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Maximal Convergence of Fourier Series with Respect to Bergman Orthogonal Polynomials via Their Strong Asymptotics

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Abstract

Let $G \subset \mathbb{C}$ be a bounded simply connected domain with rectifiable Jordan boundary. Denote by ϕ the conformal map of the exterior of G onto the exterior of the unit disk. For $R > 1$, let Γ_R be the level curve defined by $|\phi(z)| = R$, and let G_R denote its interior, so that $G \subset G_R$. Suppose that f is analytic in G_R. In this paper, we investigate the maximal convergence properties of the Fourier series of f with respect to the Bergman orthogonal polynomials of G. By employing the strong asymptotics of Bergman polynomials outside the domain G of orthogonality, determined by the boundary properties of G, we obtain estimates for the maximal convergence rate of the partial sums of the Fourier series of f in the uniform norm on $\overline{G}$. These estimates are expressed in terms of the best polynomial approximation of f in the domain G_R where f is analytic.

Keywords: bergman orthogonal polynomials; fourier series; rate of convergence; best approximation number

MSC: 30E10; 30E15; 41A10; 41A25; 42C10

1. Introduction

Bergman orthogonal polynomials play a significant role in complex approximation theory. Their strong asymptotics have been extensively studied for domains bounded by various classes of curves (see, e.g., [1–4]). On the other hand, Fourier series also play an important role in this area, particularly in terms of approximation via their partial sums. In [1,5–8], results on the approximation by Fourier series with respect to Bergman orthogonal polynomials were obtained for domains bounded by various classes of curves. In particular, convergence results for the partial sums of Fourier series with respect to these polynomials of a domain G were established by Suetin [1] for domains bounded by analytic and smooth curves, and by Abdullayev and Küçükaslan [6–8] for domains bounded by piecewise smooth and quasiconformal curves. In these works, the function f, being approximated, is generally assumed to be analytic in G, without requiring analyticity in a larger domain.

In this paper, for $R > 1$, we assume that f is analytic in a domain $G_R \supset G$, where $G \subset \mathbb{C}$ is a simply connected domain bounded by a rectifiable Jordan curve Γ, and where G_R denotes the interior of the level curve Γ_R associated with G. We investigate the approximation of f on $\overline{G}$ in connection with its analyticity in G_R, which naturally leads to the concept of *maximal convergence* introduced by Walsh [9]. In this setting, maximal convergence refers to the fact that the rate of convergence improves with the analyticity of f in G_R and is characterized in terms of the corresponding best polynomial approximation,

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while the approximation error is measured on $\overline{G}$. The foundations of maximal convergence go back to the work of Bernstein and were later systematically developed by Walsh. They also obtained direct and inverse theorems when the function f is analytic in the canonical domain G_R, $R > 1$ (see, for example, ([10] pp. 54–59)). In ([5] p. 27), Gaier proved the maximal convergence theorem of Fourier series with respect to Bergman orthogonal polynomials of a domain G bounded by a Jordan curve. In [10–15], maximal convergence problems were also investigated for Faber series.

Using the results on the strong asymptotics of Bergman orthogonal polynomials of G given in [4] by Beckermann and Stylianopoulos, we investigate the maximal convergence of the partial sums of the Fourier series of f with respect to these polynomials on $\overline{G}$ when G is bounded by certain classes of curves. We obtain results on maximal convergence rates for each of the cases where the boundary of G is analytic, piecewise analytic, or smooth. In our results, the convergence rates rapidly approach zero depending on the number R, the best approximation number $E_n^{(2)}(f, G_R)$, and the geometric properties of the boundary curve of G. Moreover, in the case of domains bounded by analytic curves, the convergence rate additionally depends on the geometric properties of the level curves approaching the boundary of G from the inside.

2. Preliminaries

Let G be a simply connected domain bounded by a rectifiable curve Γ in the complex plane $\mathbb{C}$. For $k \geq 0$, we introduce the Bergman orthogonal polynomials $P_k(z)$, defined by $P_k(z) := \lambda_k z^k +$ lower-degree terms, with $\lambda_k > 0$, satisfying

$$(P_k, P_l) := \int\int_G P_k(z)\overline{P_l(z)}dm(z) = \delta_{k,l}$$

where $dm(z)$ denotes the two dimensional Lebesque measure and $\delta_{k,l}$ is the Kronecker delta. Without loss of generality, we assume that $P_0(z) = \lambda_0 = 1$.

Let $A_2 := A_2(G)$ be the space of square-integrable analytic functions in the domain G, with respect to the norm

$$\|f\|_{A_2} := \left(\int\int_G |f(z)|^2 dm(z)\right)^{\frac{1}{2}}.$$

For every $f \in A_2(G)$, the Fourier coefficients are defined by

$$a_k := a_k(f) := \int\int_G f(z)\overline{P_k(z)}dm(z), \quad k = 0, 1, 2, \ldots$$

and the function f corresponds to the following series

$$\sum_{k=0}^{\infty} a_k P_k(z).$$

We set

$$K_n := K_n(z.f) := \sum_{k=0}^{n} a_k P_k(z)$$

for the n-th partial sums of the Fourier series. We also set the remaining term as:

$$\omega_n(z, f) := \omega_n(z) := |f(z) - K_n(z, f)|, \quad z \in G$$

It is known (see, for example, [1,5,16]) that

$$lim_{n\to\infty}\omega_n(z) = 0$$

which is uniform for compact subsets of G.

We denote by $\mathbb{D}$ the open unit disk, by $\mathbb{D}^\star$ the exterior of the closed unit disk, by Ω the exterior of the closure $\overline{G}$ of G, containing the point of infinity $z = \infty$. By the Riemann conformal mapping theorem, there exists a unique function $w = \phi(z)$, meromorphic in Ω, that maps the domain Ω conformally and univalently onto $\mathbb{D}^\star$ and satisfies the conditions

$$\phi(\infty) = \infty, \quad \phi'(\infty) = \mu > 0.$$

The second condition is sometimes written in the form

$$\lim_{z\to\infty} \frac{\phi(z)}{z} = \mu > 0.$$

The conditions above imply that the function $w = \phi(z)$, being analytic in the domain Ω without the point $z = \infty$, has a simple pole at the point $z = \infty$. Therefore, its Laurent expansion in some neighborhood of the point $z = \infty$ has the form

$$\phi(z) = \mu z + \mu_0 + \frac{\mu_1}{z} + \frac{\mu_2}{z^2} + \ldots + \frac{\mu_n}{z^n} + \ldots,$$

where $\mu > 0$ is the inverse of the logarithmic capacity of G. Let the function $z = \psi(w)$ be the inverse function of $w = \phi(z)$. This is a function that conformally and univalently maps $\mathbb{D}^\star$ to Ω and has normalization at the point ∞. Its expansion has the form

$$\psi(w) = \sigma w + \sigma_0 + \frac{\sigma_1}{w} + \frac{\sigma_2}{w^2} + \ldots + \frac{\sigma_n}{w^n} + \ldots,$$

where $\sigma := \frac{1}{\mu} > 0$. The Riemann map ϕ and its inverse ψ play a central role in the analysis. They provide the analytic link between the geometry of Γ and the polynomial expansions under consideration. In particular, the asymptotic formulas for the partial sums of Fourier series with respect to Bergman orthogonal polynomials can be expressed in terms of the conformal map ϕ. Moreover, in the study of Faber series, the inverse mapping ψ naturally arises in the formulation of their coefficients, establishing a direct connection between the series expansions and the domain geometry.

We now denote a level curve as follows:

$$\Gamma_R := \{z \in \Omega; \quad |\phi(z)| = R\}.$$

Let G_R be the interior domain of the level curve Γ_R.

Gaier dealt with the question of when the series expansion of f converges uniformly even in $\overline{G}$. In ([5] p. 27, Theorem 4), he proved that if Γ is the Jordan curve and $R > 1$ is the largest number such that f is analytic inside Γ_R, then the relation

$$max\{\omega_n(z) : z \in \overline{G}\} = O(r^{-n})$$

holds for each $r < R$, but not for $r > R$.

Definition 1. *Γ is called an analytic curve if ψ admits an analytic and univalent continuation to $|w| > \rho$ for some $\rho \in [0, 1)$ [17]. We denote by ρ the smallest such value throughout the paper. In this case, we write $\Gamma \in U(\rho)$.*

Definition 2. *A rectifiable Jordan curve Γ is said to belong to the class $C(\tilde{p}, \alpha)$, where $\tilde{p}$ is a positive integer and $0 < \alpha < 1$, if it can be represented as $z = z(s)$ with s denoting the arc length, such that the periodic function $z(s)$ is $\tilde{p}$-times continuously differentiable, and its $\tilde{p}$-th derivative $z^{(\tilde{p})}(s)$ satisfies a Lipschitz condition of order α [10].*

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Definition 3. *A Jordan curve* Γ *is called a K-quasiconformal* $(K \geq 1)$ *curve if there exists a K-quasiconformal mapping* f *of a region* G *containing* Γ *such that* $f(\Gamma)$ *is a circle [18]. As an example, a piecewise smooth Jordan curve without cusps is known to be quasiconformal [17].*

If Γ is a smooth curve, it follows that (see, for example, [17,19])

$$0 < c_1 \leq \left|\phi'(z)\right| \leq c_1' < \infty, \quad z \in \overline{\Omega} \tag{1}$$

where c_1 and c_1' are positive constants independent of $z \in \overline{\Omega}$.

Now we define the best approximation to the function $f \in A_2(G)$ as:

$$E_n^{(2)}(f, G) := inf \|f - p_n\|_{L^2(\Gamma)}$$

where inf is taken over the polynomials p_n of a degree at most n. In the main results of this paper, the assumption $f \in A_2(G_R)$ implies that the best approximation number is defined on G_R with respect to the $L^2(\Gamma_R)$ norm.

We also define the n-th Faber polynomial F_n. It represents the polynomial part of the Laurent expansion of $\phi(z)^n$ at infinity. Consequently, F_n is a polynomial of degree n, with leading coefficients μ^n. For the function f that is analytic in the domain G_R, the series

$$\sum_{k=0}^{\infty} b_k F_k(z), \quad z \in G_R, R > 1$$

converges absolutely and uniformly on G_R, where

$$b_k := b_k(f) = \frac{1}{2\pi i} \int_{|t|=R} \frac{f(\psi(t))}{t^{k+1}} dt, \quad k = 0, 1, 2, \ldots$$

are the Faber coefficients of f ([10] Chapter X). We set

$$S_n := S_n(z, f) = \sum_{k=0}^{n} b_k F_k(z)$$

for the partial sums of the Faber series. And we set

$$r_n(z, f) := |f(z) - S_n(z, f)| = \sum_{k=n+1}^{\infty} b_k F_k(z)$$

It is known that the Bergman polynomial P_n asymptotically behaves like an appropriately normalized derivative of the Faber polynomial F_{n+1}, under different assumptions on Γ, thus providing a natural link between these two families of polynomials.

In this paper, we primarily make use of the following strong asymptotics for Bergman orthogonal polynomials outside the support of orthogonality, established in [4], which serve as the main tool in our analysis.

Theorem 1 (See [4]). *Let* Γ *be quasiconformal, and assume that* $\epsilon_n = O(\frac{1}{n^\beta})$*, for some* $\beta > 0$*. Then*

$$A_n(z) := \sqrt{\frac{\pi}{n+1}} \frac{P_n(z)}{\phi'(z)\phi(z)^n} - 1 = O(\frac{\sqrt{\epsilon_n}}{n^{\frac{\beta}{2}}}), \quad n \to \infty$$

uniformly on closed subsets of Ω*.*

For a curve Γ in the class $C(p+1, \alpha)$, where $p \geq 0$, $\alpha \in (0, 1)$, Suetin established in ([1] Lemma 1.5) that ϵ_n decays as $\epsilon_n = O(1/n^\beta)$, with $\beta = 2p + 2\alpha$. When β exceeds

1, the leading coefficient of the orthogonal polynomials $P_n(z)$, $n = 0, 1, 2, \cdots$, admits an asymptotic representation given in ([1] Theorem 1.1).

ε_n in Theorem 1 is summarized in [4] as follows.

$$\varepsilon_n = \begin{cases} \mathcal{O}(\delta^{2n}), & \text{if } \Gamma \in \mathcal{U}(\delta), \\ \mathcal{O}\left(\frac{1}{n^{2(p+\alpha)}}\right), & \text{if } \Gamma \in \mathcal{C}(p+1,\ \alpha), \\ \mathcal{O}\left(\frac{1}{n}\right), & \text{if } \Gamma \text{ is piecewise analytic without cusps.} \end{cases}$$

where $\delta \in [0, 1)$, $p \geq 0$ and $\alpha \in (0, 1)$.

We also make use of the results on the maximal convergence of the Faber series. When f is analytic in the domain G_R for $R > 1$, Suetin obtained fundamental results concerning the maximal convergence of the Faber series in ([10] Chapter X). In inequality (14) in ([10] p. 208), Suetin proved that, for $f \in A_{p'}(G_R)$ (defined analogously to $A_2(G)$) with $p' > 1$, and for $z \in \Gamma_\rho$, $\rho > 1$,

$$r_n(z, f) \leq \frac{\rho^{n+1}}{R^{n+1}} E_n^{(p')}(f, G_R) \sqrt{\frac{\rho^2}{\rho^4 - 1} ln \frac{\rho^2}{\rho^2 - 1}} \left[\frac{1}{2\pi} \int_{|t|=R} \frac{|dt|}{|t - \rho|^{q'}} \right]^{\frac{1}{q'}} \tag{2}$$

such that $\frac{1}{p'} + \frac{1}{q'} = 1$. Here, $E_n^{(p')}(f, G_R)$ denotes the best approximation to the function $f \in A_{p'}(G_R)$. Assuming $\rho = 1 + \frac{1}{n}$ in the result above, for $z \in \overline{G}$, Suetin obtained Theorem 6 in ([10] p. 207).

Throughout the proofs of the main results, we denote by $c_2, c_3, \ldots$ positive constants. Each constant depends only on the parameters explicitly indicated in parentheses, if any; for example, $c_{11}(R)$, $c_{12}(p, \alpha)$, $c_{13}(R, p, \alpha)$, while some constants may not depend on any parameter. Once introduced, these constants are used throughout the proofs without further specification.

3. Main Results

3.1. The Case of Domains Bounded by Smooth Curves

In this section, we assume that the domain G is bounded by a curve Γ from the class $C(p+1, \alpha)$, $p \geq 0$, $\alpha \in (0, 1)$. Under this assumption, we obtain the maximal convergence rate of the partial sums of Fourier series of f with Bergman orthogonal polynomials of domain G, where f is analytic on $G_R \supset G$ for $R > 1$. Firstly, using the strong asymptotics of Bergman polynomials, we deduce the following lemma.

Lemma 1. *If $\Gamma \in C(p+1, \alpha)$, where $p \geq 0$, $\alpha \in (0, 1)$ with $p + \alpha > \frac{1}{2}$, for $z \in \Gamma_\rho$, $\rho > 1$, the inequality*

$$\sum_{n=1}^{N} |P_n(z)|^2 \quad \leq \quad c(p, \alpha) \frac{\rho^2}{\rho^2 - 1} \rho^{2N} N \left(1 + \frac{1}{N^{2p+2\alpha}}\right),$$

holds, where $c(p, \alpha) > 0$ is a constant independent of N and ρ.

Proof. Since $\Gamma \in C(p+1, \alpha)$, where $p \geq 0$, $\alpha \in (0, 1)$, we take $2p + 2\alpha$ instead of β in $\epsilon_n = O(\frac{1}{n^\beta})$. Hence, from Theorem 1, it follows that

$$\begin{aligned}\left|\sqrt{\frac{\pi}{n+1}}\frac{P_n(z)}{\phi^{'}(z)\phi(z)^n}\right| &\leq 1+\frac{c_2}{n^{2p+2\alpha}}\\ |P_n(z)| &\leq \sqrt{\frac{n+1}{\pi}}\left|\phi^{'}(z)\right||\phi(z)|^n(1+\frac{c_2}{n^{2p+2\alpha}})\\ |P_n(z)|^2 &\leq \frac{n+1}{\pi}\left|\phi^{'}(z)\right|^2|\phi(z)|^{2n}(1+\frac{c_2}{n^{2p+2\alpha}})^2.\end{aligned}$$

Taking $z \in \Gamma_\rho$, $\rho > 1$ and using (1), we have

$$\begin{aligned}\sum_{n=1}^{N}|P_n(z)|^2 &\leq \sum_{n=1}^{N} c_3 n\rho^{2n}(1+\frac{c_4}{n^{2p+2\alpha}})\\ &\leq c_3\sum_{n=1}^{N} n\rho^{2n} + c_5\sum_{n=1}^{N}\frac{\rho^{2n}}{n^{2p+2\alpha-1}}.\end{aligned} \tag{3}$$

We estimate the two sums separately. Since $\rho > 1$, the sequence $\{\rho^{2n}\}_{n\geq 1}$ grows exponentially. Hence,

$$\sum_{n=1}^{N} n\rho^{2n} \leq N\sum_{n=1}^{N}\rho^{2n} = N\frac{\rho^2(\rho^{2N}-1)}{\rho^2-1} \leq \frac{\rho^2}{\rho^2-1}N\rho^{2N}. \tag{4}$$

We next consider

$$\sum_{n=1}^{N}\frac{\rho^{2n}}{n^{2p+2\alpha-1}}.$$

Since the summand is the product of an exponentially growing sequence $\{\rho^{2n}\}_{n\geq 1}$, whose partial sums can be explicitly controlled, and a polynomially decaying sequence $\{n^{-(2p+2\alpha-1)}\}_{n\geq 1}$, Abel summation provides a natural tool to estimate the growth of the corresponding partial sums. Set

$$a_n := \rho^{2n}, \qquad b_n := \frac{1}{n^\gamma}, \qquad \gamma := 2p+2\alpha-1 > 0, \qquad A_n := \sum_{k=1}^{n} a_k,$$

where $\gamma > 0$ follows from the condition $p+\alpha > \frac{1}{2}$ in the hypothesis. By Abel's summation formula, it follows that

$$\sum_{n=1}^{N}\frac{\rho^{2n}}{n^\gamma} = \frac{A_N}{N^\gamma} + \sum_{n=1}^{N-1} A_n(b_n - b_{n+1}).$$

Since $\rho > 1$, we have

$$A_n = \sum_{k=1}^{n}\rho^{2k} \leq \frac{\rho^2}{\rho^2-1}\rho^{2n}, \qquad n \geq 1.$$

Moreover, since $\gamma > 0$, there exists a constant $c_6(\gamma)$ such that

$$0 < \frac{1}{n^\gamma} - \frac{1}{(n+1)^\gamma} \leq \frac{c_6(\gamma)}{n^{\gamma+1}}, \qquad n \geq 1.$$

Combining these estimates yields

$$\sum_{n=1}^{N}\frac{\rho^{2n}}{n^\gamma} \leq \frac{\rho^2}{\rho^2-1}\frac{\rho^{2N}}{N^\gamma} + c_6(\gamma)\frac{\rho^2}{\rho^2-1}\sum_{n=1}^{N-1}\frac{\rho^{2n}}{n^{\gamma+1}}.$$

Since the second sum is of lower order, it can be absorbed into the first term. Therefore, there exists a constant $c_7(p,\alpha)$ such that

$$\sum_{n=1}^{N} \frac{\rho^{2n}}{n^{2p+2\alpha-1}} \leq c_7(p,\alpha)\frac{\rho^2}{\rho^2-1}\frac{\rho^{2N}}{N^{2p+2\alpha-1}}. \tag{5}$$

Substituting (4) and (5) in place of (3), we conclude

$$\begin{aligned}\sum_{n=1}^{N}|P_n(z)|^2 &\leq c_3\frac{\rho^2}{\rho^2-1}N\rho^{2N} + c_8(p,\alpha)\frac{\rho^2}{\rho^2-1}\frac{\rho^{2N}}{N^{2p+2\alpha-1}} \\ &= c_9(p,\alpha)\frac{\rho^2}{\rho^2-1}\rho^{2N}N\left(1+\frac{1}{N^{2p+2\alpha}}\right),\end{aligned}$$

for a constant $c(p,\alpha)>0$ independent of N and ρ. □

Theorem 2. *If the boundary Γ of the domain G belongs to the class $C(p+1,\alpha)$, where $p\geq 0$, $\alpha\in(0,1)$ with $p+\alpha>\frac{1}{2}$, and a function $f(z)\in A_2(G_R), R>1$, then the remaining term $\omega_n(z,f)$ satisfies the inequality*

$$\omega_n(z,f) \leq \frac{c(R,p,\alpha)}{R^n}E_n^{(2)}(f,G_R)\left(\sqrt{\frac{nR}{R^2-1}}+n\left(1+\frac{1}{n^{2p+2\alpha}}\right)\right), \quad z\in\overline{G}$$

for a constant $c(R,p,\alpha)>0$ independent of n.

Proof. Let the boundary Γ of the domain G be of the class $C(p+1,\alpha)$ where $\alpha\in(0,1)$, $p\geq 0$, $p+\alpha>\frac{1}{2}$. Let $f(z)\in A_2(G_R)$, $R>1$ and T_n be the best approximation polynomial to $f(z)$ on G_R. We set

$$T_n := T_n(z,f) := \sum_{k=0}^{n} a_k^{(n)}P_k(z)$$

by the Fourier coefficient;

$$a_k^{(n)} := \int\int_G T_n(\zeta)\overline{P_k(\zeta)}dm(\zeta).$$

We write that

$$\omega_n(z,f) = |f(z)-K_n(z,f)| \leq |f(z)-T_n(z,f)| + |T_n(z,f)-K_n(z,f)|. \tag{6}$$

Firstly, we estimate $|T_n(z,f)-K_n(z,f)|$. Using the Cauchy–Schwarz inequality, we write

$$\begin{aligned}|T_n(z,f)-K_n(z,f)| &= \left|\sum_{k=0}^{n}a_k^{(n)}P_k(z)-\sum_{k=0}^{n}a_kP_k(z)\right| \\ &= \left|\sum_{k=0}^{n}\int\int_G(T_n(\zeta)-f(\zeta))\overline{P_k(\zeta)}dm(\zeta)P_k(z)\right| \\ &\leq \int\int_G|T_n(\zeta)-f(\zeta)|\left|\sum_{k=0}^{n}\overline{P_k(\zeta)}P_k(z)\right|dm(\zeta) \\ &\leq \left(\int\int_G|T_n(\zeta)-f(\zeta)|^2dm(\zeta)\right)^{\frac{1}{2}}\left(\int\int_G\sum_{k=0}^{n}\left|\overline{P_k(\zeta)}P_k(z)\right|^2dm(\zeta)\right)^{\frac{1}{2}}.\end{aligned}$$

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An L^2-estimate derived from the Bernstein–Walsh inequality for analytic functions on domains ([9] Chapter III, Theorem III.4) and the assumption that P_k, $k = 0, 1, 2, \ldots$ are orthonormal polynomials in G show that

$$\begin{aligned} |T_n(z,f) - K_n(z,f)| &\leq \frac{c_{10}(R)}{R^n}\left(\int\int_{G_R} |T_n(\zeta) - f(\zeta)|^2 dm(\zeta)\right)^{\frac{1}{2}} \\ &\times \left(\sum_{k=0}^{n}\left(\int\int_{G}\left|\overline{P_k(\zeta)}\right|^2 dm(\zeta)\right)|P_k(z)|^2\right)^{\frac{1}{2}} \\ &= \frac{c_{10}(R)}{R^n}E_n^{(2)}(f,G_R)\left(|P_0(z)|^2 + \sum_{k=1}^{n}|P_k(z)|^2\right)^{\frac{1}{2}}. \end{aligned}$$

Since we assume $P_0(z) = 1$ in this paper,

$$\begin{aligned} |T_n(z,f) - K_n(z,f)| &\leq \frac{c_{10}(R)}{R^n}E_n^{(2)}(f,G_R)\left(1 + \sum_{k=1}^{n}|P_k(z)|^2\right)^{\frac{1}{2}} \\ &\leq \frac{c_{11}(R)}{R^n}E_n^{(2)}(f,G_R)\left[1 + \left(\sum_{k=1}^{n}|P_k(z)|^2\right)^{\frac{1}{2}}\right] \\ &= \frac{c_{11}(R)}{R^n}E_n^{(2)}(f,G_R) + \frac{c_{11}(R)}{R^n}E_n^{(2)}(f,G_R)\left(\sum_{k=1}^{n}|P_k(z)|^2\right)^{1/2}. \end{aligned}$$

Let us take $z \in \Gamma_\rho$ such that $1 < \rho < R$. Hence, from Lemma 1, it follows

$$\begin{aligned} |T_n(z,f) - K_n(z,f)| &\leq \frac{c_{11}(R)}{R^n}E_n^{(2)}(f,G_R) \\ &+ c_{12}(R,p,\alpha)E_n^{(2)}(f,G_R)\left(\frac{\rho}{R}\right)^n\sqrt{\frac{\rho^2}{\rho^2-1}}\sqrt{n}\left(1 + \frac{1}{n^{2p+2\alpha}}\right)^{\frac{1}{2}}. \end{aligned}$$

Here, since $\rho > 1$ and $n > 1$, we have $\frac{1}{R^n} \leq C\left(\frac{\rho}{R}\right)^n\sqrt{n}$, for some constant $C > 0$; hence, the factor R^{-n} can be absorbed into $\left(\frac{\rho}{R}\right)^n\sqrt{n}$ by enlarging the constant. Taking this observation into account, we obtain

$$|T_n(z,f) - K_n(z,f)| \leq c_{13}(R,p,\alpha)E_n^{(2)}(f,G_R)\left(\frac{\rho}{R}\right)^n\sqrt{\frac{\rho^2}{\rho^2-1}}\sqrt{n}\left(1 + \frac{1}{n^{2p+2\alpha}}\right)^{\frac{1}{2}}. \quad (7)$$

On the other hand, since T_n is the best approximation polynomial to the function $f(z)$, we have

$$|f(z) - T_n(z,f)| \leq |f(z) - S_n(z,f)|.$$

Substituting (2) for $p' = q' = 2$, and (7) in place of (6), we write

$$\begin{aligned} \omega_n(z,f) &\leq \frac{\rho^{n+1}}{R^{n+1}}E_n^{(2)}(f,G_R)\sqrt{\frac{\rho^2}{\rho^4-1}ln\frac{\rho^2}{\rho^2-1}}\left[\frac{1}{2\pi}\int_{|t|=R}\frac{|dt|}{|t-\rho|^2}\right]^{\frac{1}{2}} \\ &+ c_{13}(R,p,\alpha)E_n^{(2)}(f,G_R)\left(\frac{\rho}{R}\right)^n\sqrt{\frac{\rho^2}{\rho^2-1}}\ \sqrt{n}\left(1 + \frac{1}{n^{2p+2\alpha}}\right)^{\frac{1}{2}} \\ &\leq c_{14}(R,p,\alpha)E_n^{(2)}(f,G_R)\left(\frac{\rho}{R}\right)^n\sqrt{\frac{\rho^2}{\rho^2-1}}\left(\sqrt{\frac{R}{R^2-\rho^2}} + \sqrt{n}(1 + \frac{1}{n^{2p+2\alpha}})\right) \quad (8) \end{aligned}$$

The above integral is evaluated as follows. Using the parametrization $t = Re^{i\theta}$, $0 \leq \theta < 2\pi$, we obtain

$$\frac{1}{2\pi}\int_{|t|=R}\frac{|dt|}{|t-\rho|^2} = \frac{1}{2\pi}\int_0^{2\pi}\frac{R\,d\theta}{R^2+\rho^2-2R\rho\cos\theta}.$$

Applying the classical identity

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{a-b\cos\theta} = \frac{1}{\sqrt{a^2-b^2}}, \quad a > |b|,$$

yields

$$\frac{1}{2\pi}\int_{|t|=R}\frac{|dt|}{|t-\rho|^2} = \frac{R}{|R^2-\rho^2|}.$$

Now we choose $\rho = 1+\frac{1}{n}$. This choice ensures that the convergence rate rapidly decays to zero. In this choice of ρ, when n goes to infinity, the level curve Γ_ρ approaches Γ. Hence, we can find a convergence rate for $z \in \Gamma$, and also for $z \in \overline{G}$ using the maximum modulus principle. If we replace $1+\frac{1}{n}$ instead of ρ in (8), as $n \to \infty$, for $z \in \overline{G}$ and for a constant $c(R,p,\alpha) > 0$ independent of n, we finally obtain the following result.

$$\omega_n(z,f) \quad \leq \quad \frac{c(R,p,\alpha)}{R^n}E_n^{(2)}(f,G_R)\left(\sqrt{\frac{nR}{R^2-1}}+n\left(1+\frac{1}{n^{2p+2\alpha}}\right)\right).$$

□

3.2. The Case of Domains Bounded by Piecewise Analytic Curves

In this section, we consider the case where the boundary Γ of the domain G is a piecewise analytic curve without cusps. Under this assumption, we obtain the maximal convergence rate of the partial sums of Fourier series of f with Bergman orthogonal polynomials of domain G, where f is analytic on $G_R \supset G$ for $R > 1$. We first deduce the following lemma for the sum of the Bergman orthogonal polynomials regarding their strong asymptotics.

Lemma 2. *If Γ is a piecewise analytic curve without cusps, for $z \in \Gamma_\rho$, $\rho > 1$, the inequality*

$$\sum_{n=1}^{N}|P_n(z)|^2 \leq cN\frac{\rho^2}{\rho^2-1}\rho^{2N}$$

holds, where $c > 0$ is a constant independent of N and ρ.

Proof. Since Γ is a piecewise analytic curve without cusps, we write $\beta = 1$ from Theorem 1. Hence, for $z \in \Gamma_\rho$, $\rho > 1$, we have

$$\begin{aligned}
\left|\sqrt{\frac{\pi}{n+1}}\frac{P_n(z)}{\phi'(z)\phi(z)^n}\right| &\leq 1+\frac{c_{15}}{n}\\
|P_n(z)| &\leq \sqrt{\frac{n+1}{\pi}}\left|\phi'(z)\right||\phi(z)|^n\left(1+\frac{c_{15}}{n}\right)\\
\sum_{n=1}^{N}|P_n(z)|^2 &\leq c_{16}\sum_{n=1}^{N}(n+1)\rho^{2n}\left(1+\frac{c_{15}}{n}\right)^2 \leq c_{16}\sum_{n=1}^{N}(n+1)\left(1+\frac{c_{17}}{n}\right)\rho^{2n}\\
&\leq c_{18}\sum_{n=1}^{N}\frac{n^2+1}{n}\rho^{2n}
\end{aligned}$$

Since $\rho > 1$, it follows

$$\sum_{n=1}^{N} \rho^{2n} = \rho^2 \frac{\rho^{2N} - 1}{\rho^2 - 1} \leq \frac{\rho^2}{\rho^2 - 1} \rho^{2N}.$$

Hence, we obtain

$$\sum_{n=1}^{N} |P_n(z)|^2 \leq cN \frac{\rho^2}{\rho^2 - 1} \rho^{2N},$$

for a constant $c > 0$ independent of N and ρ. □

Theorem 3. *If the boundary Γ of the domain G is a piecewise analytic curve without cusps and a function $f(z) \in A_2(G_R), R > 1$, then the remaining term $\omega_n(z, f)$ satisfies the inequality*

$$\omega_n(z, f) \leq c(R) \frac{n}{R^n} E_n^{(2)}(f, G_R) \left(\sqrt{\frac{R}{R^2 - 1}} + 1 \right), \quad z \in \overline{G}.$$

for a constant $c(R) > 0$ independent of n.

Proof. Let the boundary Γ of the domain G be a piecewise analytic curve without cusps and a function $f(z) \in A_2(G_R), R > 1$. We assume $z \in \Gamma_\rho$ such that $1 < \rho < R$. Proceeding in the same manner as in the proof of inequality (7) in Theorem 2, and employing Lemma 2 instead, for the constants depending only on R, we arrive at the following inequality.

$$\begin{aligned} |T_n(z, f) - K_n(z, f)| &\leq \frac{c_{19}(R)}{R^n} E_n^{(2)}(f, G_R) \left(\frac{\rho^2}{\rho^2 - 1} \rho^{2n} n \right)^{\frac{1}{2}} \\ &= c_{20}(R) \left(\frac{\rho}{R} \right)^n E_n^{(2)}(f, G_R) \sqrt{\frac{n\rho^2}{\rho^2 - 1}}. \end{aligned} \tag{9}$$

Using (2) for $p' = q' = 2$, and (9), we can write

$$\begin{aligned} \omega_n(z, f) &\leq |f(z) - S_n(z, f)| + |T_n(z, f) - K_n(z, f)| \\ &\leq \frac{\rho^{n+1}}{R^{n+1}} E_n^{(2)}(f, G_R) \sqrt{\frac{\rho^2}{\rho^4 - 1} ln \frac{\rho^2}{\rho^2 - 1}} \left[\frac{1}{2\pi} \int_{|t|=R} \frac{|dt|}{|t - \rho|^2} \right]^{\frac{1}{2}} \\ &+ c_{20}(R) E_n^{(2)}(f, G_R) \left(\frac{\rho}{R} \right)^n \sqrt{\frac{n\rho^2}{\rho^2 - 1}} \\ &\leq c_{21}(R) E_n^{(2)}(f, G_R) \left(\frac{\rho}{R} \right)^n \sqrt{\frac{n\rho^2}{\rho^2 - 1}} \left(\sqrt{\frac{R}{R^2 - \rho^2}} + 1 \right). \end{aligned} \tag{10}$$

If we replace $\rho = 1 + \frac{1}{n}$ instead of ρ in (10), as $n \to \infty$, we finally obtain

$$\omega_n(z, f) \leq c(R) \frac{n}{R^n} E_n^{(2)}(f, G_R) \left(\sqrt{\frac{R}{R^2 - 1}} + 1 \right), \quad z \in \overline{G}$$

for a constant $c(R) > 0$ independent of n. □

3.3. The Case of Domains Bounded by Analytic Curves

In this section, we assume that the boundary Γ of the domain G is analytic curve. Using the geometric properties of the level curves Γ_δ, $\delta < 1$ approaching Γ from the inside,

we obtain the maximal convergence rate of the partial sums of Fourier series of f with Bergman orthogonal polynomials of domain G, where f is analytic on $G_R \supset G$ for $R > 1$. We use the following result for the asymptotics of Bergman orthogonal polynomials, which improves on the geometric rate given in Theorem 1 by a factor $\max(\delta^n, 1/r^n)$, where $r > 1$ for closed subsets of Ω (see ([4] Corollary 4.1)).

Corollary 1 (See [4]). *Let $\Gamma \in U(\delta)$ for some $\delta < 1$, and define*

$$\widetilde{\beta} := \begin{cases} 2(p+\alpha) > 0 & \text{if } \Gamma_\delta \in C(p+1,\alpha), \\ 1 & \text{if } \Gamma_\delta \quad \text{is piecewise analytic without cusps}, \\ 0 & \text{otherwise}. \end{cases}$$

Then, uniformly for $z \in \overline{\Omega_r}$,

$$|A_n(z)| = \begin{cases} \mathcal{O}(\frac{(\frac{\delta}{r})^n}{n^{(\widetilde{\beta}+1)/2}}) & \text{if } r \in (\delta, \frac{1}{\delta}), \\ \mathcal{O}(\frac{\delta^{2n}}{n^{\widetilde{\beta}}}) & \text{if } r > \frac{1}{\delta}. \end{cases}$$

If the level curve Γ_δ belongs to the class of $C(p+1,\alpha)$, $p \geq 0$, $\alpha \in (0,1)$, we first give the following lemma for the sum of the Bergman polynomials according to their asymptotics.

Lemma 3. *If $\Gamma \in U(\delta)$ for some $\delta < 1$ and the level curve $\Gamma_\delta \in C(p+1,\alpha)$ where $p \geq 0$, $\alpha \in (0,1)$ with $p + \alpha > \frac{1}{2}$, then for $z \in \Gamma_\rho$ such that $\rho \in (\delta, \frac{1}{\delta})$, the inequality*

$$\sum_{n=1}^{N} |P_n(z)|^2 \leq c(p,\alpha)\left(\frac{\rho^2}{\rho^2-1} N\rho^{2N} + \frac{\delta\rho}{1-\delta\rho}\frac{1}{N^{p+\alpha-\frac{1}{2}}}\right)$$

holds, where $c(p,\alpha) > 0$ is a constant independent of N, ρ and δ.

Proof. Let $\Gamma \in U(\delta)$ for some $\delta < 1$ and the level curve $\Gamma_\delta \in C(p+1,\alpha)$, $p \geq 0, \alpha \in (0,1)$. Therefore, from Corollary 1, we take $2(p+\alpha)$ instead of $\widetilde{\beta}$. Then, we consider the case $z \in \Gamma_\rho$ with $\rho \in (\delta, \frac{1}{\delta})$. Hence, we write

$$\begin{aligned} \left|\sqrt{\frac{\pi}{n+1}}\frac{P_n(z)}{\phi'(z)\phi(z)^n}\right| &\leq 1 + c_{22}\frac{(\frac{\delta}{\rho})^n}{n^{(2p+2\alpha+1)/2}} \\ |P_n(z)|^2 &\leq \left(\frac{n+1}{\pi}\right)\left|\phi'(z)\right|^2|\phi(z)|^{2n}(1 + c_{22}\frac{(\frac{\delta}{\rho})^n}{n^{(2p+2\alpha+1)/2}})^2. \end{aligned}$$

Since $z \in \Gamma_\rho$ and $\frac{\delta}{\rho} < 1$, it follows

$$|P_n(z)|^2 \leq c_{23} n\rho^{2n}\left(1 + c_{24}\frac{(\frac{\delta}{\rho})^n}{n^{p+\alpha+\frac{1}{2}}}\right) = c_{23} n\rho^{2n} + c_{25}\frac{(\delta\rho)^n}{n^{p+\alpha-\frac{1}{2}}}.$$

Hence, we take the summation

$$\sum_{n=1}^{N} |P_n(z)|^2 \leq c_{23}\sum_{n=1}^{N} n\rho^{2n} + c_{25}\sum_{n=1}^{N}\frac{(\delta\rho)^n}{n^{p+\alpha-\frac{1}{2}}}. \tag{11}$$

From (4), we already have

$$\sum_{n=1}^{N} n\rho^{2n} \leq \frac{\rho^2}{\rho^2-1}\, N\rho^{2N}. \tag{12}$$

We now estimate the second sum. Set

$$S_N := \sum_{n=1}^{N} \frac{(\delta\rho)^n}{n^{p+\alpha-\frac{1}{2}}}.$$

Define $a_n := (\delta\rho)^n$, $b_n := \frac{1}{n^\gamma}$, and $\gamma = p+\alpha-1/2$. By Abel summation,

$$S_N = A_N b_N + \sum_{k=1}^{N-1} A_k\,(b_k - b_{k+1}), \qquad A_k := \sum_{n=1}^{k} a_n.$$

Since $0 < \delta\rho < 1$ by hypotesis $\rho \in (\delta, \frac{1}{\delta})$, we have

$$A_k = \sum_{n=1}^{k} (\delta\rho)^n = \frac{\delta\rho\,(1-(\delta\rho)^k)}{1-\delta\rho} \leq \frac{\delta\rho}{1-\delta\rho}.$$

Moreover, for $p+\alpha > \frac{1}{2}$, the discrete difference of b_k satisfies

$$0 < b_k - b_{k+1} \leq c_{26}(p,\alpha)\frac{1}{k^{p+\alpha+\frac{1}{2}}}.$$

Therefore,

$$S_N \leq \frac{\delta\rho}{1-\delta\rho}\,\frac{1}{N^{p+\alpha-\frac{1}{2}}} + c_{26}(p,\alpha)\frac{\delta\rho}{1-\delta\rho}\sum_{k=1}^{N-1}\frac{1}{k^{(p+\alpha+\frac{1}{2})}}. \tag{13}$$

Since $p+\alpha > \frac{1}{2}$, the function $x \mapsto (x+1)^{-(p+\alpha+\frac{1}{2})}$ is positive and decreasing. Hence,

$$\sum_{k=1}^{N-1} k^{-(p+\alpha+\frac{1}{2})} \leq \int_0^N (x+1)^{-(p+\alpha+\frac{1}{2})}\,dx = \frac{1-(N+1)^{-(p+\alpha-\frac{1}{2})}}{p+\alpha-\frac{1}{2}} \leq \frac{c_{27}(p,\alpha)}{N^{\,p+\alpha-\frac{1}{2}}} \tag{14}$$

From (13) and (14), we have

$$\sum_{n=1}^{N} \frac{(\delta\rho)^n}{n^{p+\alpha-\frac{1}{2}}} \leq c_{28}(p,\alpha)\frac{\delta\rho}{1-\delta\rho}\frac{1}{N^{\,p+\alpha-\frac{1}{2}}}. \qquad p+\alpha > \frac{1}{2} \tag{15}$$

If we write (12) and (15) instead of (11), we finally obtain

$$\sum_{n=1}^{N} |P_n(z)|^2 \leq c_{29}(p,\alpha)\left(\frac{\rho^2}{\rho^2-1}\,N\rho^{2N} + \frac{\delta\rho}{1-\delta\rho}\frac{1}{N^{\,p+\alpha-\frac{1}{2}}}\right), \qquad p+\alpha > \frac{1}{2}$$

for a constant $c_{29}(p,\alpha) > 0$ independent of N, ρ and δ. □

Theorem 4. *Let the boundary Γ of the domain G belong to the class $U(\delta)$ for some $\delta < 1$ and the level curve $\Gamma_\delta \in C(p+1,\alpha)$, where $p \geq 0$, $\alpha \in (0,1)$ with $p+\alpha > \frac{1}{2}$. Let also a function $f(z) \in A_2(G_R)$, $R > 1$, then the remaining term $\omega_n(z,f)$ satisfies the inequality*

$$\omega_n(z,f) \quad \leq \quad c(R,p,\alpha)\frac{n}{R^n}E_n^{(2)}(f,G_R)\left(\sqrt{\frac{R}{R^2-1}}+1+\sqrt{\frac{\delta}{(1-\delta)}}\frac{1}{n^{\frac{p}{2}+\frac{\alpha}{2}-\frac{1}{4}}}\right), \quad z \in \overline{G}$$

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for a constant $c(R,p,\alpha)>0$ independent of n and δ.

Proof. Let the boundary Γ of the domain G belong to the class $U(\delta)$ for some $\delta<1$. and the level curve Γ_δ belong to the class $C(p+1,\alpha)$ such that $p\geq 0$, $\alpha\in(0,1)$ with $p+\alpha>\frac{1}{2}$. Let also $f(z)\in A_2(G_R)$, $R>1$. We assume $z\in\Gamma_\rho$, where $1<\rho<R$. By a similar argument to that used for (7) in Theorem 2, with Lemma 3 in place, we obtain the following inequality.

$$\begin{aligned}|T_n(z,f)-K_n(z,f)| &\leq \frac{c_{30}(R,p,\alpha)}{R^n}E_n^{(2)}(f,G_R)\left(\frac{\rho^2}{\rho^2-1}n\rho^{2n}+\frac{\delta\rho}{(1-\delta\rho)n^{\,p+\alpha-\frac{1}{2}}}\right)^{\frac{1}{2}}\\ &\leq \frac{c_{31}(R,p,\alpha)}{R^n}E_n^{(2)}(f,G_R)\left(\sqrt{\frac{n\rho^2}{\rho^2-1}}\rho^n+\sqrt{\frac{\delta\rho}{(1-\delta\rho)}}\frac{1}{n^{\frac{p}{2}+\frac{\alpha}{2}-\frac{1}{4}}}\right)\end{aligned}$$

By this and (2) for $p'=q'=2$, we write

$$\begin{aligned}\omega_n(z,f) &\leq |f(z)-S_n(z,f)|+|T_n(z,f)-K_n(z,f)|\\ &\leq \frac{\rho^{n+1}}{R^{n+1}}E_n^{(2)}(f,G_R)\sqrt{\frac{\rho^2}{\rho^4-1}ln\frac{\rho^2}{\rho^2-1}}\left[\frac{1}{2\pi}\int_{|t|=R}\frac{|dt|}{|t-\rho|^2}\right]^{\frac{1}{2}}\\ &+ \frac{c_{31}(R,p,\alpha)}{R^n}E_n^{(2)}(f,G_R)\left(\sqrt{\frac{n\rho^2}{\rho^2-1}}\rho^n+\sqrt{\frac{\delta\rho}{(1-\delta\rho)}}\frac{1}{n^{\frac{p}{2}+\frac{\alpha}{2}-\frac{1}{4}}}\right)\\ &\leq c_{32}(R,p,\alpha)\left(\frac{\rho}{R}\right)^n E_n^{(2)}(f,G_R)\left(\sqrt{\frac{\rho^2}{\rho^2-1}}\sqrt{\frac{R}{R^2-\rho^2}}+\sqrt{\frac{n\rho^2}{\rho^2-1}}\right)\\ &+ c_{33}(R,p,\alpha)\left(\frac{1}{R^n}E_n^{(2)}(f,G_R)\sqrt{\frac{\delta\rho}{(1-\delta\rho)}}\frac{1}{n^{\frac{p}{2}+\frac{\alpha}{2}-\frac{1}{4}}}\right)\end{aligned}$$

If we replace $\rho=1+\frac{1}{n}$ instead of ρ, as $n\to\infty$, for $z\in\overline{G}$, we finally get

$$\omega_n(z,f) \leq c_{34}(R,p,\alpha)\frac{n}{R^n}E_n^{(2)}(f,G_R)\left(\sqrt{\frac{R}{R^2-1}}+1+\sqrt{\frac{\delta}{1-\delta}}\frac{1}{n^{\frac{p}{2}+\frac{\alpha}{2}-\frac{1}{4}}}\right),$$

where $c_{34}(R,p,\alpha)>0$ is a constant independent of n and δ. □

Lemma 4. *If $\Gamma\in U(\delta)$ for some $\delta<1$ and the level curve Γ_δ is a piecewise analytic curve without cusps, then for $z\in\Gamma_\rho$ such that $\rho\in(\delta,\frac{1}{\delta})$, the inequality*

$$\sum_{n=1}^{N}|P_n(z)|^2\leq c\left(\frac{\rho^2}{\rho^2-1}N\rho^{2N}+\frac{\delta\rho}{1-\delta\rho}\right)$$

holds for a constant $c>0$ independent of N, ρ and δ.

Proof. Let $\Gamma\in U(\delta)$ for some $\delta<1$ and let the level curve Γ_δ be a piecewise analytic curve without cusps. From Corollary 1, taking 1 instead of $\widetilde{\beta}$ and assuming $\rho\in(\delta,\frac{1}{\delta})$, we write

$$\begin{aligned}\left|\sqrt{\frac{\pi}{n+1}}\frac{P_n(z)}{\phi'(z)\phi(z)^n}\right| &\leq 1+c_{35}\frac{(\frac{\delta}{\rho})^n}{n}\\ |P_n(z)|^2 &\leq \frac{n+1}{\pi}\left|\phi'(z)\right|^2|\phi(z)|^{2n}(1+c_{35}\frac{(\frac{\delta}{\rho})^n}{n})^2\end{aligned}$$

for $z \in \overline{\Omega}_\rho$. Since $z \in \Gamma_\rho$ from the hypothesis, we have

$$|P_n(z)|^2 \leq c_{36} n\rho^{2n}\left(1 + c_{37}\frac{(\frac{\delta}{\rho})^n}{n}\right) = c_{36} n\rho^{2n} + c_{38}(\delta\rho)^n$$

Hence, it follows

$$\sum_{n=1}^{N}|P_n(z)|^2 \leq c_{36}\sum_{n=1}^{N} n\rho^{2n} + c_{38}\sum_{n=1}^{N}(\delta\rho)^n. \tag{16}$$

For the first sum, by (4), we have

$$\sum_{n=1}^{N} n\rho^{2n} \leq \frac{\rho^2}{\rho^2-1}N\rho^{2N}. \tag{17}$$

Since $\delta\rho < 1$, the second sum can be estimated as follows.

$$\sum_{n=1}^{N}(\delta\rho)^n \leq (\delta\rho)\frac{1-(\delta\rho)^n}{(1-\delta\rho)} < \frac{\delta\rho}{1-\delta\rho} \tag{18}$$

Hence, by writing (18) and (17) in (16), we obtain

$$\sum_{n=1}^{N}|P_n(z)|^2 \leq c_{39}\left(\frac{\rho^2}{\rho^2-1}N\rho^{2N} + \frac{\delta\rho}{1-\delta\rho}\right),$$

where $c_{39} > 0$ is a constant independent of N, ρ and δ. □

Theorem 5. *Let the boundary Γ of the domain G belong to the class $U(\delta)$ for some $\delta < 1$ and the level curve Γ_δ be a piecewise analytic curve without cusps. Let also a function $f(z) \in A_2(G_R)$, $R > 1$, then the remaining term $\omega_n(z, f)$ satisfies the inequality*

$$\omega_n(z,f) \leq c(R)\frac{n}{R^n}E_n^{(2)}(f, G_R)\left(\sqrt{\frac{R}{R^2-1}} + 1 + \sqrt{\frac{\delta}{1-\delta}}\right), \quad z \in \overline{G}$$

for a constant $c(R) > 0$ independent of n and δ.

Proof. Let the boundary Γ of the domain G belong to the class $U(\delta)$ for some $\delta < 1$ and the level curve Γ_δ be a piecewise analytic curve without cusps. Let also a function $f(z) \in A_2(G_R), R > 1$. We assume $z \in \Gamma_\rho$, where $1 < \rho < R$. Following the same line of reasoning as in the derivation of (7) in Theorem 2, but using Lemma 4, for the constants depending only on R, we obtain the following inequality

$$\begin{aligned}|T_n(z,f) - K_n(z,f)| &\leq \frac{c_{40}(R)}{R^n}E_n^{(2)}(f, G_R)\left(\frac{\rho^2}{\rho^2-1}n\rho^{2n} + \frac{\delta\rho}{1-\delta\rho}\right)^{\frac{1}{2}} \\ &\leq \frac{c_{41}(R)}{R^n}E_n^{(2)}(f, G_R)\left(\sqrt{\frac{n\rho^2}{\rho^2-1}}\rho^n + \sqrt{\frac{\delta\rho}{1-\delta\rho}}\right).\end{aligned}$$

Using the last inequality, and (2) for $p' = q' = 2$, we write

$$\begin{aligned}
\omega_n(z,f) &\leq |f(z) - S_n(z,f)| + |T_n(z,f) - K_n(z,f)| \\
&\leq \frac{\rho^{n+1}}{R^{n+1}} E_n^{(2)}(f,G_R) \sqrt{\frac{\rho^2}{\rho^4-1} ln \frac{\rho^2}{\rho^2-1}} \left[\frac{1}{2\pi} \int_{|t|=R} \frac{|dt|}{|t-\rho|^2} \right]^{\frac{1}{2}} \\
&+ \frac{c_{42}(R)}{R^n} E_n^{(2)}(f,G_R) \sqrt{\frac{n\rho^2}{\rho^2-1}} \rho^n + \frac{c_{43}(R)}{R^n} E_n^{(2)}(f,G_R) \sqrt{\frac{\delta\rho}{1-\delta\rho}} \\
&\leq c_{44}(R) \left(\frac{\rho}{R}\right)^n E_n^{(2)}(f,G_R) \sqrt{\frac{\rho^2}{\rho^2-1}} \sqrt{\frac{R}{R^2-\rho^2}} + c_{45}(R) \left(\frac{\rho}{R}\right)^n E_n^{(2)}(f,G_R) \sqrt{\frac{n\rho^2}{\rho^2-1}} \\
&+ \frac{c_{46}(R)}{R^n} E_n^{(2)}(f,G_R) \sqrt{\frac{\delta\rho}{1-\delta\rho}}.
\end{aligned}$$

If we replace $\rho = 1 + \frac{1}{n}$ instead of ρ, as $n \to \infty$, for $z \in \overline{G}$, we finally obtain

$$\omega_n(z,f) \leq c_{47}(R) \frac{n}{R^n} E_n^{(2)}(f,G_R) \left(\sqrt{\frac{R}{R^2-1}} + 1 + \sqrt{\frac{\delta}{1-\delta}} \right),$$

where $c_{47}(R) > 0$ is a constant independent of n and δ. □

4. Conclusions and Discussion

The approximation problems of Fourier series have been extensively studied by numerous authors in real lines or complex planes. However, so far only a few studies on the approximation of Fourier series with respect to Bergman orthogonal polynomials are available in the literature [1,5–8]. In this paper, employing the results on the strong asymptotics of Bergman orthogonal polynomials outside the domain G of orthogonality, obtained by a new matrix approach [4], we add new findings to the known results within a modern framework.

The findings in this paper extend the approximation problems for the canonical domain $G_R \supset G$ for $R > 1$, where f is analytic. Furthermore, as expected, the remaining term values rapidly approach zero as $n \to \infty$ since the exponential decay R^{-n}, $R > 1$ dominates any polynomial growth. The convergence rates we found depend on the best approximation number $E_n^{(2)}(f,G_R)$, the number R and the properties of the boundary curve of domain G, and even the properties of the level curves that approach the boundary Γ of domain G from inside for the analytic curves.

However, while the present approach suffices to establish the rapid decay of the remainder term with respect to the best approximation number $E_n^{(2)}(f,G_R)$, the number $R > 1$, and the properties of the curve Γ, improved approximation rates may be obtained by employing different techniques. And it would be of interest to investigate how the approximation rates are affected by geometric features of the boundary in the piecewise analytic case. In particular, the influence of corner angles on the asymptotic behavior of the approximation error remains an open problem. Such questions are closely related to the local singular behavior of conformal mappings near corner points. Understanding the role of corner angles in such settings may lead to refined estimates and will be considered in future work. On the other hand, the summability condition $p + \alpha > 1/2$ in Lemma 1 and Lemma 3 arises naturally from the convergence requirements in the application of the Abel transformation and is intrinsic to the present method of proof. Whether this condition can be relaxed by alternative techniques remains open.

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