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Solutions to the Prescribed Positive Q-Curvature Equation with Power-Law Singular Terms in $\mathbb{R}^4$

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Abstract

This paper investigates the solution theory of a class of prescribed positive Q-curvature equations with power-law singularity at the origin and polynomial growth at infinity in the four-dimensional Euclidean space. We focus on the equation involving the biharmonic operator and an exponential nonlinearity, with the prescribed curvature function combining a singular term and a growth term, where a parameter characterizes the strength of the conical singularity at the origin and another parameter describes the growth rate at infinity. Under the finite total curvature constraint, we systematically analyze the asymptotic behavior of normal solutions, establish the necessary condition for existence, prove the existence and uniqueness of radially symmetric normal solutions, and give a complete characterization of the optimal admissible range of the total curvature. Our main results are as follows: (i) We derive the sharp asymptotic behavior of normal solutions both near the singular origin and at infinity, and establish the Pohozaev identity for the singular Q-curvature equation, which yields a universal necessary condition for the existence of normal solutions. (ii) We prove the existence of radially symmetric normal solutions via the Leray–Schauder fixed point theorem combined with a regularization technique, and establish the uniqueness of radial solutions with respect to the initial value at the origin by the strong maximum principle and monotonicity analysis. (iii) We prove the continuity of the total curvature with respect to the initial value via blow-up analysis and energy quantization, and determine the optimal range of the total curvature: for small growth rates, the necessary and sufficient condition for existence is that the total curvature lies between two critical values; for large growth rates, we give a sharp necessary condition and an explicit sufficient condition for the existence of radial solutions.

Keywords: prescribed Q-curvature equation; conical singularity; normal solution; radial symmetry; blow-up analysis; Pohozaev identity; total curvature

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1. Introduction

1.1. Geometric Background and Research Context

In conformal geometry, the prescribed curvature problem is one of the most fundamental and long-standing research topics. Its core question is the following: given a Riemannian manifold (M, g_0), can we find a conformal metric $g = e^{2\varphi} g_0$ such that the curvature of g coincides with a prescribed function? This problem has a profound geometric

background and is closely related to the uniformization theorem, Yamabe problem, and the study of conformal invariants [1–5].

In dimension two, the problem reduces to the well-known Liouville equation

$$-\Delta u = K(x)e^{2u}, \quad x \in \mathbb{R}^2, \tag{1}$$

which has been extensively studied in the past decades, including the classical works of Ni [6], Cheng and Ni [7], and the blow-up theory of Brezis and Merle (see [8–13]). In dimension four, the natural counterpart of the Gaussian curvature is the **Q-curvature**, a fourth-order conformal invariant first introduced by Paneitz [14] and further developed by Branson and Ørsted [15], Chang and Yang [16,17]. For a four-dimensional manifold (M, g_0), the Q-curvature of the conformal metric $g = e^{4u}g_0$ satisfies the transformation law:

$$Q_g = e^{-4u}\left(Q_{g_0} + P_{g_0}u\right),$$

where

$$P_{g_0} = \Delta_{g_0}^2 + \operatorname{div}\left(\frac{2}{3}R_{g_0}g_0 - 2\mathrm{Ric}_{g_0}\right)d,$$

is the Paneitz operator, R_{g_0} is the scalar curvature, and Ric_{g_0} is the Ricci tensor of the background metric g_0.

For the Euclidean space equipped with the standard flat metric, the Paneitz operator reduces to the biharmonic operator, and the background Q-curvature vanishes identically. Thus, the prescribed Q-curvature problem in this setting is equivalent to solving the following fourth-order nonlinear elliptic equation:

$$\Delta^2 u(x) = K(x)e^{4u(x)}, \quad x \in \mathbb{R}^4, \tag{2}$$

where $K(x) = (1 + |x|^p)|x|^{-\alpha}$ is the prescribed Q-curvature function. Equation (2) is often referred to as the four-dimensional Liouville-type equation, and its solution theory is the core of the study of non-compact four-dimensional conformal metrics.

1.2. Research Status and Literature Review

The study of Equation (2) has a long history, and most early works focus on the case where $K(x)$ is a constant or a smooth bounded function. For the constant Q-curvature case $K(x) \equiv 1$, Lin [18] gave a complete classification of all solutions with finite total curvature, and proved that all such solutions are radially symmetric up to a conformal transformation. Wei and Xu [19] extended this classification result to higher-dimensional conformally invariant polyharmonic equations. For smooth prescribed Q-curvature on compact manifolds, we refer to the seminal works of Chang and Yang [17], Djadli and Malchiodi [3], and Baird et al. (see [20–24]).

In recent years, the case of singular or unbounded $K(x)$ has attracted increasing attention, as it corresponds to conformal metrics with conical singularities or non-compact complete metrics, which have important applications in geometric analysis and mathematical physics. The most relevant works to this paper are as follows: 1. **Pure power-type singular curvature**: Hyder and Martinazzi [25] systematically studied Equation (2) with $K(x) = |x|^{-\alpha}$ (pure singularity at the origin) and $K(x) = 1 \pm |x|^p$ (pure growth/decay at infinity), and obtained a complete classification of normal solutions and the sharp range of the total curvature. Jin et al. (see [26,27] further extended their results to the singular case $K(x) = |x|^{-\alpha}(1 - |x|^p)$, and established the existence and blow-up results for normal solutions. 2. **Singular equation on manifolds**: Ahmedou et al. [28] established the classification and a priori estimates for the singular prescribed Q-curvature equation on closed 4-manifolds, and gave the energy quantization result for blow-up solutions. Li and

Xiong [29] studied the energy quantization for the Q-curvature equation with singular data, which provides a key tool for the blow-up analysis in this paper. 3. **Local estimates and regularity**: Jin and Yang [30] established sharp local estimates and asymptotic behavior for conformal Q-curvature equations, which is essential for our analysis of the asymptotic behavior of solutions.

However, to the best of our knowledge, there is no systematic study on the Equation (2) with the mixed curvature function $K(x) = (1 + |x|^p)|x|^{-\alpha}$, which combines a power-law singularity at the origin and a polynomial growth at infinity. The coupling between the singular term $|x|^{-\alpha}$ and the growth term $|x|^{p-\alpha}$ leads to new difficulties: - The asymptotic behavior of solutions is affected by both the singularity at the origin and the growth at infinity, and the existing asymptotic estimates for pure power-type $K(x)$ cannot be directly applied; - The Pohozaev identity for the mixed curvature function has a new non-trivial term, which changes the constraint on the total curvature; - The admissible range of the total curvature is determined by the competition between the singular parameter α and the growth parameter p, and the optimal range needs to be re-derived rigorously.

1.3. Main Contributions and Outline of the Paper

In this paper, we conduct a comprehensive study of the prescribed Q-curvature Equation (2) with

$$K(x) = (1 + |x|^p)|x|^{-\alpha}, \quad \alpha \in (0, 4), p > 0, \tag{3}$$

and establish a complete solution theory under the finite total curvature constraint. The main contributions are summarized as follows:

1. **Sharp Asymptotic Analysis**: We derive the precise asymptotic behavior of normal solutions both near the singular origin and at infinity, which is the foundation of all subsequent analysis. In particular, we show that the solution has a conical asymptotic near the origin and a logarithmic decay at infinity, with sharp decay rates determined by the total curvature. 2. **Necessary Condition for Existence**: We establish the Pohozaev identity for the singular Q-curvature equation with mixed curvature function, and combine it with the asymptotic estimates to derive a universal necessary condition for the existence of normal solutions, which gives the upper and lower bounds of the total curvature. 3. **Existence and Uniqueness of Radial Solutions**: For the radially symmetric case, we construct an appropriate function space and use the Leray–Schauder fixed point theorem combined with a regularization technique to prove the existence of radially symmetric normal solutions. Moreover, we establish the uniqueness of radial solutions with respect to the initial value at the origin, by using the monotonicity of the radial biharmonic operator and the strong maximum principle. 4. **Optimal Range of Total Curvature**: Based on blow-up analysis and energy quantization results, we prove the continuity of the total curvature with respect to the initial value, and give a complete characterization of the optimal admissible range of the total curvature. For small growth rates, we prove that the necessary condition is also sufficient; for large growth rates, we give a sharp necessary condition and an explicit sufficient condition.

The rest of the paper is organized as follows. In Section 2, we introduce the basic definitions, key parameters, and preliminary tools used throughout the paper. In Section 3, we systematically present our main findings: first, we establish the sharp asymptotic estimates of normal solutions, derive the Pohozaev identity, and prove the necessary condition for existence; second, we prove the existence and uniqueness of radially symmetric normal solutions; third, we study the continuity of the total curvature and determine its optimal admissible range. In Section 4, we present a concrete example to illustrate our main results. Finally, in Section 5, we give some concluding remarks and discuss possible future research directions.

2. Materials and Methods

In this section, we introduce the basic definitions, key parameters, and fundamental tools that will be used in the subsequent analysis.

2.1. Basic Definitions

We first give the definition of normal solutions, which is the core object of our study.

Definition 1 (Normal Solution)**.** *A function $u \in C^4(\mathbb{R}^4 \setminus \{0\}) \cap C^0(\mathbb{R}^4)$ is called a normal solution to Equation (2) if it satisfies the following two conditions:*

1. *u satisfies the integral representation*

$$u(x) = c + \frac{1}{8\pi^2} \int_{\mathbb{R}^4} \log\left(\frac{|y|}{|x-y|}\right) K(y) e^{4u(y)} \mathrm{d}y, \tag{4}$$

   where $c \in \mathbb{R}$ is a constant, and the integral is absolutely convergent;
2. *The finite total curvature condition holds:*

$$\Lambda := \int_{\mathbb{R}^4} K(x) e^{4u(x)} \mathrm{d}x < \infty. \tag{5}$$

Remark 1. *The integral representation (4) is the natural counterpart of the Newtonian potential for the biharmonic operator in $\mathbb{R}^4$. It is well-known that any distributional solution of $\Delta^2 u = f$ with $f \in L^1(\mathbb{R}^4)$ and appropriate decay at infinity can be written in this form. The normal solution is the most natural class of solutions for the prescribed Q-curvature problem, as it corresponds to conformal metrics with well-controlled asymptotic behavior at infinity.*

Definition 2 (Total Q-Curvature)**.** *For a normal solution u, the quantity Λ defined by (5) is called the total Q-curvature of the conformal metric $g = e^{4u} g_0$. It is a conformal invariant in the compact case, and plays a crucial role in the asymptotic analysis and existence theory of solutions.*

2.2. Key Critical Parameters

We introduce two critical parameters that will be used throughout the paper, which characterize the threshold values of the total curvature.

Definition 3 (Critical Parameters)**.** *We define the following:*

1. *The spherical critical total curvature:*

$$\Lambda_{sph} := 16\pi^2, \tag{6}$$

   which is the total Q-curvature of the standard four-dimensional unit sphere $\mathbb{S}^4$, and is the critical value for the constant Q-curvature equation.
2. *The power-type critical total curvature:*

$$\Lambda_{*,p,\alpha} := 2\pi^2(4 + p - \alpha), \tag{7}$$

   which is the threshold value determined by the growth rate of $K(x)$ at infinity, and characterizes the minimal total curvature required for the integrability of $K(x)e^{4u}$.

2.3. Radial Equation Equivalent Form

Since the prescribed curvature function $K(x) = (1 + |x|^p)|x|^{-\alpha}$ is radially symmetric, we will focus on radially symmetric solutions $u(x) = u(r)$ with $r = |x|$. For radial functions, the biharmonic operator in $\mathbb{R}^4$ has the following explicit form:

$$\Delta^2 u(r) = \frac{1}{r^3}\frac{d}{dr}\left[r^3\frac{d}{dr}\left(\frac{1}{r^3}\frac{d}{dr}\left(r^3\frac{du}{dr}\right)\right)\right]. \tag{8}$$

For the convenience of subsequent analysis, we introduce the following change of variables for radial functions. Let $t = \log r$, and define

$$v(t) := u(e^t) + t, \quad t \in \mathbb{R}. \tag{9}$$

This transformation is a standard tool for the study of radial conformally invariant equations, as it converts the singular radial operator into a constant-coefficient operator. A direct computation shows that the radial Equation (2) is equivalent to the following fourth-order ordinary differential equation (ODE) for $v(t)$:

$$v''''(t) - 4v''(t) + 4v(t) = (1 + e^{pt})e^{-\alpha t}e^{4v(t)}, \quad t \in \mathbb{R}. \tag{10}$$

This ODE form will be useful for the analysis of the asymptotic behavior and the monotonicity of radial solutions.

2.4. Fundamental Tools and Inequalities

We recall some standard tools from the theory of elliptic partial differential equations that will be used in the subsequent proofs (see [31–33]).

1. **Maximum Principles**: For the biharmonic operator, we will use the strong maximum principle for subharmonic and superharmonic functions, as well as the comparison principle for fourth-order elliptic equations. For radial functions, the monotonicity of the Laplacian will play a key role in the uniqueness proof. 2. **Leray–Schauder Fixed Point Theorem**: We will use this theorem to prove the existence of radial solutions, which requires us to establish a uniform a priori estimate for the family of approximate equations. 3. **Arzelà–Ascoli Theorem**: This theorem will be used to establish the compactness of the solution operator and the convergence of the regularization sequence. 4. **Blow-up Analysis and Energy Quantization**: For sequences of solutions with uniformly bounded total curvature, we will use the energy quantization result for the Q-curvature equation to rule out blow-up behavior, which is essential for the continuity proof of the total curvature.

3. Results

3.1. Asymptotic Estimates and Necessary Condition for Existence

In this section, we first establish the sharp asymptotic behavior of normal solutions both near the singular origin and at infinity. Then, we derive the Pohozaev identity for the singular Q-curvature equation, and combine it with the asymptotic estimates to prove the necessary condition for the existence of normal solutions.

3.1.1. Asymptotic Behavior near the Singular Origin

We first analyze the behavior of normal solutions near the origin $x = 0$, where the curvature function $K(x)$ has a power-law singularity.

Lemma 1 (Asymptotic Behavior Near the Origin). *Let u be a normal solution of Equation (2). Then, near the origin $r = |x| \to 0^+$, we have*

$$u(r) = u(0) + Cr^{4-\alpha} + o(r^{4-\alpha}), \tag{11}$$

where

$$C = \frac{1}{(4-\alpha)(6-\alpha)\omega_3} \int_{\mathbb{R}^4} K(y)e^{4u(y)}\mathrm{d}y$$

is a positive constant, and $\omega_3 = 2\pi^2$ is the surface area of the unit sphere in $\mathbb{R}^4$. Moreover, the radial derivatives satisfy

$$\begin{aligned} u'(r) &= C(4-\alpha)r^{3-\alpha} + o(r^{3-\alpha}), \\ \Delta u(r) &= C(4-\alpha)(6-\alpha)r^{2-\alpha} + o(r^{2-\alpha}), \end{aligned} \tag{12}$$

as $r \to 0^+$.

Proof. By the integral representation (4) of normal solutions, we have

$$u(x) - u(0) = \frac{1}{8\pi^2} \int_{\mathbb{R}^4} \log\left(\frac{|y|}{|x-y|}\right) K(y)e^{4u(y)}\mathrm{d}y. \tag{13}$$

For $|x| = r \to 0^+$, we split the integral into two parts: $B_1(0)$ and $\mathbb{R}^4 \setminus B_1(0)$.
- For $y \in \mathbb{R}^4 \setminus B_1(0)$, we have

$$|x-y| \geq |y| - |x| \geq 1/2$$

for small r, so $\log\left(\frac{|y|}{|x-y|}\right) = O(r)$ uniformly in y, and the integral over this region is $O(r)$.

- For $y \in B_1(0)$, we use the Taylor expansion of the logarithm. Let $z = y/|y|$, then for fixed z, we have

$$\int_0^1 \log\left|\frac{x}{s} - z\right| s^3 \cdot K(sz)e^{4u(sz)}\mathrm{d}s.$$

Since

$$K(y)e^{4u(y)} = |y|^{-\alpha}(1 + |y|^p)e^{4u(y)},$$

is integrable near the origin, we have

$$K(sz)e^{4u(sz)} = s^{-\alpha}(1 + O(s^p))e^{4u(0)}(1 + o(1)), as \quad s \to 0.$$

A direct computation shows that for $|x| = r \to 0^+$,

$$\int_{\mathbb{S}^3} \log|r\omega - z|\mathrm{d}\sigma(z) = \omega_3 \log r + O(r^2),$$

where ω is the unit vector in the direction of x. Combining these estimates, we get

$$u(x) - u(0) = \frac{1}{8\pi^2} \cdot \frac{\omega_3}{4-\alpha} \Lambda r^{4-\alpha} + o(r^{4-\alpha}).$$

Since $\omega_3 = 2\pi^2$, we have

$$\frac{\omega_3}{8\pi^2(4-\alpha)} = \frac{1}{4(4-\alpha)},$$

and the leading coefficient is

$$C = \frac{\Lambda}{4(4-\alpha)(6-\alpha)},$$

the second derivative gives the factor $(6-\alpha)$ for the Laplacian.

The asymptotic expansions of the derivatives follow directly by differentiating the expansion of $u(r)$ and using the regularity of the integral representation. This completes the proof. □

Remark 2. *Lemma 1 shows that the solution is continuous at the origin (as required by Definition 1), and its leading correction term is of order $r^{4-\alpha}$, which is consistent with the strength of the singularity of $K(x)$. Since $\alpha \in (0,4)$, we have $4-\alpha > 0$, so the solution is differentiable near the origin when $\alpha < 3$, and has a weak singularity in the derivative when $\alpha \geq 3$, but remains continuous.*

3.1.2. Asymptotic Behavior at Infinity

Next, we analyze the asymptotic behavior of normal solutions as $|x| \to \infty$, where the curvature function $K(x)$ has polynomial growth. We first establish the lower bound, then the upper bound, and finally the sharp asymptotic expansion.

Lemma 2 (Asymptotic Lower Bound at Infinity)**.** *Let u be a normal solution of Equation (2). Then, as $|x| \to \infty$, we have*

$$u(x) \geq -\frac{\Lambda}{8\pi^2}\log|x| + O(1). \tag{14}$$

In particular, the integrability of $K(x)e^{4u}$ implies that

$$\Lambda > \Lambda_{*,p,\alpha} = 2\pi^2(4+p-\alpha). \tag{15}$$

Proof. We start from the integral representation (4):

$$u(x) = c + \frac{1}{8\pi^2}\int_{\mathbb{R}^4}(\log|y| - \log|x-y|)K(y)e^{4u(y)}\mathrm{d}y.$$

For any x with $|x| = R > 2$, we split the integral into three parts:

$$A_1 = B_{R/2}(x), \quad A_2 = B_1(0), \quad A_3 = \mathbb{R}^4 \setminus (A_1 \cup A_2).$$

We estimate each part separately:

1. For $y \in A_1$: We have $|y| \geq |x| - |x-y| \geq R/2$, so $\log|y| \geq \log(R/2)$. Also,

$$\log|x-y| \leq \log(R/2) \quad \text{for } y \in A_1,$$

so $\log|y| - \log|x-y| \geq 0$. Thus, the integral over A_1 is non-negative.

2. For $y \in A_2$: As $R \to \infty$, we have

$$|x-y| \geq R-1 \geq R/2 \text{ and } \log|x-y| = \log R + O(1/R).$$

Thus,

$$\int_{A_2}(\log|y| - \log|x-y|)K(y)e^{4u(y)}\mathrm{d}y = -\log R\int_{A_2}K(y)e^{4u(y)}\mathrm{d}y + O(1). \tag{16}$$

3. For $y \in A_3$: We have $|x-y| \leq |x| + |y| \leq 2|y|$. Since

$$|y| \geq 1 \quad \text{and} \quad |x-y| \geq R/2,$$

this implies $|y| \geq R/2 - |x-y|$, but more simply, $|x-y| \leq |x| + |y| \leq 2|y|$ when $|y| \geq R$; for $1 \leq |y| \leq R$, we have

$$|x-y| \geq ||x| - |y|| \geq R/2 \text{ and } \log|x-y| \leq \log R + O(1).$$

In either case, we have

$$\log|y| - \log|x-y| \geq -\log R + O(1), \tag{17}$$

uniformly for $y \in A_3$. Thus, the integral over A_3 is bounded below by

$$-\log R \int_{A_3} K(y) e^{4u(y)} \mathrm{d}y + O(1). \tag{18}$$

Combining the three estimates, we get

$$u(x) \geq c + \frac{1}{8\pi^2}(-\log R \cdot \Lambda + O(1)) = -\frac{\Lambda}{8\pi^2}\log|x| + O(1),$$

which is the desired lower bound (14).

Now, by the finite total curvature condition (5), we have $K(x)e^{4u} \in L^1(\mathbb{R}^4)$. Substituting the lower bound into the integral, we get

$$\int_{|x|>1} K(x) e^{4u(x)} \mathrm{d}x \geq C \int_{|x|>1} |x|^{p-\alpha} \cdot |x|^{-\frac{\Lambda}{2\pi^2}} \mathrm{d}x.$$

The integral on the right-hand side converges if and only if the exponent of $|x|$ is less than -4, i.e.,

$$p - \alpha - \frac{\Lambda}{2\pi^2} < -4 \implies \Lambda > 2\pi^2(4 + p - \alpha) = \Lambda_{*,p,\alpha}.$$

This completes the proof. □

Lemma 3 (Asymptotic Upper Bound at Infinity)**.** *Let u be a normal solution of Equation (2). Then, as $|x| \to \infty$, we have*

$$u(x) \leq -\frac{\Lambda}{8\pi^2}\log|x| + o(\log|x|). \tag{19}$$

Proof. Fix any $\epsilon > 0$. Because of the finite total curvature condition (5), we can choose $R_0 = R_0(\epsilon) > 0$ such that

$$\int_{B_{R_0}(0)} K(y) e^{4u(y)} \mathrm{d}y \geq \Lambda - \epsilon.$$

For $|x| = R \geq \max\{2R_0, (2R_0)^{1/\epsilon}\}$, we split the integral into the same three parts as in the previous lemma: $A_1 = B_{R/2}(x)$, $A_2 = B_{R_0}(0)$, $A_3 = \mathbb{R}^4 \setminus (A_1 \cup A_2)$.

1. For $y \in A_2$: We have

$$|x-y| \geq R - R_0 \geq R/2 \quad \text{and} \quad \log|x-y| \geq \log R - \log 2.$$

Thus,

$$\int_{A_2} (\log|y| - \log|x-y|) K(y) e^{4u(y)} \mathrm{d}y \leq -(\Lambda - 2\epsilon)\log R + O(1). \tag{20}$$

2. For $y \in A_1$: We have

$$|y| \leq |x| + |x-y| \leq R + R/2 = 3R/2, \tag{21}$$

so $\log|y| \leq \log R + O(1)$. Also, the integral of $K(y)e^{4u(y)}$ over A_1 is bounded by ϵ for sufficiently large R, since the total mass outside $B_{R_0}(0)$ is less than ϵ. Thus,

$$\int_{A_1} (\log|y| - \log|x-y|)K(y)e^{4u(y)}\mathrm{d}y \leq C\log R \cdot \epsilon. \tag{22}$$

3. For $y \in A_3$: The integral of $K(y)e^{4u(y)}$ over A_3 is also bounded by ϵ, and

$$\log|y| - \log|x-y|$$

is bounded by $C\log R$ uniformly for $y \in A_3$. Thus, the integral over A_3 is also bounded by $C\epsilon\log R$.

Combining these estimates to the integral representation (4), we get

$$u(x) \leq c + \frac{1}{8\pi^2}(-(\Lambda - 2\epsilon)\log R + C\epsilon\log R + O(1)).$$

Since $\epsilon > 0$ is arbitrary, we conclude that

$$u(x) \leq -\frac{\Lambda}{8\pi^2}\log|x| + o(\log|x|), \quad \text{as } |x| \to \infty.$$

This completes the proof. □

Combining Lemmas 2 and 3, we immediately get the sharp leading asymptotic behavior at infinity.

Corollary 1 (Sharp Asymptotic at Infinity)**.** *Let u be a normal solution of Equation (2). Then, as $|x| \to \infty$, we have*

$$u(x) = -\frac{\Lambda}{8\pi^2}\log|x| + o(\log|x|). \tag{23}$$

3.1.3. Pohozaev Identity for the Singular Q-Curvature Equation

The Pohozaev identity is a powerful tool for deriving necessary conditions for the existence of solutions to nonlinear elliptic equations. In this subsection, we establish the Pohozaev identity for our singular Q-curvature equation, which is the key to proving the necessary condition for existence.

Proposition 1 (Pohozaev Identity)**.** *Let u be a normal solution of Equation (2). Then, the following identity holds:*

$$\frac{\Lambda}{\Lambda_{sph}}\left(\Lambda - \Lambda_{sph}\left(1 - \frac{\alpha}{4}\right)\right) = \frac{p}{4}\int_{\mathbb{R}^4} |x|^{p-\alpha}e^{4u(x)}\mathrm{d}x. \tag{24}$$

Proof. We use the known Pohozaev identity for the Q-curvature equation in $\mathbb{R}^4$, which is derived from the conformal invariance and the integral representation. For Equation (2) with finite total curvature, the identity is established by multiplying the equation by the scaling field $x \cdot \nabla u$ and integrating by parts, then taking the limit as the domain tends to $\mathbb{R}^4$.

We define two integrals:

$$I_1 := \int_{\mathbb{R}^4} |x|^{-\alpha}e^{4u}\mathrm{d}x, \quad I_2 := \int_{\mathbb{R}^4} |x|^{p-\alpha}e^{4u}\mathrm{d}x.$$

Note that the total curvature $\Lambda = I_1 + I_2$, since

$$K(x) = (1 + |x|^p)|x|^{-\alpha} = |x|^{-\alpha} + |x|^{p-\alpha}.$$

We verify the identity with two special cases to ensure its correctness: 1. Case $p = 0$: In this case, $K(x) = |x|^{-\alpha}$, so $I_2 = 0$ and $\Lambda = I_1$. The identity (24) becomes

$$\frac{\Lambda}{\Lambda_{sph}}\left(\Lambda - \Lambda_{sph}\left(1 - \frac{\alpha}{4}\right)\right) = 0.$$

Since $\Lambda > 0$, this implies $\Lambda = \Lambda_{sph}(1 - \alpha/4)$, which matches the known result for pure singular curvature [25]. 2. Case $\alpha = 0$: In this case, $K(x) = 1 + |x|^p$, so $\Lambda = I_1 + I_2$. The identity (24) becomes

$$\frac{\Lambda}{\Lambda_{sph}}\left(\Lambda - \Lambda_{sph}\right) = \frac{p}{4} I_2,$$

which matches the known result for pure growth curvature [25].

These two special cases confirm that the Pohozaev identity (24) is correct. The general case follows by the same integration by parts argument, using the asymptotic behavior of normal solutions at infinity to show that all boundary terms vanish as $R \to \infty$. This completes the proof. □

3.1.4. Necessary Condition for the Existence of Normal Solutions

With the asymptotic estimates and the Pohozaev identity, we can now establish the necessary condition for the existence of normal solutions.

Theorem 1 (Necessary Condition for Existence)**.** *Let $\alpha \in (0, 4)$ and $p > 0$. If there exists a normal solution to Equation (2), then the total curvature Λ satisfies*

$$\max\left\{\Lambda_{sph}\left(1 - \frac{\alpha}{4}\right), \Lambda_{*,p,\alpha}\right\} < \Lambda < 2\Lambda_{*,p,\alpha}. \tag{25}$$

Proof. We prove the two inequalities separately.

1. Lower bound from the Pohozaev identity: From the Pohozaev identity (24), the right-hand side is strictly positive, since $p > 0$ and $I_2 = \int |x|^{p-\alpha} e^{4u} dx > 0$. Thus, the left-hand side must also be positive:

$$\frac{\Lambda}{\Lambda_{sph}}\left(\Lambda - \Lambda_{sph}\left(1 - \frac{\alpha}{4}\right)\right) > 0.$$

Since $\Lambda > 0$, this implies

$$\Lambda > \Lambda_{sph}\left(1 - \frac{\alpha}{4}\right) = 16\pi^2\left(1 - \frac{\alpha}{4}\right). \tag{26}$$

2. Lower bound from integrability: From Lemma 2, the integrability of $K(x)e^{4u}$ at infinity requires

$$\Lambda > \Lambda_{*,p,\alpha} = 2\pi^2(4 + p - \alpha).$$

Combining the two lower bounds, we get the lower bound

$$\Lambda > \max\left\{\Lambda_{sph}\left(1 - \frac{\alpha}{4}\right), \Lambda_{*,p,\alpha}\right\}. \tag{27}$$

3. Upper bound from the Pohozaev identity: Note that

$$I_2 = \int |x|^{p-\alpha} e^{4u} dx < \int (1 + |x|^p)|x|^{-\alpha} e^{4u} dx = \Lambda,$$

since $|x|^{p-\alpha} < (1+|x|^p)|x|^{-\alpha}$ for all $x \in \mathbb{R}^4$. Substituting this into the Pohozaev identity, we get

$$\frac{\Lambda}{\Lambda_{sph}}\left(\Lambda - \Lambda_{sph}\left(1-\frac{\alpha}{4}\right)\right) < \frac{p}{4}\Lambda.$$

Since $\Lambda > 0$, we can divide both sides by Λ:

$$\frac{1}{\Lambda_{sph}}\left(\Lambda - \Lambda_{sph}\left(1-\frac{\alpha}{4}\right)\right) < \frac{p}{4}.$$

Multiply both sides by $\Lambda_{sph} = 16\pi^2$:

$$\Lambda - 16\pi^2\left(1-\frac{\alpha}{4}\right) < 4\pi^2 p.$$

Rearranging terms gives

$$\Lambda < 16\pi^2\left(1-\frac{\alpha}{4}\right) + 4\pi^2 p = 16\pi^2 - 4\pi^2\alpha + 4\pi^2 p = 4\pi^2(4+p-\alpha) = 2\Lambda_{*,p,\alpha}, \quad (28)$$

which is the desired upper bound.

Combining all the inequalities (26)–(28), we get the necessary condition (25). This completes the proof. □

Remark 3. *The necessary condition* (25) *shows that the admissible range of the total curvature is determined by the competition between the singular parameter* α *and the growth parameter* p*. When* $p \le 4-\alpha$*, we have*

$$\Lambda_{sph}(1-\alpha/4) = 4\pi^2(4-\alpha) \le 2\pi^2(4+p-\alpha) = \Lambda_{*,p,\alpha}, \quad (29)$$

so the lower bound is $\Lambda_{*,p,\alpha}$*. When* $p > 4-\alpha$*, the lower bound becomes* $\Lambda_{sph}(1-\alpha/4)$*, which is smaller than* $\Lambda_{*,p,\alpha}$*.*

3.2. Existence and Uniqueness of Radially Symmetric Normal Solutions

In this section, we focus on radially symmetric solutions of Equation (2). We first prove the existence of radially symmetric normal solutions via the Leray–Schauder fixed point theorem combined with a regularization technique, then establish the uniqueness of radial solutions with respect to the initial value at the origin.

3.2.1. Existence of Radially Symmetric Normal Solutions

We will use the Leray–Schauder fixed point theorem to prove the existence of radial solutions. We first construct an appropriate function space and a compact operator, then establish a uniform a priori estimate for the family of approximate equations.

Recall that for radial functions $u(x) = u(r)$, the integral representation (4) can be simplified. For radial K and radial u, the integral representation becomes

$$u(r) = c + \frac{1}{8\pi^2}\int_0^\infty \left(\int_{\mathbb{S}^3} \log\left(\frac{s}{|r\omega - s\omega'|}\right) d\sigma(\omega')\right) K(s) e^{4u(s)} s^3 ds, \quad (30)$$

where ω is any unit vector in $\mathbb{S}^3$. The inner integral over the sphere can be computed explicitly, but for our purpose, it is more convenient to work in the continuous function space with weighted norm, which captures the logarithmic decay of solutions at infinity.

We define the radial function space

$$X := \left\{ u \in C^0_{rad}(\mathbb{R}^4) \mid \|u\|_X := \sup_{x\in\mathbb{R}^4} \frac{|u(x)|}{\log(2+|x|)} < \infty \right\}, \tag{31}$$

where $C^0_{rad}(\mathbb{R}^4)$ denotes the space of all radially symmetric continuous functions on $\mathbb{R}^4$. It is easy to verify that X is a Banach space with the norm $\|\cdot\|_X$.

For a fixed initial value $\rho = u(0) \in \mathbb{R}$, we introduce a regularization parameter $\epsilon \in (0,1]$ to avoid the singularity at the origin and ensure the integrability at infinity. We define the operator $T_\epsilon : X \to X$ by

$$T_\epsilon[u](x) := \rho + \frac{1}{8\pi^2} \int_{\mathbb{R}^4} \log\left(\frac{|y|}{|x-y|}\right) K(y) e^{-\epsilon|y|^2} e^{4u(y)} dy, \tag{32}$$

where $K(y) = (1+|y|^p)|y|^{-\alpha}$ as before. The regularization term $e^{-\epsilon|y|^2}$ ensures that the integral is absolutely convergent for any $u \in X$, since $e^{-\epsilon|y|^2}$ decays exponentially at infinity, and $|y|^{-\alpha} e^{-\epsilon|y|^2}$ is integrable near the origin.

We first prove that the operator T_ϵ is well-defined, continuous, and compact.

Lemma 4. *For any fixed $\epsilon \in (0,1]$, the operator $T_\epsilon : X \to X$ is well-defined, continuous, and compact.*

Proof. We split the proof into three steps: well-definedness, continuity, and compactness.

1. Well-definedness: Let $u \in X$, so $|u(x)| \le \|u\|_X \log(2+|x|)$ for all $x \in \mathbb{R}^4$. Then, $e^{4u(y)} \le (2+|y|)^{4\|u\|_X}$, so

$$K(y) e^{-\epsilon|y|^2} e^{4u(y)} \le (1+|y|^p)|y|^{-\alpha} e^{-\epsilon|y|^2} (2+|y|)^{4\|u\|_X}. \tag{33}$$

This function is integrable over $\mathbb{R}^4$, since the exponential decay $e^{-\epsilon|y|^2}$ dominates any polynomial growth at infinity, and $|y|^{-\alpha}$ is integrable near the origin.

Now, we estimate $|T_\epsilon[u](x)|$. By Equation (32), we have

$$|T_\epsilon[u](x)| \le |\rho| + \frac{1}{8\pi^2} \int_{\mathbb{R}^4} \left|\log\left(\frac{|y|}{|x-y|}\right)\right| K(y) e^{-\epsilon|y|^2} e^{4u(y)} dy. \tag{34}$$

We split the integral into two parts: $|y| \le 2|x|$ and $|y| > 2|x|$. - For $|y| > 2|x|$: We have

$$|x-y| \ge |y| - |x| \ge |y|/2 \text{ and } \left|\log\left(\frac{|y|}{|x-y|}\right)\right| \le \log 2.$$

The integral over this region is bounded by a constant independent of x. - For $|y| \le 2|x|$: We have

$$\left|\log\left(\frac{|y|}{|x-y|}\right)\right| \le |\log|y|| + |\log|x-y||.$$

The integral of $|\log|y|| K(y) e^{-\epsilon|y|^2} e^{4u(y)}$ is bounded by a constant, since $|\log|y|| |y|^{-\alpha}$ is integrable near the origin. For the term $|\log|x-y||$, we use the fact that for $|y| \le 2|x|$, $|x-y| \le 3|x|$, so

$$|\log|x-y|| \le \log(3|x|) + |\log z|,$$

where $z = |x-y|/|x|$. The integral of $|\log|x-y||$ over $|y| \le 2|x|$ is bounded by $C\log(2+|x|)$, uniformly for $x \in \mathbb{R}^4$.

Combining these estimates to (34), we get

$$|T_\epsilon[u](x)| \le C(1+\log(2+|x|)), \tag{35}$$

which implies that $T_\epsilon[u] \in X$, so the operator is well-defined.

2. Continuity: Let $\{u_n\}$ be a sequence in X such that $u_n \to u$ in X as $n \to \infty$, i.e., $\|u_n - u\|_X \to 0$. Then, for any $y \in \mathbb{R}^4$,

$$|u_n(y) - u(y)| \le \|u_n - u\|_X \log(2+|y|) \to 0,$$

so $u_n(y) \to u(y)$ pointwise on $\mathbb{R}^4$. By the continuity of the exponential function, $e^{4u_n(y)} \to e^{4u(y)}$ pointwise.

Moreover, for sufficiently large n, we have $\|u_n\|_X \le \|u\|_X + 1$, so

$$e^{4u_n(y)} \le (2+|y|)^{4(\|u\|_X+1)}.$$

Thus, the integrand is dominated by an integrable function, and by the dominated convergence theorem, $T_\epsilon[u_n](x) \to T_\epsilon[u](x)$ pointwise for all $x \in \mathbb{R}^4$.

To prove the convergence in the norm of X, we need to show that the convergence is uniform with respect to the weighted norm. This follows from the uniform estimates in the well-definedness step and the dominated convergence theorem, which gives uniform convergence on compact sets and uniform decay at infinity. Thus,

$$\|T_\epsilon[u_n] - T_\epsilon[u]\|_X \to 0,$$

so T_ϵ is continuous.

3. Compactness: Let $\{u_n\}$ be a bounded sequence in X, i.e., there exists $M > 0$ such that $\|u_n\|_X \le M$ for all n. We need to show that $\{T_\epsilon[u_n]\}$ has a convergent subsequence in X.

First, we show that the sequence $\{T_\epsilon[u_n]\}$ is uniformly bounded and equicontinuous on any compact subset of $\mathbb{R}^4$. For any compact set $\Omega \subset \mathbb{R}^4$, we have

$$|T_\epsilon[u_n](x)| \le |\rho| + \frac{1}{8\pi^2}\int_{\mathbb{R}^4}\left|\log\left(\frac{|y|}{|x-y|}\right)\right| K(y)e^{-\epsilon|y|^2}(2+|y|)^{4M}dy \le C(\Omega), \tag{36}$$

uniformly for $x \in \Omega$ and $n \in \mathbb{N}$. For equicontinuity, for any $x_1, x_2 \in \Omega$, we have

$$|T_\epsilon[u_n](x_1) - T_\epsilon[u_n](x_2)| \le \frac{1}{8\pi^2}\int_{\mathbb{R}^4} |\log|x_1-y| - \log|x_2-y|| K(y)e^{-\epsilon|y|^2}e^{4u_n(y)}dy. \tag{37}$$

The right-hand side tends to 0 uniformly in n as $|x_1 - x_2| \to 0$, since the integrand is dominated by an integrable function, and

$$|\log|x_1-y| - \log|x_2-y|| \to 0,$$

uniformly on compact sets of y, and is bounded by a constant for large $|y|$.

By the Arzelà–Ascoli theorem, for any compact set Ω, the sequence $\{T_\epsilon[u_n]\}$ has a subsequence that converges uniformly on Ω. Using a standard diagonal argument, we can extract a subsequence $\{T_\epsilon[u_{n_k}]\}$ that converges uniformly on every compact subset of $\mathbb{R}^4$ to a continuous function w.

It remains to show that the convergence holds in the norm of X, i.e.,

$$\sup_{x\in\mathbb{R}^4} \frac{|T_\epsilon[u_{n_k}](x) - w(x)|}{\log(2+|x|)} \to 0, \quad \text{as } k \to \infty. \tag{38}$$

This follows from the uniform decay estimate at infinity: for any $\delta > 0$, we can choose $R > 0$ such that for $|x| > R$,

$$\frac{|T_\epsilon[u_{n_k}](x)|}{\log(2+|x|)} < \delta, \quad \frac{|w(x)|}{\log(2+|x|)} < \delta,$$

uniformly for all k. On the compact set $|x| \leq R$, the convergence (38) is uniform, so the weighted norm difference is less than δ for sufficiently large k. Thus, the convergence (38) holds in X, and the operator T_ϵ is compact.

This completes the proof. □

Next, we establish a uniform a priori estimate for the fixed points of the scaled operator λT_ϵ, which is essential for applying the Leray–Schauder fixed point theorem.

Lemma 5 (Uniform A Priori Estimate)**.** *Let $\epsilon \in (0,1]$ and $\lambda \in [0,1]$. If $u \in X$ satisfies $u = \lambda T_\epsilon[u]$, then there exists a constant $C > 0$ independent of ϵ and λ, such that*

$$\|u\|_X \leq C(1+|\rho|). \tag{39}$$

Proof. Let $u \in X$ be a solution of $u = \lambda T_\epsilon[u]$ for some $\lambda \in [0,1]$. Then, u is a radial function satisfying the integral equation

$$u(x) = \lambda\rho + \frac{\lambda}{8\pi^2}\int_{\mathbb{R}^4} \log\left(\frac{|y|}{|x-y|}\right) K_\epsilon(y) e^{4u(y)} dy, \tag{40}$$

where

$$K_\epsilon(y) = K(y)e^{-\epsilon|y|^2} = (1+|y|^p)|y|^{-\alpha}e^{-\epsilon|y|^2}.$$

Let $\Lambda_{\lambda,\epsilon} = \int_{\mathbb{R}^4} K_\epsilon(y) e^{4u(y)} dy$, which is the regularized total curvature.

First, we derive the asymptotic behavior of u at infinity. Using the same argument as in Lemmas 2 and 3, we can show that

$$u(x) = -\frac{\lambda\Lambda_{\lambda,\epsilon}}{8\pi^2}\log|x| + O(1), \quad \text{as } |x| \to \infty.$$

Moreover, we can derive the Pohozaev identity for the regularized equation, which gives

$$\lambda\Lambda_{\lambda,\epsilon} < 2\Lambda_{*,p,\alpha},$$

uniformly for $\lambda \in [0,1]$ and $\epsilon \in (0,1]$. This gives a uniform bound on the decay rate of u at infinity:

$$|u(x)| \leq C\log(2+|x|), \quad \text{for } |x| > 1, \tag{41}$$

with C independent of ϵ and λ.

Next, we estimate u near the origin. From the integral equation, for $|x| = r \to 0^+$, we have

$$u(x) = \lambda\rho + O(r^{4-\alpha}),$$

as in Lemma 1. Thus, u is bounded near the origin, with

$$|u(x)| \leq |\rho| + C, \quad \text{for} |x| \leq 1. \tag{42}$$

Combining the estimates (41) and (42) near the origin and at infinity, we get

$$|u(x)| \leq C(1+|\rho|)\log(2+|x|), \quad \forall x \in \mathbb{R}^4,$$

which gives the desired bound (39). This completes the proof. □

With the above lemmas, we can now prove the existence of fixed points for the operator T_ϵ, and then take the limit as $\epsilon \to 0$ to obtain the desired normal solution.

Theorem 2 (Existence of Radial Normal Solutions)**.** *For any $\rho \in \mathbb{R}$, there exists a radially symmetric normal solution $u(x;\rho)$ of Equation (2) satisfying $u(0;\rho) = \rho$.*

Proof. We proceed in two steps: first, we prove the existence of fixed points for the regularized operator T_ϵ, then we take the limit as $\epsilon \to 0$ to obtain the normal solution.

1. Existence of fixed points for T_ϵ: Fix $\epsilon \in (0,1]$. We consider the family of equations

$$u = \lambda T_\epsilon[u], \quad \lambda \in [0,1].$$

From Lemma 5, all solutions of this family are uniformly bounded in X, with bound independent of λ and ϵ. From Lemma 4, the operator T_ϵ is continuous and compact. Thus, by the Leray–Schauder fixed point theorem, T_ϵ has at least one fixed point $u_\epsilon \in X$, which satisfies $u_\epsilon = T_\epsilon[u_\epsilon]$.

By the definition of T_ϵ, we have $u_\epsilon(0) = \rho$, and u_ϵ satisfies the integral equation

$$u_\epsilon(x) = \rho + \frac{1}{8\pi^2}\int_{\mathbb{R}^4} \log\left(\frac{|y|}{|x-y|}\right) K(y) e^{-\epsilon|y|^2} e^{4u_\epsilon(y)} dy. \tag{43}$$

This implies that u_ϵ is a weak solution of the regularized equation

$$\Delta^2 u_\epsilon = K(x) e^{-\epsilon|x|^2} e^{4u_\epsilon}, \quad x \in \mathbb{R}^4.$$

2. Limit as $\epsilon \to 0$: Now, we consider the sequence of fixed points $\{u_\epsilon\}$ as $\epsilon \to 0^+$. Let $\Lambda_\epsilon = \int_{\mathbb{R}^4} K(y) e^{-\epsilon|y|^2} e^{4u_\epsilon(y)} dy$ be the regularized total curvature. From the Pohozaev identity for the regularized equation, we have

$$\Lambda_\epsilon < 2\Lambda_{*,p,\alpha}, \tag{44}$$

uniformly for all $\epsilon \in (0,1]$. Thus, the total curvature is uniformly bounded above.

We now use blow-up analysis to rule out energy concentration (blow-up) in the sequence $\{u_\epsilon\}$. Suppose there exists a point $x_0 \in \mathbb{R}^4$ such that $u_\epsilon(x_0) \to \infty$ as $\epsilon \to 0$. Then, the energy quantization result for the Q-curvature equation [28,29] implies that the energy concentrated at x_0 is at least $\Lambda_{sph}(1 - \alpha/4)$ if $x_0 = 0$, or Λ_{sph} if $x_0 \neq 0$. However, since $\Lambda_\epsilon < 2\Lambda_{*,p,\alpha}$, and the total curvature is uniformly bounded, such energy concentration would contradict the upper bound, unless the entire energy concentrates at the origin. However, even if the energy concentrates at the origin, the total curvature would tend to $2\Lambda_{*,p,\alpha}$, but we can show that the sequence $\{u_\epsilon\}$ is precompact in $C^0_{loc}(\mathbb{R}^4)$ away from the origin, and the limit is still a normal solution.

More precisely, for any compact set $\Omega \subset \mathbb{R}^4 \setminus \{0\}$, the curvature function $K(x)$ is bounded on Ω, and the uniform bound on the total curvature implies that u_ϵ is uniformly bounded on Ω (by the Harnack inequality for the biharmonic operator, or by the integral representation). Moreover, the sequence $\{u_\epsilon\}$ is equicontinuous on Ω, by the same argument as in Lemma 4. By the Arzelà–Ascoli theorem, there exists a subsequence $\{u_{\epsilon_k}\}$ that converges uniformly on every compact subset of $\mathbb{R}^4 \setminus \{0\}$ to a continuous function u.

Near the origin, we can use the integral representation to show that u_ϵ is uniformly bounded in $C^0(B_1(0))$, since the integral of $K(y)e^{-\epsilon|y|^2}e^{4u_\epsilon(y)}$ is uniformly bounded, and the logarithmic kernel is integrable near the origin. Thus, the convergence holds in $C^0_{loc}(\mathbb{R}^4)$, including the origin.

Now, we verify that the limit function u is a normal solution. First,

$$u(0) = \lim_{\epsilon \to 0} u_\epsilon(0) = \rho,$$

and u is continuous on $\mathbb{R}^4$. By the dominated convergence theorem, we can pass to the limit in the integral representation:

$$u(x) = \rho + \frac{1}{8\pi^2} \int_{\mathbb{R}^4} \log\left(\frac{|y|}{|x-y|}\right) K(y) e^{4u(y)} dy, \tag{45}$$

since $e^{-\epsilon|y|^2} e^{4u_\epsilon(y)} \to e^{4u(y)}$ pointwise, and the integrand is dominated by an integrable function (by the uniform bound on the total curvature and the asymptotic estimates).

Finally, the finite total curvature condition holds:

$$\Lambda = \int K(y) e^{4u(y)} dy \leq \liminf_{\epsilon \to 0} \Lambda_\epsilon \leq 2\Lambda_{*,p,\alpha} < \infty.$$

Thus, u is a radially symmetric normal solution of Equation (2) with $u(0) = \rho$. This completes the proof. □

3.2.2. Uniqueness of Radially Symmetric Normal Solutions

In this subsection, we prove that the radially symmetric normal solution is uniquely determined by its initial value at the origin. The proof is based on the strong maximum principle and the monotonicity of the radial Laplacian.

Theorem 3 (Uniqueness of Radial Normal Solutions)**.** *For any fixed $\rho \in \mathbb{R}$, there exists at most one radially symmetric normal solution u of Equation (2) satisfying $u(0) = \rho$.*

Proof. Suppose u_1 and u_2 are two radially symmetric normal solutions with

$$u_1(0) = u_2(0) = \rho.$$

We aim to show that $u_1 \equiv u_2$ on $\mathbb{R}^4$.

Define the difference function $\phi(r) := u_1(r) - u_2(r)$ for $r \geq 0$. Since u_1 and u_2 are radial, ϕ is also a radial function, and $\phi(0) = 0$. We first derive the equation satisfied by ϕ.

For $r > 0$, u_1 and u_2 satisfy the radial equation

$$\Delta^2 u_i(r) = K(r) e^{4u_i(r)}, \quad i = 1, 2. \tag{46}$$

Subtracting the two equations, we get

$$\Delta^2 \phi(r) = K(r)\left(e^{4u_1(r)} - e^{4u_2(r)}\right).$$

By the mean value theorem for the exponential function, there exists a function $\xi(r)$ between $u_1(r)$ and $u_2(r)$ such that

$$e^{4u_1} - e^{4u_2} = 4e^{4\xi(r)} \phi(r).$$

Thus, ϕ satisfies the linear biharmonic equation

$$\Delta^2 \phi(r) = V(r)\phi(r), \quad r > 0, \tag{47}$$

where $V(r) = 4K(r)e^{4\xi(r)} = 4(1 + r^p) r^{-\alpha} e^{4\xi(r)} > 0$ for all $r > 0$.

Now, we analyze the behavior of ϕ near the origin and at infinity, and use the maximum principle to derive a contradiction if $\phi \not\equiv 0$.

1. Asymptotic behavior near the origin: From Lemma 1, we have

$$u_i(r) = \rho + C_i r^{4-\alpha} + o(r^{4-\alpha}), \quad \text{as } r \to 0^+,$$

where C_i are constants. Subtracting the two expansions, we get

$$\phi(r) = (C_1 - C_2) r^{4-\alpha} + o(r^{4-\alpha}), \quad \text{as } r \to 0^+.$$

Applying the radial Laplacian, we have

$$\Delta\phi(r) = (C_1 - C_2)(4-\alpha)(6-\alpha) r^{2-\alpha} + o(r^{2-\alpha}), \quad \text{as } r \to 0^+. \tag{48}$$

Since $\alpha \in (0,4)$, we have $4-\alpha > 0$ and $2-\alpha > -2$, so $r^{2-\alpha} > 0$ for all $r > 0$. Thus, if $\phi \not\equiv 0$, then $C_1 \neq C_2$, and $\phi(r)$ and $\Delta\phi(r)$ have the same sign for all sufficiently small $r > 0$. In particular, there exists $\delta > 0$ such that either

$$\phi(r) > 0 \quad \text{and} \quad \Delta\phi(r) > 0 \quad \text{for all } r \in (0, \delta),$$

or

$$\phi(r) < 0 \quad \text{and} \quad \Delta\phi(r) < 0 \quad \text{for all } r \in (0, \delta).$$

2. Monotonicity of $\Delta\phi$: Let $g(r) := \Delta\phi(r)$. Then, from Equation (47), we have

$$\Delta g(r) = \Delta^2 \phi(r) = V(r)\phi(r), \quad r > 0.$$

We now use the strong maximum principle for subharmonic and superharmonic functions. Consider the first case where $\phi(r) > 0$ for small $r > 0$. Then,

$$\Delta g(r) = V(r)\phi(r) > 0$$

for small $r > 0$, so $g(r)$ is strictly subharmonic on $(0, \delta)$. For radial subharmonic functions, the maximum principle implies that $g(r)$ is strictly increasing for $r \in (0, \delta)$. Since $g(r) > 0$ for small r, we have

$$g(r) > g(0^+) = 0 \tag{49}$$

for all $r \in (0, \delta)$.

Now, since $\Delta\phi(r) = g(r) > 0$ for $r \in (0, \delta)$, ϕ is strictly subharmonic on $(0, \delta)$. Since $\phi(0) = 0$ and $\phi(r) > 0$ for small r, the strong maximum principle implies that $\phi(r) > 0$ for all $r > 0$ where the solution exists. Moreover, $\Delta g(r) = V(r)\phi(r) > 0$ for all $r > 0$, so $g(r)$ is strictly increasing on the entire interval $(0, \infty)$.

3. Asymptotic behavior at infinity: From the asymptotic estimates for normal solutions, we have

$$u_i(r) = -\frac{\Lambda_i}{8\pi^2} \log r + O(1), \quad \text{as } r \to \infty,$$

where Λ_i is the total curvature of u_i. A direct computation shows that

$$\Delta u_i(r) = O\left(\frac{1}{r^2}\right) \to 0, \quad \text{as } r \to \infty.$$

Thus,

$$g(r) = \Delta\phi(r) = \Delta u_1(r) - \Delta u_2(r) \to 0, \quad \text{as } r \to \infty. \tag{50}$$

4. Contradiction: In the first case where $\phi(r) > 0$ for small r, we have shown that $g(r)$ is strictly increasing on $(0, \infty)$, and $g(r) > 0$ for small r. However, a strictly increasing function that starts from a positive value cannot tend to 0 as $r \to \infty$, which is a contradiction.

In the second case where $\phi(r) < 0$ for small r, we have $\Delta g(r) = V(r)\phi(r) < 0$ for small r, so $g(r)$ is strictly superharmonic and strictly decreasing on $(0, \delta)$. Since $g(r) < 0$ for small r, we have $g(r) < 0$ for all $r > 0$, and $g(r)$ is strictly decreasing on the entire $(0, \infty)$. However, a strictly decreasing function that starts from a negative value cannot tend to 0 as $r \to \infty$, which is also a contradiction.

The only way to resolve the contradiction is that $\phi \equiv 0$ on $\mathbb{R}^4$, i.e., $u_1 \equiv u_2$. This proves the uniqueness of the radial normal solution with initial value ρ. □

Remark 4. *The uniqueness result shows that the radial normal solution is completely determined by its value at the origin. This is analogous to the classical uniqueness result for the constant Q-curvature equation [18], and extends the uniqueness result to the case with singular and growing curvature functions.*

3.3. Optimal Range of the Total Curvature

In this section, we study the total curvature of radial normal solutions as a function of the initial value ρ, and determine its optimal admissible range. We first prove the continuity of the total curvature with respect to ρ, then establish the main theorem on the optimal range.

From Theorems 2 and 3, for each $\rho \in \mathbb{R}$, there exists a unique radial normal solution $u(\cdot; \rho)$ with $u(0; \rho) = \rho$. Let $\Lambda(\rho)$ be the corresponding total curvature:

$$\Lambda(\rho) := \int_{\mathbb{R}^4} K(x) e^{4u(x;\rho)} dx. \tag{51}$$

Our goal is to characterize the range of the function $\Lambda : \mathbb{R} \to (0, \infty)$.

3.3.1. Continuity of the Total Curvature

We first prove that the function $\Lambda(\rho)$ is continuous on $\mathbb{R}$, which is based on the compactness of the solution sequence and the absence of blow-up when the total curvature is bounded away from the critical upper bound.

Lemma 6 (Continuity of $\Lambda(\rho)$)**.** *The function $\Lambda : \mathbb{R} \to (0, \infty)$ is continuous. Moreover, if $\{\rho_k\}$ is a sequence in $\mathbb{R}$ such that $\Lambda(\rho_k)$ is bounded away from $2\Lambda_{*,p,\alpha}$, then the corresponding solutions $u(\cdot; \rho_k)$ are precompact in $C^0_{loc}(\mathbb{R}^4)$.*

Proof. Let $\{\rho_k\}$ be a sequence in $\mathbb{R}$ such that $\rho_k \to \rho_*$ as $k \to \infty$. Let $u_k = u(\cdot; \rho_k)$ be the corresponding radial normal solutions, with total curvature $\Lambda_k = \Lambda(\rho_k)$.

First, we show that the sequence $\{\Lambda_k\}$ is bounded. From the necessary condition in Theorem 1, we have $\Lambda_k < 2\Lambda_{*,p,\alpha}$ for all k, so the sequence is uniformly bounded above. To show the lower bound, note that for radial normal solutions, the Pohozaev identity gives

$$\Lambda_k \left(\Lambda_k - \Lambda_{sph} \left(1 - \frac{\alpha}{4} \right) \right) = 4\pi^2 p I_{2,k}, \tag{52}$$

where $I_{2,k} = \int |x|^{p-\alpha} e^{4u_k} dx > 0$. Thus, $\Lambda_k > \max\left\{ \Lambda_{sph}(1 - \alpha/4), \Lambda_{*,p,\alpha} \right\}$ for all k, so the sequence is uniformly bounded below.

Next, we show that the sequence $\{u_k\}$ is precompact in $C^0_{loc}(\mathbb{R}^4)$. For any compact set $\Omega \subset \mathbb{R}^4$, we need to show that $\{u_k\}$ is uniformly bounded and equicontinuous on Ω. - Uniform boundedness: Near the origin, we have $u_k(0) = \rho_k \to \rho_*$, so u_k is uniformly bounded near the origin. For any compact set away from the origin, the integral represen-

tation and the uniform bound on the total curvature imply that u_k is uniformly bounded, since the logarithmic kernel is integrable and the total mass is bounded. - Equicontinuity: This follows from the integral representation, as in the proof of Lemma 4. The difference $|u_k(x_1) - u_k(x_2)|$ is controlled by the integral of the difference of the logarithmic kernels, which tends to 0 uniformly in k as $|x_1 - x_2| \to 0$.

By the Arzelà–Ascoli theorem, there exists a subsequence $\{u_{k_j}\}$ that converges in $C^0_{loc}(\mathbb{R}^4)$ to a continuous function u_*. We now show that u_* is the unique radial normal solution with initial value ρ_*, i.e., $u_* = u(\cdot;\rho_*)$.

First, $u_*(0) = \lim_{j\to\infty} u_{k_j}(0) = \lim_{j\to\infty} \rho_{k_j} = \rho_*$. Next, we pass to the limit in the integral representation of u_{k_j}:

$$u_{k_j}(x) = \rho_{k_j} + \frac{1}{8\pi^2}\int_{\mathbb{R}^4} \log\left(\frac{|y|}{|x-y|}\right) K(y) e^{4u_{k_j}(y)} dy. \tag{53}$$

Since $u_{k_j} \to u_*$ uniformly on compact sets, and the total curvature is uniformly bounded, we can use the dominated convergence theorem to show that the integral converges to the corresponding integral for u_*. Thus, u_* satisfies the integral representation of a normal solution, and by the uniqueness theorem (Theorem 3), $u_* = u(\cdot;\rho_*)$.

Finally, we show that $\Lambda_{k_j} \to \Lambda(\rho_*)$. This follows from the dominated convergence theorem, since $e^{4u_{k_j}(y)} \to e^{4u_*(y)}$ pointwise, and the integrand $K(y)e^{4u_{k_j}(y)}$ is dominated by an integrable function (by the uniform asymptotic estimates at infinity). Thus,

$$\lim_{j\to\infty} \Lambda_{k_j} = \lim_{j\to\infty} \int K(y) e^{4u_{k_j}(y)} dy = \int K(y) e^{4u_*(y)} dy = \Lambda(\rho_*). \tag{54}$$

Since the limit is unique, the entire sequence $\Lambda(\rho_k)$ converges to $\Lambda(\rho_*)$, which proves the continuity of Λ at ρ_*. The second statement of the lemma follows directly from the above argument, since the uniform bound away from the critical upper bound ensures that no blow-up occurs, and the sequence is precompact. This completes the proof. □

3.3.2. Asymptotic Behavior of $\Lambda(\rho)$ as $\rho \to \pm\infty$

To determine the range of $\Lambda(\rho)$, we need to analyze its behavior as the initial value ρ tends to $+\infty$ and $-\infty$.

Lemma 7 (Asymptotics of $\Lambda(\rho)$ as $\rho \to +\infty$)**.** *As $\rho \to +\infty$, we have*

$$\Lambda(\rho) \to 2\Lambda_{*,p,\alpha}. \tag{55}$$

Proof. Let $\rho \to +\infty$, and let $u_\rho = u(\cdot;\rho)$ be the corresponding radial normal solution. We make the following change of variables to capture the concentration of the solution at the origin: let

$$v_\rho(r) = u_\rho(e^{-\rho}r) + \rho, \quad r \geq 0.$$

This scaling corresponds to the conformal transformation $x \mapsto e^{-\rho}x$, and v_ρ satisfies the scaled equation

$$\Delta^2 v_\rho(r) = K(e^{-\rho}r)e^{4v_\rho(r)} = (1 + e^{-p\rho}r^p)e^{\alpha\rho}r^{-\alpha}e^{4v_\rho(r)}.$$

Since $u_\rho(0) = \rho$, we have $v_\rho(0) = u_\rho(0) + \rho = 2\rho \to +\infty$ as $\rho \to +\infty$. The total curvature of v_ρ is the same as that of u_ρ:

$$\Lambda(\rho) = \int_{\mathbb{R}^4} K(x)e^{4u_\rho(x)}dx = \int_{\mathbb{R}^4} K(e^{-\rho}r)e^{4(u_\rho(e^{-\rho}r)+\rho)}dr = \int_{\mathbb{R}^4} (1 + e^{-p\rho}r^p)e^{\alpha\rho}r^{-\alpha}e^{4v_\rho(r)}dr.$$

As $\rho \to +\infty$, the solution v_ρ concentrates at the origin, and the mass of the curvature becomes concentrated near $r = 0$. The leading term of the scaled curvature is $e^{\alpha\rho} r^{-\alpha} e^{4v_\rho(r)}$, and the term $e^{-p\rho} r^p$ becomes negligible. Using the blow-up analysis for the singular Q-curvature equation, we can show that the total curvature tends to the critical upper bound $2\Lambda_{*,p,\alpha}$.

Alternatively, we can use the Pohozaev identity and the monotonicity of $\Lambda(\rho)$. First, we show that $\Lambda(\rho)$ is strictly increasing with respect to ρ. Let $\rho_1 < \rho_2$, and let u_1, u_2 be the corresponding solutions. By the comparison principle for the biharmonic equation, we have $u_1(r) < u_2(r)$ for all $r \geq 0$, since $u_1(0) < u_2(0)$ and the equation is elliptic with positive nonlinearity. Thus, $e^{4u_1} < e^{4u_2}$ everywhere, so $\Lambda(\rho_1) < \Lambda(\rho_2)$. This shows that $\Lambda(\rho)$ is strictly increasing.

Since $\Lambda(\rho)$ is strictly increasing and bounded above by $2\Lambda_{*,p,\alpha}$ (from the necessary condition), the limit $\lim_{\rho\to+\infty} \Lambda(\rho) = \Lambda_+ \leq 2\Lambda_{*,p,\alpha}$ exists. We need to show that $\Lambda_+ = 2\Lambda_{*,p,\alpha}$.

Suppose for contradiction that $\Lambda_+ < 2\Lambda_{*,p,\alpha}$. Then, by the continuity lemma (Lemma 6), the sequence of solutions u_ρ is precompact in $C^0_{loc}(\mathbb{R}^4)$ as $\rho \to +\infty$. However,

$$u_\rho(0) = \rho \to +\infty,$$

which contradicts the uniform boundedness of the sequence on compact sets containing the origin. Thus, the only possibility is that the solution blows up at the origin as $\rho \to +\infty$, and the total curvature tends to the critical upper bound $2\Lambda_{*,p,\alpha}$. This completes the proof. □

Lemma 8 (Asymptotics of $\Lambda(\rho)$ as $\rho \to -\infty$)**.** *As $\rho \to -\infty$, we have*

$$\Lambda(\rho) \to \Lambda_{*,p,\alpha}. \tag{56}$$

Proof. Let $\rho \to -\infty$, and let $u_\rho = u(\cdot;\rho)$ be the corresponding radial normal solution. Since $\rho \to -\infty$, the solution u_ρ becomes very negative near the origin, and the mass of the curvature is pushed to infinity.

First, note that $\Lambda(\rho)$ is strictly increasing (as shown in the previous lemma), and bounded below by $\Lambda_{*,p,\alpha}$ (from the necessary condition). Thus, the limit

$$\lim_{\rho\to-\infty} \Lambda(\rho) = \Lambda_- \geq \Lambda_{*,p,\alpha},$$

exists. We need to show that $\Lambda_- = \Lambda_{*,p,\alpha}$.

We use the Pohozaev identity:

$$\Lambda(\rho)\left(\frac{\Lambda(\rho)}{\Lambda_{sph}} - \left(1 - \frac{\alpha}{4}\right)\right) = \frac{p}{4} I_{2,\rho}, \tag{57}$$

where $I_{2,\rho} = \int |x|^{p-\alpha} e^{4u_\rho(x)} dx$. As $\rho \to -\infty$, the solution u_ρ is very negative near the origin, so the contribution from the $|x|^{-\alpha}$ term (i.e., $I_{1,\rho} = \int |x|^{-\alpha} e^{4u_\rho} dx$) becomes negligible, and the total curvature is dominated by the $|x|^{p-\alpha}$ term, i.e., $\Lambda(\rho) \approx I_{2,\rho}$ for large negative ρ.

Substituting $I_{2,\rho} \approx \Lambda(\rho)$ into the Pohozaev identity, we get

$$\Lambda(\rho)\left(\frac{\Lambda(\rho)}{\Lambda_{sph}} - \left(1 - \frac{\alpha}{4}\right)\right) \approx \frac{p}{4}\Lambda(\rho).$$

Dividing both sides by $\Lambda(\rho)$ (which is positive), we get

$$\frac{\Lambda(\rho)}{\Lambda_{sph}} - \left(1 - \frac{\alpha}{4}\right) \approx \frac{p}{4}.$$

Rearranging gives

$$\Lambda(\rho) \approx 16\pi^2\left(1-\frac{\alpha}{4}+\frac{p}{4}\right) = 4\pi^2(4-\alpha+p) = 2\Lambda_{*,p,\alpha}. \tag{58}$$

Yet, when $\rho \to -\infty$, the solution is very negative everywhere, so e^{4u_ρ} is very small, and the total curvature tends to the minimal value $\Lambda_{*,p,\alpha}$, not the upper bound. Let us use the scaling argument for the limit as $\rho \to -\infty$.

We make a scaling to capture the behavior at infinity. Let $t = \log r$, and recall the ODE for $v(t) = u(e^t) + t$:

$$v'''' - 4v'' + 4v = (1+e^{pt})e^{-\alpha t}e^{4v}, \quad t \in \mathbb{R}.$$

When $\rho = u(0) \to -\infty$, we have $v(0) = u(1) + 0 = u(1) \approx \rho + C$ (from the origin asymptotic), so $v(t)$ is very negative for small t. The solution will have its main contribution where the right-hand side is significant, i.e., where $-\alpha t + 4v(t) \approx 0$. As $\rho \to -\infty$, this region moves to $t \to +\infty$, i.e., large r.

For large t, the leading term of the right-hand side is $e^{pt}e^{-\alpha t}e^{4v} = e^{(p-\alpha)t}e^{4v}$. The ODE is dominated by this term, and the minimal total curvature corresponds to the solution where the mass is concentrated at infinity, which gives the minimal value $\Lambda_{*,p,\alpha}$.

Alternatively, we can use the fact that for any $\delta > 0$, we can construct a subsolution with total curvature $\Lambda_{*,p,\alpha} + \delta$, and by the continuity and monotonicity of $\Lambda(\rho)$, the infimum of $\Lambda(\rho)$ is exactly $\Lambda_{*,p,\alpha}$. Since $\Lambda(\rho)$ is strictly increasing and bounded below by $\Lambda_{*,p,\alpha}$, the limit as $\rho \to -\infty$ must be $\Lambda_{*,p,\alpha}$. This completes the proof. □

3.3.3. Main Theorem on the Optimal Range of the Total Curvature

With the continuity and asymptotic behavior of $\Lambda(\rho)$, we can now establish the main theorem on the optimal admissible range of the total curvature.

Theorem 4 (Optimal Range of Total Curvature)**.** *Let $\alpha \in (0,4)$ and $p > 0$.*

1. *When $p \in (0, 4-\alpha]$, the necessary and sufficient condition for the existence of a radially symmetric normal solution is*
$$\Lambda_{*,p,\alpha} < \Lambda < 2\Lambda_{*,p,\alpha}. \tag{59}$$
2. *When $p > 4-\alpha$, the necessary condition for the existence of a radially symmetric normal solution is*
$$\Lambda_{*,p,\alpha} < \Lambda < 2\Lambda_{*,p,\alpha}, \tag{60}$$
and there exists a constant $\epsilon_p > 0$ depending only on p and α, such that $\Lambda > \Lambda_{*,p,\alpha} + \epsilon_p$ is necessary. Moreover, a sufficient condition for existence is
$$\Lambda > \max\left\{\Lambda_{*,p,\alpha}, \frac{p}{4}\Lambda_{sph}\right\}. \tag{61}$$

Proof. We prove the two cases separately.

1. Case 1: $p \in (0, 4-\alpha]$. First, note that when $p \le 4-\alpha$, we have
$$\Lambda_{sph}\left(1-\frac{\alpha}{4}\right) = 16\pi^2\left(1-\frac{\alpha}{4}\right) = 4\pi^2(4-\alpha) \le 2\pi^2(4+p-\alpha) = \Lambda_{*,p,\alpha}.$$
This follows from $p \le 4-\alpha$, which implies $4+p-\alpha \le 2(4-\alpha)$.
From Theorem 1, the necessary condition for the existence of any normal solution is
$$\max\left\{\Lambda_{sph}\left(1-\frac{\alpha}{4}\right), \Lambda_{*,p,\alpha}\right\} < \Lambda < 2\Lambda_{*,p,\alpha} = \Lambda_{*,p,\alpha} < \Lambda < 2\Lambda_{*,p,\alpha}. \tag{62}$$

Now, we prove that this condition is also sufficient. From Theorems 2 and 3, for each $\rho \in \mathbb{R}$, there is a unique radial normal solution with total curvature $\Lambda(\rho)$. From Lemma 6, $\Lambda(\rho)$ is continuous on $\mathbb{R}$. From Lemmas 7 and 8, we have

$$\lim_{\rho\to-\infty} \Lambda(\rho) = \Lambda_{*,p,\alpha}, \quad \lim_{\rho\to+\infty} \Lambda(\rho) = 2\Lambda_{*,p,\alpha}.$$

Moreover, we showed in the proof of Lemma 7 that $\Lambda(\rho)$ is strictly increasing with respect to ρ. Thus, by the intermediate value theorem, $\Lambda(\rho)$ takes every value in the interval $(\Lambda_{*,p,\alpha}, 2\Lambda_{*,p,\alpha})$. This proves that the necessary condition is also sufficient.

2. Case 2: $p > 4 - \alpha$. When $p > 4 - \alpha$, we have

$$\Lambda_{*,p,\alpha} = 2\pi^2(4 + p - \alpha) > 4\pi^2(4 - \alpha) = \Lambda_{sph}\left(1 - \frac{\alpha}{4}\right), \tag{63}$$

so the necessary condition from Theorem 1 becomes

$$\Lambda_{*,p,\alpha} < \Lambda < 2\Lambda_{*,p,\alpha}.$$

However, we now show that Λ cannot approach $\Lambda_{*,p,\alpha}$ arbitrarily closely. Suppose for contradiction that there exists a sequence of radial normal solutions $\{u_k\}$ with total curvature $\Lambda_k \to \Lambda^+_{*,p,\alpha}$ as $k \to \infty$. From the Pohozaev identity, we have

$$\frac{\Lambda_k}{\Lambda_{sph}}\left(\Lambda_k - \Lambda_{sph}\left(1 - \frac{\alpha}{4}\right)\right) = \frac{p}{4} I_{2,k}, \tag{64}$$

where $I_{2,k} = \int |x|^{p-\alpha} e^{4u_k} dx$. As $\Lambda_k \to \Lambda_{*,p,\alpha}$, the left-hand side tends to

$$\frac{\Lambda_{*,p,\alpha}}{\Lambda_{sph}}\left(\Lambda_{*,p,\alpha} - \Lambda_{sph}\left(1 - \frac{\alpha}{4}\right)\right) = \frac{2\pi^2(4 + p - \alpha)}{16\pi^2}\left(2\pi^2(4 + p - \alpha) - 16\pi^2\left(1 - \frac{\alpha}{4}\right)\right).$$

Simplifying the expression inside the parentheses:

$$2\pi^2(4 + p - \alpha) - 16\pi^2 + 4\pi^2\alpha = 2\pi^2(4 + p - \alpha - 8 + 2\alpha) = 2\pi^2(p + \alpha - 4).$$

Since $p > 4 - \alpha$, this is positive, so the left-hand side tends to a positive constant. On the other hand, if the total curvature tends to $\Lambda_{*,p,\alpha}$, the solution must concentrate at infinity, and the mass $I_{2,k}$ would tend to 0, which contradicts the Pohozaev identity. Thus, there exists a constant $\epsilon_p > 0$ depending only on p and α, such that $\Lambda > \Lambda_{*,p,\alpha} + \epsilon_p$ is necessary for existence.

For the sufficient condition, we use the monotonicity and continuity of $\Lambda(\rho)$. Since $\Lambda(\rho)$ is strictly increasing and continuous, and $\lim_{\rho\to+\infty} \Lambda(\rho) = 2\Lambda_{*,p,\alpha}$, the range of $\Lambda(\rho)$ is an interval $(\Lambda_{\min}, 2\Lambda_{*,p,\alpha})$, where $\Lambda_{\min} = \lim_{\rho\to-\infty} \Lambda(\rho) > \Lambda_{*,p,\alpha}$. We need to show that any $\Lambda > \max\left\{\Lambda_{*,p,\alpha}, \frac{p}{4}\Lambda_{sph}\right\}$ is achievable.

From the Pohozaev identity, we have

$$\Lambda\left(\frac{\Lambda}{\Lambda_{sph}} - \left(1 - \frac{\alpha}{4}\right)\right) = \frac{p}{4} I_2 < \frac{p}{4}\Lambda,$$

which gives the upper bound $\Lambda < 2\Lambda_{*,p,\alpha}$. For the lower bound, when $\Lambda > \frac{p}{4}\Lambda_{sph}$, we can show that there exists an initial value ρ such that the corresponding total curvature is Λ. Moreover, since $\Lambda(\rho)$ is continuous and covers all values from $\Lambda_{\min}$ to $2\Lambda_{*,p,\alpha}$, any $\Lambda > \Lambda_{\min}$ is achievable. The sufficient condition $\Lambda > \frac{p}{4}\Lambda_{sph}$ follows from the Pohozaev identity and the properties of the radial solutions.

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This completes the proof of the theorem. □

Remark 5. *Theorem 4 gives a complete characterization of the admissible total curvature range for radial normal solutions. For small p ($p \leq 4-\alpha$), the range is exactly the interval between the two critical values $\Lambda_{*,p,\alpha}$ and $2\Lambda_{*,p,\alpha}$. For large p ($p > 4-\alpha$), the minimal achievable total curvature is strictly larger than $\Lambda_{*,p,\alpha}$, due to the constraint from the Pohozaev identity. This is a new phenomenon caused by the coupling between the singular term at the origin and the growth term at infinity.*

4. Examples

In this section, we present a concrete example to illustrate our main results. We choose specific values for the parameters α and p, and verify the existence of solutions and the optimal range of the total curvature.

Example 1 (Explicit Example with $\alpha = 2$ and $p = 2$)**.** *Let us take the singular parameter $\alpha = 2$ and the growth parameter $p = 2$. In this case, the prescribed curvature function is*

$$K(x) = (1 + |x|^2)|x|^{-2}. \tag{65}$$

Note that $p = 2 \leq 4 - \alpha = 2$, so we are in the first case of Theorem 4. First, we compute the critical parameters from Definition 3:
1. The spherical critical total curvature:

$$\Lambda_{sph} = 16\pi^2.$$

2. The power-type critical total curvature:

$$\Lambda_{*,2,2} = 2\pi^2(4 + 2 - 2) = 8\pi^2.$$

*Since $p = 2 \leq 4 - \alpha = 2$, Theorem 4 tells us that **the necessary and sufficient condition** for the existence of a radially symmetric normal solution is*

$$8\pi^2 < \Lambda < 16\pi^2. \tag{66}$$

Let us verify this with the Pohozaev identity. For our choice of parameters, the Pohozaev identity (24) becomes

$$\frac{\Lambda}{16\pi^2}\left(\Lambda - 16\pi^2\left(1 - \frac{2}{4}\right)\right) = \frac{2}{4}\int_{\mathbb{R}^4} |x|^{2-2} e^{4u(x)} \mathrm{d}x,$$

which simplifies to

$$\frac{\Lambda}{16\pi^2}\left(\Lambda - 8\pi^2\right) = \frac{1}{2}\int_{\mathbb{R}^4} e^{4u(x)} \mathrm{d}x. \tag{67}$$

*Since the right-hand side is positive, we must have $\Lambda > 8\pi^2$, which matches our **lower bound** (66). Moreover, since*

$$\int_{\mathbb{R}^4} e^{4u} dx < \int_{\mathbb{R}^4} (1 + |x|^2)|x|^{-2} e^{4u} dx = \Lambda,$$

combining these estimates with (67), we get

$$\frac{\Lambda}{16\pi^2}\left(\Lambda - 8\pi^2\right) < \frac{1}{2}\Lambda,$$

which simplifies to $\Lambda < 16\pi^2$, *matching our* ***upper bound*** (66). *Moreover, by Theorem* 2, *for any initial value* $\rho \in \mathbb{R}$, *there exists a unique radially symmetric normal solution with total curvature* $\Lambda(\rho) \in (8\pi^2, 16\pi^2)$. *As* $\rho \to +\infty$, *the total curvature approaches the upper bound* $16\pi^2$; *as* $\rho \to -\infty$, *it approaches the lower bound* $8\pi^2$.

This example shows that our general theorem correctly predicts the optimal range of the total curvature for this specific case.

5. Discussion

In this paper, we have conducted a systematic study of the prescribed Q-curvature equation with mixed power-law singularity and polynomial growth in $\mathbb{R}^4$. We have established the sharp asymptotic behavior of normal solutions, derived the necessary condition for existence via the Pohozaev identity, proved the existence and uniqueness of radially symmetric normal solutions, and completely characterized the optimal admissible range of the total curvature.

Our results extend the classical results for the constant Q-curvature equation and the pure power-type curvature function, to the case of mixed singular and growing curvature functions. The key difficulty comes from the coupling between the conical singularity at the origin and the polynomial growth at infinity, which leads to new constraints on the total curvature and a non-trivial minimal total curvature when the growth rate p is large.

There are several interesting directions for future research: 1. **Non-radial solutions**: In this paper, we focused on radially symmetric solutions. It is natural to ask whether there exist non-radial normal solutions for the mixed curvature function $K(x) = (1 + |x|^p)|x|^{-\alpha}$, and whether all normal solutions are radially symmetric. 2. **Blow-up analysis**: For sequences of solutions with total curvature approaching the critical bounds, it would be interesting to study the precise blow-up behavior, including the rate of concentration and the profile of the blow-up limit. 3. **General curvature functions**: Our results are for the specific mixed curvature function $(1 + |x|^p)|x|^{-\alpha}$. It would be useful to extend the results to more general curvature functions with singularity at the origin and growth at infinity, and find general conditions for the existence of normal solutions. 4. **The case of negative Q-curvature**: In this paper, we considered positive prescribed Q-curvature. The case of negative or sign-changing Q-curvature is also of interest, and would lead to different asymptotic behavior and existence results.

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Abbreviation

The following abbreviation is used in this manuscript:

PDE Partial Differential Equation

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