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The Cimmino Algorithm for Inverse Strongly-Monotone Mappings

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Abstract

In 2003 W. Takahashi and M. Toyodaestablished the weak convergence of an iteration process to solve a variational inequality problem induced by an inverse strongly-monotone mapping. Recently we proved that for the same iterative process, most of its exact iterates are approximate solutions of the variational inequality. It was also shown that the iteration process for solving a variational inequality problem for an inverse strongly-monotone mapping generates approximate solutions in the presence of computational errors. In this work we employ the Cimmino algorithm in order to generalize these results for common approximate solutions of a finite family of variational inequalities with inverse strongly-monotone mappings and of a finite family of fixed point problems in the presence of computational errors.

Keywords: Hilbert space; iteration; monotone operator; variational inequality

MSC: 47H05; 47H09

1. Introduction

In the seminal paper by W. Takahashi and M. Toyoda [1], an algorithm for finding a point that is a fixed point of a nonexpansive mapping and a solution of a variational inequality problem induced by an inverse strongly-monotone map was introduced and studied. W. Takahashi and M. Toyoda [1] proved the weak convergence of their method. But, in practice, only a finite number of iterates can be done. Therefore it is interesting to know that the algorithm produces an approximate solution and the number of iterations, which is needed in order to obtain an approximate solution. This goal was achieved in [2], where it was shown that for the algorithm introduced in [1], most of its exact iterates are approximate common solutions of a variational inequality induced by an inverse strongly-monotone operator. In our next work [3] we showed that the iterative process for solving the variational inequality problem induced by an inverse strongly-monotone map produces an approximate solution in the presence of small computational errors.

For other interesting results on variational inequalities, see [4,5].

In this paper our goal is to use the Cimmino algorithm [6] to extend the results of [2,3], obtained for a single variational inequality to a finite family of variational inequalities generated by inverse strongly-monotone mappings. We obtain an approximate common solution of a finite family of variational inequalities and of a finite family of fixed point problems in the presence of computational errors. Our results are established under summable and nonsummable computational errors. We prove that the algorithm produces a good approximate solution, if the sequence of computational errors is bounded from above by a small positive constant. It should be mentioned that iterative methods are very popular in the mathematical literature in order to solve variational inequalities and fixed

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point problems, but researchers in computer science prefer to work with the Cimmino algorithm because its usage allows for making calculations in parallel.

2. Preliminaries

In this section we introduce the notations, definitions and assumptions that are used throughout the paper.

Suppose that $(H, \langle \cdot, \cdot \rangle)$ is a real Hilbert space endowed with an inner product inducing the Euclidean norm

$$\|z\| = \langle z, z \rangle^{1/2}, \ z \in H,$$

$U \subset H$ is a nonempty, convex and closed set,

$$\alpha, \ \bar{c} \in (0, 1)$$

and that

$$0 < a \leq b < 2\alpha. \tag{1}$$

For every set $D \subset H$ and every $x \in H$, put

$$d(x, D) = \inf\{\|x - y\| : \ y \in D\}.$$

For every $z \in H$ and every $r > 0$ put

$$B(z, r) = \{u \in H : \ \|u - z\| \leq r\}.$$

Set $I(\xi) = \xi$, $\xi \in H$. In the sequel we employ the following useful lemma [7–9].

Lemma 1. *Assume that a set $D \subset H$ is nonempty, convex and closed. Then for any point $x \in H$ there exists a unique point $P_D(x) \in D$ for which*

$$\|x - P_D(x)\| = \inf\{\|x - v\| : \ v \in D\}.$$

Moreover, the following assertions are true.

1.

$$\|P_D(x) - P_D(y)\| \leq \|x - y\| \text{ for each pair of points } x, y \in H$$

and for every $x \in H$ and every $z \in D$,

$$\langle z - P_D(x), x - P_D(x) \rangle \leq 0,$$

$$\|z - P_D(x)\|^2 + \|x - P_D(x)\|^2 \leq \|z - x\|^2.$$

2. Assume that $x \in H$, $\xi \in D$ and that for every $z \in D$,

$$\langle z - \xi, x - \xi \rangle \leq 0.$$

Then $\xi = P_D(x)$.

3. For every $x, y \in H$,

$$\|P_D(x) - P_D(y)\|^2 + \|(I - P_D)(x) - (I - P_D)(y)\|^2 \leq \|x - y\|^2,$$

$$\|P_D(x) - P_D(y)\|^2 \leq \langle P_D(x) - P_D(y), x - y \rangle.$$

Suppose that $\mathcal{L}_1$ is a finite collection of pairs (F, K), where K is a nonempty, closed and convex subset of the set U, and $F : U \to H$ and that $\mathcal{L}_2$ is a finite collection of operators

$T : U \to U$. Assume that $\mathcal{L}_1 \cup \mathcal{L}_2$ is a nonempty set. (Note that one of the sets $\mathcal{L}_1$ or $\mathcal{L}_2$ may be empty.)

Assume that for every pair $(F, K) \in \mathcal{L}_1$ each $u, v \in K$, each $\epsilon \geq 0$,

$$\langle F(u) - F(v), u - v \rangle \geq \alpha \|F(u) - F(v)\|^2 \tag{2}$$

and define

$$S(F, K) = \{z \in K : \ \langle F(z), u - z \rangle \geq 0, \ u \in K\}, \tag{3}$$

$$S_\epsilon(F, K) = \{z \in U, \ B(z, \epsilon) \cap K \neq \emptyset,$$

$$\langle F(z), \xi - z \rangle \geq -\epsilon \|\xi - z\| - \epsilon \text{ for each point } \xi \in K\}. \tag{4}$$

If $(F, K) \in \mathcal{L}_1$, then the mapping F is called α-inverse strongly-monotone [1,10,11]. Examples of inverse strongly-monotone operators were considered in [1]. It is not difficult to show that if K is a closed, convex subset of U and $T : K \to K$ is a nonexpansive map, then $A = I - T$ is a $(1/2)$-inverse strongly-monotone mapping and $S(F, K)$ is the set of fixed points of the map T. Also, every strongly-monotone and Lipschitz continuous operator is inverse strongly-monotone [1].

For every mapping $T \in \mathcal{L}_2$ and every $\epsilon > 0$ set

$$\text{Fix}(T) := \{z \in U : \ T(z) = z\}, \tag{5}$$

$$\text{Fix}_\epsilon(T) := \{z \in U : \ \|T(z) - z\| \leq \epsilon\} \tag{6}$$

and assume that

$$\|z - x\|^2 \geq \|z - T(x)\|^2 + \bar{c}\|x - T(x)\|^2$$

$$\text{for all } x \in U \text{ and all } z \in \text{Fix}(T). \tag{7}$$

Set

$$S := [\cap_{(F,K)\in\mathcal{L}_1} S(F, K)] \cap [\cap_{T\in\mathcal{L}_2} \text{Fix}(T)]. \tag{8}$$

For each $\lambda > 0$ and each $(F, K) \in \mathcal{L}_1$ set

$$P_{F,K,\lambda} = P_K(I - \lambda F). \tag{9}$$

Set

$$l = \text{Card}(\mathcal{L}_1 \cup \mathcal{L}_2). \tag{10}$$

Fix

$$\widehat{\Delta} \in (0, l^{-1}]. \tag{11}$$

In this chapter we use the Cimmino algorithm [6].

Initialization Choose an arbitrary $x_0 \in X$.

Iterative step Assume that $k \geq 0$ is an integer and $x_k \in X$ is a current iterate. Choose

$$\alpha_T^{(k)} \geq \widehat{\Delta}, \ T \in \mathcal{L}_2,$$

$$\alpha_{F,C}^{(k)} \geq \widehat{\Delta}, \ (F, C) \in \mathcal{L}_1$$

such that

$$\sum_{(F,C)\in\mathcal{L}_1} \alpha_{F,C}^{(k)} + \sum_{T\in\mathcal{L}_2} \alpha_T^{(k)} = 1,$$

choose

$$\lambda_{k,F,C} \in [a, b], \ (F, C) \in \mathcal{L}_1$$

and set

$$x_{k+1} = \sum_{(F,C)\in\mathcal{L}_1} \alpha_{F,C}^{(k)} P_C(x_k - \lambda_{k,F,C} F(x_k)) + \sum_{T\in\mathcal{L}_2} \alpha_T^{(k)} T(x_k).$$

In this work, for a given positive number ϵ, we will find a point x that is an ϵ-approximate common solution of the problems associated with the family of mappings $\mathcal{L}_1 \cup \mathcal{L}_2$. We study inexact iterates of our method algorithm with summable and non-summable computational errors.

It is well known that algorithms for solving variational inequalities generate iterates that converge weakly to an element of S. Here, we analyze the behavior of the iterates generated by our algorithm, taking into account computational errors. More precisely, in practice the algorithm with a computational error δ generates that iterate $\{x_k\}_{k=0}^{\infty}$ such that for every nonnegative integer k,

$$\|x_{k+1} - \sum_{(F,C)\in\mathcal{L}_1} \alpha_{F,C}^{(k)} x_{k,F,C} - \sum_{T\in\mathcal{L}_2} \alpha_T^{(k)} x_{k,T}\| \le \delta,$$

where

$$\|x_{k,T} - T(x_k)\| \le \delta,\ T \in \mathcal{L}_2,$$

for each $(F,C) \in \mathcal{L}_1$,

$$\|x_{k,F,C} - P_C(x_k - \lambda_{k,F,C} F(x_k))\| \le \delta.$$

Here the constant $\delta > 0$ depends only on our computer system. Our goal is to find out what subset of H attracts all iterates $\{x_k\}_{k=0}^{\infty}$ produced by the algorithm. It is shown that this set is

$$\widehat{S}_\epsilon := \{x \in X : x \in \mathrm{Fix}_\epsilon(T) \text{ for each } T \in \mathcal{L}_2$$

$$\text{and } x \in S_\epsilon(F,C) \text{ for all } (F,C) \in \mathcal{L}_1\}$$

with some positive number $\epsilon > 0$ that depends on δ.

It should be mentioned that the assumption

$$\alpha_T^{(k)} \ge \widehat{\Delta},\ T \in \mathcal{L}_2,$$

$$\alpha_{F,C}^{(k)} \ge \widehat{\Delta},\ (F,C) \in \mathcal{L}_1$$

plays a crucial role in our proofs. It is needed in order to take into account the influence of every operator for our next iteration. Because of this our results cannot be extended to infinite families of operators. The case with infinite families of operators will be studied in our future research using a different approach. Note that the constant α appears in (2) while the inequality $b < 2\alpha$ is assumed because of Proposition 2.

3. Auxiliary Results

In this work we employ the following useful results of [2].

Proposition 1. *Let $(F,K) \in \mathcal{L}_1$, $\lambda > 0$, $u \in K$. Then $u \in S(F,K)$ if and only if*

$$P_K(u - \lambda F(u)) = u.$$

Proposition 2. *Let $(F,K) \in \mathcal{L}_1$, $u, v \in U$ and $\lambda > 0$. Then*

$$\|(I - \lambda F)(u) - (I - \lambda F)(v)\|^2 \le \|u - v\|^2 + \lambda(\lambda - 2\alpha)\|F(u) - F(v)\|^2.$$

If $\lambda \le 2\alpha$, then $I - \lambda F$ is a nonexpansive operator.

Lemma 2. *Assume that* $(F, K) \in \mathcal{L}_1$,

$$x \in U,\ \lambda \in [a, b],\ u \in K,$$

$$u = P_K(u - \lambda F(u)),\ y = P_K(x - \lambda F(x)).$$

Then

$$\|y - u\|^2 \le \|x - u\|^2 + \lambda(\lambda - 2\alpha)\|F(u) - F(x)\|^2,$$

$$\|y - u\|^2 \le \|x - u\|^2 - \|y - x + \lambda(F(x) - F(u))\|^2$$

and

$$\|y - u\|^2 \le \|x - u\|^2 - 4^{-1}\|y - x\|^2 \min\{1, b^{-2}a(2\alpha - b)\}.$$

Lemma 3. *Assume that* $(F, K) \in \mathcal{L}_1$, $\epsilon_0 > 0$, $x \in U$, $\lambda \in [a, b]$ *and*

$$\|x - P_K(x - \lambda F(x))\| \le \epsilon_0.$$

Then for each $\xi \in K$,

$$\langle Fx, \xi - x\rangle \ge -\lambda^{-1}\epsilon_0\|\xi - x\| - \lambda^{-1}\epsilon_0^2 - \|F(x)\|\epsilon_0.$$

The following lemma was obtained in [3].

Lemma 4. *Assume that* $(F, K) \in \mathcal{L}_1$, $\delta \in (0, 1)$, $M_0 > 0$, $M \ge 1$,

$$\lambda \in [a, b],$$

$u \in K$, $x \in U$ *satisfy*

$$\|u\| \le M,\ u = P_K(x - \lambda F(u)),$$

$$\|x\| \le M,$$

$$\|F(u)\| \le M_0,\ \|F(x)\| \le M_0,$$

$$y \in B(P_K(x - \lambda F(x)), \delta).$$

Then

$$\|y - u\|^2 \le \|x - u\|^2 - \lambda(2\alpha - \lambda)\|F(u) - F(x)\|^2 + \delta(4M + 1),$$

$$\|y - u\|^2 \le \|x - u\|^2 - 4^{-1}\|y - x\|^2 \min\{1, b^{-2}a(2\alpha - b\} + 2\delta(8M + 2),$$

$$\|y - u\|^2 \le \|x - u\|^2 - \|y - x + \lambda(F(x) - F(u))\|^2 + \delta(12M + 2 + 2\lambda M_0).$$

4. Inexact Iterates with Summable Errors

We prove the following theorem, which demonstrates that most of the inexact iterates of our algorithm with summable computational errors are approximate common solutions of the family of variational inequalities and approximate common fixed points for the family of mappings $\mathcal{L}_2$. Note that algorithms with summable computational errors are studied in the superiorization theory [12–14].

Theorem 1. *Let* $\{\Delta_i\}_{i=0}^{\infty} \subset (0, 1)$ *satisfy*

$$\Delta = \sum_{i=0}^{\infty} \Delta_i < \infty, \tag{12}$$

$\epsilon \in (0,1)$, $M_0 > 1$ *be such that*

$$\{u \in S \cap B(0,M_0) : \ F(u) \in B(0,M_0), \ (F,C) \in \mathcal{L}_1\} \neq \emptyset, \tag{13}$$

$$\lambda_{i,F,C} \in [a,b], \ (F,C) \in \mathcal{L}_1, \ i = 0,1,\ldots, \tag{14}$$

$\epsilon_0 > 0$ *satisfy*

$$\epsilon_0 \leq \min\{4^{-1}a, \ 4^{-1}\epsilon(M_0 + \alpha^{-1}(2M_0 + 2\Delta))^{-1}\}, \tag{15}$$

and let a natural number n_0 *satisfy for each integer* $n \geq n_0$,

$$\Delta_n \leq \epsilon_0. \tag{16}$$

Assume that

$$\{x_k\}_{k=0}^{\infty} \subset U, \ \{x_{k,T}\}_{k=0}^{\infty} \subset U, \ T \in \mathcal{L}_2,$$
$$\{x_{k,F,C}\}_{k=0}^{\infty} \subset X, \ (F,C) \in \mathcal{L}_1,$$
$$\|x_0\| \leq M_0, \tag{17))}$$

$$\{\alpha_T^{(k)}\}_{k=0}^{\infty} \subset [\widehat{\Delta}, \infty), \ T \in \mathcal{L}_2, \tag{18}$$

$$\{\alpha_{F,C}^{(k)}\}_{k=0}^{\infty} \subset [\widehat{\Delta}, \infty), \ (F,C) \in \mathcal{L}_1 \tag{19}$$

and that for each integer $i \geq 0$,

$$\sum_{(F,C)\in\mathcal{L}_1} \alpha_{F,C}^{(i)} + \sum_{T\in\mathcal{L}_2} \alpha_T^{(i)} = 1, \tag{20}$$

$$\|x_{i,T} - T(x_i)\| \leq \Delta_i, \ T \in \mathcal{L}_2, \tag{21}$$

for each $(F,C) \in \mathcal{L}_1$,

$$\|x_{i,F,C} - P_C(x_i - \lambda_{i,F,C}F(x_i))\| \leq \Delta_i \tag{22}$$

and

$$\|x_{i+1} - \sum_{(F,C)\in\mathcal{L}_1} \alpha_{F,C}^{(i)} x_{i,F,C} - \sum_{T\in\mathcal{L}_2} \alpha_T^{(i)} x_{i,T}\| \leq \Delta_i. \tag{23}$$

Then

$$\|x_i\| \leq 3M_0 + 4\Delta, \ i = 0,1,\ldots$$

and

$$Card(\{i \in \{0,1,\ldots\} : \ \max\{\|x_i - x_{i,T}\|, T \in \mathcal{L}_2, \ \|x_i - x_{i,F,C}\|, \ (F,C) \in \mathcal{L}_1\} > \epsilon_0\})$$
$$\leq n_0 + 4((2M_0 + 2\Delta)^2 + 2\Delta(28M_0 + 20\Delta + 3))\widehat{\Delta}^{-1}\epsilon_0^{-2}\min\{\bar{c}, \ b^{-2}(2\alpha - b)\}^{-1}.$$

Moreover, if $i \geq n_0$ *is an integer and for each* $T \in \mathcal{L}_2$,

$$\|x_i - x_{i,T}\| \leq \epsilon_0$$

and for each $(F,C) \in \mathcal{L}_1$,

$$\|x_i - x_{i,F,C}\| \leq \epsilon_0$$

then

$$x_i \in Fix_\epsilon(T) \text{ for all } T \in \mathcal{L}_2,$$
$$x_i \in S_\epsilon(F,C)) \text{ for all } (F,C) \in \mathcal{L}_1.$$

Proof. Set

$$M_1 = M_0 + \alpha^{-1}(2M_0 + 2\Delta). \tag{24}$$

In view of (13), there exists

$$u_* \in B(0, M_0) \cap S \tag{25}$$

such that

$$F(u_*) \in B(0, M_0),\ (F, C) \in \mathcal{L}_1. \tag{26}$$

In our proof for an integer $t \geq 0$, using Propositions 1, 2 and the convexity of the norm, we obtain

$$\|u_* - x_{t+1}\| \leq 2\Delta_t + \|x_t - u_*\|,$$

and using Lemma 4 and the convexity of the function $\|\cdot\|^2$, we obtain an estimate (42) for

$$\|u_* - x_t\|^2 - \|u_* - x_{t+1}\|.$$

Then for an arbitrary natural number $Q > n_0$, summing that estimate for all $t = n_0, \ldots, Q-1$, we obtain the bound from above for the cardinality of the set

$$\{i \in \{n_0, \ldots, Q-1\} :$$

$$\max\{\|x_i - x_{i,T}\|,\ T \in \mathcal{L}_2,\ \|x_i - x_{i,F,C}\|,\ T \in \mathcal{L}_1\} > \epsilon_0\}.$$

It turns out that this bound does not depend on Q and therefore it is also a bound from above for the cardinality of the set

$$\{i \in \{n_0, n_0+1, \ldots,\} :$$

$$\max\{\|x_i - x_{i,T}\|,\ T \in \mathcal{L}_2,\ \|x_i - x_{i,F,C}\|,\ T \in \mathcal{L}_1\} > \epsilon_0\}.$$

It follows from (17) and (25) that

$$\|u_* - x_0\| \leq 2M_0. \tag{27}$$

Let $t \geq 0$ be an integer. It follows from (7), (21) and (25) that

$$\|x_{t,T} - u_*\| \leq \|x_{t,T} - T(x_t)\| + \|T(x_t) - u_*\| \leq \Delta_t + \|x_t - u_*\|. \tag{28}$$

By (22), (25) and Propositions 1 and 2,

$$\|x_{t,F,C} - u_*\| \leq \|x_{t,F,C} - P_C(x_t - \lambda_{t,F,C}F(x_t))\|$$

$$+\|P_C(x_t - \lambda_{t,F,C}(F(x_t)) - u_*\|$$

$$\leq \Delta_t + \|x_t - u_*\|. \tag{29}$$

By the convexity of the norm, (18)–(20), (23), (28) and (29),

$$\|u_* - x_{t+1}\| \leq \|x_{t+1} - \sum_{(F,C)\in\mathcal{L}_1} \alpha_{F,C}^{(t)} x_{t,F,C} - \sum_{T\in\mathcal{L}_2} \alpha_T^{(t)} x_{t,T}\|$$

$$+\|\sum_{(F,C)\in\mathcal{L}_1} \alpha_{F,C}^{(t)} x_{t,F,C} + \sum_{T\in\mathcal{L}_2} \alpha_T^{(t)} x_{t,T} - u_*\|$$

$$\leq \Delta_t + \sum_{T\in\mathcal{L}_2} \alpha_T^{(t)} \|u_* - x_{t,T}\| + \sum_{(F,C)\in\mathcal{L}_1} \alpha_{F,C}^{(t)} \|u_* - x_{t,F,C}\|$$

$$\leq 2\Delta_t + \|x_t - u_*\|. \tag{30}$$

It follows from (28)–(30) that for each integer $\tau \geq 0$, each $T \in \mathcal{L}_2$ and each $(F, C) \in \mathcal{L}_1$,

$$\|x_{\tau+1} - u_*\| \leq \|x_0 - u_*\| + 2\sum_{t=0}^{\tau} \Delta_t \leq 2M_0 + 2\sum_{t=0}^{\infty} \Delta_t, \tag{31}$$

$$\|x_{\tau,F,C} - u_*\|,\ \|x_{\tau,T} - u_*\| \leq \|x_0 - u_*\| + 2\sum_{t=0}^{\tau} \Delta_t \leq 2M_0 + 2\sum_{t=0}^{\tau} \Delta_t, \tag{32}$$

$$\|x_{\tau,F,C}\|,\ \|x_{\tau,T}\|,\ \|x_\tau\| \leq 3M_0 + 2\Delta. \tag{33}$$

Assume that $t \geq 0$ is an integer. Let $T \in \mathcal{L}_2$. By (7) and (25),

$$\|u_* - T(x_t)\|^2 + \bar{c}\|x_t - T(x_t)\|^2 \leq \|u_* - x_t\|^2. \tag{34}$$

By (21), (31) and (34),

$$\begin{aligned} &|\|u_* - T(x_t)\|^2 - \|u_* - x_{t,T}\|^2| \\ &= |\|u_* - T(x_t)\| - \|u_* - x_{t,T}\||(\|u_* - T(x_t)\| + \|u_* - x_{t,T}\|) \\ &\leq \|T(x_t) - x_{t,T}\|(2\|u_* - T(x_t)\| + \|T(x_t) - x_{t,T}\|) \\ &\leq \Delta_t(2\|x_t - u_*\| + \Delta_t) \leq \Delta_t(4M_0 + 5\Delta). \end{aligned} \tag{35}$$

In view of (21), (31) and (34),

$$\begin{aligned} &|\|x_t - T(x_t)\|^2 - \|x_t - x_{t,T}\|^2| \\ &= |\|x_t - T(x_t)\| - \|x_t - x_{t,T}\||(\|x_t - T(x_t)\| + \|x_t - x_{t,T}\|) \\ &\leq \|T(x_t) - x_{t,T}\|(2\|x_t - T(x_t)\| + \|T(x_t) - x_{t,T}\|) \\ &\leq \Delta_t(2\|x_t - T(x_t)\| + \Delta_t) \\ &\leq \Delta_t(2\bar{c}^{-1/2}\|x_t - u_*\| + \Delta_t) \leq \Delta_t\bar{c}^{-1/2}(4M_0 + 5\Delta). \end{aligned} \tag{36}$$

It follows from (34)–(36) that

$$\begin{aligned} &\|u_* - x_t\|^2 \geq \|u_* - T(x_t)\|^2 + \bar{c}\|x_t - T(x_t)\|^2 \\ &\geq \|u_* - x_{t,T}\|^2 - \Delta_t(4M_0 + 5\Delta) + \bar{c}\|x_t - x_{t,T}\|^2 - \Delta_t(4M_0 + 5\Delta) \end{aligned} \tag{37}$$

for each $T \in \mathcal{L}_2$.

Let $(f, C) \in \mathcal{L}_1$. In view of (2), (24), (25) and (31),

$$\|F(x_t) - F(u_*)\| \leq \alpha^{-1}\|u_* - x_t\| \leq \alpha^{-1}(2M_0 + 2\Delta),$$

$$\|F(x_t)\| \leq \alpha^{-1}(2M_0 + 2\Delta) + M_0 = M_1. \tag{38}$$

By (22), (25), (33), (38) and Lemma 4 applied with

$$M = 3M_0 + 2\Delta,\ M_0 = M_1,\ \lambda = \lambda_{t,F,C}, u = u_*,$$

$$x = x_t,\ y = x_{t,F,C},\ \delta = \delta_t,\ (A, K) = (F, C)$$

we have

$$\|x_{t,F,C} - u_*\|^2$$

$$\leq \|x_t - u_*\|^2 - 4^{-1}\|x_t - x_{t,F,C}\|^2 \min\{1,\ b^{-2}a(2\alpha - b)\} + 2(24M_0 + 16\Delta + 2)\Delta_t \quad (39)$$

for each $(F, C) \in \mathcal{L}_1$.

By the convexity of the function $\|\cdot\|^2$, (18)–(20), (37) and (39),

$$\begin{aligned}
&\|u_* - \sum_{(F,C)\in\mathcal{L}_1} \alpha_{F,C}^{(t)} x_{t,f,C} - \sum_{T\in\mathcal{L}_2} \alpha_T^{(t)} x_{t,T}\|^2 \\
&\leq \sum_{T\in\mathcal{L}_2} \alpha_T^{(t)} \|u_* - x_{t,T}\|^2 + \sum_{(F,C)\in\mathcal{L}_1} \alpha_{F,C}^{(k)} \|u_* - x_{t,F,C}\|^2 \\
&\leq \sum_{T\in\mathcal{L}_2} \alpha_T^{(t)} (\|u_* - x_t\|^2 - \bar{c}\|x_t - x_{t,T}\|^2 + \Delta_t(8M_0 + 10\Delta)) \\
&\quad + \sum_{(F,C)\in\mathcal{L}_1} \alpha_{F,C}^{(t)} (\|u_* - x_t\|^2 \\
&-4^{-1}\min\{1,\ b^{-2}a(2\alpha - b)\}\|x_t - x_{t,F,C}\|^2 + 2\Delta_t(24M_0 + 16\Delta + 2)) \\
&\leq \|u_* - x_t\|^2 - \widehat{\Delta}\min\{4^{-1}\bar{c},\ 4^{-1}b^{-2}a(2\alpha - b)\} \\
&\times \max\{\|x_t - x_{t,T}\|^2,\ T \in \mathcal{L}_2,\ \|x_t - x_{t,F,C}\|^2,\ (F,C) \in \mathcal{L}_1\} \\
&\quad + 2\Delta_t(24M_0 + 16\Delta + 21). \quad (40)
\end{aligned}$$

It follows from (23) and (31) that

$$\begin{aligned}
&|\|u_* - x_{t+1}\|^2 - \|u_* - \sum_{(F,C)\in\mathcal{L}_1} \alpha_{F,C}^{(t)} x_{t,f,C} - \sum_{T\in\mathcal{L}_2} \alpha_T^{(t)} x_{t,T}\|^2| \\
&\leq |\|u_* - x_{t+1}\| - \|u_* - \sum_{(F,C)\in\mathcal{L}_1} \alpha_{F,C}^{(t)} x_{t,F,C} - \sum_{T\in\mathcal{L}_2} \alpha_T^{(t)} x_{t,T}\|| \\
&\times (|\|u_* - x_{t+1}\| + \|u_* - \sum_{(F,C)\in\mathcal{L}_1} \alpha_{F,C}^{(t)} x_{t,F,C} - \sum_{T\in\mathcal{L}_2} \alpha_T^{(t)} x_{t,T}\|) \\
&\leq \|x_{t+1} - \sum_{(F,C)\in\mathcal{L}_1} \alpha_{F,C}^{(t)} x_{t,F,C} - \sum_{T\in\mathcal{L}_2} \alpha_T^{(tk)} x_{k,T}\| \\
&\times (2\|u_* - x_{t+1}\| + \|x_{t+1} - \sum_{(F,C)\in\mathcal{L}_1} \alpha_{F,C}^{(t)} x_{t,F,C} - \sum_{T\in\mathcal{L}_2} \alpha_T^{(t)} x_{t,T}\|) \\
&\leq \Delta_t(4M_0 + 4\Delta + \Delta_t). \quad (41)
\end{aligned}$$

In view of (40) and (41),

$$\begin{aligned}
&\|u_* - x_{t+1}\|^2 \leq \Delta_t(4M_0 + 4\Delta + 1) \\
&+\|u_* - \sum_{(F,C)\in\mathcal{L}_1} \alpha_{F,C}^{(t)} x_{t,F,C} - \sum_{T\in\mathcal{L}_2} \alpha_T^{(k)} x_{t,T}\|^2 \\
&\leq \|u_* - x_t\|^2 - 4^{-1}\widehat{\Delta}\min\{\bar{c},\ b^{-2}a(2\alpha - b)\} \\
&\times \max\{\|x_t - x_{t,T}\|^2,\ T \in \mathcal{L}_2,\ \|x_t - x_{t,F,C}\|^2,\ (F,C) \in \mathcal{L}_1\} \\
&\quad + 2\Delta_t(28M_0 + 20\Delta + 3). \quad (42)
\end{aligned}$$

Set

$$\begin{aligned}
E = \{i \in \{n_0, n_0 + 1, \ldots,\} :& \\
\max\{\|x_i - x_{i,T}\|,\ T \in \mathcal{L}_2,\ \|x_i - x_{i,F,C}\|,\ T \in \mathcal{L}_1\} > \epsilon_0\}. \quad (43)&
\end{aligned}$$

Let $Q > n_0$ be a natural number. Set

$$E_Q = E \cap \{0, \dots, Q-1\}. \tag{44}$$

By (31) and (42)–(44),

$$(2M_0 + 2\Delta)^2 \geq \|x_{n_0} - u_*\|^2 \geq \|x_{n_0} - u_*\|^2 - \|x_Q - u_*\|^2$$

$$= \sum_{i=n_0}^{Q-1} (\|x_i - u_*\|^2 - \|x_{i+1} - u_*\|^2)$$

$$\geq \sum_{i=n_0}^{Q-1} (4^{-1}\widehat{\Delta} \min\{\bar{c},\ b^{-2}a(2\alpha - b)\}$$

$$\times \max\{\|x_i - x_{i,T}\|^2,\ T \in \mathcal{L}_2,\ \|x_i - x_{i,F,C}\|^2,\ (F,C) \in \mathcal{L}_1\}$$

$$-2\Delta_i(28M_0 + 20\Delta + 3)).$$

By the relation above,

$$4((2M_0 + 2\Delta)^2 + 2\Delta(28M_0 + 20\Delta + 3))\widehat{\Delta}^{-1} \min\{\bar{c},\ b^{-2}a(2\alpha - b)\}^{-1}$$

$$\geq \sum_{i=n_0}^{Q-1} \max\{\|x_i - x_{i,T}\|^2,\ T \in \mathcal{L}_2,\ \|x_i - x_{i,F,C}\|^2,\ (F,C) \in \mathcal{L}_1\}$$

$$\geq \epsilon_0^2 \mathrm{Card}(E_Q)$$

and

$$\mathrm{Card}(E_Q) \leq 4((2M_0 + 2\Delta)^2 + 2\Delta(28M_0 + 20\Delta + 3))\widehat{\Delta}^{-1} \min\{\bar{c},\ b^{-2}a(2\alpha - b)\}^{-1}\epsilon_0^{-2}.$$

Since Q is any natural number larger than n_0, in view of (43) and (44), we conclude that

$$\mathrm{Card}(\{t \in \{n_0, n_0+1, \dots\} :\ \max\{\|x_t - x_{t,T}\| :$$

$$T \in \mathcal{L}_2,\ \|x_t - x_{t,F,C}\|,\ (F,C) \in \mathcal{L}_1\} > \epsilon_0\}) = \mathrm{Card}(E)$$

$$\leq 4((2M_0 + 2\Delta)^2 + 2\Delta(28M_0 + 20\Delta + 3))\widehat{\Delta}^{-1} \min\{\bar{c},\ b^{-2}a(2\alpha - b)\}^{-1}\epsilon_0^{-2}.$$

Assume that $p \geq n_0$ is an integer and

$$\|x_p - x_{p,T}\| \leq \epsilon_0,\ T \in \mathcal{L}_2, \tag{45}$$

$$\|x_p - x_{p,F,C}\| \leq \epsilon_0,\ (F,C) \in \mathcal{L}_1. \tag{46}$$

By (16), (21) and (45), for each $T \in \mathcal{L}_2$,

$$\|x_p - T(x_p)\| \leq \|x_p - x_{p,T}\| + \|x_{p,T} - T(x_p)\| \leq \epsilon_0 + \Delta_p \leq 2\epsilon_0 < \epsilon,$$

$$x_p \in \mathrm{Fix}_\epsilon(T). \tag{47}$$

Let $(F,C) \in \mathcal{L}_1$. In view of (2), (24) and (31),

$$\|F(x_p) - F(u_*)\| \leq \alpha^{-1}\|x_p - u_*\| \leq (2M_0 + 2\Delta)\alpha^{-1},$$

$$\|F(x_p)\| \leq \|F(u_*)\| + \|F(x_p) - F(u_*)\| \leq M_0 + \alpha^{-1}(2M_0 + 2\Delta) = M_1. \tag{48}$$

By (16), (22) and (46),

$$\|x_p - P_C(x_p - \lambda_{p,F,C}F(x_p))\|$$
$$\leq \|x_p - x_{p,F,C}\| + \|x_{p,F,C} - P_C(x_p - \lambda_{p,F.C}F(x_p))\| \leq \epsilon_0 + \Delta_p \leq 2\epsilon_0. \tag{49}$$

It follows from (14), (15), (48), (49) and Lemma 3 applied with ϵ_0 replaced by $2\epsilon_0$, $x = x_p$, $\lambda = \lambda_{p,F,C}$ that for each $\xi \in C$,

$$\langle F(x_p), \xi - x_p\rangle \geq -2\lambda_{p,F,C}^{-1}\epsilon_0\|\xi - x_p\| - 4\lambda_{p,F,C}^{-1}\epsilon_0^2 - 2\|F(x_p)\|\epsilon_0$$
$$\geq -2a^{-1}\epsilon_0\|\xi - x_p\| - 4a^{-1}\epsilon_0^2 - 2\epsilon_0(M_0 + \alpha^{-1}(2M_0 + 2\Delta))$$
$$\geq -\epsilon\|\xi - x_p\| - \epsilon,$$
$$x_p \in S_\epsilon(F,C),\ (F,C) \in \mathcal{L}_1.$$

Theorem 1 is proved. □

5. Inexact Iterates with Nonsummable Computational Errors

We will prove the following result (Theorem 2), which demonstrates that an ϵ-approximate common solution is achieved after n_0 iterations of the algorithm with the presence of computational errors bounded from above by a constant δ, where δ and n_0 are constants that depend on ϵ (see (51)–(53)).

Theorem 2. *Let $\epsilon \in (0,1]$, $M_0 \geq 1$ be such that*

$$\{\xi \in S \cap B(0,M_0):\ T(\xi) \in B(0,M_0),\ (T,C) \in \mathcal{L}_1\} \neq \emptyset, \tag{50}$$

$\epsilon_0 > 0$ satisfy

$$\epsilon_0 < \min\{4^{-1}a\epsilon,\ 4^{-1}(M_0+1)^{-1}(1+4\alpha^{-1})^{-1}\epsilon\}, \tag{51}$$

a positive number δ satisfy

$$\delta \leq 16^{-1}\epsilon_0^2(18M_0+5)^{-1}\min\{\bar{c},\ b^{-2}a(2\alpha - b)\}\widehat{\Delta} \tag{52}$$

and an integer

$$n_0 > 16M_0^2\epsilon_0^{-2}\widehat{\Delta}^{-1}\min\{\bar{c},\ 4^{-1},\ 4^{-1}b^{-2}a(2\alpha - b)\}^{-1}. \tag{53}$$

Assume that

$$\{x_t\}_{t=0}^\infty \subset U,\ \{x_{t,T}\}_{t=0}^\infty \subset U,\ T \in \mathcal{L}_2,$$
$$\{x_{t,F,C}\}_{t=0}^\infty \subset U,\ (F,C) \in \mathcal{L}_1,$$
$$\|x_0\| \leq M_0, \tag{54}$$

$$\{\alpha_T^{(t)}\}_{t=0}^\infty \subset [\widehat{\Delta},\infty),\ T \in \mathcal{L}_2, \tag{55}$$

$$\{\alpha_{F,C}^{(t)}\}_{t=0}^\infty \subset [\widehat{\Delta},\infty),\ (F,C) \in \mathcal{L}_1 \tag{56}$$

and that for each integer $t \geq 0$,

$$\lambda_{t,F,C} \in [a,b],\ (F,C) \in \mathcal{L}_1, \tag{57}$$

$$\sum_{(F,C)\in\mathcal{L}_1} \alpha_{F,C}^{(t)} + \sum_{T\in\mathcal{L}_2} \alpha_T^{(t)} = 1, \tag{58}$$

$$\|x_{t,T} - T(x_t)\| \le \delta,\ T \in \mathcal{L}_2, \tag{59}$$

for each $(F, C) \in \mathcal{L}_1$,

$$\|x_{t,F,C} - P_C(x_t - \lambda_{t,F,C}F(x_t))\| \le \delta \tag{60}$$

and

$$\|x_{t+1} - \sum_{(F,C)\in\mathcal{L}_1} \alpha_{F,C}^{(t)} x_{t,F,C} - \sum_{T\in\mathcal{L}_2} \alpha_T^{(t)} x_{t,T}\| \le \delta. \tag{61}$$

Then there is an integer $q \in [0, n_0]$ *such that*

$$\|x_q\| \le 3M_0 \tag{62}$$

and

$$\|x_q - x_{q,T}\| \le \epsilon_0,\ T \in \mathcal{L}_2,\ \|x_q - x_{q,F,C}\| \le \epsilon_0,\ (F, C) \in \mathcal{L}_1. \tag{63}$$

Moreover, if an integer $q \ge 0$, *and (62) and (63) hold, then*

$$x_q \in Fix_\epsilon(T) \text{ for all } T \in \mathcal{L}_2,$$

$$x_q \in S_\epsilon(F, C) \le \epsilon \text{ for all } (F, C) \in \mathcal{L}_1.$$

Note that Theorem 2 gives the estimates for δ and n_0, following from (51)–(53). By (51)–(53), $\delta = c_1\epsilon^2$ and $n_0 = c_2\epsilon^{-2}$, where $c_1 > 0$ and $c_2 > 0$ are constants that depend only on M_0.

Let $\epsilon \in (0, 1]$, a positive number δ be defined by relations (51) and (52), and let an integer $n_0 \ge 1$ satisfy inequality (53). Assume that we apply our algorithm in the presence of computational errors bounded from above by a positive constant δ and that our goal is to find an ϵ-approximate solution. According to Theorem 2, we should find the smallest integer $q \in [0, n_0]$ such that

$$\|x_q\| \le 3M_0$$

and (63) is true.

Proof of Theorem 2. In view of (50), there exists

$$u_* \in B(0, M_0) \cap S \tag{64}$$

such that

$$F(u_*) \in B(0, M_0),\ (F, C) \in \mathcal{L}_1. \tag{65}$$

It follows from (54) and (64) that

$$\|u_* - x_0\| \le 2M_0. \tag{66}$$

Assume that a nonnegative integer $\tilde{q}$ satisfies for each $t \in \{0, \tilde{q}\}$,

$$\max\{\|x_t - x_{t,T}\|,\ T \in \mathcal{L}_2,\ \|x_t - x_{t,F,C}\|,\ (F, C) \in \mathcal{L}_1\} > \epsilon_0. \tag{67}$$

In our proof for an integer $t \in \{0, \tilde{q}\}$, using Lemma 4 and the convexity of the function $\|\cdot\|^2$, we obtain an estimate (85) for

$$\|u_* - x_t\|^2 - \|u_* - x_{t+1}\|.$$

This estimate implies that $\tilde{q} < n_0$.

Assume that $t \geq 0$ is an integer and

$$\|u_* - x_t\| \leq 2M_0. \tag{68}$$

Let $T \in \mathcal{L}_2$. By (7), (64) and (68),

$$\|u_* - x_t\|^2 \geq \|u_* - T(x_t)\|^2 + \bar{c}\|x_t - T(x_t)\|^2, \tag{69}$$

$$\|u_* - T(x_t)\| \leq \|u_* - x_t\| \leq 2M_0, \tag{70}$$

$$\|x_t - T(x_t)\| \leq \bar{c}^{-1/2}\|u_* - x_t\| \leq 2M_0\bar{c}^{-1/2}. \tag{71}$$

Equations (59) and (70) imply that

$$\begin{aligned} &|\|u_* - T(x_t)\|^2 - \|u_* - x_{t,T}\|^2| \\ &= |\|u_* - T(x_t)\| - \|u_* - x_{t,T}\||(\|u_* - T(x_t)\| + \|u_* - x_{t,T}\|) \\ &\leq \|T(x_t) - x_{t,T}\|(2\|u_* - T(x_t)\| + \|T(x_t) - x_{t,T}\|) \leq \delta(4M_0 + \delta) \leq \delta(4M_0 + 1). \end{aligned} \tag{72}$$

In view of (59) and (71),

$$\begin{aligned} &|\|x_t - T(x_t)\|^2 - \|x_t - x_{t,T}\|^2| \\ &= |\|x_t - T(x_t)\| - \|x_t - x_{t,T}\||(\|x_t - T(x_t)\| + \|x_t - x_{t,T}\|) \\ &\leq \|T(x_t) - x_{t,T}\|(2\|x_t - T(x_t)\| + \|T(x_t) - x_{t,T}\|) \\ &\leq \delta(4\bar{c}^{-1/2}M_0 + 1). \end{aligned} \tag{73}$$

It follows from (69), (72) and (73) that

$$\begin{aligned} &\|u_* - x_t\|^2 \geq \|u_* - T(x_t)\|^2 + \bar{c}\|x_t - T(x_t)\|^2 \\ &\geq \|u_* - x_{t,T}\|^2 - \delta(4M_0 + 1) + \bar{c}\|x_t - x_{t,T}\|^2 - \delta(4M_0 + 1) \end{aligned} \tag{74}$$

for each $T \in \mathcal{L}_2$.

Let $(F, C) \in \mathcal{L}_1$. Relations (2), (65) and (68) imply that

$$\|F(x_t)\| \leq \|F(x_t) - F(u_*)\| + \|F(u_*)\| \leq \alpha^{-1}\|u_* - x_t\| + M_0 \leq 2\alpha^{-1}M_0 + M_0. \tag{75}$$

Set

$$M_1 + 2M_0\alpha^{-1} + M_0. \tag{76}$$

By (57), (60), (64), (65), (68), (75), (76) and Lemma 4 applied with

$$M = 3M_0,\ M_0 = M_1,\ \lambda = \lambda_{t,F,C}, u = u_*,\ x = x_t,\ y = x_{t,F,C},\ (A, K) = (F, C),$$

we have

$$\begin{aligned} &\|x_{t,F,C} - u_*\|^2 \\ &\leq \|x_t - u_*\|^2 - 4^{-1}\|x_t - x_{t,F,C}\|^2 \min\{1,\ b^{-2}a(2\alpha - b)\} + 2\delta(24M_0 + 2). \end{aligned} \tag{77}$$

It follows from (60), (64), (66) and Propositions 1, 2 that

$$\begin{aligned} &\|x_{t,F,C} - u_*\| \leq \|x_{t,F,C} - P_C(x_t - \lambda_{t,F,C}F(x_t))\| + \|P_C(x_t - \lambda_{t,F,C}F(x_t)) - u_*\| \\ &\leq \delta + \|x_t - u_*\| \leq 2M_0 + 1. \end{aligned} \tag{78}$$

Note that (77) and (78) holds for each $(F, C) \in \mathcal{L}_1$.

By the convexity of the function $\|\cdot\|^2$, (55), (56), (58), (74) and (77),

$$\begin{aligned}
&\|u_* - \sum_{(F,C)\in\mathcal{L}_1} \alpha_{F,C}^{(t)} x_{t,f,C} - \sum_{T\in\mathcal{L}_2} \alpha_T^{(t)} x_{t,T}\|^2 \\
&\le \sum_{T\in\mathcal{L}_2} \alpha_T^{(t)} \|u_* - x_{t,T}\|^2 + \sum_{(F,C)\in\mathcal{L}_1} \alpha_{F,C}^{(k)} \|u_* - x_{t,F,C}\|^2 \\
&\le \sum_{T\in\mathcal{L}_2} \alpha_T^{(t)} (\|u_* - x_t\|^2 - \bar{c}\|x_t - x_{t,T}\|^2 + 2\delta(4M_0 + 1)) \\
&\quad + \sum_{(F,C)\in\mathcal{L}_1} \alpha_{F,C}^{(t)} (\|u_* - x_t\|^2 \\
&-4^{-1}\min\{1,\ b^{-2}a(2\alpha - b)\}\|x_t - x_{t,F,C}\|^2 + 2\delta(24M_0 + 2)) \\
&\le \|u_* - x_t\|^2 - \widehat{\Delta}\min\{\bar{c},\ 4^{-1},\ 4^{-1}b^{-2}a(2\alpha - b)\} \\
&\times \max\{\|x_t - x_{t,T}\|^2,\ T \in \mathcal{L}_2,\ \|x_t - x_{t,F,C}\|^2,\ (F,C) \in \mathcal{L}_1\} \\
&\quad + 2\delta(24M_0 + 2). \qquad (79)
\end{aligned}$$

In view of the convexity of the norm and (55), (56), (58), (70) and (78),

$$\|u_* - \sum_{(F,C)\in\mathcal{L}_1} \alpha_{F,C}^{(t)} x_{t,F,C} - \sum_{T\in\mathcal{L}_2} \alpha_T^{(t)} x_{t,T}\| \le 2M_0 + 1. \qquad (80)$$

It follows from (61) and (80) that

$$\begin{aligned}
&|\|u_* - x_{t+1}\|^2 - \|u_* - \sum_{(F,C)\in\mathcal{L}_1} \alpha_{F,C}^{(t)} x_{t,F,C} - \sum_{T\in\mathcal{L}_2} \alpha_T^{(t)} x_{t,T}\|^2| \\
&\le |\|u_* - x_{t+1}\| - \|u_* - \sum_{(F,C)\in\mathcal{L}_1} \alpha_{F,C}^{(t)} x_{t,F,C} - \sum_{T\in\mathcal{L}_2} \alpha_T^{(t)} x_{t,T}\|| \\
&\times (|\|u_* - x_{t+1}\| + \|u_* - \sum_{(F,C)\in\mathcal{L}_1} \alpha_{F,C}^{(t)} x_{t,F,C} - \sum_{T\in\mathcal{L}_2} \alpha_T^{(t)} x_{t,T}\|) \\
&\le \|x_{t+1} - \sum_{(F,C)\in\mathcal{L}_1} \alpha_{F,C}^{(t)} x_{t,F,C} - \sum_{T\in\mathcal{L}_2} \alpha_T^{(t)} x_{t,T}\| \\
&\times (2\|\sum_{(F,C)\in\mathcal{L}_1} \alpha_{F,C}^{(t)} x_{t,F,C} + \sum_{T\in\mathcal{L}_2} \alpha_T^{(t)} x_{t,T} - u_*\| \\
&+ \|x_{t+1} - \sum_{(F,C)\in\mathcal{L}_1} \alpha_{F,C}^{(t)} x_{t,F,C} - \sum_{T\in\mathcal{L}_2} \alpha_T^{(t)} x_{t,T}\|) \\
&\le \delta(4M_0 + 2 + \delta) \le \delta(4M_0 + 3).
\end{aligned}$$

In view of (78) and the relation above,

$$\begin{aligned}
&\|u_* - x_{t+1}\|^2 \le \delta(4M_0 + 3) \\
&+ \|u_* - \sum_{(F,C)\in\mathcal{L}_1} \alpha_{F,C}^{(t)} x_{t,F,C} - \sum_{T\in\mathcal{L}_2} \alpha_T^{(k)} x_{t,T}\|^2 \\
&\le \|u_* - x_t\|^2 - \widehat{\Delta}\min\{\bar{c},\ 4^{-1},\ 4^{-1}b^{-2}a(2\alpha - b)\} \\
&\times \max\{\|x_t - x_{t,T}\|^2,\ T \in \mathcal{L}_2,\ \|x_t - x_{t,F,C}\|^2,\ (F,C) \in \mathcal{L}_1\}
\end{aligned}$$

$$+2\delta(28M_0+5). \tag{81}$$

Equations (61) and (78) imply that

$$\|x_{t+1}-u_*\| \le 2M_0+2. \tag{82}$$

Assume that

$$\max\{\|x_t-x_{t,T}\|,\ T\in\mathcal{L}_2,\ \|x_t-x_{t,F,C}\|,\ (F,C)\in\mathcal{L}_1\} \ge \epsilon_0. \tag{83}$$

By (52), (81) and (83),

$$\begin{aligned}&\|u_*-x_{t+1}\|^2\\ &\le \|u_*-x_t\|^2-\widehat{\Delta}\min\{\bar{c},\ 4^{-1},\ 4^{-1}b^{-2}a(2\alpha-b)\}-2\delta(28M_0+5)\\ &\le \|u_*-x_t\|^2-2^{-1}\widehat{\Delta}\min\{\bar{c},\ 4^{-1},\ 4^{-1}b^{-2}a(2\alpha-b)\}\epsilon_0^2.\end{aligned}$$

Thus in view of (81), (82) and the relation above, the following property holds:

(a) If $t\ge 0$ is an integer,

$$\|x_t-u_*\| \le 2M_0,$$

then

$$\begin{aligned}&\|x_{t+1}-u_*\| \le 2M_0+2,\\ &\|u_*-x_{t+1}\|^2 \le \|u_*-x_t\|^2\\ &-\widehat{\Delta}\min\{\bar{c},\ 4^{-1},\ 4^{-1}b^{-2}a(2\alpha-b)\}\\ &\times\max\{\|x_t-x_{t,T}\|^2,\ T\in\mathcal{L}_2,\ \|x_t-x_{t,F,C}\|^2,\ (F,C)\in\mathcal{L}_1\}\\ &+2\delta(28M_0+5)\end{aligned}$$

and if (83) holds, then

$$\begin{aligned}&\|u_*-x_{t+1}\|^2\\ &\le \|u_*-x_t\|^2-2^{-1}\widehat{\Delta}\min\{\bar{c},\ 4^{-1},\ 4^{-1}b^{-2}a(2\alpha-b)\}\epsilon_0^2.\end{aligned}$$

By (66), (67) and using property (a) by induction, we obtain

$$\|x_t-u_*\| \le 2M_*,\ t=0,\dots,\tilde{q}+1 \tag{84}$$

and for all $t=0,\dots,\tilde{q}$,

$$\begin{aligned}&\|u_*-x_t\|^2-\|u_*-x_{t+1}\|^2\\ &\ge 2^{-1}\widehat{\Delta}\min\{\bar{c},\ 4^{-1},\ 4^{-1}b^{-2}a(2\alpha-b)\}\epsilon_0^2.\end{aligned} \tag{85}$$

In view of (66) and (85),

$$\begin{aligned}&4M_0^2 \ge \|u_*-x_0\|^2\\ &\ge \|u_*-x_0\|^2-\|u_*-x_{\tilde{q}+1}\|^2\\ &\ge \sum_{p=0}^{\tilde{q}}[\|u_*-x_p\|^2-\|u_*-x_{p+1}\|^2]\\ &\ge 2^{-1}(\tilde{q}+1)\widehat{\Delta}\epsilon_0^2\min\{\bar{c},\ 4^{-1},\ 4^{-1}b^{-2}a(2\alpha-b)\},\\ &\tilde{q}+1 \le 8M_0^2\widehat{\Delta}^{-1}\epsilon_0^{-2}\min\{\bar{c},\ 4^{-1},\ 4^{-1}b^{-2}a(2\alpha-b)\}^{-1} < n_0.\end{aligned}$$

Thus we assumed that $\tilde{q} \geq 0$ is an integer and that for each $p \in \{0, \ldots, \tilde{q}\}$ (11.67) holds; we also determined that $\tilde{q} < n_0 - 1$ and that (84) holds for all $t \in \{0, \ldots, \tilde{q}+1\}$ and (85) holds for all $t \in \{0, \ldots, \tilde{q}\}$.

This implies that there exists an integer $q \in [0, n_0 - 1]$ such that for every integer t satisfying $0 \leq t < q$, (67) holds,

$$\|x_q - x_{q,T}\| \leq \epsilon_0,\ T \in \mathcal{L}_2,\ \|x_q - x_{q,F,C}\| \leq \epsilon_0,\ (F,C) \in \mathcal{L}_1, \tag{86}$$

for all $t = 0, \ldots, q$,

$$\|x_t - u_*\| \leq 2M_0,\ \|x_t\| \leq 3M_0.$$

Assume that q is a nonnegative integer, (86) holds and

$$\|x_q\| \leq 3M_0. \tag{87}$$

Let $T \in \mathcal{L}_2$. In view of (51), (52), (59), (86),

$$\|x_q - T(x_q)\| \leq \|x_q - x_{q,T}\| + \|x_{q,T} - T(x_q)\| \leq \epsilon_0 + \delta \leq 2\epsilon_0 \leq \epsilon,$$

$$x_q \in \mathrm{Fix}_\epsilon(T).$$

Let $(F,C) \in \mathcal{L}_1$. By (51), (52), (60) and (86),

$$\|x_q - P_C(x_q - \lambda_{q,F,C}F(x_q))\|$$

$$\leq \|x_q - x_{q,F,C}\| + \|x_{q,F,C} - P_C(x_q - \lambda_{q,F.C}F(x_q))\| \leq \epsilon_0 + \delta \leq 2\epsilon_0. \tag{88}$$

In view of (2), (64), (65) and (87),

$$\|F(x_q)\| \leq \|F(x_q) - F(u_*)\| + \|F(u_*)\| \leq \alpha^{-1}\|x_q - u_*\| + M_0 \leq 4M_0\alpha^{-1} + M_0. \tag{89}$$

It follows from (51), (57), (88), (89) and Lemma 3 applied with ϵ_0 replaced by $2\epsilon_0$, $x = x_q$, $\lambda = \lambda_{q,F,C}$, $u = u_*$, $(A,K) = (F,C)$ that for each $\xi \in C$,

$$\langle F(x_q), \xi - x_q\rangle \geq -2\lambda_{p,F,C}^{-1}\epsilon_0\|\xi - x_q\| - 4\lambda_{q,F,C}^{-1}\epsilon_0^2 - 2\epsilon_0(M_0 + 4\alpha^{-1}M_0)$$

$$\geq -2a^{-1}\epsilon_0\|\xi - x_q\| - 4a^{-1}\epsilon_0^2 - 2\epsilon_0(M_0 + 4\alpha^{-1}M_0)$$

$$\geq -\epsilon\|\xi - x_q\| - \epsilon,$$

$$x_q \in S_\epsilon(F,C),\ (F,C) \in \mathcal{L}_1.$$

Theorem 2 is proved. □

6. An Example

Let m be a natural number and $C_i \subset H$, $i = 1, \ldots, m$ are nonempty, closed and convex sets such that

$$C = \cap_{i=1}^m C_i \neq \emptyset.$$

We are interested to find a point that is closed to all the sets C_i, $i = 1, \ldots, m$. This point is an approximate solution of the feasibility problem.

$$\text{Find } x \in C,$$

which has applications in the computer tomography and the radiation therapy planning. This problem belongs to the class of problems studied in the paper. Indeed, set $\mathcal{L}_2 = \emptyset$ and

$$\mathcal{L}_1 = \{(I - P_{C_i}, H) : \ i = 1, \dots, m\}.$$

Since for all $i = 1, \dots, m$ the projection map P_{C_i} is nonexpansive, the operator $I - P_{C_i}$ is $(1/2)$-inverse strongly-monotone and

$$S(I - P_{C_i}, X) = C_i$$

ref. [1].

Assume that $M_0 \geq 10$ and $C_1 \subset B(0, M_0)$. Evidently, (50) is valid. Set $a = b = 1/2$, $\widehat{\Delta} = m^{-1}$,

$$\epsilon_0 = 72^{-1}(M_0 + 1)^{-1}\epsilon, \ \delta = 320^{-1}\epsilon_0^2 M_0^{-1} m^{-1}$$

and fix an integer

$$n_0 > 64 M_0^2 \epsilon_0^{-2} m.$$

It is not difficult to see that the assumption of Theorem 2 holds. We can apply our algorithm with the computational error δ and find the smallest integer $q \in [0, n_0]$ such that

$$\|x_q - x_{q, I - P_{C_i}, H}\| \leq \epsilon_0, \ i = 1, \dots, m,$$

which exists by Theorem 2. Then x_q is an ϵ-approximate solution by Theorem 2.

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