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Second-Order Differential Inequality Convexity

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Abstract

Some equivalent statements and basic properties for the generalized convexity assumption $p(x)f(x)+q(x)f'(x)+f''(x)\geq 0$ are proved. Then based on these conclusions, various Jensen type inequalities under the generalized convexity are established. The idea is to transform such $p(x),q(x)$-convex functions to some simpler $p(x),0$-convex or $0,q(x)$-convex functions, or even convex functions. Ky Fan and Wang-Wang type inequalities are also generalized as applications.

Keywords: differential inequality; convex function; Jensen inequality

MSC: 26A51; 26D15

1. Introduction

There are some studies on Jensen or Hermite-Hadamard inequality for twice differentiable convex functions f, that extend the condition

$$f''(x)\geq 0$$

to the second order differential inequality

$$f''(x)+q(x)f'(x)+p(x)f(x)+\alpha(x)\geq 0,$$

where q,p,α are different in specific references. Such examples are Theorem 3.9.57 in [1], Theorem 5 of Chapter 1 in [2–9], Proposition 1.3 of Chapter 5 in [10].

In this paper, we focus specifically on definitions and topics mentioned in [11,12].

In [11], the generalized convexity, $a(x)$-convex is defined and several equivalent statements are proved. One of these can be expressed by the following differential inequality for twice differentiable f:

$$f''(x)-a(x)f'(x)\geq 0. \tag{1}$$

Jensen type inequality and other inequalities are established.

In [12] the following condition is considered for twice differentiable continuous f:

$$f''(x)+q(x)f'(x)+p(x)f(x)\geq 0, \tag{2}$$

then Jensen type inequality is established as the following.

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Theorem 1. *Let $p, q, f : I \to \mathbb{R}$ be continuous functions with q differentiable, f twice differentiable such that $p(x)f(x) + q(x)f'(x) + f''(x) \geq 0$ almost everywhere on I. For $a_k \in I$, $(k = 1, \ldots, n)$ and $p_k \geq 0$ with $\sum_{k=1}^{n} p_k = 1$, we have*

$$\sum_{k=1}^{n} p_k \int_{\bar{a}}^{a_k} \omega_k(x) f(x) dx + \sum_{k=1}^{n} p_k f(a_k) \geq f(\bar{a}), \tag{3}$$

where $\omega_k(x) = (a_k - x)(p(x) - q'(x)) + q(x)$, and $\bar{a}$ means $\sum_{k=1}^{n} p_k a_k$.

Let $q(x) = -a(x), p(x) \equiv 0$, it reduces to $a(x)$-convex functions. In the following sections, we abbreviate (2) as $p(x), q(x)$-convex. Properties, some inequalities in [11] and new inequalities will be considered for $p(x), q(x)$-convex functions.

The paper is arranged as the following. In the second section, we establish some basic properties and equivalent statements for $p(x), q(x)$-convex functions. These are just some generalizations or continuity of the results and ideas in the previous references, but some are essential tools for the next section. In the third main section, we establish various Jensen type inequalities. The idea is different from the previous ones, as we transform these $p(x), q(x)$-convex functions to some simpler ones, or even convex functions. Then the inequalities obtained are not just "trivial form" by just adding the corresponding parts concerning $q(x)f'(x) + p(x)f(x)$ than Jensen inequality. The fourth application section is devoted to Ky Fan and Wang-Wang inequalities. As pointed out in several Remarks, for some p, q, the second differential inequalities naturally hold (without assumed as a condition). This means we get truly extensions or more accurate forms of Ky Fan type inequalities, under the same assumptions as the classical ones.

2. Preliminary Properties

In this section, we establish some equivalent statements and properties for $p(x), q(x)$-convex functions. We suppose that p, q are continuous functions on the whole closed interval I.

Theorem 2. *If f', f'' are continuous functions and $pf + qf'$ an integrable function on the whole closed interval I, f is $p(x), q(x)$-convex on the interval I if and only if*

$$(y - x)(f'(y) - f'(x)) + (y - x) \int_{x}^{y} (q(t)f'(t) + p(t)f(t))dt \geq 0 \tag{4}$$

for every $x, y \in I$.

Proof. Let (4) holds, then for $x, y \in I$ such that $x \neq y$ we have

$$\frac{f'(y) - f'(x)}{y - x} + \frac{1}{y - x} \int_{x}^{y} (q(t)f'(t) + p(t)f(t))dt \geq 0.$$

Taking the limit as $y \to x$ we get $f''(x) + q(x)f'(x) + p(x)f(x) \geq 0$.

Conversely, let f be a $p(x), q(x)$-convex function, for $x, y \in I$ such that $x < y$ we have

$$\int_{x}^{y} (f''(t) + q(t)f'(t) + p(t)f(t))dt \geq 0.$$

Since f'' is continuous, we have $\int_{x}^{y} f''(t)dt = f'(y) - f'(x)$. Furthermore, since $x < y$ we can multiply the inequality above with $y - x$ to get (4). If $x > y$ we have

$$\int_{y}^{x} (f''(t) + q(t)f'(t) + p(t)f(t))dt \geq 0.$$

Since f'' is continuous, we have $\int_y^x f''(t)dt = f'(x) - f'(y)$. Furthermore, since $y < x$ we can multiply the inequality above with $x - y$ to get (4). □

Theorem 3. *If f, f', f'' are continuous and $pf + qf'$ an integrable function on the closed interval I, f is $p(x), q(x)$-convex on the interval I if and only if*

$$f(y) - f(x) - f'(x)(y - x) + \int_x^y (y - t)(q(t)f'(t) + p(t)f(t))dt \geq 0$$

for all $x, y \in I$.

Proof. First apply Theorem 2, suppose that $s > x$, then (4) is equivalent to

$$f'(s) - f'(x) + \int_x^s (q(t)f'(t) + p(t)f(t))dt \geq 0.$$

For every $[x, y] \subset I$ we have

$$\int_x^y f'(s)ds - \int_x^y f'(x)ds + \int_x^y \int_x^s (q(t)f'(t) + p(t)f(t))dtds \geq 0$$

i.e.,

$$f(y) - f(x) - f'(x)(y - x) + \int_x^y (y - t)(q(t)f'(t) + p(t)f(t))dt \geq 0.$$

Analogously, for $s < x$ and $[x, y] \subset I$, we have similar discussions. Since we have equivalence in each step, the proof is completed. □

Remark 1. *Due to the assumptions about the continuity, that "f is $p(x), q(x)$-convex on the interval I" is equivalent to "f is $p(x), q(x)$-convex almost everywhere on the interval I".*

Then we establish Jensen type inequality for $p(x), q(x)$-convex functions. The form is almost the same as in Theorem 1, but here we point out that, the condition that q is differentiable is unnecessary.

Theorem 4. *Let $f : I \to \mathbb{R}$ be $p(x), q(x)$-convex with f, f', f'' continuous and $pf + qf'$ integrable on the closed interval I. For $a_k \in I$, $(k = 1, \ldots, n)$ and $p_k \geq 0$ with $\sum_{k=1}^n p_k = 1$, we have*

$$\sum_{k=1}^n p_k \int_{\overline{a}}^{a_k} (a_k - x)(q(x)f'(x) + p(x)f(x))dx + \sum_{k=1}^n p_k f(a_k) \geq f(\overline{a}),$$

where $\overline{a}$ means $\sum_{k=1}^n p_k a_k$.

Proof. Apply Theorem 3 with $x = \overline{a}, y = a_k$, we obtain:

$$f(a_k) - f'(\overline{a})(a_k - \overline{a}) + \int_{\overline{a}}^{a_k} (a_k - t)(q(t)f'(t) + p(t)f(t))dt \geq f(\overline{a}),$$

multiplying this inequality with p_k and summing over k from 1 to n, we get the desired result. □

Theorem 5. *With f, f', f'' continuous and $pf + qf'$ integrable on the closed interval I, a twice differentiable continuous function f is $p(x), q(x)$-convex on interval I if and only if the continuous function $F : I \to \mathbb{R}$ defined by*

$$F(s) = f(s) + \int_{x_0}^s (s - t)(q(t)f'(t) + p(t)f(t))dt$$

is convex, where $x_0 \in I$.

Proof. If f is $p(x), q(x)$-convex, then Theorem 4 holds, which is equivalent to

$$\sum_{k=1}^{n} p_k \int_{x_0}^{a_k} (a_k - x)(q(x)f'(x) + p(x)f(x))dx + \sum_{k=1}^{n} p_k f(a_k)$$
$$\geq f(\overline{a}) + \sum_{k=1}^{n} p_k \int_{x_0}^{\overline{a}} (a_k - x)(q(x)f'(x) + p(x)f(x))dx,$$

further noticing that

$$\sum_{k=1}^{n} p_k \int_{x_0}^{\overline{a}} (a_k - x)(q(x)f'(x) + p(x)f(x))dx$$
$$= \int_{x_0}^{\overline{a}} (\overline{a} - x)(q(x)f'(x) + p(x)f(x))dx,$$

so from inequality above we get

$$\sum_{k=1}^{n} p_k F(a_k) \geq F(\overline{a}),$$

hence F is convex.

Conversely, if F is convex, as F is also continuous and twice differentiable, we may take the second order derivative and it is non-negative:

$$0 \leq F''(s) = f''(s) + q(s)f'(s) + p(s)f(s),$$

then f is $p(x), q(x)$-convex. □

Another important conclusion in [11] is worth mentioning.

Lemma 1. *Let I be an open interval. Let a be an integrable function and $h \in C^2(I)$ be such that $h'' - ah'$ is bounded by integrable functions M and m, that is, $m(x) \leq h''(x) - a(x)h'(x) \leq M(x)$, for every $x \in I$. Then functions Φ_1, Φ_2 defined by*

$$\Phi_1(x) = R_1(x) - h(x), \tag{5}$$
$$\Phi_2(x) = h(x) - R_2(x), \tag{6}$$

where

$$R_1(x) = \int \left(e^{\int a(x)dx} \int M(x) e^{-\int a(x)dx} dx \right) dx,$$
$$R_2(x) = \int \left(e^{\int a(x)dx} \int m(x) e^{-\int a(x)dx} dx \right) dx,$$

are $a(x)$-convex.

Actually this is more general than the $p(x), q(x)$-convex concept, as we can set $a(x) = -q(x)$, $m(x) = -p(x)h(x)$, then the following conclusion is deduced.

Theorem 6. *Let I be an open interval. Let q be an integrable function and $h \in C^2(I)$ be $p(x)$, $q(x)$-convex. Then the function Φ_2 defined by*

$$\Phi_2(x) = h(x) + \int \left(e^{-\int q(x)dx} \int p(x)h(x)e^{\int q(x)dx} dx \right) dx, \tag{7}$$

is $0, q(x)$-convex.

Proof. It is easy to verify

$$\begin{aligned} &\Phi_2''(x) + q(x)\Phi_2'(x) + 0\Phi_2(x) \\ =&h''(x) + q(x)h'(x) + p(x)h(x) \geq 0. \end{aligned}$$

□

Another similar situation below also derives a new $p_1(x), 0$-convex from the known $p(x), q(x)$-convex.

Theorem 7. *Let $f : I \to \mathbb{R}$ be $p(x), q(x)$-convex. Then the function y defined by*

$$y(x) = f(x)e^{\frac{1}{2}\int q(x)dx}$$

is $p_1(x), 0$-convex, where

$$p_1(x) = p(x) - \frac{1}{4}q(x)^2 - \frac{1}{2}q'(x).$$

Proof. Verify

$$\begin{aligned} &y''(x) + p_1(x)y(x) \\ =&(f''(x) + q(x)f'(x) + p(x)f(x))e^{\frac{1}{2}\int q(x)dx}. \end{aligned}$$

From the fact that f is $p(x), q(x)$-convex, the above terms is non-negative. Then y is $p_1(x), 0$-convex. □

3. Main Results

For $p(x), q(x)$-convex functions, we can establish different kinds of Jensen type inequalities than Theorems 1 and 4. The goal is to transform a $p(x), q(x)$-convex function to another function that is "closer to" convex function, or even convex function. Then we can eliminate some integral terms involving f or f' in the inequality.

Theorem 8. *Let $f : I \to \mathbb{R}$ be $p(x), q(x)$-convex with f, f', f'' continuous and $pf + qf'$ integrable on the closed interval I. For $a_k \in I$, $(k = 1, \ldots, n)$ and $p_k \geq 0$ with $\sum_{k=1}^n p_k = 1$, we have*

$$\sum_{k=1}^n p_k \int_{\overline{a}}^{a_k} (a_k - x)q(x)\Phi'(x)dx + \sum_{k=1}^n p_k\Phi(a_k) \geq \Phi(\overline{a}),$$

where $\overline{a}$ means $\sum_{k=1}^n p_k a_k$ and

$$\Phi(x) = f(x) + \int \left(e^{-\int q(x)dx} \int p(x)f(x)e^{\int q(x)dx} dx \right) dx. \tag{8}$$

Proof. For Theorem 6, we can use Theorem 4 for $0, q(x)$-convex function Φ. □

Theorem 9. *Let $f : I \to \mathbb{R}$ be $p(x), q(x)$-convex with f, f', f'' continuous and $pf + qf'$ integrable on the closed interval I. For $a_k \in I$, $(k = 1, \ldots, n)$ and $p_k \geq 0$ with $\sum_{k=1}^{n} p_k = 1$, we have*

$$\sum_{k=1}^{n} p_k \int_{\overline{a}}^{a_k} (a_k - x) p_1(x) f(x) e^{\frac{1}{2}\int_0^x q(t)dt} dx + \sum_{k=1}^{n} p_k f(a_k) e^{\frac{1}{2}\int_0^{a_k} q(t)dt} \geq f(\overline{a}) e^{\frac{1}{2}\int_0^{\overline{a}} q(t)dt},$$

where $\overline{a}$ means $\sum_{k=1}^{n} p_k a_k$ and

$$p_1(x) = p(x) - \frac{1}{4}q(x)^2 - \frac{1}{2}q'(x).$$

Proof. From Theorem 7, we can use Theorem 4 for a specific $p_1(x), 0$-convex function

$$y(x) = f(x) e^{\frac{1}{2}\int_0^x q(t)dt}.$$

□

Remark 2. *Set $p(x) = \frac{1}{4}q(x)^2 + \frac{1}{2}q'(x)$, we get Theorem 2.1 in [12] where the integral terms totally vanish.*

Recall a definition in [11].

Definition 1. *A function $f : I \to \mathbb{R}$ is convex with respect to a strictly monotone function $h : J \to \mathbb{R}, J \subseteq I$ if $f \circ h^{-1}$ is convex.*

Remark 3. *If f and h are twice differentiable functions and h is strictly increasing, then $f \circ h^{-1}$ is convex if and only if $f''(x)h'(x) - f'(x)h''(x) \geq 0$.*

Here we give a slight generalization for the concepts above. Note that for $g(x) \equiv 1$, it reduces to the situation above.

Remark 4. *A function $f : I \to \mathbb{R}$ with its positive weight function $g : I \to \mathbb{R}^+$ is convex with respect to strictly monotone function $h : J \to \mathbb{R}, J \subseteq I$ if $(fg) \circ h^{-1}$ is convex. If f, g and h are twice differentiable functions and h is strictly increasing, then $(fg) \circ h^{-1}$ is convex if and only if $g(x)[f''(x)h'(x) - f'(x)h''(x)] + f(x)[g''(x)h'(x) - g'(x)h''(x)] + 2f'g'(x)h'(x) \geq 0$.*

Proof. Take the second order derivative for $f(h^{-1}(x))g(h^{-1}(x))$, due to the non-negativity, it is equivalent to the inequality above. □

For inequalities of convex $(fg) \circ h^{-1}$, we have different perspectives to establish. The first is using Jensen inequality for convex functions in the usual sense.

Theorem 10. *A function $f : I \to \mathbb{R}$ with its positive weight function $g : I \to \mathbb{R}^+$ is convex with respect to strictly monotone function $h : J \to \mathbb{R}, J \subseteq I$. For $a_k \in J$, $(k = 1, \ldots, n)$ and $p_k \geq 0$ with $\sum_{k=1}^{n} p_k = 1$, we have*

$$g\left(h^{-1}\left(\sum_{k=1}^{n} p_k h(a_k)\right)\right) f\left(h^{-1}\left(\sum_{k=1}^{n} p_k h(a_k)\right)\right) \leq \sum_{k=1}^{n} p_k g(a_k) f(a_k).$$

Proof. Omit. □

The second perspective, for which we mainly focus on, is regarding f as a certain $p(x), q(x)$-convex function, concerning convex $(fg) \circ h^{-1}$.

Remark 5. *From Remark 4 we have that: If f, g and h are twice differentiable functions, g is positive and h is strictly increasing, then $(fg) \circ h^{-1}$ is convex if and only if*

$$f''(x) + \left[\frac{2g'(x)}{g(x)} - \frac{h''(x)}{h'(x)}\right] f'(x) + \left[\frac{g''(x)}{g(x)} - \frac{g'(x)h''(x)}{g(x)h'(x)}\right] f(x) \geq 0.$$

It is easy to observe that f is a certain $p(x), q(x)$-convex function for

$$q(x) = \frac{2g'(x)}{g(x)} - \frac{h''(x)}{h'(x)}, \tag{9}$$

$$p(x) = \frac{g''(x)}{g(x)} - \frac{g'(x)h''(x)}{g(x)h'(x)}. \tag{10}$$

This, as well as Theorem 10, shows that some $p(x), q(x)$-convex function can directly lead to a Jensen inequality, without any additional integral terms.

The above discussion points out that each $(fg) \circ h^{-1}$ convex function is also a $p(x), q(x)$-convex function for some $p(x), q(x)$. It is natural to ask, if each $p(x), q(x)$-convex function is also $(fg) \circ h^{-1}$ convex for some g, h? We need to solve g, h from (9), (10), which are equivalent to

$$-p(x)g(x)^2 + q(x)g(x)g'(x) - 2g'(x)^2 + g(x)g''(x) = 0, \tag{11}$$

$$\frac{h''(x)}{h'(x)} + q(x) - \frac{2g'(x)}{g(x)} = 0. \tag{12}$$

Solve (11) first for g, then (12) is easy to solve. If $g \neq 0$, setting $y = \frac{1}{g}$ and taking the expressions of g, g', g'' into (11), we actually get

$$y''(x) + q(x)y'(x) + p(x)y(x) = 0. \tag{13}$$

Suppose that the solution is $y = y(C_1, C_2, x) = T_{C_1,C_2}(p, q)$, then $g = \frac{1}{y(C_1,C_2,x)}$, h can also be solved. Thus we conclude, for each $p(x), q(x)$-convex function, we can establish the inequality in Theorem 10 for some g, h.

Then we give a special example of $p(x), q(x)$-convex function that (13) can be solved explicitly. Here we choose $p(x) \equiv -2, q(x) \equiv 1$.

Example 1. *Suppose that f is a $-2, 1$-convex function, by solving (13) we get*

$$y(x) = C_1 e^x + C_2 e^{-2x},$$

$$g(x) = \frac{1}{C_1 e^x + C_2 e^{-2x}},$$

with g positive restriction. Take it into (12) we get

$$h'(x) = \frac{C_3 e^{-x}}{(C_1 e^x + C_2 e^{-2x})^2}, C_3 > 0.$$

Thus,

$$h(x) = \frac{C_3}{3C_2(C_1 + C_2 e^{-3x})} + C_4,$$

$$h^{-1}(x) = -\frac{1}{3} \ln\left(\frac{C_3}{3C_2^2(x - C_4)} - \frac{C_1}{C_2}\right),$$

with feasible x. Hence, for $-2,1$-convex function, we have corresponding convex function $(fg)\circ h^{-1}$, such that the inequality in Theorem 10 holds.

With (9), (10) and using Theorem 4, we get the following inequality.

Theorem 11. *Let f, g and h be twice differentiable functions, g positive and h strictly increasing, if $(fg)\circ h^{-1}$ is convex with f, f', f'' continuous and $pf+qf'$ integrable on the closed interval I, where p, q is defined by (9) and (10). For $a_k \in I$, $(k=1,\ldots,n)$ and $p_k \geq 0$ with $\sum_{k=1}^{n} p_k = 1$, we have*

$$\sum_{k=1}^{n} p_k \int_{\bar{a}}^{a_k} (a_k - x)\left(\left[\frac{2g'(x)}{g(x)} - \frac{h''(x)}{h'(x)}\right] f'(x) + \left[\frac{g''(x)}{g(x)} - \frac{g'(x)h''(x)}{g(x)h'(x)}\right] f(x)\right) dx$$
$$\geq f(\bar{a}) - \sum_{k=1}^{n} p_k f(a_k),$$

where $\bar{a}$ means $\sum_{k=1}^{n} p_k a_k$.

This shows that a composite convex function $(fg)\circ h^{-1}$ can also satisfy Jensen type inequality with non-composite terms like $f(\bar{a}) - \sum_{k=1}^{n} p_k f(a_k)$, which differs from Theorem 10.

We may also establish a companion (reverse) Jensen type inequality similar to those in [13,14].

Theorem 12. *Let $f : I \to \mathbb{R}$ be $p(x), q(x)$-convex and increasing with f, f', f'' continuous and $pf+qf'$ integrable on the closed interval I. For $a_k \in I$, $(k=1,\ldots,n)$ and $p_k > 0$ with $\sum_{k=1}^{n} p_k = 1$, we have*

$$f(A) + \sum_{k=1}^{n} p_k \int_{a_k}^{A} (a_k - x)(q(x)f'(x) + p(x)f(x))dx \geq \sum_{k=1}^{n} p_k f(a_k),$$

where

$$A = \frac{\sum_{k=1}^{n} p_k f'(a_k) a_k}{\sum_{k=1}^{n} p_k f'(a_k)}.$$

Proof. Noticing that $A \in I$ since A is a convex combination of a_k, we can apply Theorem 3 to get

$$f(A) - f(a_k) - f'(a_k)(A - a_k) + \int_{a_k}^{A} (A - t)(q(t)f'(t) + p(t)f(t))dt \geq 0.$$

Multiply with p_k and add from 1 to n, we have

$$f(A) + \sum_{k=1}^{n} p_k \int_{a_k}^{A} (a_k - x)(q(x)f'(x) + p(x)f(x))dx \geq \sum_{k=1}^{n} p_k f(a_k) + \sum_{k=1}^{n} p_k f'(a_k)(A - a_k),$$

the conclusion is proved since $\sum_{k=1}^{n} p_k f'(a_k)(A - a_k) = 0$. □

Remark 6. *In the future, researchers may consider functions that satisfy the following differential inequality*

$$f^{(n)} + r_1(x) f^{(n-1)} + \ldots + r_n(x) f \geq 0. \tag{14}$$

If $f^{(n)} \geq 0$, then it reduces to n-convex functions. As there are various theorems about the n-convex functions, if (14) indicates another function $T(r_1,\ldots,r_n; f)x$ that is "closer to" or even

n-convex functions, like in Remark 5, then similar theorems may also apply to this $r_1(x), \ldots, r_n(x)$-n-convex function.

Under the assumption (14) we may consider some examples.

Example 2. *One example is*

$$T(r_1, \ldots, r_n; f)x = f(x) + \frac{1}{(n-1)!}\int_\alpha^x (x-t)^{n-1}(r_1(t)f^{(n-1)}(t) + \ldots + r_n(t)f(t))dt,$$

$$T^{(n)}(x) \geq 0.$$

Example 3. *The function y defined by*

$$y(x) = f(x)e^{\frac{1}{n}\int r_1(x)dx}$$

is $0, s_2(x), \ldots, s_n(x)$-n-convex that satisfies

$$y^{(n)} + s_2(x)y^{(n-2)} + \ldots + s_n(x)y \geq 0.$$

Here $s_2(x), \ldots, s_n(x)$ can be determined by comparing the coefficient functions in

$$y^{(n)} + s_2(x)y^{(n-2)} + \ldots + s_n(x)y = e^{\frac{1}{n}\int r_1(x)dx}(f^{(n)} + r_1(x)f^{(n-1)} + \ldots + r_n(x)f).$$

Readers may also consider how to define y to be an $s_1(x), \ldots, s_{i-1}(x), 0, s_{i+1}(x), \ldots, s_n(x)$-$n$-convex function. The most special case is the $s_1(x), \ldots, s_{n-1}(x), 0$-$n$-convex function below.

Example 4. *The function y defined by*

$$y(x) = f(x)e^{\int s_0(x)dx}$$

is $s_1(x), \ldots, s_{n-1}(x), 0$-n-convex that satisfies

$$y^{(n)} + s_1(x)y^{(n-1)} + \ldots + s_{n-1}(x)y' \geq 0.$$

Here $s_0(x), s_1(x), \ldots, s_{n-1}(x)$ can be determined by comparing the coefficient functions in

$$y^{(n)} + s_1(x)y^{(n-1)} + \ldots + s_{n-1}(x)y' = e^{\int s_0(x)dx}(f^{(n)} + r_1(x)f^{(n-1)} + \ldots + r_n(x)f).$$

It is easy to observe that y' is $s_1(x), \ldots, s_{n-1}(x)$-$(n-1)$-convex. Then we can repeat the discussion above for

$$y_1(x) = y'(x)e^{\int t_0(x)dx},$$

a $t_1(x), \ldots, t_{n-2}(x), 0$-$(n-1)$-convex that satisfies

$$y_1^{(n)} + s_1(x)y_1^{(n-1)} + \ldots + s_{n-2}(x)y_1' \geq 0.$$

We can continue this process to some lower order.

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4. Application

In this section, we use Jensen type inequalities $p(x), q(x)$-convex functions above, to establish some Ky Fan type inequalities. Let

$$A_n = \sum_{k=1}^{n} p_k a_k, G_n = \prod_{k=1}^{n} a_k^{p_k}, H_n = \left(\sum_{k=1}^{n} \frac{p_k}{a_k} \right)^{-1},$$
$$A_n^{'} = \sum_{k=1}^{n} p_k (1 - a_k), G_n^{'} = \prod_{k=1}^{n} (1 - a_k)^{p_k}, H_n^{'} = \left(\sum_{k=1}^{n} \frac{p_k}{1 - a_k} \right)^{-1}.$$

The arithmetic-geometric-harmonic mean inequality is

$$H_n \leq G_n \leq A_n.$$

Ky Fan inequality is

$$\frac{G_n}{G_n^{'}} \leq \frac{A_n}{A_n^{'}},$$

and Wang-Wang inequality is

$$\frac{H_n}{H_n^{'}} \leq \frac{G_n}{G_n^{'}}.$$

It is known that by setting $f(x) = \ln \frac{1-x}{x}$ in Jensen type inequalities we can get Ky Fan type inequalities. And it is pointed out in [15], by setting $f(x) = \ln \frac{1-x}{x}$ in [13] we can get Wang-Wang inequality.

The theorem below is generalized Ky Fan inequality.

Theorem 13. *Let*

$$-\frac{1}{x^2} + \frac{1}{(1-x)^2} + \frac{q(x)}{x(1-x)} + p(x) \ln \frac{x}{1-x} \leq 0 \tag{15}$$

for $x \in I \subseteq \mathbb{R}^+$ and the integral below exists. For $a_k \in I$, $(k = 1, ..., n)$ and $p_k \geq 0$ with $\sum_{k=1}^{n} p_k = 1$, we have

$$\exp \left[\sum_{k=1}^{n} p_k \int_{A_n}^{a_k} (a_k - x) \left(\frac{q(x)}{x(1-x)} + p(x) \ln \frac{x}{1-x} \right) dx \right] \leq \frac{A_n G_n^{'}}{A_n^{'} G_n}. \tag{16}$$

Proof. Using Theorem 4 for $f(x) = -\ln \frac{x}{1-x}$, then the $p(x), q(x)$-convex condition and Jensen type inequality are equivalent to the inequality condition (15) and inequality (16) in this theorem. □

There are some examples that the condition (15) naturally holds.

Remark 7. *Let $I = (0, \frac{1}{2}]$ and $p(x), q(x) \equiv 0$, we get Ky Fan inequality.*

Remark 8. *Let $I = (0, \frac{1-c}{2}]$ and $p(x) \equiv 0, q(x) = \frac{c}{x(1-x)}, -1 < c < 1$, we have:*

$$\exp \left[c \sum_{k=1}^{n} p_k \int_{A_n}^{a_k} \frac{a_k - x}{x^2 (1-x)^2} dx \right] \leq \frac{A_n G_n^{'}}{A_n^{'} G_n}$$

for $a_k \in (0, \frac{1-c}{2}]$, $(k = 1, \ldots, n)$ and $p_k \geq 0$ with $\sum_{k=1}^{n} p_k = 1$.

Remark 9. *Let* $I = (0, \frac{1}{2+c}]$ *and* $p(x) \equiv 0, q(x) = \frac{c}{1-x}, c > -2$, *we have:*

$$\exp\left[c\sum_{k=1}^{n} p_k \int_{A_n}^{a_k} \frac{a_k - x}{x(1-x)^2} dx\right] \leq \frac{A_n G_n'}{A_n' G_n}$$

for $a_k \in (0, \frac{1}{2+c}]$, $(k = 1, \ldots, n)$ *and* $p_k \geq 0$ *with* $\sum_{k=1}^{n} p_k = 1$.

The theorem below is generalized Wang-Wang inequality.

Theorem 14. *Let*

$$-\frac{1}{x^2} + \frac{1}{(1-x)^2} + \frac{q(x)}{x(1-x)} + p(x) \ln \frac{x}{1-x} \leq 0 \tag{17}$$

for $x \in I \subseteq \mathbb{R}^+$ *and the integral below exists. For* $a_k \in I$, $(k = 1, \ldots, n)$ *and* $p_k \geq 0$ *with* $\sum_{k=1}^{n} p_k = 1$, *we have*

$$\exp\left[\sum_{k=1}^{n} p_k \int_{a_k}^{H_n/(H_n+H_n')} (a_k - x)\left(\frac{q(x)}{x(1-x)} + p(x) \ln \frac{x}{1-x}\right) dx\right] \leq \frac{H_n' G_n}{G_n' H_n}. \tag{18}$$

Proof. Using Theorem 12 for $f(x) = -\ln \frac{x}{1-x}$, then the $p(x), q(x)$-convex condition and Jensen type inequality are equivalent to the inequality condition (17) and inequality (18) in this theorem, since

$$A = \frac{\sum_{k=1}^{n} p_k f'(a_k) a_k}{\sum_{k=1}^{n} p_k f'(a_k)} = \frac{H_n}{H_n + H_n'}.$$

□

There are some examples that the condition (17) naturally holds.

Remark 10. *Let* $I = (0, \frac{1}{2}]$ *and* $p(x), q(x) \equiv 0$, *we get Wang-Wang inequality.*

Remark 11. *Let* $I = \mathbb{R}^+$ *and* $p(x) \equiv 0, q(x) = -\frac{cx}{1-x}, c \geq 1$, *we have:*

$$\exp\left[c\sum_{k=1}^{n} p_k \int_{a_k}^{H_n/(H_n+H_n')} \frac{a_k - x}{(1-x)^2} dx\right] \leq \frac{H_n' G_n}{G_n' H_n}$$

for $a_k \in \mathbb{R}^+$, $(k = 1, \ldots, n)$ *and* $p_k \geq 0$ *with* $\sum_{k=1}^{n} p_k = 1$.

5. Declaration

AI tool is only used to examine, not to generate content in this article.

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