A]}} + \frac{(1+o(1))T_1}{(R-r_n)^{\varrho_1}} - \frac{T[A]\varrho[A]}{(R-r_n)^{\varrho[A]}} +$$

$$+O\left(L\left(\frac{1}{(R-r_n)^{\varrho[A]}}\right)\right) = \frac{T[A]}{(R-r_n)^{\varrho[A]}} + \frac{(1+o(1))T_1}{(R-r_n)^{\varrho_1}} =$$

$$= \frac{T[A]}{(R-r_{n+1})^{\varrho[A]}} + \frac{(1+o(1))T_1}{(R-r_{n+1})^{\varrho_1}} + T[A]\left(\frac{1}{(R-r_n)^{\varrho[A]}} - \frac{1}{(R-r_{n+1})^{\varrho[A]}}\right) +$$

$$+(1+o(1))T_1\left(\frac{1}{(R-r_n)^{\varrho_1}} - \frac{1}{(R-r_{n+1})^{\varrho_1}}\right) =$$

$$= \frac{T[A]}{(R-r_{n+1})^{\varrho[A]}} + \frac{(1+o(1))T_1}{(R-r_{n+1})^{\varrho_1}} \geq \frac{T[A]}{(R-r)^{\varrho[A]}} + \frac{(1+o(1))T_1}{(R-r)^{\varrho_1}}, \quad r\uparrow R[A],$$

i.e., in view of (15) we obtain (18). Theorem 6 is proved. □

5. Conclusions

In this paper, we have investigated the asymptotic growth properties of functions analytic in a finite disk that are represented by series in systems of functions of the form $A(z) = \sum_{n=1}^{\infty} a_n f(\lambda_n z)$, where f is an entire transcendental function and (λ_n) is a sequence of positive numbers tending to infinity. The analysis has been carried out under natural and fairly general assumptions on the growth of the function f and on the distribution of the sequence (λ_n).

First, we established that under appropriate regularity conditions on the function Γ_f and the counting function of the sequence (λ_n), the order and lower order of the function A coincide with the corresponding characteristics of its maximal term. In particular, Theorem 1 shows that the order and lower order of A can be completely determined via the asymptotic behavior of the maximal term $\mu(r, A)$. This reduces the study of the global growth of A to the analysis of a single dominant term of the defining series.

Next, explicit coefficientwise formulas for the order and lower order were obtained. Theorem 2 expresses these growth characteristics in terms of the limits involving $\ln\ln\left(|a_n|e^{R[A]\lambda_n}\right)$ and $\ln\lambda_n$, thereby revealing a precise quantitative connection between the growth of A near the boundary of its disk of convergence and the asymptotic behavior of its coefficients. These results extend and refine earlier coefficient criteria known for power series and Dirichlet series.

Furthermore, we derived analogous results for the type and lower type of the function A. Theorems 3 and 4 show that, under suitable additional assumptions, the type and lower type of A are also determined by the maximal term and can be calculated explicitly in terms of coefficient sequences. These formulas provide a complete description of the fine growth scale of $\ln M(r, A)$ in terms of the coefficients a_n and the sequence (λ_n).

Finally, the two-member power asymptotic of $\ln M(r, A)$ was studied in detail. Theorems 5 and 6 give necessary and sufficient conditions, formulated purely in terms of the coefficients a_n, under which a two-term asymptotic expansion holds as $r \uparrow R[A]$. These results can be viewed as precise analogues of the Lockhart–Straus-type inequalities for series in systems of functions and significantly sharpen known upper estimates.

Overall, the paper provides a unified and systematic approach to determining the main growth characteristics of analytic functions represented by series in general systems of functions. The obtained formulas demonstrate that the boundary growth of such functions is governed in a precise way by the asymptotic behavior of their coefficients. The methods developed here can be further applied to study growth on finer scales, such as logarithmic or generalized orders, and to other classes of functional series, offering a promising direction for future research.

The methods used in this article can be applied to study of the growth of the function $\ln \mathfrak{M}(r, A)$ on various scales, i.e., we can study, for example, the asymptotic properties of series (1) in the system of functions, regularly converging in a finite disc in terms of k-logarithmic orders, generalized α-orders and the like.

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