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Entire Solutions of Painlevé V-Type Differential-Difference Equation

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Abstract

This research focuses on the following equation: $(\bar{g}g-1)(g\underline{g}-1)+B(z)\frac{g'(z)}{g(z)}=R(z,g)=\frac{P(z,g)}{Q(z,g)}$, where $P(z,g)$ and $Q(z,g)$ represent polynomials in g, possessing rational coefficients in z and lacking shared roots, $B(z)$ is rational, and $\bar{g}\equiv g(z+1), \underline{g}\equiv g(z-1)$. Reduced forms arise when this equation permits an entire solution with a hyper-order strictly below one. This paper investigates the forms and value distribution of entire solutions to the reduced Painlevé-type differential-difference equation. Furthermore, the existence of entire solutions to the above equation is examined when $\deg_g(P)\geq 5$ and $\deg_g(Q)=0$ or $\deg_g(Q)\geq 1$ and $Q(z,0)\not\equiv 0$.

Keywords: Painlevé-type differential-difference equation; hyper-order; entire solution

MSC: 30D35; 34M55

1. Introduction

Differential-difference equations find extensive utility across various scientific domains, particularly within physics. To illustrate, evaluating singularity confinement offers a dependable integrability condition for discrete frameworks, which helps in formulating discrete counterparts to the Painlevé differential equations [1,2]. Through the lens of theoretical physics, a singularity manifests in a discrete model whenever a specific iterative phase leads to the elimination of a degree of freedom.

To assist readers who may be unfamiliar with this field, we briefly outline several core concepts of Nevanlinna theory (refer to [3–5]). For any meromorphic function $g(z)$, the characteristic function $T(r,g)$ serves as a measure of its global growth bounded by the region $|z|\leq r$. This metric is conventionally defined as

$$T(r,g)=m(r,g)+N(r,g).$$

In this context, the proximity function $m(r,g)$ is evaluated via

$$m(r,g)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|g(re^{i\theta})|d\theta,$$

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with the notation $\log^+ x = \max\{0,\ \log x\}$. Additionally, $n(r,g)$ and $n\left(r,\frac{1}{g}\right)$ are utilized to represent the total count of poles and zeros of g, respectively, located within the circle $|z| \le r$ (counting multiplicities).

The integrated counting functions, which track the poles and zeros of g, are defined as

$$N(r,g) = \int_0^r \frac{n(t,g) - n(0,g)}{t} dt + n(0,g)\log r,$$

$$N\left(r,\frac{1}{g}\right) = \int_0^r \frac{n\left(t,\frac{1}{g}\right) - n\left(0,\frac{1}{g}\right)}{t} dt + n\left(0,\frac{1}{g}\right)\log r.$$

The classical order $\sigma(g)$ and hyper-order $\sigma_2(g)$ associated with $g(z)$ are defined, respectively, via

$$\sigma(g) := \limsup_{r\to\infty} \frac{\log^+ T(r,g)}{\log r},\ \sigma_2(g) := \limsup_{r\to\infty} \frac{\log^+\log^+ T(r,g)}{\log r}.$$

The exponent of convergence concerning the zeros of $g(z)$, symbolized by $\lambda(g)$, alongside that of the poles of $g(z)$, symbolized by $\lambda\left(\frac{1}{g}\right)$, are formulated as

$$\lambda(g) := \limsup_{r\to\infty} \frac{\log^+ n(r,\frac{1}{g})}{\log r} = \limsup_{r\to\infty} \frac{\log^+ N(r,\frac{1}{g})}{\log r},$$

$$\lambda\left(\frac{1}{g}\right) := \limsup_{r\to\infty} \frac{\log^+ n(r,g)}{\log r} = \limsup_{r\to\infty} \frac{\log^+ N(r,g)}{\log r}.$$

These metrics evaluate the asymptotic density of the roots and poles of g, respectively.

We call a meromorphic function g subnormal whenever $\limsup_{r\to\infty} \frac{\log T(r,g)}{r} = 0$. Then one directly obtains $\sigma_2(g) \le 1$.

Furthermore, we define a meromorphic function $\gamma(z)$ to be small with respect to $g(z)$ provided that $T(r,\gamma(z)) = S(r,g)$. Here, the notation $S(r,g)$ stands for any term that fulfills $S(r,g) = o(T(r,g))$ as $r \to \infty$, omitting a potential exceptional set with zero upper density.

The difference version of Nevanlinna theory was pioneered by a number of professors (see [6–8]). This theoretical framework has been employed to investigate the characteristics of solutions to difference equations. Numerous significant findings have been achieved through this approach, for example, see [6,9,10] and so on.

The Nevanlinna theory is extensively applied in the examination of differential-difference equations (see [11–20]). In particular, Quispel, Capel and Sahadevan [17] (1992) analyzed

$$g(z)(\bar{g} - \underline{g}) + ag'(z) = bg(z),$$

where $\bar{g} \equiv g(z+1)$ and $\underline{g} \equiv g(z-1)$, while a and b represent constant values. Originating from a symmetry reduction applied to the Kac–Van Moerbeke equation, this relation possesses a formal continuum limit that connects it with the first Painlev*é* Equation $g''(z) = 6g^2(z) + t$. For convenience, we will employ the notations $\bar{g} \equiv g(z+1)$ and $\underline{g} \equiv g(z-1)$. Throughout the remainder of this paper, all entire solutions to the equations mentioned refer to transcendental entire solutions.

Drawing inspiration from Quispel's work, Halburd's examined various attributes of differential-difference equations given by $\bar{g} - \underline{g} + B(z)g'/g = P(z,g)/Q(z,g)$. A key result from [13] is as follows.

Theorem 1 ([13]). *Let g be a non-rational meromorphic solution to*

$$\overline{g} - \underline{g} + B(z)\frac{g'(z)}{g(z)} = R(z,g) = \frac{P(z,g)}{Q(z,g)},$$

where $B(z)$ is rational. Here, $P(z,g)$ is a polynomial in g with rational coefficients, $Q(z,g)$ is a polynomial in g whose roots are nonzero rational functions of z without sharing any roots with $P(z,g)$. If $\sigma_2(g) < 1$, then

$$\deg_g(P) = \deg_g(Q) + 1 \leq 3 \text{ or } \deg_g(R) \leq 1.$$

Subsequently, Du and Zhang [12] investigated the fifth Painlevé differential-difference equation. They proved Theorem 2 as follows.

Theorem 2 ([12]). *Let g denote a meromorphic solution to the following equation:*

$$(\bar{g}g - 1)(g\underline{g} - 1) + B(z)\frac{g'(z)}{g(z)} = R(z,g) = \frac{P(z,g)}{Q(z,g)}. \tag{1}$$

If $\sigma_2(g) < 1$, then one of the possibilities below occurs:

(i) $\deg_g(Q) = 0$, $\deg_g(P) \leq 4$;

(ii) $\deg_g(Q) = 1$, *either* $Q = g(z)$ *and* $\deg_g(P) \leq 6$, *or* $Q = g(z) - b_1(z)$ *and* $\deg_g(P) = 3$ *or 4;*

(iii) $\deg_g(Q) = 2$, *either* $Q = (g(z) - b_1(z))^2$ *or* $Q = (g(z) - b_1(z))(g(z) - b_2(z))$, *and* $\deg_g(P) = 4$, *or* $Q = g(z)(g(z) - b_1(z))$ *and* $\deg_g(P) \leq 5$;

(iv) $\deg_g(Q) = 3$, $Q = g(z)(g(z) - b_1(z))^2$ *or* $Q = g(z)(g(z) - b_1(z))(g(z) - b_2(z))$, *and* $\deg_g(P) = 5$. *with $b_1(z)$ and $b_2(z)$ above being distinct nonzero rational functions in z.*

Subsequently, by exploring meromorphic solutions to a higher-order generalization of Equation (1), Xia, Long, and Xiang [18] established Theorem 3 below.

Theorem 3 ([18]). *Let g be a non-rational subnormal meromorphic solution of*

$$(\bar{g}g - 1)(g\underline{g} - 1) + B(z)\frac{g^{(n)}(z)}{g(z)} = R(z,g) = \frac{P(z,g)}{Q(z,g)},$$

where n is a positive integer, $B(z)$ is small relative to g, and $P(z,g)$ and $Q(z,g)$ are polynomials in g with meromorphic coefficients that are also small relative to g in z. Moreover, the roots of $Q(z,g)$ are nonzero meromorphic functions of z and are distinct from those of $P(z,g)$. Then, one of the following cases occurs:

(i) If $\deg_g(Q) = 0$, *then* $\deg_g(P) \leq n + 3$.

(ii) $\deg_g(Q) \geq 1$, *then* $\deg_g(P) \leq \deg_g(Q) + 2$ *or* $\deg_g(Q) + 2 < \deg_g(P) \leq \deg_g(Q) + n + 2$.

Moreover, when $n = 1$, *we have* $\deg_g(P) = \deg_g(Q) + 2$; *and when* $n > 2$ *is odd,* $\deg_g(Q) + 2 < \deg_g(P) \leq \deg_g(Q) + n + 2$.

Through an analysis of the equation in Theorem 1, Zhang and Huang [20] derived the following results in 2020.

Theorem 4 ([20]). *Let $R(z,g) \not\equiv 0$ denote an irreducible rational function in g with coefficients that are rational in z, and let $B(z)$ be rational. If*

$$\bar{g} - \underline{g} + B(z)\frac{g'(z)}{g(z)} = R(z,g) = \frac{P(z,g)}{Q(z,g)} \tag{2}$$

permits an entire transcendental solution g satisfying $\sigma_2(g) < 1$, then Equation (2) transforms into

$$\bar{g} - \underline{g} + B(z)\frac{g'(z)}{g(z)} = B_1(z)g + B_0(z), \tag{3}$$

or

$$\bar{g} - \underline{g} + B(z)\frac{g'(z)}{g(z)} = \frac{B_2(z)g^2 + B_1(z)g + B_0(z)}{g},$$

where $B_j(z)(j = 0, 1, 2)$ are rational.

Theorem 5 ([20]). *Let $B(z)$, $B_0(z)$ and $B_1(z)$ be rational functions with $B_1(z) \not\equiv 0$ or $B_0(z) \not\equiv 0$. Suppose g is an entire transcendental solution to (3) with $\sigma_2(g) < 1$.*

(i) When $B(z) \equiv 0$, then $\sigma(g) \geq 1$.

(ii) If $B(z) \not\equiv 0$, then one has $g(z) = L(z)e^{lz}$, with $L(z) \not\equiv 0$ a polynomial and $l \neq 0$ a constant. Specifically, in the case where $B_1(z)$ is a polynomial and $B_1(z) \not\equiv \pm 2i$, we obtain $g(z) = C_0 e^{lz}$, where $C_0 \in \mathbb{C}/\{0\}$; when $B_1(z) \equiv \pm 2i$, then $g(z) = (A_1 z + A_0)e^{(2n \pm \frac{1}{2})\pi i z}$, with $n \in \mathbb{Z}$ and $A_1, A_0 \in \mathbb{C}$ satisfying $|A_1| + |A_0| \neq 0$.

Naturally, we pose two problems related to Theorems 2–5.

Problem 1. What are the reduced forms of Equation (1) when it permits an entire solution?

Problem 2. Does Equation (1) have transcendental entire solutions, and if they exist, what form do they take?

In Sections 2 and 4, we answer Problems 1 and 2, and in Section 3, we find the forms of entire solutions of (1).

2. Reduced Forms of Painlevé V-Type Differential-Difference Equation

The present section focuses on resolving Problem 1. We characterize all reduced forms for (1) which admits an entire solution of hyper-order strictly below one, and the subsequent theorem arises.

Theorem 6. *Let $P(z,g)$ and $Q(z,g)$ be coprime polynomials in g with coefficients that are rational functions of z, and let $B(z) \not\equiv 0$ be rational. Suppose that (1) admits an entire solution g with $\sigma_2(g) < 1$.*

(i) If g possesses finite many zeros, then Equation (1) transforms into

$$(\bar{g}g - 1)(g\underline{g} - 1) + B(z)\frac{g'(z)}{g(z)} = B_4(z)g^4 + B_2(z)g^2 + B_0(z), \tag{4}$$

where $B_4(z)$, $B_2(z)$, $B_0(z)$ are rational functions.

(ii) If g possesses infinitely many zeros, then Equation (1) transforms into

$$(\bar{g}g - 1)(g\underline{g} - 1) + B(z)\frac{g'(z)}{g(z)} = \frac{B_5(z)g^5 + B_4(z)g^4 + B_3(z)g^3 + B_2(z)g^2 + B_1(z)g + B_0(z)}{g}, \tag{5}$$

where $B_i(z)$ (for $i = 0, 1, \cdots, 5$) are rational.

Remark 1. *(i) Considering $R(z,g)(\not\equiv 0)$ as an irreducible rational function in g, we can infer that $B_4(z)$ in (4) does not vanish, and neither $B_5(z)$ nor $B_0(z)$ in (5) disappear.*

(ii) In Theorem 2, if $Q(z,g)=g(z)$, then $=\deg_g(P)\leq 6$. In Theorem 6, $\deg_g(P)=5$ in (5). In fact, it was derived by Xia et al. in [18] that $\deg_g(R)\leq 5$.

The Examples 1 and 2 provided below illustrate the existence of (4) as stated in Theorem 6.

Example 1. *The equation*

$$(\bar{g}g-1)(g\underline{g}-1)+B(z)\frac{g'(z)}{g(z)}=\frac{z^2-1}{z^2}g^4-\left(\frac{z+1}{z}e+\frac{z-1}{ez}\right)g^2+B(z)\frac{z+1}{z}+1$$

possesses an entire solution given by $g(z)=ze^z$, where $B(z)$ denotes an arbitrary rational function. It is clear that $\sigma(g)=1$ and $\sigma_2(g)=0$.

Example 2. *The equation*

$$(\bar{g}g-1)(g\underline{g}-1)+B(z)\frac{g'(z)}{g(z)}=g^4-(e+e^{-1})g^2+1-B(z)$$

possesses an entire solution given by $g(z)=\frac{1}{e^z}$, where $B(z)$ denotes an arbitrary rational function. It is clear that $\sigma(g)=1$, $\sigma_2(g)=0$.

To support our findings, we need some auxiliary results.

Lemma 1 ([10]). *Suppose $g(z)$ is a subnormal meromorphic function. Then $T(r,g(z+\eta))=T(r,g)$, $m\left(r,\frac{g(z+\eta)}{g(z)}\right)=o(T(r,g))$ hold as $r\to\infty, r\notin E$, where E is a subset of $[1,+\infty)$ with the zero upper density and $\eta\in\mathbb{C}/\{0\}$.*

Let g be meromorphic. A differential-difference polynomial evaluated at $g(z)$ is formally structured as

$$P(z,g)=\sum_{s=1}^{m}\alpha_s(z)\prod_{i=0}^{k}(g(z+\eta_i))^{\gamma_{i,0}^s}(g'(z+\eta_i))^{\gamma_{i,1}^s}\cdots(g^{(k_i)}(z+\eta_i))^{\gamma_{i,k_i}^s},$$

where $\eta_0,\cdots,\eta_k$ denote distinct constants, the exponents $\gamma_{i,0}^s,\cdots,\gamma_{i,k_i}^s,k_i$ denote non-negative integers, and $\alpha_s(z)$ operate as small functions relative to g. If $\gamma_{i,0}^s=\cdots=\gamma_{i,k_i}^s=0$, then $P(z,g)$ simplifies into a pure difference polynomial.

The Lemma 2 establishes a differential-difference adaptation of the classical Clunie lemma.

Lemma 2 ([21]). *Suppose g is a non-rational subnormal meromorphic solution of*

$$U(z,g)P_1(z,g)=Q_1(z,g),$$

where U is a single-term difference polynomial of highest total degree in g and its shifts, while P_1 and Q_1 are ordinary differential-difference polynomials. Assume their coefficients are small functions, with $\deg_g(U)=n$, $\deg_g(Q_1)\leq n$. Then

$$m(r,P_1)=S(r,g).$$

Lemma 3 ([5,22]). *Assume we have meromorphic functions $\alpha_i(z)$ (for $i=1,\cdots,n$) and entire functions $b_i(z)$ (for $i=1,\cdots,n$) that meet the following requirements:*

(i) $\sum_{i=1}^{n} \alpha_i(z)e^{b_i(z)} \equiv 0$;
(ii) the differences $b_i(z) - b_k(z)$ *are never constant for* $1 \le i < k \le n$;
(iii) considering $1 \le i \le n$ *and* $1 \le h < k \le n$, *the relation*

$$T(r, \alpha_i) = o\{T(r, e^{b_h - b_k})\} \ \ (r \to \infty, r \notin E)$$

holds, where the set $E \subset (1, \infty)$ *exhibits either a finite linear measure or a finite logarithmic measure.*

Then $\alpha_i(z) \equiv 0$ $(i = 1, 2, \cdots, n)$.

Lemma 4, universally recognized as due to Valiron and Mohonko, is crucially significant when examining differential, difference, and their associated delay differential equations.

Lemma 4 ([4]). *Suppose* g *is meromorphic and* $R(z, g)$ *denotes a rational function in* g *that remains meromorphic in* z. *If every coefficient of* $R(z, g)$ *is relatively small in comparison to* g, *then*

$$T(r, R(z, g)) = \deg_g(R)T(r, g) + S(r, g).$$

Next, we establish an auxiliary lemma directly linked to the structural reduction form (4).

Lemma 5. *Assume that* $R(z, g) \not\equiv 0$ *is an irreducible rational function in* g *with rational coefficients in* z *satisfying* $\deg_g(R) \le 5$. *Additionally, let* $B(z) \not\equiv 0$ *denote another rational function. Suppose* g *is an entire solution of (1) with* $\sigma_2(g) < 1$. *If* g *has finitely many zeros, then Equation (1) can be represented in the form of (4), where* $B_4(z)$, $B_2(z)$ *and* $B_0(z)$ *denote rational functions satisfying* $B_4(z) \not\equiv 0$.

Proof. By the hypotheses and the Hadamard factorization theorem, g necessarily admits the form

$$g(z) = L(z)e^{l(z)}, \tag{6}$$

where $L(z)$ is a nonzero polynomial, and $l(z)$ is a nonconstant entire function satisfying $\sigma_2(g) = \sigma_2(e^l) = \sigma(l) < 1$. Substituting (6) into Equation (1) yields

$$(\bar{L}Le^{\bar{l}+l} - 1)(\underline{L}Le^{\underline{l}+l} - 1) + B(z)\left(\frac{L'}{L} + l'\right) = \frac{P(z, g)}{Q(z, g)}, \tag{7}$$

where $L = L(z)$, $\bar{L} = L(z+1)$, $\underline{L} = L(z-1)$ and $l = l(z)$, $\bar{l} = l(z+1)$, $\underline{l} = (z-1)$. By (7), we have

$$L^2\bar{L}\underline{L}e^{\bar{l}+\underline{l}-2l}e^{4l} - (L\bar{L}e^{\bar{l}-l} + L\underline{L}e^{\underline{l}-l})e^{2l} + 1 + B(z)\left(\frac{L'}{L} + l'\right) = \frac{P(z, g)}{Q(z, g)}. \tag{8}$$

Setting $G_2(z) = L^2\bar{L}\underline{L}e^{\bar{l}+\underline{l}-2l}$ and $G_1(z) = L\bar{L}e^{\bar{l}-l} + L\underline{L}e^{\underline{l}-l}$. Obviously, $G_2(z) \not\equiv 0$. Lemma 1 and $\sigma_2(e^l) < 1$ lead to

$$T(r, G_i) = m(r, G_i) \le m\left(r, e^{\bar{l}-l}\right) + m\left(r, e^{\underline{l}-l}\right) + O(\log r) = S(r, e^l)$$

for $i = 1, 2$.

Utilizing (8), we derive

$$G_2(z)e^{4l} - G_1(z)e^{2l} + 1 + B(z)\left(\frac{L'}{L} + l'\right) = \frac{P(z, g)}{Q(z, g)}. \tag{9}$$

(9) implies

$$T\left(r, \frac{P(z,g}{Q(z,g}\right) \leq T(r, e^{4l}) + S(r, e^{l}) = 4T(r,g) + S(r,g).$$

Based on Lemma 4 and the above inequality, we obtain $\deg_g(P) \leq 4$ and $\deg_g(Q) \leq 4$. Hence, expression (9) takes the form

$$G_2(z)e^{4l} - G_1(z)e^{2l} + 1 + B(z)\left(\frac{L'}{L} + l'\right) = \frac{\hat{a}_4(z)L^4e^{4l} + \cdots + \hat{a}_1(z)Le^{l} + \hat{a}_0(z)}{\hat{b}_4(z)L^4e^{4l} + \cdots + \hat{b}_1(z)Le^{l} + \hat{b}_0(z)}, \quad (10)$$

where $\hat{a}_i(z), \hat{b}_i(z)$ $(i = 0, 1, \cdots, 4)$ are rational. Expression (10) shows

$$\hat{b}_4(z)L^4G_2(z)e^{8l} + \hat{b}_3(z)L^3G_2(z)e^{7l} + S_6e^{6l} + S_5e^{5l} + S_4e^{4l} + S_3e^{3l} + S_2e^{2l} + S_1e^{l} + S_0 = 0, \quad (11)$$

where

$$\begin{aligned}
S_6 &= \hat{b}_2(z)L^2G_2(z) - \hat{b}_4(z)L^4G_1(z),\\
S_5 &= \hat{b}_1(z)LG_2(z) - \hat{b}_3(z)L^3G_1(z),\\
S_4 &= \hat{b}_4(z)L^4[1 + B(z)\left(\tfrac{L'}{L} + l'\right)] + \hat{b}_0(z)G_2(z) - \hat{b}_2(z)L^2G_1(z) - \hat{a}_4(z)L^4,\\
S_3 &= \hat{b}_3(z)L^3[1 + B(z)\left(\tfrac{L'}{L} + l'\right)] - \hat{b}_1(z)LG_1(z) - \hat{a}_3(z)L^3,\\
S_2 &= \hat{b}_2(z)L^2[1 + B(z)\left(\tfrac{L'}{L} + l'\right)] - \hat{b}_0(z)G_1(z) - \hat{a}_2(z)L^2,\\
S_1 &= \hat{b}_1(z)L[1 + B(z)\left(\tfrac{L'}{L} + l'\right)] - \hat{a}_1(z)L,\\
S_0 &= \hat{b}_0(z)[1 + B(z)\left(\tfrac{L'}{L} + l'\right)] - \hat{a}_0(z).
\end{aligned}$$

(11) and Lemma 3 lead to $\hat{b}_4(z) \equiv 0$ and $\hat{b}_3(z) \equiv 0$. By $S_6 \equiv 0$ and $S_5 \equiv 0$, we get $\hat{b}_1(z) \equiv 0$ and $\hat{b}_2(z) \equiv 0$. Furthermore, we get $\hat{a}_1(z) \equiv 0$ and $\hat{a}_3(z) \equiv 0$. From (10), we deduce that Equation (1) is of the form

$$(\bar{g}g - 1)(g\underline{g} - 1) + B(z)\frac{g'(z)}{g(z)} = B_4(z)g^4 + B_2(z)g^2 + B_0(z), \quad (12)$$

where $B_4(z)$, $B_2(z)$ and $B_0(z)$ represent rational functions.

If $B_4(z) \equiv 0$, then we deduce from (12) that

$$g^3\bar{g}\underline{g} = B_2(z)g^3 + B_0(z)g - B(z)g' + g^2\bar{g} + g^2\underline{g} - g.$$

Using the equality mentioned above in conjunction with Lemma 2, the contradiction $T(r,g) = m(r,g) = S(r,g)$ follows. So, $B_4(z) \not\equiv 0$. □

Proof of Theorem 6. Assume that $g(z)$ is an entire solution of (1) with $\sigma_2(g) < 1$.

(i) Based on (1) and Lemmas 1 and 4, we derive

$$\begin{aligned}
T(r, P(z,g)/Q(z,g)) &\leq T\Big(r, (\bar{g}g - 1)(g\underline{g} - 1)\Big) + T(r, B(z)g'/g)\\
&\leq 4T(r,g) + N\left(r, \frac{g'}{g}\right) + S(r,g)\\
&\leq 4T(r,g) + N\left(r, \frac{1}{g}\right) + S(r,g)\\
&\leq 5T(r,g) + S(r,g).
\end{aligned}$$

Lemma 4 leads to $\deg_g(P) \leq 5$ and $\deg_g(Q) \leq 5$. Therefore,

$$\frac{P(z,g)}{Q(z,g)} = \frac{\hat{a}_5(z)g^5 + \hat{a}_4(z)g^4 + \hat{a}_3(z)g^3 + \hat{a}_2(z)g^2 + \hat{a}_1(z)g + \hat{a}_0(z)}{\hat{b}_5(z)g^5 + \hat{b}_4(z)g^4 + \hat{b}_3(z)g^3 + \hat{b}_2(z)g^2 + \hat{b}_1(z)g + \hat{b}_0(z)}, \quad (13)$$

where $\hat{a}_i(z)$ and $\hat{b}_i(z)$ $(i = 0, 1, \cdots, 5)$ are rational.

When g possesses a finite number of zeros, Lemma 5 demonstrates that Equation (1) assumes the structure of (4) and $B_4(z) \not\equiv 0$.

(ii) Now, let us assume that g has an infinite number of zeros. Consider z_0 as one of the zeros of g, and suppose that neither $B(z)$ nor any coefficient in $\frac{P(z,g)}{Q(z,g)}$ exhibits a pole or zero at z_0. If $\hat{b}_0(z) \not\equiv 0$, then z_0 becomes a simple pole of $(\bar{g}g - 1)(g\underline{g} - 1) + B(z)g'/g$, as well as a finite value of $\frac{P(z,g)}{Q(z,g)}$, a contradiction. So, $\hat{b}_0(z) \equiv 0$.

Assuming $\hat{b}_5(z) \not\equiv 0$, through Equations (1) and (13), we derive

$$(\bar{g}g - 1)(g\underline{g} - 1) + B(z)\frac{g'(z)}{g(z)} = \frac{\hat{a}_5(z)g^5 + \cdots + \hat{a}_1(z)g + \hat{a}_0(z)}{\hat{b}_5(z)g^5 + \hat{b}_4(z)g^4 + \hat{b}_3(z)g^3 + \hat{b}_2(z)g^2 + \hat{b}_1(z)g}. \quad (14)$$

(14) implies

$$\hat{b}_5(z)g^7\bar{g}\underline{g} = \hat{Q}(z, g), \quad (15)$$

where $\hat{Q}(z, g) = -\hat{b}_4(z)g^6\bar{g}\underline{g} + \hat{b}_5(z)g^6\bar{g} + \hat{b}_5(z)g^6\underline{g} + \cdots + \hat{a}_0(z)$ represents a polynomial involving both differential and difference operators with respect to g, and $\deg_g(\hat{Q}) \leq 8$.

Lemma 2 and (15) lead to $T(r, g) = m(r, g) = S(r, g)$, a contradiction. So, $\hat{b}_5(z) \equiv 0$. Thus, (14) becomes

$$(\bar{g}g - 1)(g\underline{g} - 1) + B(z)\frac{g'(z)}{g(z)} = \frac{\hat{a}_5(z)g^5 + \cdots + \hat{a}_1(z)g + \hat{a}_0(z)}{\hat{b}_4(z)g^4 + \hat{b}_3(z)g^3 + \hat{b}_2(z)g^2 + \hat{b}_1(z)g}. \quad (16)$$

When $\hat{b}_4(z) \not\equiv 0$, employing Equation (16), we obtain

$$\hat{b}_4(z)g^6\bar{g}\underline{g} = \hat{Q}_1(z, g),$$

where $\hat{Q}_1(z, g) = -\hat{b}_3(z)g^5\bar{g}\underline{g} + \hat{b}_4(z)g^5\bar{g} + \hat{b}_4(z)g^5\underline{g} + \cdots + \hat{a}_0(z)$ represents a polynomial involving both differential and difference operators with respect to g, and $\deg_g(\hat{Q}_1) \leq 7$.

Similarly as above, we get $\hat{b}_4(z) \equiv 0$. Thus, (16) is of the form

$$(\bar{g}g - 1)(g\underline{g} - 1) + B(z)\frac{g'(z)}{g(z)} = \frac{\hat{a}_5(z)g^5 + \cdots + \hat{a}_1(z)g + \hat{a}_0(z)}{\hat{b}_3(z)g^3 + \hat{b}_2(z)g^2 + \hat{b}_1(z)g}.$$

By a similar approach, we get $\hat{b}_3(z) \equiv 0$ and $\hat{b}_2(z) \equiv 0$. Hence, (1) assumes the form (5), with $B_i(z)$ (for $i = 0, 1, \cdots, 5$) being rational functions, where $B_0(z) \not\equiv 0$.

If $B_5(z) \equiv 0$, (5) implies

$$\bar{g}\underline{g}g^3 = B_4(z)g^4 + B_3(z)g^3 + B_2(z)g^2 + B_1(z)g + B_0(z) + g^2\bar{g} + g^2\underline{g} - g - B(z)g'. \quad (17)$$

According to (17) and Lemma 2, the contradiction $T(r, g) = m(r, g) = S(r, g)$ follows. So $B_5(z) \not\equiv 0$.

Thus, Theorem 6 has been successfully proven. □

3. Forms and Value Distribution of Entire Solutions for Reduced Painlevé V-Type Differential-Difference Equation

Inspired by the results established in Theorem 5 for a different class of equations, we now investigate the entire solutions to the reduced form (4) derived in Section 2, and aim to determine the specific forms that such solutions must take.

Theorem 7. *Let $B_4(z), B_2(z), B_0(z)$ and $B(z)$ be rational functions appearing in Equation (4), with $B_4(z)B(z) \not\equiv 0$. Suppose $g(z)$ is an entire solution of Equation (4) satisfying $\sigma_2(g) < 1$. Then $g(z)$ must take the form $g(z) = L(z)e^{Cz}$, where $L(z)$ is a nonzero polynomial and C is a nonzero*

constant. In particular, if $B_4(z)$ is a polynomial, then $g(z) = Le^{Cz}$ for some constant $L \in \mathbb{C} \setminus \{0\}$, and $B_4(z) \equiv 1$.

Theorem 7 establishes the explicit form of transcendental entire solutions to Equation (4). However, characterizing its meromorphic solutions remains an open problem.

Furthermore, an analysis of the characteristics of entire solutions of Equation (5) leads to Theorem 8, as described below.

Theorem 8. *Let $B_i(z)$ $(i = 0, 1, \cdots, 5)$ and $B(z)$ (with $B(z) \not\equiv 0$) be rational functions. If g is an entire solution to (5) with $\sigma_2(g) < 1$, then $\sigma(g) \geq 1$ and $\lambda(g) = \sigma(g)$.*

Next we give Examples 3 and 4 to illustrate that the situation described in Theorems 7 and 8 can, indeed, occur.

Example 3. *The equation*

$$(\bar{g}g - 1)(g\underline{g} - 1) + B(z)\frac{g'(z)}{g(z)} = g^4 - (e + e^{-1})g^2 + B(z)$$

admits an entire solution given by $g(z) = e^z$, where $B(z)$ denotes an arbitrary rational function. It is evident that g possesses finitely many zeros, and $\sigma(g) = 1$.

Example 4. *The equation*

$$(\bar{g}g - 1)(g\underline{g} - 1) + B(z)\frac{g'(z)}{g(z)} = \frac{g^5 - 2g^3 + (1 + 2\pi iB(z))g - 2\pi iB(z)}{g}$$

admits an entire solution given by $g(z) = e^{2\pi iz} + 1$, where $B(z)$ denotes an arbitrary rational function. It is evident that g possesses infinite number of zeros, and $\lambda(g) = \sigma(g) = 1$.

In [20], Zhang and Huang established Lemma 6, which is crucial for proving Theorem 8.

Lemma 6 ([20])**.** *Suppose g represents a meromorphic solution of a specific type of delay differential equation, with finite order σ, given by*

$$g^n P_1(z, g) = Q_1(z, g),$$

where P_1 and Q_1 are differential-difference polynomials in $g(z)$ with rational coefficients and $\deg_g(Q_1) \leq n$. Then, for every $\varepsilon > 0$, it follows that

$$m(r, P_1) = O(r^{\sigma - 1 + \varepsilon}) + O(\log r).$$

Lemma 7 serves as the explicit differential-difference counterpart to Mohon'ko's theorem, organically descending from Lemma 2.2 in [19].

Lemma 7 ([23])**.** *Assume g is a non-rational subnormal meromorphic solution of $P(z, g) = 0$, where $P(z, g)$ denotes a differential-difference polynomial in $g(z)$ having rational coefficients. If a corresponding small meromorphic function $b(z)$ guarantees $P(z, b) \not\equiv 0$, we obtain*

$$m\left(r, \frac{1}{g - b}\right) = S(r, g).$$

Proof of Theorem 7. Assume g is an entire solution to Equation (4) with $\sigma_2(g) < 1$.

Because $B(z) \not\equiv 0$, we can infer that g possesses only a finite number of zeros. By Hadamard factorization theorem, g could be represented as

$$g = L(z)e^{l(z)}. \tag{18}$$

In (18), $L(z)$ denotes a nonzero polynomial, and $l(z)$ is an entire function satisfying $\sigma_2(g) = \sigma_2(e^{l(z)}) = \sigma(l) < 1$. Substituting (18) into Equation (4) yields

$$G_2(z)e^{4l} - G_1(z)e^{2l} + 1 + B(z)\left(\frac{L'}{L} + l'\right) = B_4(z)L^4e^{4l} + B_2(z)L^2e^{2l} + B_0(z), \tag{19}$$

where $G_2(z) = L^2\bar{L}\underline{L}e^{\bar{l}+\underline{l}-2l}$, $G_1(z) = L\bar{L}e^{\bar{l}-l} + L\underline{L}e^{\underline{l}-l}$. Obviously, $G_2(z) \not\equiv 0$. Lemma 1 implies that $T(r, G_i) = m(r, G_i) = S(r, G_i)$ for $i = 1, 2$. Expression (19) shows

$$[G_2(z) - B_4(z)L^4]e^{4l} - [G_1(z) + B_2(z)L^2]e^{2l} + B(z)\left(\frac{L'}{L} + l'\right) + 1 - B_0(z) = 0. \tag{20}$$

If l is transcendental entire, we observe from [24] that $\bar{l} - l$ and $\underline{l} - l$ are also transcendental entire functions, since $\sigma(l) < 1$. Utilizing Lemma 3 and (20), we derive $G_2(z) - B_4(z)L^4 \equiv 0$, leading to a contradiction. Thus, l must be a polynomial. Setting $h(z) = l - \underline{l}$. If $\deg h \geq 2$, then $\deg(\bar{h} - h) = \deg[(\bar{l} - l) - (l - \underline{l})] = \deg(\bar{l} + \underline{l} - 2l) \geq 1$. Thus, $G_2(z)$ being transcendental entire contradicts the assertion $G_2(z) - B_4(z)L^4 \equiv 0$. So, $\deg h \leq 1$. We infer $\deg l \leq 2$. Therefore, g can be given by

$$g(z) = L(z)e^{Az^2+Cz}. \tag{21}$$

If $G_1(z) = L\bar{L}e^{\bar{l}-l} + L\underline{L}e^{\underline{l}-l} \equiv 0$, then $\bar{L}e^{2Az+C} = -\underline{L}e^{-2Az-C}$. Thus, we get $A = 0$. If $G_1(z) \not\equiv 0$, then we get $A = 0$ since $G_1(z) + B_2(z)L^2 \equiv 0$. So, (21) implies

$$g(z) = L(z)e^{Cz}, \tag{22}$$

where C is some nonzero complex constant.

By (22), $G_2(z) = L^2\bar{L}\underline{L}e^{\bar{l}+\underline{l}-2l}$ and $G_2(z) - B_4(z)L^4 \equiv 0$, we get

$$\bar{L}\underline{L} = B_4(z)L^2. \tag{23}$$

Suppose $B_4(z)$ is a polynomial. Expression (23) shows that $B_4(z)$ is a constant. Let us assume

$$L(z) = d_nz^n + d_{n-1}z^{n-1} + d_{n-2}z^{n-2} + \cdots + d_0, (d_n \neq 0). \tag{24}$$

If $n \geq 2$, then by (23) and (24), we have

$$d_nd_n = d_nd_nB_4(z) \tag{25}$$

and

$$[2d_n(d_nC_n^2 + d_{n-2}) + d_{n-1}^2 - (d_nC_n^1)^2] = (2d_nd_{n-2} + d_{n-1}^2)B_4(z). \tag{26}$$

By (25) and $d_n \neq 0$, we get $B_4(z) = 1$. Furthermore, we get $n(n-1) = n^2$ from (26), a contradiction. So, $n \leq 1$.

If $n = 1$, then we have by (23) and (24) that

$$(d_1^2z^2 + 2d_1d_0z + d_0^2 - d_1^2) = (d_1^2z^2 + 2d_1d_0z + d_0^2)B_4(z).$$

From the above equality, we have $d_1 = 0$. This leads to a contradiction. So, we have $L(z) = d_0 \neq 0$ and $B_4(z) = 1$. Hence, g takes the form $g(z) = Le^{Cz}$, where $L \in \mathbb{C}$ and $L \neq 0$.

Thus, Theorem 7 has been successfully proven. □

Proof of Theorem 8. Suppose that g is an entire solution of (5) with $\sigma_2(g) < 1$.

In contrast, we take $\sigma(g) < 1$; (5) yields

$$gP_1(z,g) = B_0(z) - B(z)g', \tag{27}$$

where $P_1(z,g) = (g\bar{g}-1)(g\underline{g}-1) - B_5(z)g^4 - B_4(z)g^3 - B_3(z)g^2 - B_2(z)g - B_1(z)$. The transcendence of g and rationality of $B(z) \not\equiv 0$ imply $B_0(z) - B(z)g' \not\equiv 0$. Thus, $P_1(z,g) \not\equiv 0$. Utilizing Lemma 6 for Equation (27), we find

$$T(r,P_1) = m(r,P_1) = O(\log r). \tag{28}$$

Rewrite (27) as

$$-\frac{P_1(z,g)}{B(z)} = \frac{g'(z)}{g(z)} - \frac{B_0(z)}{B(z)g}. \tag{29}$$

Considering (28), (29), and the rationality of $B(z)$, we find that $-\frac{P_1(z,g)}{B(z)} \not\equiv 0$ is rational. Thus, we find

$$-\frac{P_1(z,g)}{B(z)} = Hz^n(1+o(1)), \tag{30}$$

where $z \to \infty$, H is a nonzero constant and $n \in \mathbb{Z}$. According to the Wiman–Valiron theory (see [4], p. 51, [25], pp. 28–32 or [26], pp. 103–105), we obtain

$$\frac{g'(z)}{g(z)} = \frac{\nu(r)}{z}(1+o(1)), \tag{31}$$

where $|z| = r \notin [0,1] \cup E, E \subset (1,\infty)$ is a set with finite logarithmic measure, satisfying $|g(z)| = M(r,g)$, and $\nu(r)$ represents the central index of $g(z)$. By (29)–(31), and continuing with the same approach as outlined in [20], we derive

$$\nu(r) = |H|r^{n+1}(1+o(1)), (r \notin [0,1] \cup E, r \to \infty).$$

It follows that $\sigma(g) = \limsup_{r\to\infty} \frac{\log \nu(r)}{\log r} = n+1 \leq 0$, a contradiction. Hence, we derive $\sigma(g) \geq 1$.

Set

$$P(z,g) = g(g\bar{g}-1)(g\underline{g}-1) + B(z)g' - (B_5(z)g^5 + \cdots + B_1(z)g + B_0(z)) = 0.$$

Then $P(z,0) = -B_0(z) \not\equiv 0$. Using Lemma 7, we find that $m(r,\frac{1}{g}) = S(r,g)$. So, $N(r,\frac{1}{g}) = T(r,g) + S(r,g)$. So, $\lambda(g) = \sigma(g)$r.

Therefore, Theorem 8 has been successfully proven. □

4. Necessary Conditions and Non-Existence of Solutions for Painlevé V-Type Differential-Difference Equation

When $\deg_g(Q) = 0$, Theorem 9, due to Du and Zhang, reads as follows.

Theorem 9 ([12]). *Let $g(z)$ be a non-rational meromorphic solution of*

$$(\bar{g}g-1)(g\underline{g}-1) + B(z)\frac{g'(z)}{g(z)} = P(z,g). \tag{32}$$

If $\sigma_2(g) < 1$, *then* $\deg_g(P) \leq 4$.

Theorem 9 further implies that if $\deg_g(P) \geq 5$ and g serves as a non-rational meromorphic solution to (32) featuring rational coefficients, then $\sigma_2(g) \geq 1$. Consequently, we need only address the existence of entire solutions to (32), which yields Theorem 10.

Theorem 10. *Let* $P(z, g)$ *be a polynomial in* g *possessing rational coefficients and* $\deg_g(P) \geq 5$, *and let* $B(z) \not\equiv 0$ *be rational. Then (32) does not admit any transcendental entire solution.*

Theorem 11 is a consequence of Theorems 2, 3, and 9.

Theorem 11. *Let rational functions* $a_0, \cdots, a_p$, $b_0, \cdots, b_d$, *and* B *be in terms of* z, *satisfying* $a_p(z)b_d(z) \not\equiv 0$. *Assume that* g *represents a non-rational meromorphic solution of*

$$(\bar{g}g - 1)(g\underline{g} - 1) + B(z)\frac{g'(z)}{g(z)} = \frac{P(z,g)}{Q(z,g)} = \frac{a_0(z) + a_1(z)g + \cdots + a_p(z)g^p}{b_0(z) + b_1(z)g + \cdots + b_d(z)g^d}, \tag{33}$$

where $P(z, g)$ *and* $Q(z, g)$ *are polynomials relatively prime in* g. *If* g *is subnormal, then* $\deg_g(P) - \deg_g(Q) = p - d \leq 4$.

Example 5 below demonstrates that (33) can indeed have a subnormal solution when $\deg_g(P) - \deg_g(Q) = p - d \leq 4$. Example 5 also shows that Theorem 11 and Theorem 3 are independent.

Example 5. *The equation*

$$(\bar{g}g - 1)(g\underline{g} - 1) + \frac{g'(z)}{g(z)} = \frac{g(g^2 - 1)^2 + \pi g^2 + \pi}{g}$$

has a meromorphic solution $g(z) = \tan(\pi z)$.

The same outcome is also discovered for the case of $\deg_g(Q) \geq 1$ and $Q(z, 0) \not\equiv 0$ in Equation (33).

Theorem 12. *Let* $P(z, g)$ *and* $Q(z, g)$ *be prime polynomials in* g *with rational coefficients, such that* $\deg_g(Q) \geq 1$ *and* $Q(z, 0) \not\equiv 0$. *Furthermore, let* $B(z) \not\equiv 0$ *be rational. Then Equation (33) admits no transcendental entire solution.*

Lemma 8 ([5])**.** *Assume we have meromorphic functions* $\alpha_j(z)(j = 0, 1, \cdots, n)$ *and entire functions* $f_j(z)(j = 1, 2, \cdots, n)$ *such that*

$$T(r, \alpha_j) = o\left(\sum_{k=1}^{n} T(r, e^{f_k(z)})\right), \ r \to \infty, r \notin E, j = 0, 1, \cdots, n.$$

If

$$\sum_{j=1}^{n} \alpha_j(z)e^{f_j(z)} \equiv \alpha_0(z),$$

then there exist constants $d_j, j = 1, 2, \cdots, n$, *not all zero, for which*

$$\sum_{j=1}^{n} d_j\alpha_j(z)e^{f_j(z)} \equiv 0.$$

Lemma 9 ([5])**.** *Let* $g_1(z), g_2(z), \cdots, g_n(z)(n \geq 2)$ *be meromorphic satisfying the following:*

(i) The identity $\sum_{j=1}^{n} d_j g_j(z) \equiv 0$ *holds, involving constants* $d_j (j = 1, 2, \cdots, n)$;

(ii) $g_j(z) \not\equiv 0 (j = 1, 2, \cdots, n)$*; furthermore, the quotients* $\frac{g_j(z)}{g_k(z)}$ *remain nonconstant for all* $1 \le j < k \le n$;

(iii) The asymptotic relation $\sum_{j=1}^{n}(N(r, g_j) + N(r, \frac{1}{g_j})) = o(\tau(r))(r \to \infty, r \notin E)$ *is valid, wherein* $\tau(r) = \min\{T\left(r, \frac{g_j(z)}{g_k(z)}\right)\}$ $(1 \le j < k \le n)$.

Then $d_j = 0 (j = 1, 2, \cdots, n)$.

Lemma 10 ([14]). *Assume* $g(z) = L(z)e^{l(z)}$ *with* $L(z)$ *a nontrivial small function of* $e^{l(z)}$*. Let* $P(z, g)$ *and* $Q(z, g)$ *be polynomials in* g *(coefficients small relative to* g*) and such that* $Q(z, 0) \not\equiv 0$*. If* $p = \deg_g(P) \le \deg_g(Q) = d$*, we derive*

$$m\left(r, \frac{P(z, g)}{Q(z, g)}\right) = S(r, e^l) = S(r, g).$$

Proof of Theorem 10. Set $P(z, g) = \sum_{j=0}^{p} a_j(z) g^j(z)$. (32) becomes $(\bar{g}g - 1)(g\underline{g} - 1) + B(z)\frac{g'(z)}{g(z)} = \sum_{j=0}^{p} a_j(z) g^j(z)$. Assume $g(z)$ is an entire transcendental solution of (32). If $g(z)$ possesses infinitely many zeros, then $B(z)\frac{g'(z)}{g(z)}$ would necessarily give rise to infinitely many simple poles, whereas $(\bar{g}g - 1)(g\underline{g} - 1) - \sum_{j=0}^{p} a_j(z) g^j(z)$ has finitely many poles, a contradiction. Consequently, $g(z)$ must be expressible as

$$g(z) = L(z)e^{l(z)}, \tag{34}$$

with $L(z) \neq 0$ a polynomial and $l(z)$ a nonconstant entire function

Plugging (34) into (32) leads to

$$L^2 \bar{L}\underline{L} e^{\bar{l}+\underline{l}+2l} - L\bar{L}e^{\bar{l}+l} - L\underline{L}e^{\underline{l}+l} - \sum_{j=1}^{p} a_j(z) L^j(z) e^{jl(z)} = H(z), \tag{35}$$

where $H(z) = a_0(z) - B(z)\left(\frac{L'(z)}{L(z)} + l'(z)\right) - 1$, $\bar{L} = L(z+1)$, $\underline{L} = L(z-1)$, $\bar{l} = l(z+1)$ and $\underline{l} = l(z-1)$.

Obviously,

$$T(r, H(z)) \le O(T(r, l(z))) + O(\log r) = S(r, g).$$

The demonstration is divided into the subsequent distinct cases.

Case 1. Assume that both $\bar{l} - l(z)$ and $\underline{l} - l(z)$ are nonconstant.

Applying Lemma 8 to (35), there exist constants $d_1, \cdots, d_{p+3}$, not all of which are zero, such that

$$d_{p+3} L^2 \bar{L}\underline{L} e^{\bar{l}+\underline{l}+2l} - d_{p+2} L\bar{L}e^{\bar{l}+l} - d_{p+1} L\underline{L}e^{\underline{l}+l} - \sum_{j=1}^{p} d_j a_j(z) L^j(z) e^{jl(z)} = 0. \tag{36}$$

Set

$f_{p+3}(z) = L^2 \bar{L}\underline{L} e^{\bar{l}+\underline{l}+2l}$, $f_{p+2}(z) = L\bar{L}e^{\bar{l}+l}$, $f_{p+1}(z) = L\underline{L}e^{\underline{l}+l}$,
$f_j(z) = a_j(z) L^j(z) e^{jl(z)}, j = 1, 2, \cdots, p$.

Thus, (36) becomes

$$d_{p+3} f_{p+3}(z) - d_{p+2} f_{p+2}(z) - d_{p+1} f_{p+1}(z) - d_p f_p(z) - \cdots - d_1 f_1(z) = 0. \tag{37}$$

Obviously,

$$\frac{f_j(z)}{f_k(z)} = \frac{a_j(z)}{a_k(z)} L^{j-k}(z) e^{(j-k)l(z)}, 1 \le j < k \le p,$$

and

$$\frac{f_{p+1}(z)}{f_{p+2}(z)} = \frac{L\underline{L}e^{\underline{l}+l}}{L\bar{L}e^{\bar{l}+l}} = \frac{\underline{L}}{\bar{L}}e^{\underline{l}-l}e^{l-\bar{l}},$$

$$\frac{f_{p+1}(z)}{f_{p+3}(z)} = \frac{L\underline{L}e^{\underline{l}+l}}{L^2\bar{L}\underline{L}e^{\bar{l}+\underline{l}+2l}} = \frac{1}{L\bar{L}}e^{-2l}e^{l-\bar{l}},$$

$$\frac{f_{p+2}(z)}{f_{p+3}(z)} = \frac{L\bar{L}e^{\bar{l}+l}}{L^2\bar{L}\underline{L}e^{\bar{l}+\underline{l}+2l}} = \frac{1}{L\underline{L}}e^{-2l}e^{l-\underline{l}},$$

$$\frac{f_j(z)}{f_{p+1}(z)} = \frac{a_j(z)L^j(z)e^{jl(z)}}{L\underline{L}e^{\underline{l}+l}} = \frac{a_j(z)L^j(z)}{L\underline{L}}e^{(j-2)l(z)}e^{l-\underline{l}},$$

$$\frac{f_j(z)}{f_{p+2}(z)} = \frac{a_j(z)L^j(z)e^{jl(z)}}{L\bar{L}e^{\bar{l}+l}} = \frac{a_j(z)L^j(z)}{L\bar{L}}e^{(j-2)l(z)}e^{l-\bar{l}},$$

$$\frac{f_j(z)}{f_{p+3}(z)} = \frac{a_j(z)L^j(z)e^{jl(z)}}{L^2\bar{L}\underline{L}e^{\bar{l}+\underline{l}+2l}} = \frac{a_j(z)L^j(z)}{L^2\bar{L}\underline{L}}e^{(j-4)l(z)}e^{l-\bar{l}}e^{l-\underline{l}}$$

$(j = 1, 2, \cdots, p)$ are non-rational. Thus,

$$\sum_{j=1}^{p+3}\left(N(r, f_j(z)) + N\left(r, \frac{1}{f_j(z)}\right)\right) = O(\log r) = o(\tau(r)),$$

with $\tau(r) = \min\{T(r, \frac{f_j}{f_k}) : 1 \le j < k \le p+3\}$.

From (37) and Lemma 9, we obtain $d_j = 0$ for $j = 1, 2, \ldots, p+3$, which contradicts the assumption.

Case 2. Let $\bar{l} - l(z) = d$. Then

$$\bar{l} = l(z) + d,\ \underline{l} = l(z) - d. \tag{38}$$

Substituting (38) into (35) yields

$$L^2\bar{L}\underline{L}e^{4l(z)} - L\bar{L}e^{d}e^{2l(z)} - L\underline{L}e^{-d}e^{2l(z)} + 1 + B(z)\left(\frac{L'(z)}{L(z)} + l'(z)\right) = \sum_{j=0}^{p} a_j(z)L^j(z)e^{jl(z)}. \tag{39}$$

In view of Lemma 4 and (39), we obtain

$$\begin{aligned} pT(r, e^{l(z)}) &= T\left(r, \sum_{j=0}^{p} a_j(z)L^j(z)e^{jl(z)}\right) + S(r, e^{l(z)}) \\ &\le 4T(r, e^{l(z)}) + S(r, e^{l(z)}), \end{aligned}$$

which yields $p \le 4$, contradicting $p \ge 5$.

Case 3. Assume that $\underline{l} - l(z) = -d$. Then $\bar{l} - l(z) = d$ is likewise constant. Case 2 leads to a contradiction.

Therefore, Theorem 10 has been successfully proven. □

Proof of Theorem 11. Let us assume that $g(z)$ represents a non-rational subnormal meromorphic solution to Equation (33).

Conversely, let us assume that $s = \deg_g(P) - \deg_g(Q) = p - d \geq 5$. Then by (33), we obtain

$$a_p(z)g^p g = H(z,g), \tag{40}$$

where $H(z,g) = b_d(z)g^{d+1}(g\bar{g}-1)(g\underline{g}-1) + \cdots + b_0(z)B(z)g' - a_0(z)g$. Obviously, $\deg_g(H) \leq \max\{d+5, p\} \leq p$ since $p - d \geq 5$. By (40) and Lemma 2, we get $m(r,g) = S(r,g)$. So, $N(r,g) = T(r,g) + S(r,g)$, we see that g possesses infinitely many poles and $\lambda(\frac{1}{g}) = \sigma(g)$.

Proceeding as in [18], let z_0 be a pole of g of order $k_0 \geq 1$, and assume z_0 avoids the zeros and poles of the coefficients in (33). Then (33) forces a pole of $g(z)$ at $z_0 + 1$ or $z_0 - 1$ (or both). If the pole occurs at $z = z_0 + 1$ with order k_1 satisfying $k_1 \geq \frac{(s-2)k_0}{2} > k_0$, then shifting (33) gives

$$[g(z+2)g(z+1)-1][g(z+1)g(z)-1] + B(z+1)\frac{g'(z+1)}{g(z+1)} = \frac{P(z+1,g(z+1))}{Q(z+1,g(z+1))}. \tag{41}$$

By $s = p - d \geq 5$ and (41), we have that $z_0 + 2$ serves as a pole of $g(z)$ with an order $k_2 = sk_1 - 2k_1 - k_0 > (s-2)k_1 - k_1 = (s-3)k_1$.

Using (41) and the previous reasoning, one obtains that $z_0 + 3$ constitutes a pole of $g(z)$ whose order is $k_3 = sk_2 - 2k_2 - k_1 > (s-2)k_2 - k_2 \geq (s-3)^2 k_1$.

Proceeding as before, we obtain

$$n(|z_0| + m, g) \geq (s-3)^{m-1}k_1 + O(1)$$

for all positive integer m, and then

$$\limsup_{r\to\infty} \frac{\log^+ T(r,g)}{r} \geq \limsup_{r\to\infty} \frac{\log^+ n(r,g)}{r} \geq \limsup_{m\to\infty} \frac{\log k_1(s-3)^{(m-1)}}{|z_0| + m} \geq \log 2 > 0.$$

This is a contradiction. Thus, $\deg_g(P) - \deg_g(Q) \leq 4$.

Alternatively, when the pole occurs at $z_0 - 1$, the argument proceeds analogously to the case where $z_0 + 1$ is a pole.

Therefore, Theorem 11 has been successfully proven. □

Proof of Theorem 12. Suppose $g(z)$ is an entire transcendental solution of (33). Following the previous discussion in Theorem 10, $g(z)$ has only finitely many zeros. Hence, it can be expressed as $g(z) = L(z)e^{l(z)}$ with $L(z) \neq 0$ a polynomial and $l(z)$ a nonconstant entire function.

We split the proof into the following cases.

Case 1. $p \leq d$. Applying Lemma 10, we get

$$m\left(r, \frac{P(z,g)}{Q(z,g)}\right) = S(r,g)$$

and

$$N\left(r, \frac{P(z,g)}{Q(z,g)}\right) = N\left(r, (\bar{g}g-1)(g\underline{g}-1) + B(z)\frac{g'(z)}{g(z)}\right) = O(\log r) = S(r,g).$$

Then, $T(r, R(z,g)) = S(r,g)$. Let $R(z,g) = (\bar{g}g-1)(g\underline{g}-1) + B(z)\frac{g'(z)}{g(z)} = R(z)$. Applying Lemma 4 to equation $P(z,g) = R(z)Q(z,g)$, we obtain $p = q$, and $a_i(z) = R(z)b_i(z)$, $i = 0, 1, ..., p$. This runs counter to the coprimality of $P(z,g)$ and $Q(z,g)$.

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Case 2. $p > d$. Applying the division algorithm, (33) becomes

$$(\bar{g}g - 1)(g\underline{g} - 1) + B(z)\frac{g'(z)}{g(z)} = P_1(z, g) + \frac{P_2(z, g)}{Q(z, g)}. \tag{42}$$

where $\deg_g(P_2) = p_2 < q$. Expression (42) becomes

$$H(z, g) = \frac{P_2(z, g)}{Q(z, g)},$$

where $H(z, g) = \bar{g}\underline{g} + B(z)\frac{g'(z)}{g(z)} - P_1(z, g)$. Similarly as Case 1, we have $T(r, H(z, g)) = S(r, g)$. Applying Lemma 4 to $P_2(z, g) = H(z, g)Q(z, g)$, we obtain $p_2 = q$. This contradicts the fact that $p_2 < q$.

Therefore, Theorem 12 has been successfully proven. □

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