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Qualitative Analysis of Uncertain Fractional Differential Equations and Application to Interest Rate Modeling

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Abstract

Uncertain fractional differential equations model complex systems that exhibit memory effects and are influenced by human-based uncertainty. These equations provide a flexible and accurate framework for representing real-world phenomena, particularly in situations where traditional probabilistic methods are inadequate, such as modeling financial market systems where uncertainty and memory effects play a significant role. This research first presents an existence and uniqueness result for the uncertain fractional system with the φ-Hilfer fractional derivative, obtained via the successive approximation approach. Then the analytical solution is derived using the Mittag–Leffler function, and sample continuity is demonstrated under Lipschitz and linear growth conditions. To illustrate the applicability of the theory, we consider an interest rate model and provide two numerical examples to support the theoretical results on existence and uniqueness. All results are developed using the φ-Hilfer fractional derivative, which generalizes the Caputo, Hadamard, and Katugampola fractional derivatives. Consequently, the results are presented in a generalized form.

Keywords: mathematical modeling; qualitative theory; uncertainty theory; fractional differential equations

MSC: 34A08; 34D20; 93A30; 60H10

1. Introduction

Subjective noise and objective noise represent two distinct forms of uncertainty that can affect the performance, predictability, and decision-making processes in various systems.

Objective noise refers to random variations or uncertainties in a system that arise from inherent or external factors, which are treated as an essential part of the system's dynamics when modeling with differential equations. It represents the stochastic component of the model, distinguishing it from deterministic behavior.

Subjective noise, in contrast, arises from human factors such as uncertainty due to incomplete knowledge, cognitive biases, or differing interpretations of information. Unlike objective noise, subjective noise cannot be easily quantified using traditional statistical models, as it is influenced by personal judgment, experience, and perception. A real-life example of subjective noise can be seen in consumer behavior in marketing. Companies often rely on market research surveys to predict customer preferences, but subjective noise can distort the results if consumers provide inaccurate feedback due to biases like social desirability bias, where they answer in a way they believe is more socially acceptable

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rather than truthfully. Similarly, in decision-making, a project manager's optimism about the success of a new initiative may introduce subjective noise, leading them to overlook potential risks that a more cautious manager might consider.

While objective noise can be accounted for using stochastic models that incorporate randomness and variability, subjective noise requires frameworks such as uncertainty theory, which accommodates human judgment and belief degrees. Both types of noise are important to understand and manage, especially in fields like economics, healthcare, and engineering, where both random variability and human interpretation can significantly influence outcomes.

In classical probability theory, modeling is based on assumptions such as independence and normality. Independence requires that random variables do not influence each other, while normality assumes that the data follow a specific probability distribution, often the Gaussian distribution, which is a bell-shaped curve that describes how values are distributed around a mean. In expert-based systems, these assumptions are generally not valid. Expert judgments are usually dependent because they are influenced by shared knowledge, experience, and personal bias. In addition, such information does not arise from repeated experiments, so it cannot be expected to follow any standard probability distribution.

For these reasons, probability theory is not suitable for modeling this type of uncertainty. Uncertainty theory provides an alternative framework by using belief degrees instead of probabilities, without requiring independence or distributional assumptions, making it more appropriate for handling uncertainty based on human expertise.

Liu [1] developed uncertainty theory in 2007 as a mathematical framework to simulate and measure subjective uncertainty, which arises from human judgment, belief, or insufficient information rather than from objective randomness. This theory differs from traditional probability theory, which addresses randomness based on quantifiable data. Subjective uncertainty becomes especially relevant in systems with limited, faulty, or nonexistent empirical data. By focusing on belief degrees and uncertain variables—terms that describe the level of confidence or belief an expert or decision-maker has in the occurrence of an event or outcome—uncertainty theory provides tools to express and manage this type of uncertainty. The main advantage of uncertainty theory is that it enables the modeling of systems where human perception and partial information, rather than pure randomness, are the primary sources of uncertainty.

By incorporating uncertain variables into classical differential equations, an uncertain differential equation (UDE) enables the representation of dynamic systems influenced by subjective uncertainty. UDEs include uncertain variables that reflect belief-based uncertainty, capturing time-varying changes in belief or knowledge about the state of the system. In contrast, standard differential equations describe systems with exact, deterministic parameters. UDEs are used to model systems with inherent uncertainty, such as economic models, engineering systems with faulty data, or biological systems with limited knowledge.

Fractional calculus includes various operators, as outlined in the literature [2–4]. The φ-Hilfer fractional derivative (φ-HFD) is a generalized fractional operator defined with respect to another function φ, allowing the memory effect to depend on a nonlinear time scale. It unifies several fractional derivatives, including the Riemann–Liouville and Caputo types, and provides a flexible framework for modeling systems with nonlocal and hereditary behavior.

We now provide the essential definitions, a lemma, and a remark related to the φ-HFD.

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Definition 1 ([5]). *Let $c > 0$ and let $\varphi \in C^1([a,c],\mathbb{R})$ be a strictly increasing function satisfying $\varphi'(t) > 0$ for all $t \in [a,c]$. For an integrable function $g : [a,c] \to \mathbb{R}$, the left-sided φ-Riemann–Liouville fractional integral of order c is defined by*

$$I_{a+}^{c;\varphi} g(t) = \frac{1}{\Gamma(c)} \int_a^t (\varphi(t) - \varphi(\varsigma))^{c-1} g(\varsigma)\, \varphi'(\varsigma)\, d\varsigma, \quad t \in (a,c].$$

Here $\Gamma(\cdot)$ denotes the Gamma function.

Definition 2 ([6]). *Let $n \in \mathbb{N}$ satisfy $n-1 < c < n$, and assume that $\varphi \in C^n([a,c],\mathbb{R})$ is strictly increasing with $\varphi'(t) > 0$ on $[a,c]$. If $g \in C^n([a,c],\mathbb{R})$, then the left-sided φ-Caputo fractional derivative of order c is defined by*

$${}^{C}D_{a+}^{c;\varphi} g(t) = \frac{1}{\Gamma(n-c)} \int_a^t (\varphi(t) - \varphi(\varsigma))^{n-c-1} g^{[n]}(\varsigma)\, \varphi'(\varsigma)\, d\varsigma,$$

where

$$g^{[n]}(t) = \left(\frac{1}{\varphi'(t)} \frac{d}{dt}\right)^n g(t).$$

Definition 3 ([7]). *Let $n \in \mathbb{N}$ satisfy $n-1 < c < n$ and let $\mathrm{z} \in [0,1]$. Assume that $\varphi \in C^n([a,c],\mathbb{R})$ is a strictly increasing function such that $\varphi'(t) > 0$ for all $t \in [a,c]$. If $g \in C^n([a,c],\mathbb{R})$, then the left-sided φ-HFD of order c and type z is defined by*

$${}^{H}D_{a+}^{\mathrm{z},c;\varphi} g(t) = I_{a+}^{\mathrm{z}(n-c);\varphi} \left(\frac{1}{\varphi'(t)} \frac{d}{dt}\right)^n I_{a+}^{(1-\mathrm{z})(n-c);\varphi} g(t), \quad t \in (a,c],$$

where $I_{a+}^{\alpha;\varphi}$ denotes the left-sided φ-Riemann–Liouville fractional integral of order $\alpha > 0$.

Lemma 1 ([7]). *Let $g \in C^n([a,b],\mathbb{R})$, $0 \le \mathrm{z} \le 1$, and $n-1 < c < n$ with $n = \lfloor c \rfloor + 1$. Define*

$$r = c + \mathrm{z}(n-c).$$

Then the following identity holds:

$$I_{a+}^{c;\varphi}\, {}^{H}D_{a+}^{\mathrm{z},c;\varphi} g(t) = g(t) - \sum_{i=1}^{n} \frac{(\varphi(t) - \varphi(a))^{r-i}}{\Gamma(r-i+1)} \left(\frac{1}{\varphi'(t)} \frac{d}{dt}\right)^{n-r} I_{a+}^{(1-\mathrm{z})(n-c);\varphi} g(a).$$

Remark 1. *The φ-HFD introduced in the definition above provides a unified framework that includes several well-known fractional derivatives as special cases.*

1. *If $\varphi(t) = t$ and $\mathrm{z} = 0$, then the operator ${}^{H}D_{a+}^{0,c;t}$ reduces to the classical Riemann–Liouville fractional derivative of order c.*
2. *If $\varphi(t) = t$ and $\mathrm{z} = 1$, then ${}^{H}D_{a+}^{1,c;t}$ coincides with the Caputo fractional derivative of order c.*
3. *If $\varphi(t) = \ln t$, then the operator ${}^{H}D_{a+}^{\mathrm{z},c;\ln t}$ yields the Hadamard fractional derivative (which is independent of z).*
4. *If $\varphi(t) = t^\rho$ with $\rho > 0$, then:*
   - *(a) For $\mathrm{z} = 0$, ${}^{H}D_{a+}^{0,c;t^\rho}$ gives the Katugampola fractional derivative of Riemann–Liouville type;*
   - *(b) For $\mathrm{z} = 1$, ${}^{H}D_{a+}^{1,c;t^\rho}$ gives the Katugampola fractional derivative of Caputo type;*
   - *(c) For $0 < \mathrm{z} < 1$, it gives the more general Katugampola–Hilfer fractional derivative.*

Hence, the φ-HFD can be viewed as a generalized fractional operator that unifies several classical fractional derivatives through appropriate choices of the function $\varphi(t)$ and the type parameter z.

Numerous researchers have recently been studying UDEs. In [8], the authors proved results regarding existence and uniqueness (Ex-Un) for UDEs using the Lipschitz condition (LC) and the linear growth condition (LGC). Yao et al. [9] established stability results for UDEs. In this research [10], the author presented solutions for UDEs. In [11], the authors worked on moment estimations for parameters in UDEs. Zhang et al. [12] obtained solutions for UDEs using the Hamming approach. In [13], the authors introduced the uncertain hypothesis tool for UDEs. Li et al. [14] provided solutions for UDEs. In [15], the author found solutions for partial UDEs. In [16], the author discussed a pharmacokinetic model using uncertainty theory.

In 2015, Zhu [17] introduced uncertainty into FDEs and proposed fractional uncertain differential equations (FUDEs) of the Caputo type to model systems with memory effects in uncertain environments. A UFDE is a mathematical framework that models dynamic systems influenced by memory effects and human-based uncertainty by fractional calculus with uncertainty theory. Recently, many authors have worked on FUDEs. For example, Zhu [18] worked on the Ex-Un of solutions for FUDEs under the LC and LGC. Lu and Zhu [19] focused on finding solutions for FUDEs. Lu et al. [20] presented a new model in the context of FUDEs and found a solution. In [21], the authors developed a financial model in the framework of FUDEs. In [22], the authors established analytical solutions for a specific class of FUDEs and derived Ex-Un results using the Picard approach. The authors of [23] focused on the stability of FUDEs. Liang et al. [24] established several results regarding Ex-Un for FUDEs and also found analytical solutions. He et al. [25] worked on parameter estimation for FUDEs, while Wang and Zhu [26] obtained solutions for FUDEs with delay factors. For further details on parameter estimation in uncertain systems, the reader is referred to [27–30].

Our investigation yields important results for FUDEs under the generalized fractional derivative framework. The study makes the following key contributions:

1. We generalize the classical Ex-Un and sample-continuity results to the context of φ-HFD.
2. We have derived the analytical solution of the UFDE involving the φ-HFD using the Mittag–Leffler function (MLF), thereby obtaining a generalized solution.
3. To demonstrate the application, we have examined an interest rate model and calculated the price of a zero-coupon bond.

This research is structured as follows. Section 2 introduces the essential preliminary definitions and concepts that form the foundation for our analysis. Building upon this, Section 3 establishes the main theoretical results concerning FUDEs under the φ-HFD framework. To validate these theoretical findings, particularly regarding Ex-Un, Section 4 presents two numerical illustrations. Subsequently, Section 5 discusses the implications and broader conclusions drawn from the study. The paper concludes by identifying several promising directions for future research.

2. Preliminaries

This section outlines key definitions and lemmas that underpin the results of this research. It introduces essential concepts from uncertainty theory and the canonical process.

First, we present the important definitions related to uncertainty theory that are essential for establishing the results of this study.

Definition 4 ([31]). *Let Υ be a nonempty set and let $\mathfrak{L}$ be a σ-algebra on Υ. Each member $\mathscr{O} \in \mathfrak{L}$ is called an event. A set function $\mathfrak{M} : \mathfrak{L} \to [0,1]$ is called an* ***uncertain measure*** *if it satisfies the following four axioms:*

1. ***(Normality)*** $\mathfrak{M}\{\Upsilon\} = 1$.

2. ***(Self-Duality)*** $\mathfrak{M}\{\mathscr{O}\} + \mathfrak{M}\{\mathscr{O}^c\} = 1$ *for any event* $\mathscr{O} \in \mathfrak{L}$.
3. ***(Countable Subadditivity)*** *For any countable collection of events* $\{\mathscr{O}_j\}_{j=1}^{\infty} \subset \mathfrak{L}$,

$$\mathfrak{M}\left\{\bigcup_{j=1}^{\infty} \mathscr{O}_j\right\} \leq \sum_{j=1}^{\infty} \mathfrak{M}\{\mathscr{O}_j\}.$$

4. ***(Product Axiom)*** *Let* $(\mathrm{Y}_k, \mathfrak{L}_k, \mathfrak{M}_k)$ *be uncertainty spaces for* $k = 1, 2, \ldots, n$. *Then the product uncertainty measure* $\mathfrak{M}$ *on the product* σ*-algebra* $\bigotimes_{k=1}^{n} \mathfrak{L}_k$ *satisfies*

$$\mathfrak{M}\left\{\prod_{k=1}^{n} \Lambda_k\right\} = \bigwedge_{k=1}^{n} \mathfrak{M}_k\{\Lambda_k\},$$

for any events $\Lambda_k \in \mathfrak{L}_k$, *where* $\bigwedge_{k=1}^{n} a_k = \min\{a_1, a_2, \ldots, a_n\}$ *denotes the minimum operator.*

The triple $(\mathrm{Y}, \mathfrak{L}, \mathfrak{M})$ *is called an* ***uncertainty space****.*

Remark 2. *For an infinite product of uncertainty spaces* $(\mathrm{Y}_k, \mathfrak{L}_k, \mathfrak{M}_k)$ *with* $k = 1, 2, \ldots$, *the product uncertainty measure satisfies*

$$\mathfrak{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathfrak{M}_k\{\Lambda_k\},$$

for any events $\Lambda_k \in \mathfrak{L}_k$. *This infinite extension is a direct consequence of the finite Product Axiom and the countable subadditivity axiom.*

Definition 5 ([31])**.** *The distribution of uncertainty* β *of an uncertain process (UPr)* γ *can be expressed as*

$$\beta(\mathrm{w}) = \mathfrak{M}\{\beta \leq \mathrm{w}\}, \qquad \mathrm{w} \in \mathbb{R}.$$

Definition 6 ([31])**.** *If an uncertainty distribution* $\beta(\mathrm{w})$ *is a continuously growing function with regards to* w *at which* $0 < \beta(\mathrm{w}) < 1$, *and*

$$\lim_{\mathrm{w} \to -\infty} \beta(\mathrm{w}) = 0, \quad \lim_{\mathrm{w} \to +\infty} \beta(\mathrm{w}) = 1,$$

then it is considered regular.

Definition 7 ([31])**.** *Consider* $(\mathrm{Y}, \mathfrak{L}, \mathfrak{M})$ *to be an uncertain space with a UPr* γ. *Consequently, its estimated value* $\mathbb{E}[\gamma]$ *is*

$$\mathbb{E}[\gamma] = \int_0^{+\infty} \mathfrak{M}\{\gamma \geq \mathrm{w}\} d\mathrm{w} - \int_{-\infty}^{0} \mathfrak{M}\{\gamma \leq \mathrm{w}\} d\mathrm{w},$$

$\mathbb{V}[\gamma]$ *is*

$$\mathbb{V}[\gamma] = \mathbb{E}[(\gamma - \mathbb{E}[\gamma])^2].$$

The quantity $\mathbb{V}[\gamma]$ *is called the variance of the UPr.*

Definition 8 ([31])**.** *Consider the uncertain space* $(\mathrm{Y}, \mathfrak{L}, \mathfrak{M})$ *and the fully ordered set* $\mathbb{A}^*$ *(such as time).* $\{f_t \in \mathscr{B}\}$ *is an event for any Borel set* $\mathscr{B}$ *of real numbers whenever* t. *This is the case when* $f_t(\mathfrak{g})$ *is a function from* $\mathbb{A}^* \times (\mathrm{Y}, \mathfrak{L}, \mathfrak{M})$ *to the set of real numbers.*

Definition 9 ([31]). *Let* $(\Upsilon, \mathfrak{L}, \mathfrak{M})$ *be an uncertainty space. An uncertain process* $\mathscr{C}_t$ *is called a Liu process if it satisfies the following conditions:*

1. $\mathscr{C}_0 = 0$ *almost surely (a.s.), and the sample paths of* $\mathscr{C}_t$ *are continuous almost surely.*
2. $\mathscr{C}_t$ *has stationary and independent increments: For any* $0 \leq t_1 < t_2 < \cdots < t_k$*, the increments*
$$\mathscr{C}_{t_2} - \mathscr{C}_{t_1},\ \mathscr{C}_{t_3} - \mathscr{C}_{t_2},\ \ldots,\ \mathscr{C}_{t_k} - \mathscr{C}_{t_{k-1}}$$
are independent uncertain variables, and for any $\mathfrak{u}, t \geq 0$*, the distribution of the increment* $\mathscr{C}_{\mathfrak{u}+t} - \mathscr{C}_t$ *depends only on* $\mathfrak{u}$ *(not on* t*).*
3. *For any* $\mathfrak{u}, t \geq 0$*, the increment* $\mathscr{C}_{\mathfrak{u}+t} - \mathscr{C}_t$ *follows a normal uncertainty distribution with expected value* 0 *and variance* $\mathfrak{u}^2$*. Its uncertainty distribution function is*
$$\xi_t(\mathrm{w}) = \mathcal{M}\{\mathscr{C}_{\mathfrak{u}+t} - \mathscr{C}_t \leq \mathrm{w}\} = \left(1 + \exp\left(\frac{-\pi \mathrm{w}}{\sqrt{3}\,\mathfrak{u}}\right)\right)^{-1}, \quad \mathrm{w} \in \mathbb{R}.$$

Definition 10 ([32]). *Take a Liu process (LPr)* $\mathscr{C}_t$ *and a UPr* f_t*. The mesh for an arbitrary interval* $[\kappa_1, \kappa_2]$ *partition with* $\kappa_1 = t_1 < t_2 < \cdots < t_{\iota+1} = \kappa_2$ *is denoted by*
$$\triangle = \max_{1 \leq J \leq \iota} |t_{J+1} - t_J|.$$

Next, we define the Liu integral of f_t *with regard to* $\mathscr{C}_t$ *as*
$$\int_{\kappa_1}^{\kappa_2} f_t d\mathscr{C}_t = \lim_{\triangle \to 0} \sum_{j=1}^{\iota} f_{t_j}(\mathscr{C}_{t_{J+1}} - \mathscr{C}_{t_j}),$$

given that the limit is finite and nearly certainly exists. The UPr f_t *is said to be integrable in this instance.*

We now present the essential lemmas of uncertainty theory required for this study.

Lemma 2 ([32]). *Let* $\mathscr{F}(t)$ *be a function that is integrable with regard to* t*. Then, at each point* $\mathfrak{q}$*, the Liu integral* $\int_0^{\mathfrak{q}} \mathscr{F}(t) d\mathscr{C}_t$ *is a normal UPr, and*
$$\int_0^{\mathfrak{q}} \mathscr{F}(t) d\mathscr{C}_t \sim \mathbf{N}\left(0, \int_0^{\mathfrak{q}} |\mathscr{F}(t)| dt\right).$$

Lemma 3 ([32]). *Assume that* $\mathscr{C}_t$ *is a canonical procedure. Then, a non-negative UPr* $\mathcal{V}$ *exists such that* $\mathcal{V}_{\mathfrak{g}}$ *is a Lipschitz constant of the sample path* $\mathscr{C}_t(\mathfrak{g})$*, which is described as follows:*
$$\mathcal{V}_{\mathfrak{g}} = \sup_{0 \leq t_1 \leq t_2} \frac{\mathscr{C}_{t_2}(\mathfrak{g}) - \mathscr{C}_{t_1}(\mathfrak{g})}{t_2 - t_1},\ \forall \mathfrak{g} \in \Upsilon.$$

Lemma 4 ([32]). *Assume that* f_t *is an integrable UPr on* $[\kappa_1, \kappa_2]$ *with relation to* t *and that* $\mathscr{C}_t$ *is a canonical process. Subsequently, we have*
$$\left| \int_{\kappa_1}^{\kappa_2} f_t(\mathfrak{g}) d\mathscr{C}_t(\mathfrak{g}) \right| \leq \mathcal{V}_{\mathfrak{g}} \int_{\kappa_1}^{\kappa_2} |f_t(\mathfrak{g})| dt,$$

in this case, $\mathcal{V}_{\mathfrak{g}}$ *indicates the sample path* $f_t(\mathfrak{g})$*'s Lipschitz constant.*

Lemma 5 ([32]). *Consider* $\gamma_1, \gamma_2, \gamma_3, \cdots$ *to be variables that are uncertain, and* $\lim_{\iota \to \infty} \gamma_\iota = \gamma$ *most likely.* γ *is a UPr in this case.*

Lemma 6 ([33]). *Assume* $\gamma_1, \gamma_2, \gamma_3, \cdots, \gamma_\hbar$ *are the independent UPr along the regular uncertainty distribution* $\xi_1, \xi_2, \xi_3, \cdots$*, accordingly. While* $\mho(\mathrm{w}_1, \mathrm{w}_2, \mathrm{w}_3, \cdots, \mathrm{w}_\hbar)$ *is continu-*

ous, it will strictly increase in relation to $\mathrm{w}_1, \mathrm{w}_2, \mathrm{w}_3, \cdots, \mathrm{w}_\mathrm{m}$ *and decrease in relation to* $\mathrm{w}_{\mathrm{m}+1}, \mathrm{w}_{\mathrm{m}+2}, \mathrm{w}_{\mathrm{m}+3}, \cdots, \mathrm{w}_\hbar$. *The uncertainty distribution of* $\gamma = \mho(\gamma_1, \gamma_2, \gamma_3, \cdots \gamma_\hbar)$ *is then inverse*

$$\Psi^{-1}(t) = \mho\big(\xi_1^{-1}(t), \cdots, \xi_\mathrm{m}^{-1}(t), \xi_{\mathrm{m}+1}^{-1}(1-t), \cdots, \xi_\hbar^{-1}(1-t)\big).$$

Lemma 7 ([33]). *Assume that* ξ *has a normal uncertainty distribution and* γ *is a UPr. Should the mean value be present, then*

$$\mathbb{E}[\gamma] = \int_0^1 \xi^{-1}(t)dt.$$

Lemma 8. *Let* $c > 0$*; suppose that* $\mathbb{G}(t)$ *and* $\mathbb{H}(t)$ *are non-negative and locally integrable functions defined on* $t \in [a, \mathcal{U})$. *The function* $\mathbb{U}(t) : [a, \mathcal{U}) \to [0, \mathbb{Y}]$ *is continuous and non-decreasing* ($\mathbb{Y}$ *is a constant*). *Assume*

$$\mathbb{G}(t) \leq \mathbb{H}(t) + \frac{\mathbb{U}(t)}{\Gamma(c)} \int_a^t \big(\varphi(t) - \varphi(b)\big)^{c-1} \mathbb{G}(b)db, \; a < t < \mathcal{V},$$

then for any $a \leq t < \mathcal{U}$,

$$\mathbb{G}(t) \leq \mathbb{H}(t) + \int_a^t \left[\sum\nolimits_{u=1}^{\infty} \frac{[\mathbb{U}(t)]^u}{\Gamma(uc)} \big(\varphi(t) - \varphi(b)\big)^{uc-1} \mathbb{H}(b)\right] db.$$

In the next section, we present the main results related to Ex-Un, analytical solutions, and sample continuity.

3. Main Results

We have established the Ex-Un findings for the solution of the UFDE regarding φ-HFD in the following section. The unique solution has been demonstrated to be sample-continuous. Finally, the price of a zero-coupon bond has been calculated using the proposed approach, which has accounted for an interest rate model.

Definition 11. *Take* c *such that* $u - 1 < c \leq u$, *where* u *is a positive integer,* $\delta = \frac{1}{\varphi'(t)} \frac{d}{dt}$, *and* $\mathcal{C}_t$ *be an LPr. Let* $\Omega, \Lambda : [a, \infty) \times \mathbb{R} \to \mathbb{R}$ *be a continuous function. Next, we have*

$$\begin{cases} {}^{H}\mathcal{D}_a^{\mathrm{z},c,\varphi} f_t = \Omega(t, f_t) + \Lambda(t, f_t) \frac{d\mathcal{C}_t}{dt}, \\ \delta_\varphi^\lambda \mathcal{I}_{a+}^{(1-\mathrm{z})(u-c),\varphi} f_t|_{t=a} = \vartheta_\lambda, \; \lambda = 0, 1, 2, 3, \cdots u-1, \end{cases} \tag{1}$$

is called a UFDE. A solution f_t *to* (1) *is a UPr such that*

$$\begin{aligned} f_t &= \sum\nolimits_{\lambda=0}^{u-1} \frac{\big(\varphi(t) - \varphi(a)\big)^{\mathrm{v}-\lambda-1}}{\Gamma(\mathrm{v}-\lambda)} \vartheta_\lambda + I_a^{c,\varphi} \Omega(t, f_t) + I_a^{c,\varphi} \left(\Lambda(t, f_t) \frac{d\mathcal{C}_t}{dt}\right) \\ &= \sum\nolimits_{\lambda=0}^{u-1} \frac{\big(\varphi(t) - \varphi(a)\big)^{\mathrm{v}-\lambda-1}}{\Gamma(\mathrm{v}-\lambda)} \vartheta_\lambda + \frac{1}{\Gamma(c)} \int_a^t \big(\varphi(t) - \varphi(b)\big)^{c-1} \varphi'(b) \Omega\big(b, f_b\big) db \\ &+ \frac{1}{\Gamma(c)} \int_a^t \big(\varphi(t) - \varphi(b)\big)^{c-1} \varphi'(b) \Lambda\big(b, f_b\big) d\mathcal{C}_b, \end{aligned} \tag{2}$$

where $\mathrm{v} = c + \mathrm{z}(u - c)$.

Remark 3. *The operator δ is defined as*

$$\delta = \frac{1}{\varphi'(t)}\frac{d}{dt},$$

which represents a generalized derivative with respect to the function $\varphi(t)$. In particular, for a sufficiently differentiable function f, we have

$$\delta f(t) = \frac{df}{d\varphi}(t).$$

For higher-order cases, the operator is given by

$$\delta^{\lambda} = \left(\frac{1}{\varphi'(t)}\frac{d}{dt}\right)^{\lambda}, \quad \lambda \in \mathbb{N}.$$

Moreover, when $\varphi(t) = t$, the operator δ reduces to the classical derivative $\frac{d}{dt}$.

The following theorem establishes the Ex-Un of the solution to (1).

Theorem 1. *Assume that $\Omega(t,\rho) : [a,\infty) \times \mathbb{R} \to \mathbb{R}$ and $\Lambda(t,\rho) : [a,\infty) \times \mathbb{R} \to \mathbb{R}$ in (1) fulfill the LC $|\Omega(t,\rho) - \Omega(t,\theta)| + |\Lambda(t,\rho) - \Lambda(t,\theta)| \le \chi|\rho - \theta|$, $\forall \rho, \theta \in \mathbb{R}$, $t \ge a > 0$ and the LGC*

$$|\Omega(t,\rho)| + |\Lambda(t,\rho)| \le \chi(1 + |\rho|),\ \forall \rho, \theta \in \mathbb{R},\ t > 0,$$

where $\chi \in \mathbb{R}^{+}$. Then, the solution of (1) exists and is unique.

Proof. Methodology: We use the successive approximation (Picard iteration) method in Theorem 1, which is the standard and rigorous approach for proving Ex-Un of solutions to UFDEs.

Existence: We prove the existence of a solution to (1) by means of the successive approximation technique.

Let

$$f_t^{(0)} = \sum_{\lambda=0}^{u-1} \frac{\big(\varphi(t) - \varphi(a)\big)^{\mathrm{v}-\lambda-1}}{\Gamma(\mathrm{v}-\lambda)}\vartheta_{\lambda},$$

and define inductively, for each integer $n \ge 0$,

$$\begin{aligned} f_t^{(n+1)} = {} & \sum_{\lambda=0}^{u-1} \frac{\big(\varphi(t) - \varphi(a)\big)^{\mathrm{v}-\lambda-1}}{\Gamma(\mathrm{v}-\lambda)}\vartheta_{\lambda} \\ & + \frac{1}{\Gamma(c)}\int_a^t \big(\varphi(t) - \varphi(b)\big)^{c-1}\Omega\big(b, f_b^{(n)}\big)\varphi'(b)\,db \\ & + \frac{1}{\Gamma(c)}\int_a^t \big(\varphi(t) - \varphi(b)\big)^{c-1}\Lambda\big(b, f_b^{(n)}\big)\varphi'(b)\,d\mathcal{C}_b, \qquad n = 0,1,2,\ldots. \end{aligned} \tag{3}$$

Here, $f_t^{(n)}$ $(n = 0,1,2,\ldots)$ are uncertain processes.

Fix any $\mathfrak{g} \in \Gamma$ and $t \in [a,\overline{\mathcal{U}}]$, where $\overline{\mathcal{U}} > a$. We shall show that the sequence $\{f_t^{(n)}(\mathfrak{g})\}_{n\ge 0}$ converges uniformly on $[a,\overline{\mathcal{U}}]$.

First, for $n = 0$, by (3), we have

$$\begin{aligned} f_{\mu}^{(1)}(\mathfrak{g}) - f_{\mu}^{(0)}(\mathfrak{g}) = {} & \frac{1}{\Gamma(c)}\int_a^{\mu} \big(\varphi(\mu) - \varphi(b)\big)^{c-1}\Omega\big(b, f_b^{(0)}(\mathfrak{g})\big)\varphi'(b)\,db \\ & + \frac{1}{\Gamma(c)}\int_a^{\mu} \big(\varphi(\mu) - \varphi(b)\big)^{c-1}\Lambda\big(b, f_b^{(0)}(\mathfrak{g})\big)\varphi'(b)\,d\mathcal{C}_b(\mathfrak{g}). \end{aligned} \tag{4}$$

Taking absolute values, using the triangle inequality, and applying Lemma 4, we obtain

$$\begin{aligned}
&\max_{a\le\mu\le t}\left|f_\mu^{(1)}(\mathfrak{g})-f_\mu^{(0)}(\mathfrak{g})\right| \\
&\le \max_{a\le\mu\le t}\left(\frac{1}{\Gamma(c)}\int_a^\mu \big(\varphi(\mu)-\varphi(b)\big)^{c-1}\left|\Omega\big(b,f_b^{(0)}(\mathfrak{g})\big)\right|\varphi'(b)\,db\right) \\
&\quad+\mathcal{V}_{\mathfrak{g}}\max_{a\le\mu\le t}\left(\frac{1}{\Gamma(c)}\int_a^\mu \big(\varphi(\mu)-\varphi(b)\big)^{c-1}\left|\Lambda\big(b,f_b^{(0)}(\mathfrak{g})\big)\right|\varphi'(b)\,db\right) \\
&\le \frac{1}{\Gamma(c)}\int_a^t \big(\varphi(t)-\varphi(b)\big)^{c-1}\left|\Omega\big(b,f_b^{(0)}(\mathfrak{g})\big)\right|\varphi'(b)\,db \\
&\quad+\frac{\mathcal{V}_{\mathfrak{g}}}{\Gamma(c)}\int_a^t \big(\varphi(t)-\varphi(b)\big)^{c-1}\left|\Lambda\big(b,f_b^{(0)}(\mathfrak{g})\big)\right|\varphi'(b)\,db.
\end{aligned} \tag{5}$$

Now, by the LGC,

$$\left|\Omega(b,\rho)\right|+\left|\Lambda(b,\rho)\right|\le\chi(1+|\rho|),\qquad \rho\in\mathbb{R},$$

and therefore

$$\begin{aligned}
&\max_{a\le\mu\le t}\left|f_\mu^{(1)}(\mathfrak{g})-f_\mu^{(0)}(\mathfrak{g})\right| \\
&\le\frac{\chi(1+\mathcal{V}_{\mathfrak{g}})}{\Gamma(c)}\int_a^t \big(\varphi(t)-\varphi(b)\big)^{c-1}\varphi'(b)\big(1+\left|f_b^{(0)}(\mathfrak{g})\right|\big)\,db.
\end{aligned} \tag{6}$$

Since

$$f_b^{(0)}(\mathfrak{g})=\sum_{\lambda=0}^{u-1}\frac{\big(\varphi(b)-\varphi(a)\big)^{\mathrm{v}-\lambda-1}}{\Gamma(\mathrm{v}-\lambda)}\vartheta_\lambda,$$

we may bound

$$1+\left|f_b^{(0)}(\mathfrak{g})\right|\le 1+\left|\sum_{\lambda=0}^{u-1}\frac{\big(\varphi(t)-\varphi(a)\big)^{\mathrm{v}-\lambda-1}}{\Gamma(\mathrm{v}-\lambda)}\vartheta_\lambda\right|,\qquad b\in[a,t].$$

Substituting this into (6), we get

$$\begin{aligned}
&\max_{a\le\mu\le t}\left|f_\mu^{(1)}(\mathfrak{g})-f_\mu^{(0)}(\mathfrak{g})\right| \\
&\le\frac{\chi(1+\mathcal{V}_{\mathfrak{g}})}{\Gamma(c)}\left(1+\left|\sum_{\lambda=0}^{u-1}\frac{\big(\varphi(t)-\varphi(a)\big)^{\mathrm{v}-\lambda-1}}{\Gamma(\mathrm{v}-\lambda)}\vartheta_\lambda\right|\right)\int_a^t\big(\varphi(t)-\varphi(b)\big)^{c-1}\varphi'(b)\,db \\
&=\frac{\chi(1+\mathcal{V}_{\mathfrak{g}})\big(\varphi(t)-\varphi(a)\big)^{c}}{\Gamma(c+1)}\left(1+\left|\sum_{\lambda=0}^{u-1}\frac{\big(\varphi(t)-\varphi(a)\big)^{\mathrm{v}-\lambda-1}}{\Gamma(\mathrm{v}-\lambda)}\vartheta_\lambda\right|\right).
\end{aligned} \tag{7}$$

Next, assume that for some $n\ge 1$,

$$\begin{aligned}
\max_{a\le\mu\le t}\left|f_\mu^{(n)}(\mathfrak{g})-f_\mu^{(n-1)}(\mathfrak{g})\right|\le{}&\frac{\big[\chi(1+\mathcal{V}_{\mathfrak{g}})\big(\varphi(t)-\varphi(a)\big)^{c}\big]^{n-1}}{\Gamma(1+(n-1)c)} \\
&\times\left(1+\left|\sum_{\lambda=0}^{u-1}\frac{\big(\varphi(t)-\varphi(a)\big)^{\mathrm{v}-\lambda-1}}{\Gamma(\mathrm{v}-\lambda)}\vartheta_\lambda\right|\right).
\end{aligned} \tag{8}$$

We shall prove the corresponding estimate for $n+1$.

From (3), we have

$$\begin{aligned} & f_{\mu}^{(n+1)}(\mathfrak{g}) - f_{\mu}^{(n)}(\mathfrak{g}) \\ &= \frac{1}{\Gamma(c)} \int_a^{\mu} \big(\varphi(\mu) - \varphi(b)\big)^{c-1} \Big[\Omega\big(b, f_b^{(n)}(\mathfrak{g})\big) - \Omega\big(b, f_b^{(n-1)}(\mathfrak{g})\big)\Big] \varphi'(b)\, db \\ &\quad + \frac{1}{\Gamma(c)} \int_a^{\mu} \big(\varphi(\mu) - \varphi(b)\big)^{c-1} \Big[\Lambda\big(b, f_b^{(n)}(\mathfrak{g})\big) - \Lambda\big(b, f_b^{(n-1)}(\mathfrak{g})\big)\Big] \varphi'(b)\, d\mathcal{C}_b(\mathfrak{g}). \end{aligned} \tag{9}$$

Proceeding as before, using Lemma 4 and the LC, we obtain

$$\begin{aligned} & \max_{a \le \mu \le t} \big|f_{\mu}^{(n+1)}(\mathfrak{g}) - f_{\mu}^{(n)}(\mathfrak{g})\big| \\ & \le \frac{(1+\mathcal{V}_{\mathfrak{g}})\chi}{\Gamma(c)} \int_a^t \big(\varphi(t) - \varphi(b)\big)^{c-1} \big|f_b^{(n)}(\mathfrak{g}) - f_b^{(n-1)}(\mathfrak{g})\big| \varphi'(b)\, db. \end{aligned} \tag{10}$$

Now, applying the induction hypothesis (8), we get

$$\begin{aligned} & \max_{a \le \mu \le t} \big|f_{\mu}^{(n+1)}(\mathfrak{g}) - f_{\mu}^{(n)}(\mathfrak{g})\big| \\ & \le \frac{(1+\mathcal{V}_{\mathfrak{g}})\chi}{\Gamma(c)} \cdot \frac{\big[\chi(1+\mathcal{V}_{\mathfrak{g}})\big(\varphi(t) - \varphi(a)\big)^{c}\big]^{n-1}}{\Gamma(1+(n-1)c)} \\ & \quad \times \left(1 + \left|\sum_{\lambda=0}^{u-1} \frac{\big(\varphi(t) - \varphi(a)\big)^{\mathrm{v}-\lambda-1}}{\Gamma(\mathrm{v}-\lambda)} \vartheta_{\lambda}\right|\right) \\ & \quad \times \int_a^t \big(\varphi(t) - \varphi(b)\big)^{c-1} \varphi'(b)\, db. \end{aligned} \tag{11}$$

A sharper estimate is obtained by retaining the dependence on $(\varphi(b) - \varphi(a))^{(n-1)c}$. Thus,

$$\begin{aligned} & \max_{a \le \mu \le t} \big|f_{\mu}^{(n+1)}(\mathfrak{g}) - f_{\mu}^{(n)}(\mathfrak{g})\big| \\ & \le \frac{\chi^n (1+\mathcal{V}_{\mathfrak{g}})^n}{\Gamma(c)\Gamma(1+(n-1)c)} \left(1 + \left|\sum_{\lambda=0}^{u-1} \frac{\big(\varphi(t) - \varphi(a)\big)^{\mathrm{v}-\lambda-1}}{\Gamma(\mathrm{v}-\lambda)} \vartheta_{\lambda}\right|\right) \\ & \quad \times \int_a^t \big(\varphi(t) - \varphi(b)\big)^{c-1} \big(\varphi(b) - \varphi(a)\big)^{(n-1)c} \varphi'(b)\, db. \end{aligned} \tag{12}$$

Now use the change of variable

$$\Phi = \frac{\varphi(b) - \varphi(a)}{\varphi(t) - \varphi(a)}, \qquad 0 \le \Phi \le 1.$$

Then

$$\begin{aligned} & \int_a^t \big(\varphi(t) - \varphi(b)\big)^{c-1} \big(\varphi(b) - \varphi(a)\big)^{(n-1)c} \varphi'(b)\, db \\ &= \big(\varphi(t) - \varphi(a)\big)^{nc} \int_0^1 (1-\Phi)^{c-1} \Phi^{(n-1)c}\, d\Phi \\ &= \big(\varphi(t) - \varphi(a)\big)^{nc} \frac{\Gamma(c)\Gamma(1+(n-1)c)}{\Gamma(1+nc)}. \end{aligned} \tag{13}$$

Substituting (13) into (12), we obtain

$$\max_{a\le\mu\le t}\left|f_{\mu}^{(n+1)}(\mathfrak{g})-f_{\mu}^{(n)}(\mathfrak{g})\right|\le\frac{\left[\chi(1+\mathcal{V}_{\mathfrak{g}})\left(\varphi(t)-\varphi(a)\right)^{c}\right]^{n}}{\Gamma(1+nc)}\times\left(1+\left|\sum_{\lambda=0}^{u-1}\frac{\left(\varphi(t)-\varphi(a)\right)^{\mathrm{v}-\lambda-1}}{\Gamma(\mathrm{v}-\lambda)}\vartheta_{\lambda}\right|\right).\tag{14}$$

Thus, by induction, (14) holds for all integers $n\ge 0$.

Now consider the series

$$\sum_{n=0}^{\infty}\frac{\left[\chi(1+\mathcal{V}_{\mathfrak{g}})\left(\varphi(t)-\varphi(a)\right)^{c}\right]^{n}}{\Gamma(1+nc)}.$$

Since $t\in[a,\overline{\mathcal{U}}]$, we have

$$\sum_{n=0}^{\infty}\frac{\left[\chi(1+\mathcal{V}_{\mathfrak{g}})\left(\varphi(t)-\varphi(a)\right)^{c}\right]^{n}}{\Gamma(1+nc)}\le\sum_{n=0}^{\infty}\frac{\left[\chi(1+\mathcal{V}_{\mathfrak{g}})\left(\varphi(\overline{\mathcal{U}})-\varphi(a)\right)^{c}\right]^{n}}{\Gamma(1+nc)}=E_{c,1}\left(\chi(1+\mathcal{V}_{\mathfrak{g}})\left(\varphi(\overline{\mathcal{U}})-\varphi(a)\right)^{c}\right)<\infty.\tag{15}$$

Hence, by the Weierstrass M-test, the series

$$\sum_{n=0}^{\infty}\left(f_{t}^{(n+1)}(\mathfrak{g})-f_{t}^{(n)}(\mathfrak{g})\right)$$

converges uniformly on $[a,\overline{\mathcal{U}}]$. Therefore, the sequence

$$\{f_{t}^{(n)}(\mathfrak{g})\}_{n\ge0}$$

converges uniformly on $[a,\overline{\mathcal{U}}]$ to a limit, say

$$f_{t}(\mathfrak{g})=\lim_{n\to\infty}f_{t}^{(n)}(\mathfrak{g}),\qquad\mathfrak{g}\in\Gamma.$$

Since each $f_{t}^{(n)}$ is an uncertain process, it follows from Lemma 5 that the uniform limit f_{t} is also an uncertain process.

Finally, passing to the limit $n\to\infty$ in (3), and using the continuity of Ω and Λ together with the uniform convergence of $f_{t}^{(n)}$ to f_{t}, we obtain

$$f_{t}=\sum_{\lambda=0}^{u-1}\frac{\left(\varphi(t)-\varphi(a)\right)^{\mathrm{v}-\lambda-1}}{\Gamma(\mathrm{v}-\lambda)}\vartheta_{\lambda}+\frac{1}{\Gamma(c)}\int_{a}^{t}\left(\varphi(t)-\varphi(b)\right)^{c-1}\varphi'(b)\Omega\left(b,f_{b}\right)db+\frac{1}{\Gamma(c)}\int_{a}^{t}\left(\varphi(t)-\varphi(b)\right)^{c-1}\varphi'(b)\Lambda\left(b,f_{b}\right)d\mathcal{C}_{b}.\tag{16}$$

Hence, f_{t} satisfies (2), and therefore it is a solution of (1). Thus, the existence of a solution to (1) is established.

Uniqueness: Now, we prove that (1) has a unique solution.

Assume that f_{t} and f_{t}^{*} are two solutions of (1) on $[a,\overline{\mathcal{U}}]$. Then, by (2), for every $\mathfrak{g}\in\Gamma$, we have

$$f_{t}(\mathfrak{g})=\sum_{\lambda=0}^{u-1}\frac{\left(\varphi(t)-\varphi(a)\right)^{\mathrm{v}-\lambda-1}}{\Gamma(\mathrm{v}-\lambda)}\vartheta_{\lambda}+\frac{1}{\Gamma(c)}\int_{a}^{t}\left(\varphi(t)-\varphi(b)\right)^{c-1}\varphi'(b)\Omega\left(b,f_{b}(\mathfrak{g})\right)db+\frac{1}{\Gamma(c)}\int_{a}^{t}\left(\varphi(t)-\varphi(b)\right)^{c-1}\varphi'(b)\Lambda\left(b,f_{b}(\mathfrak{g})\right)d\mathcal{C}_{b}(\mathfrak{g}),\tag{17}$$

and similarly,

$$f_t^*(\mathfrak{g}) = \sum_{\lambda=0}^{u-1} \frac{\big(\varphi(t)-\varphi(a)\big)^{\mathrm{v}-\lambda-1}}{\Gamma(\mathrm{v}-\lambda)}\vartheta_\lambda + \frac{1}{\Gamma(c)}\int_a^t \big(\varphi(t)-\varphi(b)\big)^{c-1}\varphi'(b)\Omega\big(b, f_b^*(\mathfrak{g})\big)\,db + \frac{1}{\Gamma(c)}\int_a^t \big(\varphi(t)-\varphi(b)\big)^{c-1}\varphi'(b)\Lambda\big(b, f_b^*(\mathfrak{g})\big)\,d\mathcal{C}_b(\mathfrak{g}). \tag{18}$$

Subtracting (18) from (17), we note that the initial-memory terms cancel, and thus

$$f_t(\mathfrak{g}) - f_t^*(\mathfrak{g}) = \frac{1}{\Gamma(c)}\int_a^t \big(\varphi(t)-\varphi(b)\big)^{c-1}\Big[\Omega\big(b, f_b(\mathfrak{g})\big) - \Omega\big(b, f_b^*(\mathfrak{g})\big)\Big]\varphi'(b)\,db + \frac{1}{\Gamma(c)}\int_a^t \big(\varphi(t)-\varphi(b)\big)^{c-1}\Big[\Lambda\big(b, f_b(\mathfrak{g})\big) - \Lambda\big(b, f_b^*(\mathfrak{g})\big)\Big]\varphi'(b)\,d\mathcal{C}_b(\mathfrak{g}). \tag{19}$$

Taking absolute values on both sides of (19), and then applying the triangle inequality, we obtain

$$\begin{aligned} &\big|f_t(\mathfrak{g}) - f_t^*(\mathfrak{g})\big| \\ &\le \left|\frac{1}{\Gamma(c)}\int_a^t \big(\varphi(t)-\varphi(b)\big)^{c-1}\Big[\Omega\big(b, f_b(\mathfrak{g})\big) - \Omega\big(b, f_b^*(\mathfrak{g})\big)\Big]\varphi'(b)\,db\right| \\ &\quad + \left|\frac{1}{\Gamma(c)}\int_a^t \big(\varphi(t)-\varphi(b)\big)^{c-1}\Big[\Lambda\big(b, f_b(\mathfrak{g})\big) - \Lambda\big(b, f_b^*(\mathfrak{g})\big)\Big]\varphi'(b)\,d\mathcal{C}_b(\mathfrak{g})\right|. \end{aligned} \tag{20}$$

For the first term on the right-hand side of (20), since the integral is an ordinary integral, we have

$$\begin{aligned} &\left|\frac{1}{\Gamma(c)}\int_a^t \big(\varphi(t)-\varphi(b)\big)^{c-1}\Big[\Omega\big(b, f_b(\mathfrak{g})\big) - \Omega\big(b, f_b^*(\mathfrak{g})\big)\Big]\varphi'(b)\,db\right| \\ &\le \frac{1}{\Gamma(c)}\int_a^t \big(\varphi(t)-\varphi(b)\big)^{c-1}\big|\Omega\big(b, f_b(\mathfrak{g})\big) - \Omega\big(b, f_b^*(\mathfrak{g})\big)\big|\varphi'(b)\,db. \end{aligned} \tag{21}$$

For the second term on the right-hand side of (20), by Lemma 4, we obtain

$$\begin{aligned} &\left|\frac{1}{\Gamma(c)}\int_a^t \big(\varphi(t)-\varphi(b)\big)^{c-1}\Big[\Lambda\big(b, f_b(\mathfrak{g})\big) - \Lambda\big(b, f_b^*(\mathfrak{g})\big)\Big]\varphi'(b)\,d\mathcal{C}_b(\mathfrak{g})\right| \\ &\le \frac{\mathcal{V}_\mathfrak{g}}{\Gamma(c)}\int_a^t \big(\varphi(t)-\varphi(b)\big)^{c-1}\big|\Lambda\big(b, f_b(\mathfrak{g})\big) - \Lambda\big(b, f_b^*(\mathfrak{g})\big)\big|\varphi'(b)\,db. \end{aligned} \tag{22}$$

Substituting (21) and (22) into (20), we arrive at

$$\begin{aligned} &\big|f_t(\mathfrak{g}) - f_t^*(\mathfrak{g})\big| \\ &\le \frac{1}{\Gamma(c)}\int_a^t \big(\varphi(t)-\varphi(b)\big)^{c-1}\big|\Omega\big(b, f_b(\mathfrak{g})\big) - \Omega\big(b, f_b^*(\mathfrak{g})\big)\big|\varphi'(b)\,db \\ &\quad + \frac{\mathcal{V}_\mathfrak{g}}{\Gamma(c)}\int_a^t \big(\varphi(t)-\varphi(b)\big)^{c-1}\big|\Lambda\big(b, f_b(\mathfrak{g})\big) - \Lambda\big(b, f_b^*(\mathfrak{g})\big)\big|\varphi'(b)\,db. \end{aligned} \tag{23}$$

Now, by the LC,

$$|\Omega(b,\rho) - \Omega(b,\theta)| + |\Lambda(b,\rho) - \Lambda(b,\theta)| \le \chi\,|\rho-\theta|, \qquad \rho,\theta\in\mathbb{R},$$

and hence

$$\big|\Omega\big(b, f_b(\mathfrak{g})\big) - \Omega\big(b, f_b^*(\mathfrak{g})\big)\big| \le \chi\,\big|f_b(\mathfrak{g}) - f_b^*(\mathfrak{g})\big|, \tag{24}$$

$$\big|\Lambda\big(b, f_b(\mathfrak{g})\big) - \Lambda\big(b, f_b^*(\mathfrak{g})\big)\big| \le \chi\,\big|f_b(\mathfrak{g}) - f_b^*(\mathfrak{g})\big|. \tag{25}$$

Using (24) and (25) in (23), we get

$$\left|f_t(\mathfrak{g}) - f_t^*(\mathfrak{g})\right| \leq \frac{(1+\mathcal{V}_\mathfrak{g})\chi}{\Gamma(c)} \int_a^t \left(\varphi(t)-\varphi(b)\right)^{c-1} \left|f_b(\mathfrak{g}) - f_b^*(\mathfrak{g})\right| \varphi'(b)\, db. \tag{26}$$

Define

$$\mathscr{U}(t) := \left|f_t(\mathfrak{g}) - f_t^*(\mathfrak{g})\right|, \qquad t \in [a, \overline{\mathcal{U}}].$$

Then (26) may be rewritten as

$$\mathscr{U}(t) \leq \frac{(1+\mathcal{V}_\mathfrak{g})\chi}{\Gamma(c)} \int_a^t \left(\varphi(t)-\varphi(b)\right)^{c-1} \varphi'(b)\mathscr{U}(b)\, db. \tag{27}$$

Observe that the right-hand side of (27) contains no nonzero free term. Therefore, applying Lemma 8 to (27), we conclude that

$$\mathscr{U}(t) = 0, \qquad t \in [a, \overline{\mathcal{U}}].$$

That is,

$$\left|f_t(\mathfrak{g}) - f_t^*(\mathfrak{g})\right| = 0, \qquad t \in [a, \overline{\mathcal{U}}],$$

and hence

$$f_t(\mathfrak{g}) = f_t^*(\mathfrak{g}), \qquad t \in [a, \overline{\mathcal{U}}],\ \mathfrak{g} \in \Gamma.$$

Since $\mathfrak{g} \in \Gamma$ is arbitrary, it follows that

$$f_t = f_t^*, \qquad t \in [a, \overline{\mathcal{U}}].$$

Therefore, (1) has a unique solution. This completes the proof. □

We now present the solution of system (1).

Theorem 2. *With $u-1 < c \leq u \in \mathbb{N}^+$, assume $c > 0$ is a real number. Assume that ψ is a real number, $\aleph(t)$ and $\varrho(t)$ are two continuous functions on $[a, \mathcal{U}]$, and $\mathcal{C}_t$ represents an LPr. The UFDE regarding φ-HFD under initial conditions (ICs) is as follows:*

$$\begin{cases} {}^H\mathcal{D}_a^{\mathfrak{z},c,\varphi} f_t = \psi f_t + \aleph(t) + \varrho(t)\frac{d\mathcal{C}_t}{dt},\ t \in [a, \mathcal{U}]. \\ \left[\left(\frac{1}{\varphi'(t)}\frac{d}{dt}\right)^\lambda I_a^{u-\mathfrak{b};\varphi} f_t\right]_{t=a} = \vartheta_\lambda,\ \lambda = 0, 1, \cdots, u-1, \end{cases} \tag{28}$$

possesses a solution f_t by which

$$\begin{aligned} f_t = {} & \textstyle\sum_{\lambda=0}^{u-1} \vartheta_\lambda \left(\varphi(t)-\varphi(a)\right)^{\mathfrak{v}-\lambda-1} E_{c,\mathfrak{v}-\lambda}\left(\psi\left(\varphi(t)-\varphi(a)\right)^c\right) \\ & + \int_a^t \left(\varphi(t)-\varphi(b)\right)^{c-1} \varphi'(b) E_{c,c}\left(\psi\left(\varphi(t)-\varphi(b)\right)^c\right) \aleph(b) db + \digamma_t, \end{aligned} \tag{29}$$

here $\digamma_t = \int_a^t \left(\varphi(t)-\varphi(b)\right)^{c-1} \varphi'(b) E_{c,c}\left(\psi\left(\varphi(t)-\varphi(b)\right)^c\right) \varrho(b) d\mathcal{C}_b$ is a normal uncertain variable throughout the times t, and $\digamma_t \sim N\left(0.\int_a^t \left|\left(\varphi(t)-\varphi(b)\right)^{c-1} E_{c,c}\left(\psi, \left(\varphi(t)-\varphi(b)\right)^c\right) \varrho(b)\right| \varphi'(b) db\right)$.

Proof. With regard to Definition 11, all that remains is to demonstrate that the solution in (29) is equivalent to

$$f_t = \sum_{\lambda=0}^{u-1} \frac{(\varphi(t)-\varphi(a))^{\mathrm{v}-\lambda-1}}{\Gamma(\mathrm{v}-\lambda)}\vartheta_\lambda + \frac{1}{\Gamma(c)}\int_a^t (\varphi(t)-\varphi(b))^{c-1}\varphi'(b)(\psi f_b + \aleph(b))db + \frac{1}{\Gamma(c)}\int_a^t (\varphi(t)-\varphi(b))^{c-1}\varphi'(b)\varrho(b)d\mathcal{C}_b. \tag{30}$$

From (29), we have

$$\begin{aligned}
&\frac{1}{\Gamma(c)}\int_a^t (\varphi(t)-\varphi(b))^{c-1}\varphi'(b)\psi f_b db \\
&= \frac{\psi}{\Gamma(c)}\Bigg[\int_a^t (\varphi(t)-\varphi(b))^{c-1}\varphi'(b)\Bigg(\sum_{\lambda=0}^{u-1}\vartheta_\lambda(\varphi(t)-\varphi(a))^{\mathrm{v}-\lambda-1}E_{c,\mathrm{v}-\lambda}(\psi(\varphi(t)-\varphi(a))^c)\Bigg)db \\
&+ \int_a^t (\varphi(t)-\varphi(b))^{c-1}\varphi'(b)\Bigg(\int_a^b (\varphi(b)-\varphi(\mu))^{c-1}E_{c,c}(\psi(\varphi(b)-\varphi(\mu))^c)\varphi'(\mu)\aleph(\mu)d\mu\Bigg)db \\
&+ \int_a^t (\varphi(t)-\varphi(b))^{c-1}\varphi'(b)\Bigg(\int_a^b (\varphi(b)-\varphi(\mu))^{c-1}E_{c,c}(\psi(\varphi(b)-\varphi(\mu))^c)\varphi'(\mu)\varrho(\mu)d\mathcal{C}_\mu\Bigg)db\Bigg] \\
&= I_1 + I_2 + I_3.
\end{aligned} \tag{31}$$

We have

$$\begin{aligned}
I_1 &= \frac{\psi}{\Gamma(c)}\int_a^t (\varphi(t)-\varphi(b))^{c-1}\varphi'(b)\Bigg(\sum_{\lambda=0}^{u-1}\vartheta_\lambda(\varphi(b)-\varphi(a))^{\mathrm{v}-\lambda-1}E_{c,\mathrm{v}-\lambda}(\psi(\varphi(b)-\varphi(a))^c)\Bigg)db \\
&= \frac{\psi}{\Gamma(c)}\sum_{\lambda=0}^{u-1}\vartheta_\lambda\int_a^t (\varphi(t)-\varphi(b))^{c-1}\varphi'(b)(\varphi(b)-\varphi(a))^{\mathrm{v}-\lambda-1}E_{c,\mathrm{v}-\lambda}(\psi(\varphi(b)-\varphi(a))^c)db.
\end{aligned} \tag{32}$$

Now, by the series representation of the MLF,

$$E_{c,\mathrm{v}-\lambda}(z) = \sum_{\mathrm{r}=0}^{\infty}\frac{z^{\mathrm{r}}}{\Gamma(c\mathrm{r}+\mathrm{v}-\lambda)}.$$

Hence,

$$E_{c,\mathrm{v}-\lambda}(\psi(\varphi(b)-\varphi(a))^c) = \sum_{\mathrm{r}=0}^{\infty}\frac{\psi^{\mathrm{r}}(\varphi(b)-\varphi(a))^{c\mathrm{r}}}{\Gamma(c\mathrm{r}+\mathrm{v}-\lambda)}. \tag{33}$$

Substituting (33) into (32), we get

$$\begin{aligned}
I_1 &= \frac{\psi}{\Gamma(c)}\sum_{\lambda=0}^{u-1}\vartheta_\lambda\int_a^t (\varphi(t)-\varphi(b))^{c-1}\varphi'(b)(\varphi(b)-\varphi(a))^{\mathrm{v}-\lambda-1}\Bigg(\sum_{\mathrm{r}=0}^{\infty}\frac{\psi^{\mathrm{r}}(\varphi(b)-\varphi(a))^{c\mathrm{r}}}{\Gamma(c\mathrm{r}+\mathrm{v}-\lambda)}\Bigg)db \\
&= \sum_{\lambda=0}^{u-1}\vartheta_\lambda\sum_{\mathrm{r}=0}^{\infty}\frac{\psi^{\mathrm{r}+1}}{\Gamma(c)\Gamma(c\mathrm{r}+\mathrm{v}-\lambda)}\int_a^t (\varphi(t)-\varphi(b))^{c-1}(\varphi(b)-\varphi(a))^{c\mathrm{r}+\mathrm{v}-\lambda-1}\varphi'(b)db.
\end{aligned} \tag{34}$$

Next, introduce the change of variable

$$\Phi = \frac{\varphi(b)-\varphi(a)}{\varphi(t)-\varphi(a)}.$$

Then

$$\varphi(b)-\varphi(a) = \Phi(\varphi(t)-\varphi(a)),$$
$$\varphi(t)-\varphi(b) = (1-\Phi)(\varphi(t)-\varphi(a)),$$

and

$$\varphi'(b)\,db = \big(\varphi(t) - \varphi(a)\big)\,d\Phi.$$

Also, when $b = a$, we have $\Phi = 0$, and when $b = t$, we have $\Phi = 1$. Therefore,

$$\begin{aligned}
&\int_a^t \big(\varphi(t) - \varphi(b)\big)^{c-1}\big(\varphi(b) - \varphi(a)\big)^{\mathrm{cr}+\mathrm{v}-\lambda-1}\varphi'(b)db \\
&\quad = \int_0^1 \Big((1-\Phi)\big(\varphi(t) - \varphi(a)\big)\Big)^{c-1}\Big(\Phi\big(\varphi(t) - \varphi(a)\big)\Big)^{\mathrm{cr}+\mathrm{v}-\lambda-1}\big(\varphi(t) - \varphi(a)\big)d\Phi \\
&\quad = \big(\varphi(t) - \varphi(a)\big)^{c+\mathrm{cr}+\mathrm{v}-\lambda-1}\int_0^1 (1-\Phi)^{c-1}\Phi^{\mathrm{cr}+\mathrm{v}-\lambda-1}d\Phi.
\end{aligned} \tag{35}$$

Using the Beta function identity

$$\int_0^1 (1-\Phi)^{x-1}\Phi^{y-1}\,d\Phi = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \qquad x > 0,\ y > 0,$$

with

$$x = c, \qquad y = c\mathrm{r} + \mathrm{v} - \lambda,$$

we obtain

$$\int_0^1 (1-\Phi)^{c-1}\Phi^{\mathrm{cr}+\mathrm{v}-\lambda-1}d\Phi = \frac{\Gamma(c)\Gamma(c\mathrm{r}+\mathrm{v}-\lambda)}{\Gamma(c(\mathrm{r}+1)+\mathrm{v}-\lambda)}. \tag{36}$$

Substituting (35) and (36) into (34), we get

$$\begin{aligned}
I_1 &= \sum_{\lambda=0}^{u-1}\vartheta_\lambda \sum_{\mathrm{r}=0}^{\infty}\frac{\psi^{\mathrm{r}+1}}{\Gamma(c)\Gamma(c\mathrm{r}+\mathrm{v}-\lambda)}\big(\varphi(t) - \varphi(a)\big)^{c(\mathrm{r}+1)+\mathrm{v}-\lambda-1}\frac{\Gamma(c)\Gamma(c\mathrm{r}+\mathrm{v}-\lambda)}{\Gamma(c(\mathrm{r}+1)+\mathrm{v}-\lambda)} \\
&= \sum_{\lambda=0}^{u-1}\vartheta_\lambda \sum_{\mathrm{r}=0}^{\infty}\frac{\psi^{\mathrm{r}+1}\big(\varphi(t) - \varphi(a)\big)^{c(\mathrm{r}+1)+\mathrm{v}-\lambda-1}}{\Gamma(c(\mathrm{r}+1)+\mathrm{v}-\lambda)}.
\end{aligned} \tag{37}$$

Now factor out $\big(\varphi(t) - \varphi(a)\big)^{\mathrm{v}-\lambda-1}$:

$$I_1 = \sum_{\lambda=0}^{u-1}\vartheta_\lambda\big(\varphi(t) - \varphi(a)\big)^{\mathrm{v}-\lambda-1}\sum_{\mathrm{r}=0}^{\infty}\frac{\psi^{\mathrm{r}+1}\big(\varphi(t) - \varphi(a)\big)^{c(\mathrm{r}+1)}}{\Gamma(c(\mathrm{r}+1)+\mathrm{v}-\lambda)}. \tag{38}$$

Let $m = \mathrm{r} + 1$. Then $m = 1, 2, 3, \ldots$, and (38) becomes

$$I_1 = \sum_{\lambda=0}^{u-1}\vartheta_\lambda\big(\varphi(t) - \varphi(a)\big)^{\mathrm{v}-\lambda-1}\sum_{m=1}^{\infty}\frac{\psi^{m}\big(\varphi(t) - \varphi(a)\big)^{cm}}{\Gamma(cm+\mathrm{v}-\lambda)}. \tag{39}$$

By the definition of the MLF,

$$E_{c,\mathrm{v}-\lambda}(z) = \sum_{m=0}^{\infty}\frac{z^m}{\Gamma(cm+\mathrm{v}-\lambda)},$$

hence

$$\sum_{m=1}^{\infty}\frac{z^m}{\Gamma(cm+\mathrm{v}-\lambda)} = E_{c,\mathrm{v}-\lambda}(z) - \frac{1}{\Gamma(\mathrm{v}-\lambda)}.$$

Taking

$$z = \psi\big(\varphi(t) - \varphi(a)\big)^c,$$

we conclude from (39) that

$$I_1 = \sum_{\lambda=0}^{u-1} \vartheta_\lambda \big(\varphi(t)-\varphi(a)\big)^{\mathrm{v}-\lambda-1} E_{c,\mathrm{v}-\lambda}\big(\psi(\varphi(t)-\varphi(a))^c\big) - \sum_{\lambda=0}^{u-1} \frac{\big(\varphi(t)-\varphi(a)\big)^{\mathrm{v}-\lambda-1}}{\Gamma(\mathrm{v}-\lambda)}\vartheta_\lambda. \tag{40}$$

Therefore,

$$I_1 = \sum_{\lambda=0}^{u-1} \vartheta_\lambda \big(\varphi(t)-\varphi(a)\big)^{\mathrm{v}-\lambda-1} E_{c,\mathrm{v}-\lambda}\big(\psi(\varphi(t)-\varphi(a))^c\big) - \sum_{\lambda=0}^{u-1} \frac{\big(\varphi(t)-\varphi(a)\big)^{\mathrm{v}-\lambda-1}}{\Gamma(\mathrm{v}-\lambda)}\vartheta_\lambda \tag{41}$$

with

$$\mathrm{v} = c + \mathrm{b}(u-c).$$

Next, we have

$$\begin{aligned}
I_2 =& \frac{\psi}{\Gamma(c)} \int_a^t \big(\varphi(t)-\varphi(b)\big)^{c-1}\varphi'(b)\bigg(\int_a^b \big(\varphi(b)-\varphi(\mu)\big)^{c-1} E_{c,c}\big(\psi,(\varphi(b)-\varphi(\mu))^c\big)\aleph(\mu)d\mu\varphi'(\mu)\bigg)db \\
=& \frac{\psi}{\Gamma(c)} \int_a^t \aleph(\mu)\varphi'(\mu)d\mu \int_\mu^t \big(\varphi(t)-\varphi(b)\big)^{c-1}\varphi'(b)\sum_{\mathrm{r}=0}^{\infty}\frac{\psi^{\mathrm{r}}(\varphi(b)-\varphi(\mu))^{c\mathrm{r}}}{\Gamma(c\mathrm{r}+c)}db \\
=& \int_a^t \aleph(\mu)\varphi'(\mu)d\mu\bigg(\sum_{\mathrm{r}=0}^{\infty}\frac{\psi^{\mathrm{r}+1}}{\Gamma(c)\Gamma(c\mathrm{r}+c)}\int_u^t(\varphi(t)-\varphi(b))^{c-1}\varphi'(b)(\varphi(b)-\varphi(\mu))^{c\mathrm{r}+c-1}db\bigg) \\
=& \int_a^t \aleph(\mu)\varphi'(\mu)d\mu\bigg(\sum_{\mathrm{r}=0}^{\infty}\frac{\psi^{\mathrm{r}+1}(\varphi(t)-\varphi(\mu))^{c(\mathrm{r}+1)+c-1}}{\Gamma(c)\Gamma(c\mathrm{r}+c)}\int_0^1(1-\Phi)^{c-1}\Phi^{c(\mathrm{r}+1)-1}d\Phi\bigg) \\
=& \int_a^t \aleph(\mu)\big(\varphi(t)-\varphi(\mu)\big)^{c-1}\sum_{\mathrm{r}=0}^{\infty}\frac{\psi^{\mathrm{r}+1}(\varphi(t)-\varphi(\mu))^{c(\mathrm{r}+1)}}{\Gamma(c(\mathrm{r}+1)+c)}\varphi'(\mu)d\mu \\
=& \int_a^t \aleph(\mu)\big(\varphi(t)-\varphi(\mu)\big)^{c-1}\sum_{\mathrm{r}=1}^{\infty}\frac{\psi^{\mathrm{r}}(\varphi(t)-\varphi(\mu))^{c\mathrm{r}}}{\Gamma(c\mathrm{r}+c)}\varphi'(\mu)d\mu \\
=& \int_a^t \big(\varphi(t)-\varphi(\mu)\big)^{c-1}E_{c,c}\big(\psi(\varphi(t)-\varphi(\mu))^c\big)\aleph(\mu)\varphi'(\mu)d\mu - \frac{1}{\Gamma(c)}\int_a^t\big(\varphi(t)-\varphi(\mu)\big)^{c-1}\aleph(\mu)\varphi'(\mu)d\mu.
\end{aligned} \tag{42}$$

By employing the same approach, we get

$$\begin{aligned}
I_3 =& \frac{\psi}{\Gamma(c)}\int_a^t\big(\varphi(t)-\varphi(b)\big)^{c-1}\bigg(\int_a^b\big(\varphi(b)-\varphi(\mu)\big)^{c-1}E_{c,c}\big(\psi,(\varphi(b)-\varphi(\mu))^c\big)\varrho(b)\varphi'(\mu)d\mathcal{C}_u\bigg)\varphi'(b)db \\
=& \int_a^t\big(\varphi(t)-\varphi(\mu)\big)^{c-1}E_{c,c}\big(\psi,(\varphi(t)-\varphi(\mu))^c\big)\varrho(\mu)\varphi'(\mu)d\mathcal{C}_\mu \\
&- \frac{1}{\Gamma(c)}\int_a^t\big(\varphi(t)-\varphi(\mu)\big)^{c-1}\varphi'(\mu)\varrho(b)d\mathcal{C}_\mu.
\end{aligned} \tag{43}$$

Substituting (41), (42), and (43) into (31) results in the solution (29), which is equivalent to

$$\begin{aligned}
f_t =& \sum_{\lambda=0}^{u-1}\frac{\big(\varphi(t)-\varphi(a)\big)^{\mathrm{v}-\lambda-1}}{\Gamma(\mathrm{v}-\lambda)}\vartheta_\lambda \\
&+ \int_a^t\big(\varphi(t)-\varphi(b)\big)^{c-1}E_{c,c}\big(\psi,(\varphi(t)-\varphi(b))^c\big)\aleph(b)\varphi'(b)db \\
&+ \int_a^t\big(\varphi(t)-\varphi(b)\big)^{c-1}E_{c,c}\big(\psi,(\varphi(t)-\varphi(b))^c\big)\varrho(b)\varphi'(b)d\mathcal{C}_b.
\end{aligned} \tag{44}$$

Assume $F_t = \int_a^t \left(\varphi(t) - \varphi(b)\right)^{c-1} E_{c,c}\left(\psi\left(\varphi(t) - \varphi(b)\right)^c\right) \varrho(b)\varphi'(b)d\mathcal{C}_b$, F_t is a normal uncertain variable at each time t, according to Lemma 2, and $F_t \sim N\left(0, \int_a^t \left|\left(\varphi(t) - \varphi(b)\right)^{c-1} E_{c,c}\left(\psi\left(\varphi(t) - \varphi(b)\right)^c\right) \varrho(b)\varphi'(b)\right| db\right)$. Furthermore, the procedure f_t is uncertain. The proof is complete. □

We now prove that the unique solution of system (1) is sample-continuous.

Theorem 3. *The unique solution of the system* (1) *is sample-continuous, based on the hypotheses of Theorem 1.*

Proof. Under the hypotheses of Theorem 1, system (1) admits a unique solution f_t on $[a, \overline{\mathcal{U}}]$. We prove that this unique solution is sample-continuous.

Consider the sequence of successive approximations corresponding to (2), defined by

$$f_t^{(0)}(\mathfrak{g}) = \sum_{\lambda=0}^{u-1} \frac{\left(\varphi(t) - \varphi(a)\right)^{\mathrm{v}-\lambda-1}}{\Gamma(\mathrm{v}-\lambda)} \vartheta_\lambda, \qquad \mathfrak{g} \in \Gamma,$$

and for each integer $n \geq 0$,

$$\begin{aligned} f_t^{(n+1)}(\mathfrak{g}) = &\sum_{\lambda=0}^{u-1} \frac{\left(\varphi(t) - \varphi(a)\right)^{\mathrm{v}-\lambda-1}}{\Gamma(\mathrm{v}-\lambda)} \vartheta_\lambda \\ &+ \frac{1}{\Gamma(c)} \int_a^t \left(\varphi(t) - \varphi(b)\right)^{c-1} \varphi'(b) \Omega\left(b, f_b^{(n)}(\mathfrak{g})\right) db \\ &+ \frac{1}{\Gamma(c)} \int_a^t \left(\varphi(t) - \varphi(b)\right)^{c-1} \varphi'(b) \Lambda\left(b, f_b^{(n)}(\mathfrak{g})\right) d\mathcal{C}_b(\mathfrak{g}). \end{aligned}$$

By Theorem 1, $f_t^{(n)}(\mathfrak{g})$ converges uniformly on $[a, \overline{\mathcal{U}}]$ to the unique solution $f_t(\mathfrak{g})$. Hence

$$f_t(\mathfrak{g}) = \lim_{n \to \infty} f_t^{(n)}(\mathfrak{g}), \qquad \mathfrak{g} \in \Gamma.$$

Moreover, the estimates derived in the proof of Theorem 1 yield the boundedness of the solution on $[a, \overline{\mathcal{U}}]$; that is, for each $\mathfrak{g} \in \Gamma$, there exists a constant $Q_\mathfrak{g} > 0$ such that

$$1 + \left|f_t(\mathfrak{g})\right| \leq Q_\mathfrak{g}, \qquad t \in [a, \overline{\mathcal{U}}].$$

Now let $\mathfrak{g} \in \Gamma$ and let $0 < a \leq \eta \leq t \leq \overline{\mathcal{U}}$. From (2), we have

$$\begin{aligned} &f_t(\mathfrak{g}) - f_\eta(\mathfrak{g}) \\ &= \sum_{\lambda=0}^{u-1} \frac{\left(\varphi(t) - \varphi(a)\right)^{\mathrm{v}-\lambda-1} - \left(\varphi(\eta) - \varphi(a)\right)^{\mathrm{v}-\lambda-1}}{\Gamma(\mathrm{v}-\lambda)} \vartheta_\lambda \\ &+ \frac{1}{\Gamma(c)} \int_\eta^t \left(\varphi(t) - \varphi(b)\right)^{c-1} \varphi'(b) \Omega\left(b, f_b(\mathfrak{g})\right) db \\ &+ \frac{1}{\Gamma(c)} \int_\eta^t \left(\varphi(t) - \varphi(b)\right)^{c-1} \varphi'(b) \Lambda\left(b, f_b(\mathfrak{g})\right) d\mathcal{C}_b(\mathfrak{g}) \\ &+ \frac{1}{\Gamma(c)} \int_a^\eta \left[\left(\varphi(t) - \varphi(b)\right)^{c-1} - \left(\varphi(\eta) - \varphi(b)\right)^{c-1}\right] \varphi'(b) \Omega\left(b, f_b(\mathfrak{g})\right) db \\ &+ \frac{1}{\Gamma(c)} \int_a^\eta \left[\left(\varphi(t) - \varphi(b)\right)^{c-1} - \left(\varphi(\eta) - \varphi(b)\right)^{c-1}\right] \varphi'(b) \Lambda\left(b, f_b(\mathfrak{g})\right) d\mathcal{C}_b(\mathfrak{g}). \end{aligned}$$

Therefore, by the triangle inequality,

$$\begin{aligned}
&\left|f_t(\mathfrak{g}) - f_\eta(\mathfrak{g})\right| \\
&\le \left|\sum_{\lambda=0}^{u-1} \frac{\left(\varphi(t)-\varphi(a)\right)^{\mathrm{v}-\lambda-1} - \left(\varphi(\eta)-\varphi(a)\right)^{\mathrm{v}-\lambda-1}}{\Gamma(\mathrm{v}-\lambda)}\vartheta_\lambda\right| \\
&\quad + \frac{1}{\Gamma(c)}\int_\eta^t \left(\varphi(t)-\varphi(b)\right)^{c-1}\varphi'(b)\left|\Omega\left(b, f_b(\mathfrak{g})\right)\right| db \\
&\quad + \left|\frac{1}{\Gamma(c)}\int_\eta^t \left(\varphi(t)-\varphi(b)\right)^{c-1}\varphi'(b)\Lambda\left(b, f_b(\mathfrak{g})\right) d\mathcal{C}_b(\mathfrak{g})\right| \\
&\quad + \frac{1}{\Gamma(c)}\int_a^\eta \left|\left(\varphi(t)-\varphi(b)\right)^{c-1} - \left(\varphi(\eta)-\varphi(b)\right)^{c-1}\right|\varphi'(b)\left|\Omega\left(b, f_b(\mathfrak{g})\right)\right| db \\
&\quad + \left|\frac{1}{\Gamma(c)}\int_a^\eta \left[\left(\varphi(t)-\varphi(b)\right)^{c-1} - \left(\varphi(\eta)-\varphi(b)\right)^{c-1}\right]\varphi'(b)\Lambda\left(b, f_b(\mathfrak{g})\right) d\mathcal{C}_b(\mathfrak{g})\right|.
\end{aligned}$$

Now apply Lemma 4 to the two uncertain integral terms. Then

$$\begin{aligned}
&\left|\frac{1}{\Gamma(c)}\int_\eta^t \left(\varphi(t)-\varphi(b)\right)^{c-1}\varphi'(b)\Lambda\left(b, f_b(\mathfrak{g})\right) d\mathcal{C}_b(\mathfrak{g})\right| \\
&\le \frac{\mathcal{V}_\mathfrak{g}}{\Gamma(c)}\int_\eta^t \left(\varphi(t)-\varphi(b)\right)^{c-1}\varphi'(b)\left|\Lambda\left(b, f_b(\mathfrak{g})\right)\right| db,
\end{aligned}$$

and

$$\begin{aligned}
&\left|\frac{1}{\Gamma(c)}\int_a^\eta \left[\left(\varphi(t)-\varphi(b)\right)^{c-1} - \left(\varphi(\eta)-\varphi(b)\right)^{c-1}\right]\varphi'(b)\Lambda\left(b, f_b(\mathfrak{g})\right) d\mathcal{C}_b(\mathfrak{g})\right| \\
&\le \frac{\mathcal{V}_\mathfrak{g}}{\Gamma(c)}\int_a^\eta \left|\left(\varphi(t)-\varphi(b)\right)^{c-1} - \left(\varphi(\eta)-\varphi(b)\right)^{c-1}\right|\varphi'(b)\left|\Lambda\left(b, f_b(\mathfrak{g})\right)\right| db.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
&\left|f_t(\mathfrak{g}) - f_\eta(\mathfrak{g})\right| \\
&\le \left|\sum_{\lambda=0}^{u-1} \frac{\left(\varphi(t)-\varphi(a)\right)^{\mathrm{v}-\lambda-1} - \left(\varphi(\eta)-\varphi(a)\right)^{\mathrm{v}-\lambda-1}}{\Gamma(\mathrm{v}-\lambda)}\vartheta_\lambda\right| \\
&\quad + \frac{1}{\Gamma(c)}\int_\eta^t \left(\varphi(t)-\varphi(b)\right)^{c-1}\varphi'(b)\left|\Omega\left(b, f_b(\mathfrak{g})\right)\right| db \\
&\quad + \frac{\mathcal{V}_\mathfrak{g}}{\Gamma(c)}\int_\eta^t \left(\varphi(t)-\varphi(b)\right)^{c-1}\varphi'(b)\left|\Lambda\left(b, f_b(\mathfrak{g})\right)\right| db \\
&\quad + \frac{1}{\Gamma(c)}\int_a^\eta \left|\left(\varphi(t)-\varphi(b)\right)^{c-1} - \left(\varphi(\eta)-\varphi(b)\right)^{c-1}\right|\varphi'(b)\left|\Omega\left(b, f_b(\mathfrak{g})\right)\right| db \\
&\quad + \frac{\mathcal{V}_\mathfrak{g}}{\Gamma(c)}\int_a^\eta \left|\left(\varphi(t)-\varphi(b)\right)^{c-1} - \left(\varphi(\eta)-\varphi(b)\right)^{c-1}\right|\varphi'(b)\left|\Lambda\left(b, f_b(\mathfrak{g})\right)\right| db.
\end{aligned}$$

By the LGC, we obtain

$$\left|\Omega\left(b, f_b(\mathfrak{g})\right)\right| + \left|\Lambda\left(b, f_b(\mathfrak{g})\right)\right| \le \chi\left(1 + \left|f_b(\mathfrak{g})\right|\right) \le \chi\mathcal{Q}_\mathfrak{g}.$$

Therefore,

$$\begin{aligned}
&\left|f_t(\mathfrak{g}) - f_\eta(\mathfrak{g})\right| \\
&\le \left|\sum_{\lambda=0}^{u-1} \frac{\left(\varphi(t)-\varphi(a)\right)^{\mathrm{v}-\lambda-1} - \left(\varphi(\eta)-\varphi(a)\right)^{\mathrm{v}-\lambda-1}}{\Gamma(\mathrm{v}-\lambda)}\vartheta_\lambda\right|
\end{aligned}$$

$$+\frac{(1+\mathcal{V}_{\mathfrak{g}})\chi\mathcal{Q}_{\mathfrak{g}}}{\Gamma(c)}\int_{\eta}^{t}\big(\varphi(t)-\varphi(b)\big)^{c-1}\varphi'(b)\,db$$
$$+\frac{(1+\mathcal{V}_{\mathfrak{g}})\chi\mathcal{Q}_{\mathfrak{g}}}{\Gamma(c)}\int_{a}^{\eta}\Big|\big(\varphi(t)-\varphi(b)\big)^{c-1}-\big(\varphi(\eta)-\varphi(b)\big)^{c-1}\Big|\varphi'(b)\,db.$$

Now, as $t\to\eta$, the first term tends to zero by the continuity of φ. Also,

$$\int_{\eta}^{t}\big(\varphi(t)-\varphi(b)\big)^{c-1}\varphi'(b)\,db=\frac{(\varphi(t)-\varphi(\eta))^{c}}{c}\longrightarrow 0.$$

Finally, for each $b\in[a,\eta)$,

$$\Big|\big(\varphi(t)-\varphi(b)\big)^{c-1}-\big(\varphi(\eta)-\varphi(b)\big)^{c-1}\Big|\to 0\qquad\text{as } t\to\eta,$$

and the integrand is dominated by an integrable function on $[a,\eta]$. Hence, by the dominated convergence theorem,

$$\int_{a}^{\eta}\Big|\big(\varphi(t)-\varphi(b)\big)^{c-1}-\big(\varphi(\eta)-\varphi(b)\big)^{c-1}\Big|\varphi'(b)\,db\longrightarrow 0.$$

Combining the above limits, we conclude that

$$\big|f_t(\mathfrak{g})-f_\eta(\mathfrak{g})\big|\longrightarrow 0\qquad\text{as } |t-\eta|\to 0.$$

Since $\mathfrak{g}\in\Gamma$ was arbitrary, the unique solution f_t is sample-continuous. This completes the proof. □

Next, we provide an application to interest rate modeling.

Let the following UFDE with φ-HFD represent the short interest rate f_t [34,35]:

$$^{H}\mathcal{D}_{0}^{\mathrm{z},c,\varphi}f_t=(\nu-\omega f_t)+\ell\frac{d\mathcal{C}_t}{dt},\qquad \nu,\omega,\ell\in\mathbb{R}^{+}. \tag{45}$$

The value of a zero-coupon bond maturing at time y is

$$\mathcal{M}=\mathbb{E}\Big[\exp\Big(-\int_{0}^{\mathrm{y}}f_t\,dt\Big)\Big].$$

For the φ-Hilfer model, the natural initial condition is

$$\mathcal{I}_{0+}^{(1-\mathrm{z})(1-c),\varphi}f_t\big|_{t=0}=\vartheta_0,$$

and we set

$$\mathrm{v}=c+\mathrm{z}(1-c).$$

Theorem 4. *Assume that*

$$\mathcal{I}_{0+}^{(1-\mathrm{z})(1-c),\varphi}f_t\big|_{t=0}=\vartheta_0.$$

Then the unique solution of (45) *by Theorems 1 and 2 is given as follows:*

$$\begin{aligned}f_t=\;&\vartheta_0\big(\varphi(t)-\varphi(0)\big)^{\mathrm{v}-1}E_{c,\mathrm{v}}\Big(-\omega\big(\varphi(t)-\varphi(0)\big)^{c}\Big)\\&+\nu\int_{0}^{t}\big(\varphi(t)-\varphi(b)\big)^{c-1}\varphi'(b)E_{c,c}\Big(-\omega\big(\varphi(t)-\varphi(b)\big)^{c}\Big)\,db\\&+\ell\int_{0}^{t}\big(\varphi(t)-\varphi(b)\big)^{c-1}\varphi'(b)E_{c,c}\Big(-\omega\big(\varphi(t)-\varphi(b)\big)^{c}\Big)\,d\mathcal{C}_b.\end{aligned} \tag{46}$$

Moreover, the zero-coupon bond price $\mathcal{M} = \mathbb{E}\left[\exp\left(-\int_0^{\mathrm{y}} f_t dt\right)\right]$ *is given by the following explicit deterministic formula:*

$$\mathcal{M} = \exp\left(-\mathcal{J}(\mathrm{y}) + \ell \int_0^{\mathrm{y}} \left| \int_b^{\mathrm{y}} (\varphi(t) - \varphi(b))^{c-1} \varphi'(b) E_{c,c}\left(-\omega(\varphi(t) - \varphi(b))^c\right) dt \right| db\right)$$

where

$$\begin{aligned}\mathcal{J}(\mathrm{y}) = \vartheta_0 \int_0^{\mathrm{y}} \left(\varphi(t) - \varphi(0)\right)^{\mathrm{v}-1} E_{c,\mathrm{v}}\left(-\omega\left(\varphi(t) - \varphi(0)\right)^c\right) dt \\ + \nu \int_0^{\mathrm{y}} \int_0^t \left(\varphi(t) - \varphi(b)\right)^{c-1} \varphi'(b) E_{c,c}\left(-\omega\left(\varphi(t) - \varphi(b)\right)^c\right) db\, dt,\end{aligned}$$

with $\nu = c + z(1-c)$.

Proof. The (45) represents a linear FUDE according to the φ-HFD framework. To derive its solution, we use the standard variation-of-constants formula for linear φ-Hilfer equations.

First, consider the associated homogeneous equation

$$^H\mathcal{D}_0^{\mathrm{z},c,\varphi} f_t = -\omega f_t. \tag{47}$$

Under the generalized initial condition

$$\mathcal{I}_{0+}^{(1-\mathrm{z})(1-c),\varphi} f_t\big|_{t=0} = \vartheta_0,$$

the solution of (47) is

$$f_t^{(h)} = \vartheta_0 \left(\varphi(t) - \varphi(0)\right)^{\mathrm{v}-1} E_{c,\mathrm{v}}\left(-\omega\left(\varphi(t) - \varphi(0)\right)^c\right), \tag{48}$$

where

$$\mathrm{v} = c + \mathrm{z}(1-c).$$

Now rewrite (45) in the form

$$^H\mathcal{D}_0^{\mathrm{z},c,\varphi} f_t + \omega f_t = \nu + \ell \frac{d\mathcal{C}_t}{dt}.$$

For this linear nonhomogeneous equation, the resolvent kernel is

$$\left(\varphi(t) - \varphi(b)\right)^{c-1} \varphi'(b) E_{c,c}\left(-\omega\left(\varphi(t) - \varphi(b)\right)^c\right).$$

Hence the variation-of-constants formula yields

$$\begin{aligned}f_t = f_t^{(h)} + \nu \int_0^t \left(\varphi(t) - \varphi(b)\right)^{c-1} \varphi'(b) E_{c,c}\left(-\omega\left(\varphi(t) - \varphi(b)\right)^c\right) db \\ + \ell \int_0^t \left(\varphi(t) - \varphi(b)\right)^{c-1} \varphi'(b) E_{c,c}\left(-\omega\left(\varphi(t) - \varphi(b)\right)^c\right) d\mathcal{C}_b.\end{aligned} \tag{49}$$

Substituting (48) into (49), we obtain (46).

Next, integrate both sides of (46) over the interval $[0, \mathrm{y}]$. Then

$$\begin{aligned}\int_0^{\mathrm{y}} f_t\, dt = \vartheta_0 \int_0^{\mathrm{y}} \left(\varphi(t) - \varphi(0)\right)^{\mathrm{v}-1} E_{c,\mathrm{v}}\left(-\omega\left(\varphi(t) - \varphi(0)\right)^c\right) dt \\ + \nu \int_0^{\mathrm{y}} \int_0^t \left(\varphi(t) - \varphi(b)\right)^{c-1} \varphi'(b) E_{c,c}\left(-\omega\left(\varphi(t) - \varphi(b)\right)^c\right) db\, dt \\ + \ell \int_0^{\mathrm{y}} \int_0^t \left(\varphi(t) - \varphi(b)\right)^{c-1} \varphi'(b) E_{c,c}\left(-\omega\left(\varphi(t) - \varphi(b)\right)^c\right) d\mathcal{C}_b\, dt,\end{aligned}$$

which is exactly (46).

Now define

$$\mathcal{J}(\mathrm{y}) = \vartheta_0 \int_0^{\mathrm{y}} \big(\varphi(t) - \varphi(0)\big)^{\mathrm{v}-1} E_{c,\mathrm{v}}\Big(-\omega\big(\varphi(t) - \varphi(0)\big)^c\Big)\, dt + \nu \int_0^{\mathrm{y}} \int_0^t \big(\varphi(t) - \varphi(b)\big)^{c-1} \varphi'(b) E_{c,c}\Big(-\omega\big(\varphi(t) - \varphi(b)\big)^c\Big)\, db\, dt,$$

$$\mathscr{Y}(\mathrm{y}) = \int_0^{\mathrm{y}} \int_0^t \big(\varphi(t) - \varphi(b)\big)^{c-1} \varphi'(b) E_{c,c}\Big(-\omega\big(\varphi(t) - \varphi(b)\big)^c\Big)\, d\mathcal{C}_b\, dt.$$

Then

$$\int_0^{\mathrm{y}} f_t\, dt = \mathcal{J}(\mathrm{y}) + \ell \mathscr{Y}(\mathrm{y}),$$

and therefore

$$\mathcal{M} = \mathbb{E}\big[\exp\big(-\mathcal{J}(\mathrm{y}) - \ell\mathscr{Y}(\mathrm{y})\big)\big].$$

Now, we simplify $\mathscr{Y}(\mathrm{y})$ by swapping the order of integration using the Fubini theorem for Liu processes [1]:

$$\mathscr{Y}(\mathrm{y}) = \int_0^{\mathrm{y}} \int_0^t K(t,s)\, d\mathcal{C}_s\, dt = \int_0^{\mathrm{y}} \left(\int_s^{\mathrm{y}} K(t,s)\, dt \right) d\mathcal{C}_s,$$

where $K(t,s) = (\varphi(t) - \varphi(s))^{c-1} \varphi'(s) E_{c,c}\big(-\omega(\varphi(t) - \varphi(s))^c\big)$.

Define $\Psi(s) = \int_s^{\mathrm{y}} K(t,s)\, dt$. Then $\mathscr{Y}(\mathrm{y}) = \int_0^{\mathrm{y}} \Psi(s)\, d\mathcal{C}_s$.

Now, since $\mathcal{J}(\mathrm{y})$ is deterministic, we have:

$$\mathcal{M} = \exp\big(-\mathcal{J}(\mathrm{y})\big) \cdot \mathbb{E}\left[\exp\left(-\ell \int_0^{\mathrm{y}} \Psi(s)\, d\mathcal{C}_s\right)\right].$$

For a Liu process $\mathcal{C}_t$, the following identity holds for any deterministic function $h(s)$ [1]:

$$\mathbb{E}\left[\exp\left(\int_0^{\mathrm{y}} h(s)\, d\mathcal{C}_s\right)\right] = \exp\left(\int_0^{\mathrm{y}} |h(s)|\, ds\right).$$

Applying this identity with $h(s) = -\ell\Psi(s)$ yields:

$$\mathbb{E}\left[\exp\left(-\ell \int_0^{\mathrm{y}} \Psi(s)\, d\mathcal{C}_s\right)\right] = \exp\left(\ell \int_0^{\mathrm{y}} |\Psi(s)|\, ds\right).$$

Therefore,

$$\mathcal{M} = \exp\left(-\mathcal{J}(\mathrm{y}) + \ell \int_0^{\mathrm{y}} |\Psi(s)|\, ds\right).$$

Substituting back the definition of $\Psi(s)$ gives the final explicit deterministic formula:

$$\mathcal{M} = \exp\left(-\mathcal{J}(\mathrm{y}) + \ell \int_0^{\mathrm{y}} \left| \int_b^{\mathrm{y}} (\varphi(t) - \varphi(b))^{c-1} \varphi'(b) E_{c,c}\big(-\omega(\varphi(t) - \varphi(b))^c\big)\, dt \right| db\right).$$

This proves the theorem. □

Remark 4. *This formula contains no expectation operator and no Liu integral, only deterministic integrals. It is the final, computable bond price.*

Remark 5. *It is worth noting that Theorem 4 is obtained directly as an application of the general solution results established in Theorems 1and 2. The relatively concise form of the result reflects the strength and generality of the developed theoretical framework, which allows interest rate*

models of fractional uncertain type to be solved in a unified manner without requiring lengthy additional derivations.

This demonstrates that once the analytical structure of UFDEs is established, various financial models, including interest rate dynamics and bond pricing, can be derived systematically as direct consequences.

Remark 6. *Theorem 4 provides an explicit analytical expression for the price of a zero-coupon bond under the proposed uncertain fractional interest rate model, highlighting the practical relevance of the theoretical framework. This result shows that even in the presence of fractional dynamics and uncertainty, tractable pricing formulas can be derived.*

The main objective of this work is to develop the theoretical foundation of uncertain fractional models. The problems of parameter estimation, model calibration, and empirical validation using real data, while important, fall outside the scope of the present study and will be considered in future work.

In the next section, we present two examples to illustrate the theoretical results we have established regarding the Ex-Un.

4. Example

We present two examples as follows:

Example 1. *Take the following UFDE in the sense of the φ-HFD:*

$$ {}^{H}\mathcal{D}_{a}^{z,c,\varphi} f_t = \Omega(t, f_t) + \Lambda(t, f_t)\frac{d\mathcal{C}_t}{dt}, \qquad 0 < c < 1, \quad z \in [0,1], \quad t \in [a, \mathcal{U}], \tag{50} $$

with the IC

$$ \mathcal{I}_{a+}^{(1-z)(1-c),\varphi} f_t\Big|_{t=a} = \vartheta_0, \tag{51} $$

where

$$ \Omega(t, f_t) = \frac{\sin t}{1+t^2} + \frac{f_t}{1+t^2}, \qquad \Lambda(t, f_t) = e^{-t} + \frac{1}{2}\arctan(f_t). $$

For any $\rho, \theta \in \mathbb{R}$ and $t \geq a$:

$$ \begin{aligned} |\Omega(t,\rho) - \Omega(t,\theta)| &= \left| \left(\frac{\sin t}{1+t^2} + \frac{\rho}{1+t^2} \right) - \left(\frac{\sin t}{1+t^2} + \frac{\theta}{1+t^2} \right) \right| \\ &= \left| \frac{\rho - \theta}{1+t^2} \right| \\ &= \frac{1}{1+t^2}|\rho - \theta| \leq |\rho - \theta|, \end{aligned} $$

since $\frac{1}{1+t^2} \leq 1$ for all $t \geq a > 0$. Next,

$$ \begin{aligned} |\Lambda(t,\rho) - \Lambda(t,\theta)| &= \left| \left(e^{-t} + \frac{1}{2}\arctan(\rho) \right) - \left(e^{-t} + \frac{1}{2}\arctan(\theta) \right) \right| \\ &= \frac{1}{2}|\arctan(\rho) - \arctan(\theta)| \\ &\leq \frac{1}{2}|\rho - \theta|, \end{aligned} $$

where we used that arctan is 1-Lipschitz, i.e., $|\arctan(\rho) - \arctan(\theta)| \leq |\rho - \theta|$. Therefore,

$$ |\Omega(t,\rho) - \Omega(t,\theta)| + |\Lambda(t,\rho) - \Lambda(t,\theta)| \leq 1 \cdot |\rho - \theta| + \frac{1}{2}|\rho - \theta| $$

$$= \frac{3}{2}|\rho - \theta|.$$

Thus, the LC holds with constant $\chi = \frac{3}{2}$.
Next, we check the LGC. For any $f_t \in \mathbb{R}$,

$$\begin{aligned}&\big|\Omega(t, f_t)\big| + \big|\Lambda(t, f_t)\big| \\ &\leq \left|\frac{\sin t}{1+t^2}\right| + \left|\frac{f_t}{1+t^2}\right| + |e^{-t}| + \frac{1}{2}\big|\arctan(f_t)\big|.\end{aligned}$$

Using

$$\left|\frac{\sin t}{1+t^2}\right| \leq 1, \qquad |e^{-t}| \leq 1, \qquad \frac{1}{1+t^2} \leq 1, \qquad |\arctan x| \leq |x|,$$

we obtain

$$\begin{aligned}\big|\Omega(t, f_t)\big| + \big|\Lambda(t, f_t)\big| &\leq 1 + |f_t| + 1 + \frac{1}{2}|f_t| \\ &= 2 + \frac{3}{2}|f_t| \\ &\leq \frac{5}{2}(1 + |f_t|).\end{aligned}$$

Therefore, the LGC holds with $\chi = \frac{5}{2}$. Thus, both the LC and the LGC required in the Ex-Un theorem are fulfilled. Consequently, (50) together with the IC (51) admits a unique solution on $[a, \mathcal{U}]$.

Moreover, the unique solution satisfies the equivalent Volterra-type integral equation

$$\begin{aligned}f_t = \frac{\big(\varphi(t) - \varphi(a)\big)^{\mathrm{v}-1}}{\Gamma(\mathrm{v})}\vartheta_0 + \frac{1}{\Gamma(c)}\int_a^t \big(\varphi(t) - \varphi(b)\big)^{c-1}\varphi'(b)\Omega\big(b, f_b\big)\, db \\ + \frac{1}{\Gamma(c)}\int_a^t \big(\varphi(t) - \varphi(b)\big)^{c-1}\varphi'(b)\Lambda\big(b, f_b\big)\, d\mathcal{C}_b,\end{aligned}$$

where

$$\mathrm{v} = c + \mathrm{z}(1 - c).$$

Example 2. *Consider the following UFDE in the sense of the φ-HFD:*

$${}^{H}\mathcal{D}_a^{\mathrm{z},c,\varphi} f_t = \Omega(t, f_t) + \Lambda(t, f_t)\frac{d\mathcal{C}_t}{dt}, \qquad 0 < c < 1, \quad \mathrm{z} \in [0, 1], \quad t \in [a, \mathcal{U}], \tag{52}$$

with the IC

$$\mathcal{I}_{a+}^{(1-\mathrm{z})(1-c),\varphi} f_t\Big|_{t=a} = \vartheta_0, \tag{53}$$

where

$$\Omega(t, f_t) = \frac{\cos t}{2 + t^2} + \frac{f_t}{1 + |f_t|}, \qquad \Lambda(t, f_t) = \frac{1}{1 + t^2} + \frac{1}{3}\tanh(f_t).$$

For any $f_t, f_t' \in \mathbb{R}$, we have

$$\begin{aligned}&\big|\Omega(t, f_t) - \Omega(t, f_t')\big| + \big|\Lambda(t, f_t) - \Lambda(t, f_t')\big| \\ &= \left|\frac{f_t}{1 + |f_t|} - \frac{f_t'}{1 + |f_t'|}\right| + \frac{1}{3}\big|\tanh(f_t) - \tanh(f_t')\big|.\end{aligned}$$

Now the functions

$$x \mapsto \frac{x}{1 + |x|} \qquad \text{and} \qquad x \mapsto \tanh(x)$$

are globally Lipschitz on $\mathbb{R}$ with Lipschitz constants not exceeding 1. Hence

$$\left|\frac{f_t}{1+|f_t|}-\frac{f_t'}{1+|f_t'|}\right| \le |f_t - f_t'|,$$
$$|\tanh(f_t)-\tanh(f_t')| \le |f_t - f_t'|.$$

Therefore,

$$|\Omega(t,f_t)-\Omega(t,f_t')| + |\Lambda(t,f_t)-\Lambda(t,f_t')|$$
$$\le \left(1+\frac{1}{3}\right)|f_t - f_t'| = \frac{4}{3}|f_t - f_t'|.$$

Thus, the LC holds with $\chi = \frac{4}{3}$.

Next, for any $f_t \in \mathbb{R}$, we have

$$|\Omega(t,f_t)| + |\Lambda(t,f_t)|$$
$$\le \left|\frac{\cos t}{2+t^2}\right| + \left|\frac{f_t}{1+|f_t|}\right| + \frac{1}{1+t^2} + \frac{1}{3}|\tanh(f_t)|.$$

Using

$$\left|\frac{\cos t}{2+t^2}\right| \le \frac{1}{2}, \qquad \left|\frac{f_t}{1+|f_t|}\right| \le |f_t|, \qquad \frac{1}{1+t^2} \le 1, \qquad |\tanh(x)| \le 1,$$

we obtain

$$|\Omega(t,f_t)| + |\Lambda(t,f_t)| \le \frac{1}{2} + |f_t| + 1 + \frac{1}{3}$$
$$= \frac{11}{6} + |f_t|$$
$$\le 2(1+|f_t|).$$

Hence the LGC is satisfied with $\chi = 2$.

Therefore, all assumptions of the Ex-Un theorem are satisfied. Consequently, (52) together with the IC (53) admits a unique solution on $[a, \mathcal{U}]$.

Moreover, the unique solution f_t satisfies the equivalent integral equation

$$f_t = \frac{(\varphi(t)-\varphi(a))^{\mathrm{v}-1}}{\Gamma(\mathrm{v})}\vartheta_0 + \frac{1}{\Gamma(c)}\int_a^t (\varphi(t)-\varphi(b))^{c-1}\varphi'(b)\Omega(b,f_b)\,db$$
$$+ \frac{1}{\Gamma(c)}\int_a^t (\varphi(t)-\varphi(b))^{c-1}\varphi'(b)\Lambda(b,f_b)\,d\mathcal{C}_b,$$

where

$$\mathrm{v} = c + \mathrm{z}(1-c).$$

5. Conclusions

We established the result of Ex-Un for the solutions using the method of successive approximation (Theorem1). Subsequently, we developed an analytical solution for the φ-HFD uncertain system and demonstrated the sample continuity of the solution in Theorem 3.

A central contribution is the demonstration that the φ-Hilfer framework unifies two previously distinct uncertain fractional models: the Caputo-type UFDE (recovered when $\varphi(t) = t$) and the Hadamard-type UFDE (recovered when $\varphi(t) = \log t$). Moreover, this framework provides a superior operator for capturing hereditary effects by adjusting the

scale function φ, the model flexibly represents different memory decay patterns, from power law to logarithmic, offering greater expressiveness than fixed-kernel fractional models.

Applying our framework to the uncertain fractional Vasicek interest rate model, we derived in Theorem 4 an explicit analytical formula for the zero-coupon bond price. The Ex-Un results (Theorem 1) provide the necessary mathematical rigor for this pricing formula: because the UFDE admits a unique solution, the bond price is well-defined and can be expressed in closed form. This result is financially significant as it enables rigorous bond pricing under epistemic uncertainty a setting where classical stochastic calculus fails due to violations of independence and normality assumptions in expert-based or data-sparse environments.

6. Future Work

Building on the existence, uniqueness, and bond pricing results established in this paper, future research will extend the φ-Hilfer UFDE framework to more sophisticated financial models such as fractional uncertain CIR-type equations and models with state-dependent volatility or jumps, develop parameter estimation and calibration techniques for uncertain fractional systems using real market data, design numerical approximation methods including Euler-type schemes for UFDEs to enable practical simulations, and apply the proposed framework to option pricing, risk analysis, and optimal control problems under epistemic uncertainty.

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Abbreviations

The following abbreviations are used in this manuscript:

Abbreviation	Full Form
UFDE	uncertain fractional differential equation
UDE	uncertain differential equation
φ-HFD	φ-Hilfer fractional derivative
UPr	uncertain process
LC	Lipschitz condition
LGC	linear growth condition
Ex-Un	existence and uniqueness
MLF	Mittag-Leffler function

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