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Envelopes of One-Parameter Family of Frontals Related to Two Classes of Legendrian Dualities

Pengfei Zhang [1], Zhigang Wang [2,*] and Yudi Zhang [3,*]

[1] School of Teacher Education, Harbin Normal University, Harbin 150025, China; zhangpengfei313@hrbnu.edu.cn
[2] School of Mathematical Sciences, Harbin Normal University, Harbin 150025, China
[3] Primary Education Department, Liaoyuan Branch of Jilin Normal University, Liaoyuan 136200, China
* Correspondence: wangzg2003205@163.com (Z.W.); zhangydm2019@163.com (Y.Z.)

Abstract

Giving the definitions of one-parameter family of frontals in lightcone and one-parameter family of spacelike Legendrian curves in Δ_2 and Δ_3, and further using the variability condition and the tangency condition, the definitions of envelopes of these geometric objects are presented. The aim of this work is to explore the criterion conditions on the envelopes of a one-parameter family of frontals related to Δ_2 and Δ_3 in three pseudo-spheres. Thereby the characterizations of these envelopes are described via Envelope Theorems. The geometric relations among these envelopes are discussed in detail. It is demonstrated that the Δ_2-duality or the Δ_3-duality of one-parameter family of frontals among three pseudo-spheres leads to the fact that the one-parameter family of frontals that are Δ_2-duality or Δ_3-duality each other share the same envelope. In addition, the hyperbolic and de Sitter parallels of the one-parameter family of spacelike frontals are also defined, and the existence conditions of the envelopes of such parallels are investigated correspondingly. Finally, two examples are provided to understand the theoretical results.

Keywords: frontals; envelope; Legendrian duality; singularity

MSC: 57R45; 53A35; 53B30; 53D10; 57R17

1. Introduction

The concept of an envelope, understood as a curve that makes tangential contact with each member of a prescribed one-parameter family, stands as a cornerstone of classical differential geometry [1–8]. Its practical relevance spans a wide array of disciplines, including the design of gears, the verification of CNC machining processes, signal processing, collision-avoidance algorithms in robotics, and error analysis in CAD systems. This broad applicability has sparked interest from researchers in astrophysics, particle physics, and geometry, prompting investigations into envelope behavior within non-Euclidean settings such as hyperbolic, de Sitter, and Minkowski spaces [8–11].

The field of singularity theory has undergone significant evolution in recent decades, broadening its focus from regular manifolds to include singular objects known as frontals, and extending its ambient spaces from Euclidean to non-Euclidean frameworks. This expansion furnishes powerful instruments for probing the geometric structure of manifolds. Inspired by the foundational work of V. I. Arnold, a substantial body of research has been dedicated to the study of fronts through the prism of singularity theory; key contributions are documented in [5,9,12–18] and the references therein, while regular planar curves

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possess well-defined tangents, the appearance of singularities introduces considerable complexity. To address this, Takahashi formulated an envelope definition tailored for parameter-dependent families of Legendrian curves residing in the unit tangent bundle over $\mathbb{R}^2$, subsequently analyzing their characteristics [7]. Building on this, Li and co-authors extended the framework to encompass families of singular spherical curves parametrized by a single variable, providing definitions and exploring their properties [5]. Concurrently, Pei et al. investigated families of framed curves in Euclidean 3-space depending on a parameter; their work involved constructing moving frames, formulating curvature functions, establishing existence and uniqueness theorems predicated on these curvatures, and concurrently defining envelope concepts for such families [19].

A novel perspective was introduced by Chen and his collaborators, who harnessed the Δ_1 and Δ_5 dualities from Legendrian theory to scrutinize smooth curves with singularities in Minkowski 3-space. This approach led to the formulation of parameter-dependent families of Legendrian curves and the definition of envelopes for frontals situated in hyperbolic and de Sitter 2-spaces [2]. A subsequent, more systematic investigation [9] delved into the envelopes of frontal families parametrized by a single variable within these spaces via duality principles. This work introduced families of Legendrian curves associated with Δ_1 and Δ_5 dualities and, departing from classical envelope theories applicable only to regular curves, devised new methodologies for defining envelopes applicable to singular frontals, accompanied by a thorough examination of their properties and interrelations.

The current study serves to complement and extend the findings of Chen et al. [9] regarding envelope families linked to Δ_1 and Δ_5 dualities. The Δ_2 and Δ_3 dualities occupy a distinctive position within Legendrian theory due to the inherent degeneracy of the light cone. When dealing with singular curves confined to the light cone, conventional techniques for constructing adapted frames prove insufficient, thereby necessitating the development of alternative constructions. The second author, along with various collaborators, has previously established several moving frames suitable for singular curves and lightlike submanifolds [20–23].

In this paper, we introduce precise definitions for parameter-dependent families of Legendrian curves and their associated envelopes, specifically for frontals exhibiting Δ_2 and Δ_3 dualities. Novel frames are constructed along these families, which permits a smooth definition and analysis of envelopes for frontal families parametrized by a single variable within the light cone, facilitated by the Δ_2 and Δ_3 dualities. This approach parallels the treatment of Δ_1 and Δ_5 duality-related envelope families in hyperbolic and de Sitter spaces previously explored in [9]. Furthermore, we conduct a detailed examination of the geometric properties characterizing envelopes derived from Δ_2 and Δ_3-dual frontal families depending on a parameter.

The structure of the paper is as follows. Section 2 provides a review of Legendrian duality theory and formally introduces parameter-dependent families of spacelike frontals associated with Δ_2 and Δ_3 dualities. We define the corresponding families of spacelike Legendrian curves within Δ_2 and Δ_3, establish their moving frames, derive the associated curvature functions, and investigate the fundamental properties of these frontal families. Section 3 employs variability and tangency conditions to define envelopes for these Δ_2 and Δ_3-dual families, demonstrating that the resulting envelopes themselves constitute frontals in hyperbolic 2-space, de Sitter 2-space, or the light cone. Equivalent conditions for envelope existence are derived, alongside an examination of hyperbolic and de Sitter parallels and the conditions for the existence of their envelopes. Section 4 explores the relationships among envelopes of spacelike Legendrian curve families parametrized by a single variable associated with Δ_2 and Δ_3 dualities, confirming that mutually dual frontal

families share identical pre-envelopes. Section 5 presents two illustrative examples to elucidate the theoretical findings.

Throughout this exposition, all mappings and manifolds are assumed to be of class C^∞ unless explicitly stated otherwise.

2. Dualities for Parameter-Dependent Families of Spacelike Frontals

This section is dedicated to the differential-geometric study of families of smooth curves depending on a parameter, residing in hyperbolic 2-space, de Sitter 2-space, and the light cone. We commence by recalling essential concepts from Minkowski space, following the presentation in [24].

Let $\mathbb{E}^3 = \{(a_1, a_2, a_3) \mid a_i \in \mathbb{R}, i = 1, 2, 3\}$ denote the familiar 3-dimensional real vector space. For vectors $\boldsymbol{a} = (a_1, a_2, a_3)$ and $\boldsymbol{b} = (b_1, b_2, b_3)$ in $\mathbb{E}^3$, the *pseudo-scalar product* is defined by $\langle \boldsymbol{a}, \boldsymbol{b} \rangle = -a_1 b_1 + a_2 b_2 + a_3 b_3$. Endowed with this product, the pair $(\mathbb{E}^3, \langle \cdot, \cdot \rangle)$ is termed *Minkowski 3-space*, denoted henceforth as $\mathbb{E}^3_1$.

A non-zero vector $\boldsymbol{a} \in \mathbb{R}^3_1$ is categorized as *spacelike*, *lightlike*, or *timelike* according to the sign of its pseudo-norm, i.e., if $\langle \boldsymbol{a}, \boldsymbol{a} \rangle > 0$, $\langle \boldsymbol{a}, \boldsymbol{a} \rangle = 0$, or $\langle \boldsymbol{a}, \boldsymbol{a} \rangle < 0$, respectively. Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are said to be *pseudo-orthogonal* if their product vanishes: $\langle \boldsymbol{a}, \boldsymbol{b} \rangle = 0$. The term *causal* collectively refers to vectors that are either timelike or lightlike. The *norm* of a vector $\boldsymbol{a} \in \mathbb{E}^3_1$ is given by $\|\boldsymbol{a}\| = \sqrt{|\langle \boldsymbol{a}, \boldsymbol{a} \rangle|}$.

For given vectors $\boldsymbol{a} = (a_1, a_2, a_3)$ and $\boldsymbol{b} = (b_1, b_2, b_3)$ in $\mathbb{E}^3_1$, their wedge product $\boldsymbol{a} \wedge \boldsymbol{b}$ is defined via the formal determinant

$$\boldsymbol{a} \wedge \boldsymbol{b} = \begin{vmatrix} -\boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix},$$

where $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ constitutes the canonical basis of $\mathbb{E}^3_1$. For an arbitrary vector $\boldsymbol{c} \in \mathbb{E}^3_1$, a straightforward verification yields the identity

$$\langle \boldsymbol{c}, \boldsymbol{a} \wedge \boldsymbol{b} \rangle = \det(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{b}),$$

which confirms that $\boldsymbol{a} \wedge \boldsymbol{b}$ is pseudo-orthogonal to both $\boldsymbol{a}$ and $\boldsymbol{b}$. If $\boldsymbol{a} = (a_1, a_2, a_3)$ is causal, it satisfies $a_1^2 \geq a_2^2 + a_3^2$. When both $\boldsymbol{a} = (a_1, a_2, a_3)$ and $\boldsymbol{b} = (b_1, b_2, b_3)$ are causal, the Schwarz inequality leads to

$$a_1 b_1 \geq \sqrt{a_2^2 + a_3^2}\sqrt{b_2^2 + b_3^2} \geq a_2 b_2 + a_3 b_3,$$

demonstrating that two non-zero causal vectors cannot be pseudo-orthogonal unless they are both lightlike and proportional to one another.

Consider $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{E}^3_1$ such that $\boldsymbol{a} \wedge \boldsymbol{b} = \boldsymbol{c}$ and the triple is linearly independent. In this situation, if at least one of $\boldsymbol{a}$ or $\boldsymbol{b}$ is causal, then $\boldsymbol{c}$ is necessarily spacelike. Conversely, if both $\boldsymbol{a}$ and $\boldsymbol{b}$ are spacelike, then $\boldsymbol{c}$ is causal.

We now define the following fundamental geometric objects within $\mathbb{E}^3_1$:

(i) *Hyperbolic 2-space*: $\mathbb{H}^2_0 = \{\boldsymbol{a} \in \mathbb{E}^3_1 \mid \langle \boldsymbol{a}, \boldsymbol{a} \rangle = -1\}$;
(ii) *de Sitter 2-space*: $\mathbb{S}^2_1 = \{\boldsymbol{a} \in \mathbb{E}^3_1 \mid \langle \boldsymbol{a}, \boldsymbol{a} \rangle = 1\}$;
(iii) *(Open) lightcone* at the origin: $\mathbb{LC}^* = \{\boldsymbol{a} \in \mathbb{E}^3_1 \setminus \{\boldsymbol{0}\} \mid \langle \boldsymbol{a}, \boldsymbol{a} \rangle = 0\}$.

The spherical tangent bundle is denoted

$$T\mathbb{E}^3_1(1) = \{(a, v) \in T\mathbb{E}^3_1 \mid \langle v, v \rangle = 1, \ -1, \text{ or } 0\},$$

i.e., v is a unit spacelike vector ($\langle v, v\rangle = 1$), a unit timelike vector ($\langle v, v\rangle = -1$), or a lightlike vector ($\langle v, v\rangle = 0$).

Definition 1. *A pair $(\boldsymbol{\gamma}, \boldsymbol{\nu}) : \mathcal{I} \to T\mathbb{E}^3_1(1)$ is called a Legendrian curve on the spherical tangent bundle $T\mathbb{E}^3_1(1)$ if it satisfies the condition $\langle \boldsymbol{\gamma}'(t), \boldsymbol{\nu}(t)\rangle = 0$ for every $t \in \mathcal{I}$. In this context, $\boldsymbol{\gamma}$ is referred to as a frontal curve. If, moreover, the map $(\boldsymbol{\gamma}, \boldsymbol{\nu})$ is an immersion (a Legendre immersion), then $\boldsymbol{\gamma}$ is specifically designated a front curve (or a wavefront).*

2.1. The Legendrian Duality Theorem

We begin by reviewing fundamental notions concerning contact geometry and Legendrian submanifolds; for a comprehensive treatment, the reader may consult [12,25]. Let N be a smooth manifold of odd dimension $2n+1$, and let $K \subset TN$ be a field of tangent hyperplanes. Locally, such a field can be expressed as the kernel of a non-vanishing 1-form α. The hyperplane field K is said to be *non-degenerate* if $\alpha \wedge (d\alpha)^n \neq 0$ holds at every point of N. When K is non-degenerate, the pair (N, K) is called a *contact manifold*; K itself is referred to as a *contact structure*, and any local 1-form α with $K = \ker \alpha$ is a *contact form*. A diffeomorphism $\phi : N \to N'$ between two contact manifolds (N, K) and (N', K') is a *contact diffeomorphism* if it satisfies $d\phi(K) = K'$; the two manifolds are said to be *contact diffeomorphic* when such a map exists.

A submanifold $i : L \hookrightarrow N$ of a contact manifold (N, K) is termed *Legendrian* provided its dimension equals n (the maximal possible for an integral submanifold) and its tangent space at every point is contained in the contact hyperplane: $di_x(T_xL) \subset K_{i(x)}$ for all $x \in L$. A smooth fiber bundle $\pi : E \to M$ is called a *Legendrian fibration* if its total space E is equipped with a contact structure for which the fibers are Legendrian submanifolds. For a Legendrian submanifold $i : L \to E$, the composition $\pi \circ i : L \to M$ is a *Legendrian map* and its image $W(L) = \pi(i(L))$ is the *wavefront set* of L. Around any point $z \in E$ one can introduce local coordinates $(x, p, y) = (x_1, \ldots, x_m, p_1, \ldots, p_m, y)$ such that the projection takes the form $\pi(x, p, y) = (x, y)$ and the contact structure is given by the canonical 1-form

$$\alpha = dy - \sum_{i=1}^{m} p_i \, dx_i.$$

Consider now the product space $\mathbb{E}^3_1 \times \mathbb{E}^3_1$. On it we define two 1-forms

$$\langle d\boldsymbol{u}, \boldsymbol{v}\rangle = -v_1 \, du_1 + \sum_{i=2}^{3} v_i \, du_i, \qquad \langle \boldsymbol{u}, d\boldsymbol{v}\rangle = -u_1 \, dv_1 + \sum_{i=2}^{3} u_i \, dv_i.$$

These forms give rise to the following five double fibrations, which are fundamental for Legendrian duality:

(i) $\Delta_1 = \{(\boldsymbol{u}, \boldsymbol{v}) \in \mathbb{H}^2_0 \times \mathbb{S}^2_1 \mid \langle \boldsymbol{u}, \boldsymbol{v}\rangle = 0\}$, with projections $\pi_{11} : \Delta_1 \to \mathbb{H}^2_0$, $\pi_{12} : \Delta_1 \to \mathbb{S}^2_1$, and contact forms $\theta_{11} = \langle d\boldsymbol{u}, \boldsymbol{v}\rangle|_{\Delta_1}$, $\theta_{12} = \langle \boldsymbol{u}, d\boldsymbol{v}\rangle|_{\Delta_1}$.

(ii) $\Delta_2 = \{(\boldsymbol{u}, \boldsymbol{v}) \in \mathbb{H}^2_0 \times \mathbb{LC}^* \mid \langle \boldsymbol{u}, \boldsymbol{v}\rangle = -1\}$, with $\pi_{21} : \Delta_2 \to \mathbb{H}^2_0$, $\pi_{22} : \Delta_2 \to \mathbb{LC}^*$, $\theta_{21} = \langle d\boldsymbol{u}, \boldsymbol{v}\rangle|_{\Delta_2}$, $\theta_{22} = \langle \boldsymbol{u}, d\boldsymbol{v}\rangle|_{\Delta_2}$.

(iii) $\Delta_3 = \{(\boldsymbol{u}, \boldsymbol{v}) \in \mathbb{S}^2_1 \times \mathbb{LC}^* \mid \langle \boldsymbol{u}, \boldsymbol{v}\rangle = 1\}$, with $\pi_{31} : \Delta_3 \to \mathbb{S}^2_1$, $\pi_{32} : \Delta_3 \to \mathbb{LC}^*$, $\theta_{31} = \langle d\boldsymbol{u}, \boldsymbol{v}\rangle|_{\Delta_3}$, $\theta_{32} = \langle \boldsymbol{u}, d\boldsymbol{v}\rangle|_{\Delta_3}$.

(iv) $\Delta_4 = \{(\boldsymbol{u}, \boldsymbol{v}) \in \mathbb{LC}^* \times \mathbb{LC}^* \mid \langle \boldsymbol{u}, \boldsymbol{v}\rangle = -2\}$, with $\pi_{41} : \Delta_4 \to \mathbb{LC}^*$, $\pi_{42} : \Delta_4 \to \mathbb{LC}^*$, $\theta_{41} = \langle d\boldsymbol{u}, \boldsymbol{v}\rangle|_{\Delta_4}$, $\theta_{42} = \langle \boldsymbol{u}, d\boldsymbol{v}\rangle|_{\Delta_4}$.

(v) $\Delta_5 = \{(\boldsymbol{u}, \boldsymbol{v}) \in \mathbb{S}^2_1 \times \mathbb{S}^2_1 \mid \langle \boldsymbol{u}, \boldsymbol{v}\rangle = 0\}$, with $\pi_{51} : \Delta_5 \to \mathbb{S}^2_1$, $\pi_{52} : \Delta_5 \to \mathbb{S}^2_1$, $\theta_{51} = \langle d\boldsymbol{u}, \boldsymbol{v}\rangle|_{\Delta_5}$, $\theta_{52} = \langle \boldsymbol{u}, d\boldsymbol{v}\rangle|_{\Delta_5}$.

For each $i = 1, \ldots, 5$, the zero level sets $\theta_{i1}^{-1}(0)$ and $\theta_{i2}^{-1}(0)$ define the same tangent hyperplane field on Δ_i; we denote this field by K_i. As established in [25], each pair (Δ_i, K_i) is a contact manifold and both projections π_{i1}, π_{i2} are Legendrian fibrations. If a point $(\boldsymbol{u}, \boldsymbol{v})$

lies in (Δ_i, K_i), we say that $\boldsymbol{u}$ is Δ_i-dual to $\boldsymbol{v}$. The contact manifolds (Δ_i, K_i) for $i = 1, 2, 3, 4$ are mutually contact diffeomorphic; further details can be found in [12]. The fibration Δ_5 was introduced by Chen and Izumiya in [26]; there it is shown that (Δ_5, K_5) is a contact manifold with Legendrian fibrations π_{51}, π_{52}, but it is not contact diffeomorphic to any (Δ_i, K_i) with $i = 1, 2, 3, 4$.

2.2. Δ_2-Duality for Families of Spacelike Frontals

This subsection is devoted to introducing parameter-dependent families of spacelike frontals in hyperbolic 2-space that are associated with Δ_2-duality. We also define the corresponding families of spacelike Legendrian curves in Δ_2 and introduce the concept of hyperbolic parallels, subsequently examining their geometric attributes.

Consider a smooth map $\boldsymbol{\Gamma}_h : J \times \Sigma \to \mathbb{H}^2_0$, where J and Σ are intervals (or possibly $\mathbb{R}$). This map is termed a *parameter-dependent family of spacelike frontals* in $\mathbb{H}^2_0$ if, for every fixed $\sigma \in \Sigma$, the derivative with respect to the first argument, $\partial_t \boldsymbol{\Gamma}_h(t, \sigma)$, is spacelike at all regular points t of the curve $\boldsymbol{\Gamma}_h(\cdot, \sigma)$, and there exists a smooth map $\boldsymbol{\Gamma}^\ell_h : J \times \Sigma \to \mathbb{LC}^*$ such that the pair $(\boldsymbol{\Gamma}_h, \boldsymbol{\Gamma}^\ell_h)$ maps into Δ_2 and satisfies the Legendrian condition

$$\langle \partial_t \boldsymbol{\Gamma}_h(t, \sigma), \boldsymbol{\Gamma}^\ell_h(t, \sigma) \rangle = 0 \quad \forall (t, \sigma) \in J \times \Sigma.$$

Here, $\Delta_2 = \{(\boldsymbol{u}, \boldsymbol{v}) \in \mathbb{H}^2_0 \times \mathbb{LC}^* \mid \langle \boldsymbol{u}, \boldsymbol{v} \rangle = -1\}$ embodies the Δ_2-duality introduced by Izumiya [25]. The resulting map $(\boldsymbol{\Gamma}_h, \boldsymbol{\Gamma}^\ell_h) : J \times \Sigma \to \Delta_2$ is called a *family of spacelike Legendrian curves parametrized by a single variable* in Δ_2.

We now define an auxiliary vector field $\boldsymbol{\Xi}_h(t, \sigma) = \boldsymbol{\Gamma}_h(t, \sigma) \wedge \boldsymbol{\Gamma}^\ell_h(t, \sigma) \in \mathbb{S}^2_1$. The triple $\{\boldsymbol{\Gamma}_h, \boldsymbol{\Gamma}^\ell_h, \boldsymbol{\Xi}_h\}$ then provides a moving frame for $\mathbb{R}^3_1$ along the frontal family $\boldsymbol{\Gamma}_h$ (equivalently, along the Legendrian family $(\boldsymbol{\Gamma}_h, \boldsymbol{\Gamma}^\ell_h)$). Standard differential-geometric computations lead to the following relations, which we refer to as *spacelike hyperbolic Legendrian Frenet–Serret type formulas*:

$$\begin{pmatrix} \partial_t \boldsymbol{\Gamma}_h \\ \partial_t \boldsymbol{\Gamma}^\ell_h \\ \partial_t \boldsymbol{\Xi}_h \end{pmatrix} = \begin{pmatrix} 0 & 0 & \alpha_h \\ 0 & 0 & \beta_h \\ \beta_h & \alpha_h - \beta_h & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\Gamma}_h \\ \boldsymbol{\Gamma}^\ell_h \\ \boldsymbol{\Xi}_h \end{pmatrix},$$
$$\begin{pmatrix} \partial_\sigma \boldsymbol{\Gamma}_h \\ \partial_\sigma \boldsymbol{\Gamma}^\ell_h \\ \partial_\sigma \boldsymbol{\Xi}_h \end{pmatrix} = \begin{pmatrix} \gamma_h & -\gamma_h & \delta_h \\ 0 & -\gamma_h & \varepsilon_h \\ \varepsilon_h & \delta_h - \varepsilon_h & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\Gamma}_h \\ \boldsymbol{\Gamma}^\ell_h \\ \boldsymbol{\Xi}_h \end{pmatrix},$$

where the curvature functions are defined by the following inner products:

$$\begin{aligned} &\alpha_h = \langle \partial_t \boldsymbol{\Gamma}_h, \boldsymbol{\Xi}_h \rangle, \quad \beta_h = \langle \partial_t \boldsymbol{\Gamma}^\ell_h, \boldsymbol{\Xi}_h \rangle, \\ &\gamma_h = -\langle \partial_\sigma \boldsymbol{\Gamma}_h, \boldsymbol{\Gamma}^\ell_h \rangle, \quad \delta_h = \langle \partial_\sigma \boldsymbol{\Gamma}_h, \boldsymbol{\Xi}_h \rangle, \\ &\varepsilon_h = \langle \partial_\sigma \boldsymbol{\Gamma}^\ell_h, \boldsymbol{\Xi}_h \rangle. \end{aligned}$$

(For brevity, the arguments (t, σ) are omitted here and in similar contexts where no confusion arises).

By assembling the coefficient matrices,

$$A_h = \begin{pmatrix} 0 & 0 & \alpha_h \\ 0 & 0 & \beta_h \\ \beta_h & \alpha_h - \beta_h & 0 \end{pmatrix}, \quad B_h = \begin{pmatrix} \gamma_h & -\gamma_h & \delta_h \\ 0 & -\gamma_h & \varepsilon_h \\ \varepsilon_h & \delta_h - \varepsilon_h & 0 \end{pmatrix},$$

the integrability conditions $\partial_\sigma \partial_t \mathbf{\Gamma}_h = \partial_t \partial_\sigma \mathbf{\Gamma}_h$, $\partial_\sigma \partial_t \mathbf{\Gamma}_h^\ell = \partial_t \partial_\sigma \mathbf{\Gamma}_h^\ell$, and $\partial_\sigma \partial_t \mathbf{\Xi}_h = \partial_t \partial_\sigma \mathbf{\Xi}_h$ are equivalent to the matrix equation

$$\partial_\sigma A_h - \partial_t B_h = B_h A_h - A_h B_h,$$

which, upon expansion, yields the following system of partial differential equations for the curvature functions:

$$\begin{aligned}
\partial_t \gamma_h &= \alpha_h \varepsilon_h - \delta_h \beta_h, \\
\partial_t \delta_h &= \partial_\sigma \alpha_h - \gamma_h \alpha_h + \gamma_h \beta_h, \\
\partial_t \varepsilon_h &= \partial_\sigma \beta_h + \gamma_h \beta_h.
\end{aligned}$$

The ordered quintuple $(\alpha_h, \beta_h, \gamma_h, \delta_h, \varepsilon_h)$, subject to the integrability conditions above, is designated the *spacelike hyperbolic Legendrian curvature* of the family $(\mathbf{\Gamma}_h, \mathbf{\Gamma}_h^\ell)$.

A few observations are in order. If $(\mathbf{\Gamma}_h, \mathbf{\Gamma}_h^\ell)$ is a family with curvature $(\alpha_h, \beta_h, \gamma_h, \delta_h, \varepsilon_h)$, then the modified families $(\mathbf{\Gamma}_h, -\mathbf{\Gamma}_h^\ell)$, $(-\mathbf{\Gamma}_h, \mathbf{\Gamma}_h^\ell)$, and $(-\mathbf{\Gamma}_h, -\mathbf{\Gamma}_h^\ell)$ are also parameter-dependent families of spacelike Legendrian curves. Their respective curvatures are $(-\alpha_h, \beta_h, -\gamma_h, -\delta_h, \varepsilon_h)$, $(\alpha_h, -\beta_h, -\gamma_h, \delta_h, -\varepsilon_h)$, and $(-\alpha_h, -\beta_h, \gamma_h, -\delta_h, -\varepsilon_h)$.

For a fixed real number ϕ, we introduce a new family $\mathbf{\Gamma}_h^\phi : J \times \Sigma \to \mathbb{H}_0^2$, called the *hyperbolic parallel* of $\mathbf{\Gamma}_h$, defined by

$$\mathbf{\Gamma}_h^\phi(t, \sigma) = -\frac{\phi}{2} \mathbf{\Gamma}_h(t, \sigma) + \frac{\phi^2 - 4}{4\phi} \mathbf{\Gamma}_h^\ell(t, \sigma).$$

The following proposition elucidates the properties of this parallel family.

Proposition 1. *Let $(\mathbf{\Gamma}_h, \mathbf{\Gamma}_h^\ell) : J \times \Sigma \to \Delta_2$ be a family of spacelike Legendrian curves depending on a parameter, possessing spacelike hyperbolic Legendrian curvature $(\alpha_h, \beta_h, \gamma_h, \delta_h, \varepsilon_h)$. For any fixed real number ϕ, the pair $(\mathbf{\Gamma}_h^\phi, (\mathbf{\Gamma}_h^\phi)^\ell)$, where*

$$(\mathbf{\Gamma}_h^\phi)^\ell(t, \sigma) = -\phi \mathbf{\Gamma}_h(t, \sigma) + \frac{\phi}{2} \mathbf{\Gamma}_h^\ell(t, \sigma) \in \mathbb{LC}^*,$$

also forms a parameter-dependent family of spacelike Legendrian curves mapping into Δ_2. Its spacelike hyperbolic Legendrian curvature $(\alpha_h^\phi, \beta_h^\phi, \gamma_h^\phi, \delta_h^\phi, \varepsilon_h^\phi)$ is given by

$$\begin{aligned}
\alpha_h^\phi &= -\frac{\phi}{2} \alpha_h + \frac{\phi^2 - 4}{4\phi} \beta_h, & \beta_h^\phi &= -\phi \alpha_h + \frac{\phi}{2} \beta_h, \\
\gamma_h^\phi &= -\gamma_h, & \delta_h^\phi &= -\frac{\phi}{2} \delta_h + \frac{\phi^2 - 4}{4\phi} \varepsilon_h, & \varepsilon_h^\phi &= -\phi \delta_h + \frac{\phi}{2} \varepsilon_h.
\end{aligned}$$

Proof. We first verify that $(\mathbf{\Gamma}_h^\phi, (\mathbf{\Gamma}_h^\phi)^\ell)$ lies in Δ_2 and satisfies the Legendrian condition. Using the definitions and the properties of the inner product, we compute

$$\begin{aligned}
\langle \mathbf{\Gamma}_h^\phi, (\mathbf{\Gamma}_h^\phi)^\ell \rangle &= \Big\langle -\frac{\phi}{2} \mathbf{\Gamma}_h + \frac{\phi^2 - 4}{4\phi} \mathbf{\Gamma}_h^\ell, \ -\phi \mathbf{\Gamma}_h + \frac{\phi}{2} \mathbf{\Gamma}_h^\ell \Big\rangle \\
&= \frac{\phi^2}{2} \langle \mathbf{\Gamma}_h, \mathbf{\Gamma}_h \rangle - \frac{\phi}{4} \langle \mathbf{\Gamma}_h, \mathbf{\Gamma}_h^\ell \rangle - \frac{\phi(\phi^2 - 4)}{4\phi} \langle \mathbf{\Gamma}_h^\ell, \mathbf{\Gamma}_h \rangle + \frac{\phi^2 - 4}{4\phi} \cdot \frac{\phi}{2} \langle \mathbf{\Gamma}_h^\ell, \mathbf{\Gamma}_h^\ell \rangle.
\end{aligned}$$

Since $\langle \mathbf{\Gamma}_h, \mathbf{\Gamma}_h \rangle = -1$, $\langle \mathbf{\Gamma}_h^\ell, \mathbf{\Gamma}_h^\ell \rangle = 0$, and $\langle \mathbf{\Gamma}_h, \mathbf{\Gamma}_h^\ell \rangle = -1$, we obtain

$$\langle \mathbf{\Gamma}_h^\phi, (\mathbf{\Gamma}_h^\phi)^\ell \rangle = -\frac{\phi^2}{2} + \frac{\phi}{4} + \frac{\phi^2 - 4}{4} = -1,$$

so $(\mathbf{\Gamma}_h^\phi, (\mathbf{\Gamma}_h^\phi)^\ell) \in \Delta_2$. Next, for the derivative with respect to t,

$$\begin{aligned}\langle \partial_t \mathbf{\Gamma}_h^\phi, (\mathbf{\Gamma}_h^\phi)^\ell \rangle &= \langle -\frac{\phi}{2}\partial_t \mathbf{\Gamma}_h + \frac{\phi^2-4}{4\phi}\partial_t \mathbf{\Gamma}_h^\ell,\ -\phi \mathbf{\Gamma}_h + \frac{\phi}{2}\mathbf{\Gamma}_h^\ell \rangle \\ &= \frac{\phi^2}{2}\langle \partial_t \mathbf{\Gamma}_h, \mathbf{\Gamma}_h\rangle - \frac{\phi}{4}\langle \partial_t \mathbf{\Gamma}_h, \mathbf{\Gamma}_h^\ell\rangle - \frac{\phi(\phi^2-4)}{4\phi}\langle \partial_t \mathbf{\Gamma}_h^\ell, \mathbf{\Gamma}_h\rangle + \frac{\phi^2-4}{4\phi}\cdot\frac{\phi}{2}\langle \partial_t \mathbf{\Gamma}_h^\ell, \mathbf{\Gamma}_h^\ell\rangle.\end{aligned}$$

Using $\langle \partial_t \mathbf{\Gamma}_h, \mathbf{\Gamma}_h\rangle = 0$, $\langle \partial_t \mathbf{\Gamma}_h^\ell, \mathbf{\Gamma}_h^\ell\rangle = 0$, and the identity $\langle \partial_t \mathbf{\Gamma}_h, \mathbf{\Gamma}_h^\ell\rangle = -\langle \mathbf{\Gamma}_h, \partial_t \mathbf{\Gamma}_h^\ell\rangle$ (which follows from differentiating $\langle \mathbf{\Gamma}_h, \mathbf{\Gamma}_h^\ell\rangle = -1$), we find that all terms cancel, yielding zero. Hence, the Legendrian condition holds.

A straightforward computation also shows that $\mathbf{\Gamma}_h^\phi \wedge (\mathbf{\Gamma}_h^\phi)^\ell = \mathbf{\Xi}_h$, so the moving frame for the new family can be taken as $\{\mathbf{\Gamma}_h^\phi, (\mathbf{\Gamma}_h^\phi)^\ell, \mathbf{\Xi}_h\}$.

Now we differentiate this frame. Using the Frenet–Serret formulas for the original family and then employing the following formula (expressing $\mathbf{\Gamma}_h$, $\mathbf{\Gamma}_h^\ell$ by $\mathbf{\Gamma}_h^\phi$, $(\mathbf{\Gamma}_h^\phi)^\ell$):

$$\begin{aligned}\mathbf{\Gamma}_h &= -\frac{\phi}{2}\mathbf{\Gamma}_h^\phi + \frac{\phi^2-4}{4\phi}(\mathbf{\Gamma}_h^\phi)^\ell, \\ \mathbf{\Gamma}_h^\ell &= -\phi\mathbf{\Gamma}_h^\phi + \frac{\phi}{2}(\mathbf{\Gamma}_h^\phi)^\ell,\end{aligned}$$

we have

$$\begin{aligned}\partial_t \mathbf{\Gamma}_h^\phi &= -\frac{\phi}{2}\partial_t \mathbf{\Gamma}_h + \frac{\phi^2-4}{4\phi}\partial_t \mathbf{\Gamma}_h^\ell = \big(-\frac{\phi}{2}\alpha_h + \frac{\phi^2-4}{4\phi}\beta_h\big)\mathbf{\Xi}_h, \\ \partial_t (\mathbf{\Gamma}_h^\phi)^\ell &= -\phi\partial_t \mathbf{\Gamma}_h + \frac{\phi}{2}\partial_t \mathbf{\Gamma}_h^\ell = \big(-\phi\alpha_h + \frac{\phi}{2}\beta_h\big)\mathbf{\Xi}_h, \\ \partial_t \mathbf{\Xi}_h &= \beta_h \mathbf{\Gamma}_h + (\alpha_h - \beta_h)\mathbf{\Gamma}_h^\ell \\ &= \big(-\phi\alpha_h + \frac{\phi}{2}\beta_h\big)\mathbf{\Gamma}_h^\phi + \big(\frac{\phi}{2}\alpha_h - \frac{\phi^2+4}{4\phi}\beta_h\big)(\mathbf{\Gamma}_h^\phi)^\ell, \\ \partial_\sigma \mathbf{\Gamma}_h^\phi &= -\frac{\phi}{2}\partial_\sigma \mathbf{\Gamma}_h + \frac{\phi^2-4}{4\phi}\partial_\sigma \mathbf{\Gamma}_h^\ell \\ &= -\frac{\phi}{2}\big(\gamma_h \mathbf{\Gamma}_h - \gamma_h \mathbf{\Gamma}_h^\ell + \delta_h \mathbf{\Xi}_h\big) + \frac{\phi^2-4}{4\phi}\big(-\gamma_h \mathbf{\Gamma}_h^\ell + \varepsilon_h \mathbf{\Xi}_h\big) \\ &= \big(-\frac{\phi}{2}\gamma_h\big)\mathbf{\Gamma}_h + \big(\frac{\phi}{2}\gamma_h - \frac{\phi^2-4}{4\phi}\gamma_h\big)\mathbf{\Gamma}_h^\ell + \big(-\frac{\phi}{2}\delta_h + \frac{\phi^2-4}{4\phi}\varepsilon_h\big)\mathbf{\Xi}_h \\ &= -\gamma_h \mathbf{\Gamma}_h^\phi + \gamma_h (\mathbf{\Gamma}_h^\phi)^\ell + \big(-\frac{\phi}{2}\delta_h + \frac{\phi^2-4}{4\phi}\varepsilon_h\big)\mathbf{\Xi}_h, \\ \partial_\sigma (\mathbf{\Gamma}_h^\phi)^\ell &= -\phi\partial_\sigma \mathbf{\Gamma}_h + \frac{\phi}{2}\partial_\sigma \mathbf{\Gamma}_h^\ell \\ &= -\phi\big(\gamma_h \mathbf{\Gamma}_h - \gamma_h \mathbf{\Gamma}_h^\ell + \delta_h \mathbf{\Xi}_h\big) + \frac{\phi}{2}\big(-\gamma_h \mathbf{\Gamma}_h^\ell + \varepsilon_h \mathbf{\Xi}_h\big) \\ &= (-\phi\gamma_h)\mathbf{\Gamma}_h + \big(\phi\gamma_h - \frac{\phi}{2}\gamma_h\big)\mathbf{\Gamma}_h^\ell + \big(-\phi\delta_h + \frac{\phi}{2}\varepsilon_h\big)\mathbf{\Xi}_h \\ &= \gamma_h (\mathbf{\Gamma}_h^\phi)^\ell + \big(-\phi\delta_h + \frac{\phi}{2}\varepsilon_h\big)\mathbf{\Xi}_h\end{aligned}$$

and

$$\begin{aligned}\partial_\sigma \mathbf{\Xi}_h &= \varepsilon_h \mathbf{\Gamma}_h + (\delta_h - \varepsilon_h)\mathbf{\Gamma}_h^\ell \\ &= \big(-\phi\delta_h + \frac{\phi}{2}\varepsilon_h\big)\mathbf{\Gamma}_h^\phi + \big(\frac{\phi}{2}\delta_h - \frac{\phi^2+4}{4\phi}\varepsilon_h\big)(\mathbf{\Gamma}_h^\phi)^\ell.\end{aligned}$$

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By comparing these expressions with the Frenet–Serret-type formulas for the moving frame $\{\mathbf{\Gamma}_h^{\phi}, (\mathbf{\Gamma}_h^{\phi})^{\ell}, \Xi_h\}$, the curvature relations stated in the proposition become evident. □

2.3. Δ_3-Duality for Families of Spacelike Frontals

This subsection introduces parameter-dependent families of spacelike frontals in de Sitter 2-space and the lightcone that are associated with Δ_3-duality. We define the corresponding Legendrian curve families in Δ_3 and the concept of de Sitter parallels, and we investigate their geometric properties.

Let $\mathbf{\Gamma}_d : J \times \Sigma \to \mathbb{S}_1^2$ be a smooth map, with J and Σ as before. This map is called a *parameter-dependent family of spacelike frontals* in $\mathbb{S}_1^2$ if, for each fixed $\sigma \in \Sigma$, the derivative $\partial_t \mathbf{\Gamma}_d(t,\sigma)$ is spacelike at every regular point t, and there exists a smooth map $\mathbf{\Gamma}_d^L : J \times \Sigma \to \mathbb{LC}^*$ such that the pair $(\mathbf{\Gamma}_d, \mathbf{\Gamma}_d^L)$ maps into Δ_3 and fulfills the Legendrian condition

$$\langle \partial_t \mathbf{\Gamma}_d(t,\sigma), \mathbf{\Gamma}_d^L(t,\sigma) \rangle = 0 \quad \forall (t,\sigma) \in J \times \Sigma.$$

Here, $\Delta_3 = \{(\boldsymbol{u},\boldsymbol{v}) \in \mathbb{S}_1^2 \times \mathbb{LC}^* \mid \langle \boldsymbol{u},\boldsymbol{v} \rangle = 1\}$ is the Δ_3-duality [25]. The resulting map $(\mathbf{\Gamma}_d, \mathbf{\Gamma}_d^L) : J \times \Sigma \to \Delta_3$ is termed a *family of spacelike Legendrian curves indexed by a parameter* in Δ_3.

Conversely, consider a smooth map $\mathbf{\Gamma}^{\ell} : J \times \Sigma \to \mathbb{LC}^*$. We call $\mathbf{\Gamma}^{\ell}$ a *parameter-dependent family of spacelike frontals* in $\mathbb{LC}^*$ if there exists a smooth map $\mathbf{\Gamma} : J \times \Sigma \to \mathbb{H}_0^2$ (or $\mathbb{S}_1^2$) such that the pair $(\mathbf{\Gamma}, \mathbf{\Gamma}^{\ell})$ maps into Δ_2 (or Δ_3, respectively) and satisfies $\langle \partial_t \mathbf{\Gamma}^{\ell}(t,\sigma), \mathbf{\Gamma}(t,\sigma) \rangle = 0$ for all (t,σ). It is worth noting that, by definition, both $\mathbf{\Gamma}_h^{\ell}$ and $\mathbf{\Gamma}_d^L$ are instances of parameter-dependent families of spacelike frontals residing in $\mathbb{LC}^*$.

For the family $\mathbf{\Gamma}_d$, we define a companion vector field $\Xi_d(t,\sigma) = \mathbf{\Gamma}_d(t,\sigma) \wedge \mathbf{\Gamma}_d^L(t,\sigma) \in \mathbb{S}_1^2$. The triplet $\{\mathbf{\Gamma}_d, \mathbf{\Gamma}_d^L, \Xi_d\}$ constitutes a moving frame along $\mathbf{\Gamma}_d$. The associated *spacelike de Sitter Legendrian Frenet–Serret type formulas* are

$$\begin{pmatrix} \partial_t \mathbf{\Gamma}_d \\ \partial_t \mathbf{\Gamma}_d^L \\ \partial_t \Xi_d \end{pmatrix} = \begin{pmatrix} 0 & 0 & \mu_d \\ 0 & 0 & \nu_d \\ -\nu_d & \nu_d - \mu_d & 0 \end{pmatrix} \begin{pmatrix} \mathbf{\Gamma}_d \\ \mathbf{\Gamma}_d^L \\ \Xi_d \end{pmatrix},$$
$$\begin{pmatrix} \partial_\sigma \mathbf{\Gamma}_d \\ \partial_\sigma \mathbf{\Gamma}_d^L \\ \partial_\sigma \Xi_d \end{pmatrix} = \begin{pmatrix} \xi_d & -\xi_d & o_d \\ 0 & -\xi_d & \pi_d \\ -\pi_d & \pi_d - o_d & 0 \end{pmatrix} \begin{pmatrix} \mathbf{\Gamma}_d \\ \mathbf{\Gamma}_d^L \\ \Xi_d \end{pmatrix},$$

with the curvature functions defined by

$$\begin{aligned} \mu_d &= \langle \partial_t \mathbf{\Gamma}_d, \Xi_d \rangle, \quad \nu_d = \langle \partial_t \mathbf{\Gamma}_d^L, \Xi_d \rangle, \\ \xi_d &= \langle \partial_\sigma \mathbf{\Gamma}_d, \mathbf{\Gamma}_d^L \rangle, \quad o_d = \langle \partial_\sigma \mathbf{\Gamma}_d, \Xi_d \rangle, \\ \pi_d &= \langle \partial_\sigma \mathbf{\Gamma}_d^L, \Xi_d \rangle. \end{aligned}$$

Defining the coefficient matrices

$$A_d = \begin{pmatrix} 0 & 0 & \mu_d \\ 0 & 0 & \nu_d \\ -\nu_d & \nu_d - \mu_d & 0 \end{pmatrix}, \quad B_d = \begin{pmatrix} \xi_d & -\xi_d & o_d \\ 0 & -\xi_d & \pi_d \\ -\pi_d & \pi_d - o_d & 0 \end{pmatrix},$$

the integrability conditions lead to the matrix equation $\partial_\sigma A_d - \partial_t B_d = B_d A_d - A_d B_d$, which is equivalent to the following system:

$$\begin{aligned}\partial_t \xi_d &= o_d \nu_d - \mu_d \pi_d,\\ \partial_t o_d &= \partial_\sigma \mu_d - \xi_d \mu_d + \xi_d \nu_d,\\ \partial_t \pi_d &= \partial_\sigma \nu_d + \xi_d \nu_d.\end{aligned}$$

The quintuple $(\mu_d, \nu_d, \xi_d, o_d, \pi_d)$, satisfying these relations, is called the *spacelike de Sitter Legendrian curvature* of the family $(\mathbf{\Gamma}_d, \mathbf{\Gamma}_d^L)$.

Analogous to the hyperbolic case, if $(\mathbf{\Gamma}_d, \mathbf{\Gamma}_d^L)$ has curvature $(\mu_d, \nu_d, \xi_d, o_d, \pi_d)$, then the families $(\mathbf{\Gamma}_d, -\mathbf{\Gamma}_d^L)$, $(-\mathbf{\Gamma}_d, \mathbf{\Gamma}_d^L)$, and $(-\mathbf{\Gamma}_d, -\mathbf{\Gamma}_d^L)$ are also spacelike Legendrian families depending on a parameter, with curvatures $(-\mu_d, \nu_d, -\xi_d, -o_d, \pi_d)$, $(\mu_d, -\nu_d, -\xi_d, o_d, -\pi_d)$, and $(-\mu_d, -\nu_d, \xi_d, -o_d, -\pi_d)$, respectively.

For a fixed real ϕ, we define the *de Sitter parallel* of $\mathbf{\Gamma}_d$ as the family $\mathbf{\Gamma}_d^\phi : J \times \Sigma \to \mathbb{S}_1^2$ given by

$$\mathbf{\Gamma}_d^\phi(t,\sigma) = -\frac{\phi}{2}\mathbf{\Gamma}_d(t,\sigma) + \frac{\phi^2-4}{4\phi}\mathbf{\Gamma}_d^L(t,\sigma).$$

Proposition 2. *Let $(\mathbf{\Gamma}_d, \mathbf{\Gamma}_d^L) : J \times \Sigma \to \Delta_3$ be a family of spacelike Legendrian curves parametrized by a single variable, endowed with spacelike de Sitter Legendrian curvature $(\mu_d, \nu_d, \xi_d, o_d, \pi_d)$. For any fixed real ϕ, the pair $(\mathbf{\Gamma}_d^\phi, (\mathbf{\Gamma}_d^\phi)^L)$, where*

$$(\mathbf{\Gamma}_d^\phi)^L(t,\sigma) = -\phi\mathbf{\Gamma}_d(t,\sigma) + \frac{\phi}{2}\mathbf{\Gamma}_d^L(t,\sigma) \in \mathbb{LC}^*,$$

forms a parameter-dependent family of spacelike Legendrian curves in Δ_3 with curvature $(\mu_d^\phi, \nu_d^\phi, \xi_d^\phi, o_d^\phi, \pi_d^\phi)$ given by

$$\begin{aligned}\mu_d^\phi &= -\frac{\phi}{2}\mu_d + \frac{\phi^2-4}{4\phi}\nu_d, & \nu_d^\phi &= -\phi\mu_d + \frac{\phi}{2}\nu_d,\\ \xi_d^\phi &= -\xi_d, & o_d^\phi &= -\frac{\phi}{2}o_d + \frac{\phi^2-4}{4\phi}\pi_d, & \pi_d^\phi &= -\phi o_d + \frac{\phi}{2}\pi_d.\end{aligned}$$

Proof. The proof follows the same pattern as that of Proposition 1. One first verifies that $(\mathbf{\Gamma}_d^\phi, (\mathbf{\Gamma}_d^\phi)^L)$ belongs to Δ_3 and satisfies the Legendrian condition by direct inner product calculations similar to those above, using $\langle \mathbf{\Gamma}_d, \mathbf{\Gamma}_d\rangle = 1$, $\langle \mathbf{\Gamma}_d^L, \mathbf{\Gamma}_d^L\rangle = 0$, $\langle \mathbf{\Gamma}_d, \mathbf{\Gamma}_d^L\rangle = 1$, and the relations obtained from differentiating these. A straightforward computation also shows $\mathbf{\Gamma}_d^\phi \wedge (\mathbf{\Gamma}_d^\phi)^L = \mathbf{\Xi}_d$, so $\{\mathbf{\Gamma}_d^\phi, (\mathbf{\Gamma}_d^\phi)^L, \mathbf{\Xi}_d\}$ is a valid moving frame.

Differentiating this frame and employing the de Sitter Frenet–Serret formulas for the original family yields the expressions for the curvature functions. For instance,

$$\begin{aligned}\partial_t \mathbf{\Gamma}_d^\phi &= -\frac{\phi}{2}\partial_t\mathbf{\Gamma}_d + \frac{\phi^2-4}{4\phi}\partial_t\mathbf{\Gamma}_d^L = \big(-\frac{\phi}{2}\mu_d + \frac{\phi^2-4}{4\phi}\nu_d\big)\mathbf{\Xi}_d,\\ \partial_t (\mathbf{\Gamma}_d^\phi)^L &= -\phi\partial_t\mathbf{\Gamma}_d + \frac{\phi}{2}\partial_t\mathbf{\Gamma}_d^L = \big(-\phi\mu_d + \frac{\phi}{2}\nu_d\big)\mathbf{\Xi}_d,\end{aligned}$$

which directly give μ_d^ϕ and ν_d^ϕ. The σ-derivatives are computed analogously, leading to the remaining curvature functions after expressing $\mathbf{\Gamma}_d$, $\mathbf{\Gamma}_d^L$ in terms of $\mathbf{\Gamma}_d^\phi$, $(\mathbf{\Gamma}_d^\phi)^L$. □

3. Envelopes of Parameter-Dependent Frontal Families in Pseudo-Spheres

This section is dedicated to a detailed investigation of the differential-geometric properties of envelopes for families of frontals parametrized by a single variable, living in hyperbolic 2-space, de Sitter 2-space, and the lightcone. We establish the precise conditions that characterize envelopes associated with Δ_2 and Δ_3 dualities. Furthermore, we provide

sufficient conditions for the existence of envelopes for the hyperbolic parallels of the frontal families $\mathbf{\Gamma}_h$ and $\mathbf{\Gamma}_d$.

3.1. Envelopes of Δ_2-Dual Spacelike Frontal Families in Hyperbolic Space

Let us consider a parameter-dependent family of spacelike Legendrian curves $(\mathbf{\Gamma}_h, \mathbf{\Gamma}_h^\ell) : J \times \Sigma \to \Delta_2$. We introduce a smooth map $\boldsymbol{p} : \mathcal{V} \to J \times \Sigma$, where $\mathcal{V} \subset \mathbb{R}$ is an interval, and write its components as $\boldsymbol{p}(s) = (t(s), \sigma(s))$. Using this map, we define the restricted curves along the family

$$\begin{aligned} \boldsymbol{F}_h(s) &= \mathbf{\Gamma}_h(\boldsymbol{p}(s)) = \mathbf{\Gamma}_h(t(s), \sigma(s)), \\ \boldsymbol{F}_h^\ell(s) &= \mathbf{\Gamma}_h^\ell(\boldsymbol{p}(s)) = \mathbf{\Gamma}_h^\ell(t(s), \sigma(s)). \end{aligned}$$

The definition of an envelope relies on the following three conditions:

(H1) Variability condition: The function σ is not constant on any non-trivial subinterval of $\mathcal{V}$.

(H2) Tangency condition (for $\mathbf{\Gamma}_h$): The derivative $\boldsymbol{F}_h'(s)$ is linearly dependent on the frame vector $\mathbf{\Xi}_h(\boldsymbol{p}(s))$ for every $s \in \mathcal{V}$.

(H3) Tangency condition (for $\mathbf{\Gamma}_h^\ell$): The derivative $(\boldsymbol{F}_h^\ell)'(s)$ is linearly dependent on $\mathbf{\Xi}_h(\boldsymbol{p}(s))$ for every $s \in \mathcal{V}$.

When conditions **(H1)** and **(H2)** are met, we call $\boldsymbol{F}_h$ a *hyperbolic envelope* of the frontal family $\mathbf{\Gamma}_h$, and the map $\boldsymbol{p}$ is referred to as a *pre-envelope*. Similarly, if **(H1)** and **(H3)** hold, then $\boldsymbol{F}_h^\ell$ is termed a *lightcone envelope* of the frontal family $\mathbf{\Gamma}_h^\ell$, with $\boldsymbol{p}$ again being a pre-envelope.

Proposition 3. *Assume $(\mathbf{\Gamma}_h, \mathbf{\Gamma}_h^\ell) : J \times \Sigma \to \Delta_2$ is a family of spacelike Legendrian curves depending on a parameter, whose spacelike hyperbolic Legendrian curvature is $(\alpha_h, \beta_h, \gamma_h, \delta_h, \varepsilon_h)$. If $\boldsymbol{p} : \mathcal{V} \to J \times \Sigma$ constitutes a pre-envelope, then the induced pair $(\boldsymbol{F}_h, \boldsymbol{F}_h^\ell) : \mathcal{V} \to \Delta_2$ itself forms a spacelike Legendrian curve. Its curvature functions are*

$$\big(t'(s)\,\alpha_h(\boldsymbol{p}(s)) + \sigma'(s)\,\delta_h(\boldsymbol{p}(s)),\ t'(s)\,\beta_h(\boldsymbol{p}(s)) + \sigma'(s)\,\varepsilon_h(\boldsymbol{p}(s))\big).$$

Moreover, a necessary condition for $\boldsymbol{p}$ to be a pre-envelope is the vanishing of γ_h along its image: $\gamma_h(\boldsymbol{p}(s)) = 0$ for all $s \in \mathcal{V}$.

Proof. Condition **(H2)** directly implies that $\boldsymbol{F}_h'(s)$ is parallel to $\mathbf{\Xi}_h(\boldsymbol{p}(s))$, from which it follows that $\langle \boldsymbol{F}_h'(s), \boldsymbol{F}_h^\ell(s) \rangle = 0$. Since $(\boldsymbol{F}_h(s), \boldsymbol{F}_h^\ell(s)) \in \Delta_2$, this is precisely the Legendrian condition, establishing $(\boldsymbol{F}_h, \boldsymbol{F}_h^\ell)$ as a Legendrian curve in Δ_2.

A direct computation, employing the chain rule and the Frenet–Serret formulas, yields

$$\begin{aligned} \boldsymbol{F}_h'(s) &= \sigma'(s)\gamma_h(\boldsymbol{p}(s))\boldsymbol{F}_h(s) - \sigma'(s)\gamma_h(\boldsymbol{p}(s))\boldsymbol{F}_h^\ell(s) \\ &\quad + \big(t'(s)\alpha_h(\boldsymbol{p}(s)) + \sigma'(s)\delta_h(\boldsymbol{p}(s))\big)\mathbf{\Xi}_h(\boldsymbol{p}(s)), \\ (\boldsymbol{F}_h^\ell)'(s) &= -\sigma'(s)\gamma_h(\boldsymbol{p}(s))\boldsymbol{F}_h^\ell(s) + \big(t'(s)\beta_h(\boldsymbol{p}(s)) + \sigma'(s)\varepsilon_h(\boldsymbol{p}(s))\big)\mathbf{\Xi}_h(\boldsymbol{p}(s)). \end{aligned}$$

Condition **(H2)** forces the terms not parallel to $\mathbf{\Xi}_h$ in the expression for $\boldsymbol{F}_h'(s)$ to vanish. Given the variability condition **(H1)**, which ensures $\sigma'(s) \not\equiv 0$, we deduce $\gamma_h(\boldsymbol{p}(s)) = 0$ for all $s \in \mathcal{V}$. Substituting this back simplifies the expressions and allows us to read off the curvature functions of the restricted curve $(\boldsymbol{F}_h, \boldsymbol{F}_h^\ell)$ as the coefficients of $\mathbf{\Xi}_h(\boldsymbol{p}(s))$ in $\boldsymbol{F}_h'(s)$ and $(\boldsymbol{F}_h^\ell)'(s)$. □

The preceding proposition and its proof lead directly to the following fundamental characterization.

Theorem 1 (Envelope Theorem for Δ_2)**.** *Let $(\mathbf{\Gamma}_h, \mathbf{\Gamma}_h^\ell) : J \times \Sigma \to \Delta_2$ be a parameter-dependent family of spacelike Legendrian curves possessing spacelike hyperbolic Legendrian curvature $(\alpha_h, \beta_h, \gamma_h, \delta_h, \varepsilon_h)$. Suppose $\boldsymbol{p} : \mathcal{V} \to J \times \Sigma$ is a smooth map fulfilling the variability condition (H1). Then the following assertions are pairwise equivalent:*

1. *$\boldsymbol{p}$ is a pre-envelope of $\mathbf{\Gamma}_h$;*
2. *$\langle \partial_\sigma \mathbf{\Gamma}_h(\boldsymbol{p}(s)), \mathbf{\Gamma}_h^\ell(\boldsymbol{p}(s)) \rangle = 0$ holds for every $s \in \mathcal{V}$;*
3. *$\gamma_h(\boldsymbol{p}(s)) = 0$ for all $s \in \mathcal{V}$.*

Proof. We establish the equivalence by proving a cycle of implications.

(1) $\Rightarrow$ (2): If $\boldsymbol{e}$ is a pre-envelope, condition **(H2)** holds, so there exists a function $c(s)$ such that $\boldsymbol{F}_h'(s) = c(s)\mathbf{\Xi}_h(\boldsymbol{p}(s))$. Differentiating $\boldsymbol{F}_h = \mathbf{\Gamma}_h \circ \boldsymbol{p}$ and using $\partial_t \mathbf{\Gamma}_h = \alpha_h \mathbf{\Xi}_h$, we obtain $t'(s)\alpha_h(\boldsymbol{p}(s))\mathbf{\Xi}_h(\boldsymbol{p}(s)) + \sigma'(s)\partial_\sigma \mathbf{\Gamma}_h(\boldsymbol{p}(s)) = c(s)\mathbf{\Xi}_h(\boldsymbol{p}(s))$. Taking the inner product of this equation with $\mathbf{\Gamma}_h^\ell(\boldsymbol{p}(s))$ and noting that $\mathbf{\Xi}_h$ is orthogonal to $\mathbf{\Gamma}_h^\ell$, we get $\sigma'(s)\langle \partial_\sigma \mathbf{\Gamma}_h(\boldsymbol{p}(s)), \mathbf{\Gamma}_h^\ell(\boldsymbol{p}(s)) \rangle = 0$. Since $\sigma'(s) \not\equiv 0$ by **(H1)**, the desired inner product must vanish pointwise.

(2) $\Rightarrow$ (3): This follows directly from the expression for $\partial_\sigma \mathbf{\Gamma}_h$ given by the Frenet–Serret formulas: $\partial_\sigma \mathbf{\Gamma}_h = \gamma_h \mathbf{\Gamma}_h - \gamma_h \mathbf{\Gamma}_h^\ell + \delta_h \mathbf{\Xi}_h$. Taking the inner product with $\mathbf{\Gamma}_h^\ell$ yields $\langle \partial_\sigma \mathbf{\Gamma}_h, \mathbf{\Gamma}_h^\ell \rangle = \gamma_h \langle \mathbf{\Gamma}_h, \mathbf{\Gamma}_h^\ell \rangle = \gamma_h$, because $\langle \mathbf{\Gamma}_h, \mathbf{\Gamma}_h^\ell \rangle = -1$ for points in Δ_2. Thus condition (2) is equivalent to $\gamma_h = 0$.

(3) $\Rightarrow$ (1): If $\gamma_h(\boldsymbol{p}(s)) = 0$, the expression for $\boldsymbol{F}_h'(s)$ computed in the proof of Proposition 3 reduces to $\boldsymbol{F}_h'(s) = (t'\alpha_h + \sigma'\delta_h)\mathbf{\Xi}_h(\boldsymbol{p}(s))$, which is clearly parallel to $\mathbf{\Xi}_h(\boldsymbol{p}(s))$. Hence, condition **(H2)** is satisfied, and with **(H1)** holding by hypothesis, $\boldsymbol{p}$ is a pre-envelope. □

Remark 1. *For Δ_2-dual families, the tangency conditions **(H2)** and **(H3)** are equivalent. This follows from the Frenet–Serret formulas and the fact that $\gamma_h = 0$ implies both $\boldsymbol{F}_h'$ and $(\boldsymbol{F}_h^\ell)'$ are parallel to $\mathbf{\Xi}_h$ (see Proposition 3). Therefore, a pre-envelope defined by **(H1)** and **(H2)** automatically satisfies **(H3)**, and vice versa. Theorem 1 lists both for completeness.*

Proposition 4. *Let $(\mathbf{\Gamma}_h, \mathbf{\Gamma}_h^\ell) : J \times \Sigma \to \Delta_2$ be a family of spacelike Legendrian curves indexed by a parameter, with spacelike hyperbolic Legendrian curvature $(\alpha_h, \beta_h, \gamma_h, \delta_h, \varepsilon_h)$. If $\boldsymbol{p} : \mathcal{V} \to J \times \Sigma$ is a pre-envelope of the original family $(\mathbf{\Gamma}_h, \mathbf{\Gamma}_h^\ell)$, then $\boldsymbol{p}$ is also a pre-envelope of the related families $(-\mathbf{\Gamma}_h, \mathbf{\Gamma}_h^\ell)$ and $(\mathbf{\Gamma}_h, -\mathbf{\Gamma}_h^\ell)$.*

Proof. Consider the family $(-\mathbf{\Gamma}_h, \mathbf{\Gamma}_h^\ell)$. Its associated third frame vector is $\widetilde{\mathbf{\Xi}}_h = (-\mathbf{\Gamma}_h) \wedge \mathbf{\Gamma}_h^\ell = -\mathbf{\Xi}_h$. Define $\widetilde{\boldsymbol{F}}_h(s) = -\mathbf{\Gamma}_h(\boldsymbol{p}(s))$. Then $\widetilde{\boldsymbol{F}}_h'(s) = -\boldsymbol{F}_h'(s)$. Since $\boldsymbol{p}$ is a pre-envelope for the original family, $\boldsymbol{F}_h'(s)$ is parallel to $\mathbf{\Xi}_h(\boldsymbol{p}(s))$. Consequently, $\widetilde{\boldsymbol{F}}_h'(s)$ is parallel to $-\mathbf{\Xi}_h(\boldsymbol{p}(s)) = \widetilde{\mathbf{\Xi}}_h(\boldsymbol{p}(s))$, satisfying the tangency condition for the new family. The variability condition **(H1)** is unchanged, so $\boldsymbol{p}$ is a pre-envelope for $(-\mathbf{\Gamma}_h, \mathbf{\Gamma}_h^\ell)$. A completely analogous argument applies to the family $(\mathbf{\Gamma}_h, -\mathbf{\Gamma}_h^\ell)$. □

Proposition 5. *Let $(\mathbf{\Gamma}_h, \mathbf{\Gamma}_h^\ell) : J \times \Sigma \to \Delta_2$ be a parameter-dependent family of spacelike Legendrian curves endowed with spacelike hyperbolic Legendrian curvature $(\alpha_h, \beta_h, \gamma_h, \delta_h, \varepsilon_h)$. Suppose $\boldsymbol{p} : \mathcal{V} \to J \times \Sigma$ is a smooth map satisfying the variability condition **(H1)** . If the set of regular points of $\mathbf{\Gamma}_h$ along $\boldsymbol{p}(\mathcal{V})$ is dense and the entire trace of $\boldsymbol{p}$ lies within the singular set of $\mathbf{\Gamma}_h$, then $\boldsymbol{p}$ is a pre-envelope of $\mathbf{\Gamma}_h$.*

Proof. From the Frenet–Serret formulas, we have $\partial_t \mathbf{\Gamma}_h = \alpha_h \mathbf{\Xi}_h$ and $\partial_\sigma \mathbf{\Gamma}_h = \gamma_h \mathbf{\Gamma}_h - \gamma_h \mathbf{\Gamma}_h^\ell + \delta_h \mathbf{\Xi}_h$. The condition that $\boldsymbol{p}(\mathcal{V})$ is contained in the singular set of $\mathbf{\Gamma}_h$ means that the partial derivative vectors are linearly dependent at those points, i.e., $\operatorname{rank}(\partial_t \mathbf{\Gamma}_h(\boldsymbol{p}(s)), \partial_\sigma \mathbf{\Gamma}_h(\boldsymbol{p}(s))) < 2$ for all $s \in \mathcal{V}$. Because the regular points are dense, $\alpha_h(\boldsymbol{p}(s))$ is non-zero on a dense subset

of $\mathcal{V}$. On this dense subset, $\partial_t \mathbf{\Gamma}_h(\boldsymbol{p}(s))$ provides a non-zero direction parallel to $\mathbf{\Xi}_h$. For $\partial_\sigma \mathbf{\Gamma}_h(\boldsymbol{p}(s))$ to be linearly dependent with this direction, its components along $\mathbf{\Gamma}_h$ and $\mathbf{\Gamma}_h^\ell$ must vanish, forcing $\gamma_h(\boldsymbol{p}(s)) = 0$. By continuity, $\gamma_h(\boldsymbol{p}(s)) = 0$ for all $s \in \mathcal{V}$. Theorem 1 then implies that $\boldsymbol{e}$ is a pre-envelope. □

Remark 2. *The converse of Proposition 5 is not generally true. A pre-envelope $\boldsymbol{p}$ indeed forces the derivatives $\partial_t \mathbf{\Gamma}_h$ and $\partial_\sigma \mathbf{\Gamma}_h$ to be linearly dependent (as $\gamma_h = 0$), placing $\boldsymbol{p}(\mathcal{V})$ within the singular set. However, it is possible that $\partial_t \mathbf{\Gamma}_h$ itself vanishes identically along $\boldsymbol{p}(\mathcal{V})$, which is a more degenerate situation.*

Remark 3. *The density assumption is generic. If regular points were not dense, then Γ_h would be singular on an open interval, a degenerate case that is usually excluded in the general theory of singularities (cf. [18]).*

We now turn to the relationship between the envelopes of a family and those of its hyperbolic parallel. For a given $\boldsymbol{p}$, define $\boldsymbol{F}_h^\phi(s) = \mathbf{\Gamma}_h^\phi(\boldsymbol{p}(s))$.

Proposition 6. *Let $(\mathbf{\Gamma}_h, \mathbf{\Gamma}_h^\ell) : J \times \Sigma \to \Delta_2$ be a family of spacelike Legendrian curves parametrized by a single variable, carrying spacelike hyperbolic Legendrian curvature $(\alpha_h, \beta_h, \gamma_h, \delta_h, \varepsilon_h)$. For a fixed real ϕ, let $\mathbf{\Gamma}_h^\phi$ be the hyperbolic parallel of $\mathbf{\Gamma}_h$. Suppose $\boldsymbol{p} : \mathcal{V} \to J \times \Sigma$ satisfies the variability condition* ***(H1)*** *. Then $\boldsymbol{F}_h$ is a hyperbolic envelope of $\mathbf{\Gamma}_h$ if and only if $\boldsymbol{F}_h^\phi$ is a hyperbolic envelope of $\mathbf{\Gamma}_h^\phi$.*

Proof. By Theorem 1, $\boldsymbol{F}_h$ is an envelope of Γ_h precisely when $\boldsymbol{p}$ is a pre-envelope, i.e., when the curvature function γ_h vanishes along $\boldsymbol{p}$

$$\gamma_h(\boldsymbol{p}(s)) = 0 \qquad \text{for all } s \in \mathcal{V}.$$

For the parallel family, a direct computation using the formulas from Proposition 1 (or following the same derivation as in Proposition 3) yields the derivative of $\boldsymbol{F}_h^\phi = \Gamma_h^\phi \circ \boldsymbol{p}$:

$$\begin{aligned}(\boldsymbol{F}_h^\phi)'(s) =& \sigma'(s)\gamma_h^\phi(\boldsymbol{p}(s))\, \boldsymbol{F}_h^\phi(s) - \sigma'(s)\gamma_h^\phi(\boldsymbol{p}(s))\, (\boldsymbol{F}_h^\phi)^\ell(s) \\ &+ \left(t'(s)\alpha_h^\phi(\boldsymbol{p}(s)) + \sigma'(s)\delta_h^\phi(\boldsymbol{p}(s))\right) \mathbf{\Xi}_h(\boldsymbol{p}(s)).\end{aligned} \tag{1}$$

Here, $(\boldsymbol{F}_h^\phi)^\ell$ is the dual lightlike vector associated with Γ_h^ϕ and $\mathbf{\Xi}_h$ is the common frame vector $\Gamma_h \wedge \Gamma_h^\ell = \Gamma_h^\phi \wedge (\Gamma_h^\phi)^\ell$. The curvature functions $\alpha_h^\phi, \delta_h^\phi$ are given in Proposition 1, and importantly Proposition 1 also states that

$$\gamma_h^\phi = -\gamma_h.$$

Now assume that $\boldsymbol{F}_h$ is an envelope of Γ_h. Then $\gamma_h(\boldsymbol{p}(s)) = 0$ for all s. From $\gamma_h^\phi = -\gamma_h$ we obtain $\gamma_h^\phi(\boldsymbol{p}(s)) = 0$ as well. Substituting this into (1), the first two terms vanish, leaving

$$(\boldsymbol{F}_h^\phi)'(s) = \left(t'(s)\alpha_h^\phi(\boldsymbol{p}(s)) + \sigma'(s)\delta_h^\phi(\boldsymbol{p}(s))\right) \mathbf{\Xi}_h(\boldsymbol{p}(s)).$$

Thus $(\boldsymbol{F}_h^\phi)'(s)$ is everywhere parallel to $\mathbf{\Xi}_h(\boldsymbol{p}(s))$, which is exactly the tangency condition **(H2)** for the parallel family. Since the variability condition **(H1)** holds by hypothesis, $\boldsymbol{p}$ is a pre-envelope of Γ_h^ϕ, and consequently $\boldsymbol{F}_h^\phi$ is a hyperbolic envelope of Γ_h^ϕ.

Conversely, if $\boldsymbol{F}_h^\phi$ is an envelope of Γ_h^ϕ, then $\gamma_h^\phi(\boldsymbol{p}(s)) = 0$ by Theorem 1 applied to the parallel family. Using $\gamma_h = -\gamma_h^\phi$ we get $\gamma_h(\boldsymbol{p}(s)) = 0$, so $\boldsymbol{F}_h$ is an envelope of Γ_h. This completes the proof. □

3.2. Envelopes of Δ_3-Dual Spacelike Frontal Families in De Sitter Space

We now develop the analogous theory for families associated with Δ_3-duality. Consider a parameter-dependent family $(\mathbf{\Gamma}_d, \mathbf{\Gamma}_d^L) : J \times \Sigma \to \Delta_3$ equipped with spacelike de Sitter Legendrian curvature $(\mu_d, \nu_d, \xi_d, o_d, \pi_d)$. For a smooth map $\boldsymbol{p} : \mathcal{V} \to J \times \Sigma$, we set $\boldsymbol{F}_d(s) = \mathbf{\Gamma}_d(\boldsymbol{p}(s))$ and $\boldsymbol{F}_d^L(s) = \mathbf{\Gamma}_d^L(\boldsymbol{p}(s))$.

The relevant conditions are:

(H1) Variability condition: σ is non-constant on any non-trivial subinterval of $\mathcal{V}$.

($\widetilde{\mathbf{H}}$2) Tangency condition (for $\mathbf{\Gamma}_d$): $\boldsymbol{F}_d'(s)$ is linearly dependent on $\Xi_d(\boldsymbol{p}(s))$ for all $s \in \mathcal{V}$.

($\widetilde{\mathbf{H}}$3) Tangency condition (for $\mathbf{\Gamma}_d^L$): $(\boldsymbol{F}_d^L)'(s)$ is linearly dependent on $\Xi_d(\boldsymbol{p}(s))$ for all $s \in \mathcal{V}$.

When **(H1)** and ($\widetilde{\mathbf{H}}$3) are satisfied, $\boldsymbol{F}_d$ is a *de Sitter envelope* (with $\boldsymbol{p}$ a pre-envelope) of $\mathbf{\Gamma}_d$. When **(H1)** and ($\widetilde{\mathbf{H}}$3) are satisfied, $\boldsymbol{F}_d^L$ is a *lightcone envelope* (with $\boldsymbol{p}$ a pre-envelope) of $\mathbf{\Gamma}_d^L$.

The following results are the counterparts of those in the previous subsection. Their proofs are structurally similar to the Δ_2 case, relying on the appropriate de Sitter Frenet–Serret formulas.

Proposition 7. *Assume $(\mathbf{\Gamma}_d, \mathbf{\Gamma}_d^L) : J \times \Sigma \to \Delta_3$ is a family of spacelike Legendrian curves depending on a parameter, whose spacelike de Sitter Legendrian curvature is $(\mu_d, \nu_d, \xi_d, o_d, \pi_d)$. If $\boldsymbol{p} : \mathcal{V} \to J \times \Sigma$ constitutes a pre-envelope, then the induced pair $(\boldsymbol{F}_d, \boldsymbol{F}_d^L) : \mathcal{V} \to \Delta_3$ itself forms a spacelike Legendrian curve. Its curvature functions are*

$$\big(t'(s)\,\mu_d(\boldsymbol{p}(s)) + \sigma'(s)\,o_d(\boldsymbol{p}(s)),\ t'(s)\,\nu_d(\boldsymbol{p}(s)) + \sigma'(s)\,\pi_d(\boldsymbol{p}(s))\big).$$

Moreover, a necessary condition for $\boldsymbol{p}$ to be a pre-envelope is the vanishing of ξ_d along its image: $\xi_d(\boldsymbol{p}(s)) = 0$ for all $s \in \mathcal{V}$.

Proof. Condition ($\widetilde{\mathbf{H}}$2) directly implies that $\boldsymbol{F}_d'(s)$ is parallel to $\Xi_d(\boldsymbol{p}(s))$, from which it follows that $\langle \boldsymbol{F}_d'(s), \boldsymbol{F}_d^L(s) \rangle = 0$. Since $(\boldsymbol{F}_d(s), \boldsymbol{F}_d^L(s)) \in \Delta_3$, this is precisely the Legendrian condition, establishing $(\boldsymbol{F}_d, \boldsymbol{F}_d^L)$ as a Legendrian curve in Δ_3.

A direct computation, employing the chain rule and the Frenet–Serret formulas for Δ_3, yields

$$\begin{aligned}
\boldsymbol{F}_d'(s) &= t'(s)\,\partial_t \mathbf{\Gamma}_d(\boldsymbol{p}(s)) + \sigma'(s)\,\partial_\sigma \mathbf{\Gamma}_d(\boldsymbol{p}(s)) \\
&= t'(s)\,\mu_d(\boldsymbol{p}(s))\,\Xi_d(\boldsymbol{p}(s)) + \sigma'(s)\big(\xi_d(\boldsymbol{p}(s))\,\mathbf{\Gamma}_d^L(\boldsymbol{p}(s)) + o_d(\boldsymbol{p}(s))\,\Xi_d(\boldsymbol{p}(s))\big) \\
&= \sigma'(s)\xi_d(\boldsymbol{p}(s))\,\boldsymbol{F}_d^L(s) + \big(t'(s)\mu_d(\boldsymbol{p}(s)) + \sigma'(s)o_d(\boldsymbol{p}(s))\big)\,\Xi_d(\boldsymbol{p}(s)), \\
(\boldsymbol{F}_d^L)'(s) &= t'(s)\,\partial_t \mathbf{\Gamma}_d^L(\boldsymbol{p}(s)) + \sigma'(s)\,\partial_\sigma \mathbf{\Gamma}_d^L(\boldsymbol{p}(s)) \\
&= t'(s)\,\nu_d(\boldsymbol{p}(s))\,\Xi_d(\boldsymbol{p}(s)) + \sigma'(s)\big(\xi_d(\boldsymbol{p}(s))\,\mathbf{\Gamma}_d(\boldsymbol{p}(s)) + \pi_d(\boldsymbol{p}(s))\,\Xi_d(\boldsymbol{p}(s))\big) \\
&= \sigma'(s)\xi_d(\boldsymbol{p}(s))\,\boldsymbol{F}_d(s) + \big(t'(s)\nu_d(\boldsymbol{p}(s)) + \sigma'(s)\pi_d(\boldsymbol{p}(s))\big)\,\Xi_d(\boldsymbol{p}(s)).
\end{aligned}$$

Condition ($\widetilde{\mathbf{H}}$2) forces the terms not parallel to Ξ_d in the expression for $\boldsymbol{F}_d'(s)$ to vanish. Given the variability condition **(H1)**, which ensures $\sigma'(s) \not\equiv 0$, we deduce $\xi_d(\boldsymbol{p}(s)) = 0$ for all $s \in \mathcal{V}$. Substituting this back simplifies the expressions to

$$\boldsymbol{F}_d'(s) = \big(t'(s)\mu_d(\boldsymbol{p}(s)) + \sigma'(s)o_d(\boldsymbol{p}(s))\big)\,\Xi_d(\boldsymbol{p}(s)),$$

$$(\boldsymbol{F}_d^L)'(s) = \big(t'(s)\nu_d(\boldsymbol{p}(s)) + \sigma'(s)\pi_d(\boldsymbol{p}(s))\big)\,\Xi_d(\boldsymbol{p}(s)).$$

Thus $(\boldsymbol{F}_d, \boldsymbol{F}_d^L)$ is a spacelike Legendrian curve whose curvature functions are the coefficients of $\Xi_d(\boldsymbol{p}(s))$ in $\boldsymbol{F}_d'(s)$ and $(\boldsymbol{F}_d^L)'(s)$, namely $t'(s)\mu_d(\boldsymbol{p}(s)) + \sigma'(s)o_d(\boldsymbol{p}(s))$ and $t'(s)\nu_d(\boldsymbol{p}(s)) + \sigma'(s)\pi_d(\boldsymbol{p}(s))$. The remaining curvature functions (e.g., $\widetilde{\xi}_d$, $\widetilde{o}_d$, $\widetilde{\pi}_d$) are either

zero or redundant; in particular, $\widetilde{\xi}_d = 0$ follows from the orthogonality $\langle \Xi_d, \boldsymbol{F}_d^L \rangle = 0$. This completes the proof. □

The following results are the counterparts of those in the previous subsection. Their proofs are structurally similar to the Δ_2 case, relying on the appropriate de Sitter Frenet–Serret formulas. Since we have provided a complete proof of Proposition 7 as an illustration, the proofs of the remaining results (Theorem 2, Proposition 8, etc.) are omitted for brevity as they follow the same pattern.

Theorem 2 (Envelope Theorem for Δ_3)**.** *Let $(\boldsymbol{\Gamma}_d, \boldsymbol{\Gamma}_d^L) : J \times \Sigma \to \Delta_3$ be a parameter-dependent family of spacelike Legendrian curves possessing spacelike de Sitter Legendrian curvature $(\mu_d, \nu_d, \xi_d, o_d, \pi_d)$. Suppose $\boldsymbol{p} : \mathcal{V} \to J \times \Sigma$ satisfies the variability condition **(H1)**. Then the following are equivalent:*

1. *$\boldsymbol{p}$ is a pre-envelope of $\boldsymbol{\Gamma}_d$;*
2. *$\langle \partial_\sigma \boldsymbol{\Gamma}_d(\boldsymbol{p}(s)), \boldsymbol{\Gamma}_d^L(\boldsymbol{p}(s)) \rangle = 0$ for all $s \in \mathcal{V}$;*
3. *$\xi_d(\boldsymbol{p}(s)) = 0$ for all $s \in \mathcal{V}$.*

For the de Sitter parallel $\boldsymbol{\Gamma}_d^\phi$ of $\boldsymbol{\Gamma}_d$, defined in Proposition 2, and with $\boldsymbol{F}_d^\phi(s) = \boldsymbol{\Gamma}_d^\phi(\boldsymbol{p}(s))$, we have:

Proposition 8. *Let $(\boldsymbol{\Gamma}_d, \boldsymbol{\Gamma}_d^L) : J \times \Sigma \to \Delta_3$ be a family of spacelike Legendrian curves parametrized by a single variable, endowed with spacelike de Sitter Legendrian curvature $(\mu_d, \nu_d, \xi_d, o_d, \pi_d)$. For fixed real ϕ, let $\boldsymbol{\Gamma}_d^\phi$ be the de Sitter parallel of $\boldsymbol{\Gamma}_d$. Suppose $\boldsymbol{p} : \mathcal{V} \to J \times \Sigma$ satisfies the variability condition **(H1)**. Then $\boldsymbol{F}_d$ is a de Sitter envelope of $\boldsymbol{\Gamma}_d$ if and only if $\boldsymbol{F}_d^\phi$ is a de Sitter envelope of $\boldsymbol{\Gamma}_d^\phi$.*

4. Relations Among Envelopes of Dual Frontal Families

This section explores the deep connections that exist between the envelopes of parameter-dependent frontal families that are linked by Legendrian duality.

Theorem 3. *Let $\boldsymbol{\Gamma}_h : J \times \Sigma \to \mathbb{H}_0^2$ and $\boldsymbol{\Gamma}_d : J \times \Sigma \to \mathbb{S}_1^2$ be families of spacelike frontals depending on a parameter. Define a new family in the lightcone by $\boldsymbol{\Gamma}_h^\ell(t, \sigma) = \boldsymbol{\Gamma}_h(t, \sigma) + \boldsymbol{\Gamma}_d(t, \sigma)$. If the pair $(\boldsymbol{\Gamma}_h, \boldsymbol{\Gamma}_d) : J \times \Sigma \to \Delta_1$ is a parameter-dependent family of spacelike Legendrian curves with spacelike hyperbolic Legendrian curvature $(\alpha_h, \beta_h, \gamma_h, \delta_h, \varepsilon_h)$, then:*

(1) $(\boldsymbol{\Gamma}_h, \boldsymbol{\Gamma}_h^\ell) : J \times \Sigma \to \Delta_2$ is a family of spacelike Legendrian curves indexed by a parameter. Its curvature $(\widetilde{\alpha}_h, \widetilde{\beta}_h, \widetilde{\gamma}_h, \widetilde{\delta}_h, \widetilde{\varepsilon}_h)$ is given by $\widetilde{\alpha}_h = \alpha_h$, $\widetilde{\beta}_h = \alpha_h + \beta_h$, $\widetilde{\gamma}_h = -\gamma_h$, $\widetilde{\delta}_h = \delta_h$, $\widetilde{\varepsilon}_h = \delta_h + \varepsilon_h$;

(2) The swapped pair $(\boldsymbol{\Gamma}_h^\ell, \boldsymbol{\Gamma}_h) : J \times \Sigma \to \Delta_2$ is also a parameter-dependent family of spacelike Legendrian curves. Its curvature $(\widehat{\alpha}_h, \widehat{\beta}_h, \widehat{\gamma}_h, \widehat{\delta}_h, \widehat{\varepsilon}_h)$ is given by

$$\widehat{\alpha}_h = -\alpha_h - \beta_h, \quad \widehat{\beta}_h = -\alpha_h, \quad \widehat{\gamma}_h = \gamma_h, \quad \widehat{\delta}_h = -\delta_h - \varepsilon_h, \quad \widehat{\varepsilon}_h = -\delta_h.$$

Proof. The proof proceeds by direct computation using the given data and the definitions. Since $(\boldsymbol{\Gamma}_h, \boldsymbol{\Gamma}_d)$ lies in Δ_1, we have a moving frame $\{\boldsymbol{\Gamma}_h, \boldsymbol{\Gamma}_d, \boldsymbol{\Xi}_h\}$ with $\boldsymbol{\Xi}_h = \boldsymbol{\Gamma}_h \wedge \boldsymbol{\Gamma}_d$, satisfying the hyperbolic Frenet–Serret formulas. The condition $\boldsymbol{\Gamma}_h^\ell = \boldsymbol{\Gamma}_h + \boldsymbol{\Gamma}_d$ is used to compute its derivatives. For instance, $\partial_t \boldsymbol{\Gamma}_h^\ell = \partial_t \boldsymbol{\Gamma}_h + \partial_t \boldsymbol{\Gamma}_d = \alpha_h \boldsymbol{\Xi}_h + \beta_h \boldsymbol{\Xi}_h = (\alpha_h + \beta_h)\boldsymbol{\Xi}_h$. The inner products defining the new curvature functions are then evaluated using the known orthonormality relations among $\boldsymbol{\Gamma}_h$, $\boldsymbol{\Gamma}_d$, and $\boldsymbol{\Xi}_h$, and the fact that $\langle \boldsymbol{\Gamma}_h, \boldsymbol{\Gamma}_d \rangle = 0$ and $\langle \boldsymbol{\Gamma}_h, \boldsymbol{\Gamma}_h \rangle = -1$, $\langle \boldsymbol{\Gamma}_d, \boldsymbol{\Gamma}_d \rangle = 1$. The calculations for the second family $(\boldsymbol{\Gamma}_h^\ell, \boldsymbol{\Gamma}_h)$ are analogous, noting that its third frame vector is $-\boldsymbol{\Xi}_h$. □

This theorem has an immediate and important corollary regarding the pre-envelopes of Δ_2-dual families.

Corollary 1. *Let $(\mathbf{\Gamma}_h, \mathbf{\Gamma}_h^\ell) : J \times \Sigma \to \Delta_2$ be a parameter-dependent family of spacelike Legendrian curves possessing spacelike hyperbolic Legendrian curvature $(\alpha_h, \beta_h, \gamma_h, \delta_h, \varepsilon_h)$. Suppose $\boldsymbol{p} : \mathcal{V} \to J \times \Sigma$ satisfies the variability condition **(H1)**. Then $\boldsymbol{p}$ is a pre-envelope of $\mathbf{\Gamma}_h$ if and only if $\boldsymbol{p}$ is a pre-envelope of $\mathbf{\Gamma}_h^\ell$.*

Proof. From Theorem 3, the curvature function corresponding to γ for the pair $(\mathbf{\Gamma}_h, \mathbf{\Gamma}_h^\ell)$ is $\widetilde{\gamma}_h = -\gamma_h$, while for the swapped pair $(\mathbf{\Gamma}_h^\ell, \mathbf{\Gamma}_h)$ it is $\widehat{\gamma}_h = -\widetilde{\gamma}_h = \gamma_h$. The Envelope Theorem 1 states that $\boldsymbol{p}$ is a pre-envelope for a given family iff the corresponding γ-function vanishes along $\boldsymbol{p}(\mathcal{V})$. Therefore, $\boldsymbol{p}$ is a pre-envelope for $(\mathbf{\Gamma}_h, \mathbf{\Gamma}_h^\ell)$ iff $\gamma_h(\boldsymbol{p}(s)) = 0$, which is exactly the condition for it to be a pre-envelope for $(\mathbf{\Gamma}_h^\ell, \mathbf{\Gamma}_h)$. □

A completely analogous relationship holds for Δ_3-dual families.

Theorem 4. *Let $\mathbf{\Gamma}_h : J \times \Sigma \to \mathbb{H}_0^2$ and $\mathbf{\Gamma}_d : J \times \Sigma \to \mathbb{S}_1^2$ be families of spacelike frontals parametrized by a single variable. Define a lightcone family by $\mathbf{\Gamma}_d^L(t,\sigma) = \mathbf{\Gamma}_d(t,\sigma) + \mathbf{\Gamma}_h(t,\sigma)$. If $(\mathbf{\Gamma}_d, \mathbf{\Gamma}_h) : J \times \Sigma \to \Delta_1$ is a parameter-dependent family of spacelike Legendrian curves with spacelike de Sitter Legendrian curvature $(\mu_d, \nu_d, \xi_d, o_d, \pi_d)$, then:*

(1) $(\mathbf{\Gamma}_d, \mathbf{\Gamma}_d^L) : J \times \Sigma \to \Delta_3$ is a family of spacelike Legendrian curves indexed by a parameter. Its curvature $(\widetilde{\mu}_d, \widetilde{\nu}_d, \widetilde{\xi}_d, \widetilde{o}_d, \widetilde{\pi}_d)$ is given by $\widetilde{\mu}_d = \mu_d$, $\widetilde{\nu}_d = \mu_d + \nu_d$, $\widetilde{\xi}_d = -\xi_d$, $\widetilde{o}_d = o_d$, $\widetilde{\pi}_d = o_d + \pi_d$;

(2) The swapped pair $(\mathbf{\Gamma}_d^L, \mathbf{\Gamma}_d) : J \times \Sigma \to \Delta_3$ is also a parameter-dependent family of spacelike Legendrian curves. Its curvature $(\widehat{\mu}_d, \widehat{\nu}_d, \widehat{\xi}_d, \widehat{o}_d, \widehat{\pi}_d)$ is given by

$$\widehat{\mu}_d = -\mu_d - \nu_d, \quad \widehat{\nu}_d = -\mu_d, \quad \widehat{\xi}_d = \xi_d, \quad \widehat{o}_d = -o_d - \pi_d, \quad \widehat{\pi}_d = -o_d.$$

Proof. The proof mirrors that of Theorem 3, but uses the de Sitter Frenet–Serret formulas for the frame $\{\mathbf{\Gamma}_d, \mathbf{\Gamma}_h, \mathbf{\Xi}_d\}$, where $\mathbf{\Xi}_d = \mathbf{\Gamma}_d \wedge \mathbf{\Gamma}_h = -\mathbf{\Xi}_h$. □

Corollary 2. *Let $(\mathbf{\Gamma}_d, \mathbf{\Gamma}_d^L) : J \times \Sigma \to \Delta_3$ be a family of spacelike Legendrian curves depending on a parameter, endowed with spacelike de Sitter Legendrian curvature $(\mu_d, \nu_d, \xi_d, o_d, \pi_d)$. Suppose $\boldsymbol{p} : \mathcal{V} \to J \times \Sigma$ satisfies the variability condition **(H1)**. Then $\boldsymbol{p}$ is a pre-envelope of $\mathbf{\Gamma}_d$ if and only if $\boldsymbol{p}$ is a pre-envelope of $\mathbf{\Gamma}_d^L$.*

Remark 4. *The corollaries above encapsulate a key finding: the Δ_2 duality between $\mathbf{\Gamma}_h$ and $\mathbf{\Gamma}_h^\ell$, and the Δ_3 duality between $\mathbf{\Gamma}_d$ and $\mathbf{\Gamma}_d^L$, imply that these pairs of frontal families share precisely the same pre-envelopes. Consequently, their envelopes, whether in hyperbolic space, de Sitter space, or the lightcone, are determined by a common pre-envelope map $\boldsymbol{p}$.*

5. Illustrative Examples

This section presents two concrete examples that illustrate the theoretical concepts developed in the paper. The accompanying figures provide a visual representation of the frontal families, their envelopes, and the duality relations.

Example 1 (A Δ_2-dual family and its envelopes). *Let $\mathcal{U} \subset \mathbb{R}^2$ be the open set defined by*

$$\mathcal{U} = \{(t,\sigma) \in \mathbb{R}^2 \mid V(t,\sigma) > 0,\ W(t,\sigma) > 0\},$$

where V and W are given below. Consider the maps $\mathbf{\Gamma}_h, \mathbf{\Gamma}_h^\ell : \mathcal{U} \to \mathbb{E}_1^3$ defined by

$$\mathbf{\Gamma}_h(t,\sigma) = \big(t^2+\sigma,\ t^3,\ \sqrt{t^4+2t^2\sigma+\sigma^2-t^6-1}\,\big),$$
$$\mathbf{\Gamma}_h^{\ell}(t,\sigma) = \Big(t^2+\sigma+\frac{-3t+t^5+4t^3\sigma+3t\sigma^2}{\sqrt{W}},\ t^3+\frac{-2+t^6+3t^4\sigma}{\sqrt{W}},\ \sqrt{V}+\frac{t^3\sqrt{V}+3t\sigma\sqrt{V}}{\sqrt{W}}\Big),$$

where the auxiliary functions are

$$V = t^4+2t^2\sigma+\sigma^2-t^6-1, \qquad W = 4+t^6+6t^4\sigma+9t^2(-1+\sigma^2).$$

(These expressions are well-defined on the open subset of $\mathbb{R}^2$ *where* $V \geq 0$ *and* $W > 0$*; for our choice of* $\boldsymbol{p}$ *below this condition is satisfied.) A direct but lengthy computation verifies that* $(\mathbf{\Gamma}_h, \mathbf{\Gamma}_h^{\ell})$ *takes values in* Δ_2 *and satisfies the Legendrian condition, thus forming a parameter-dependent family of spacelike Legendrian curves. The associated frame vector* $\mathbf{\Xi}_h = \mathbf{\Gamma}_h \wedge \mathbf{\Gamma}_h^{\ell}$ *is found to be*

$$\mathbf{\Xi}_h(t,\sigma) = \Big(\frac{2\sqrt{V}}{\sqrt{W}},\ \frac{3t\sqrt{V}}{\sqrt{W}},\ \frac{2t^2-3t^4+2\sigma}{\sqrt{W}}\Big) \in \mathbb{S}_1^2.$$

The spacelike hyperbolic Legendrian curvature functions $(\alpha_h, \beta_h, \gamma_h, \delta_h, \varepsilon_h)$ *can be derived from the Frenet–Serret formulas as follows*

$$\begin{aligned}
\alpha_h(t,\sigma) &= \frac{t\sqrt{W}}{\sqrt{V}},\\
\beta_h(t,\sigma) &= \frac{t^{10}+24t^2\sigma+9t^8\sigma-6+6\sigma^2-15t^6-27t^6\sigma^2+10t^4-27t^4\sigma}{W\sqrt{V}}\\
&\quad+\frac{27t^4\sigma^3+(4t+t^7+6t^5\sigma-9t^3+9t^3\sigma^2)\sqrt{W}}{W\sqrt{V}},\\
\gamma_h(t,\sigma) &= \frac{-3t}{\sqrt{W}},\\
\delta_h(t,\sigma) &= \frac{2-t^6-3t^4\sigma}{\sqrt{V}\sqrt{W}},\\
\varepsilon_h(t,\sigma) &= \frac{(-2+t^6+3t^4\sigma)(t^3+3t\sigma+\sqrt{W})}{W\sqrt{V}}.
\end{aligned}$$

Now, choose the map $\boldsymbol{p} : \mathcal{V} \to \mathbb{R}\times\mathbb{R}$ *given by* $\boldsymbol{p}(s) = (0,s)$*, where we take* $\mathcal{V} = (1,\infty)$ *(the symmetric branch* $\mathcal{V} = (-\infty,-1)$ *yields a similar envelope by reflection). The restricted curves are*

$$\boldsymbol{F}_h(s) = \mathbf{\Gamma}_h(0,s) = (s,0,\sqrt{s^2-1}),$$
$$\boldsymbol{F}_h^{\ell}(s) = \mathbf{\Gamma}_h^{\ell}(0,s) = (s,-1,\sqrt{s^2-1}),$$

and the frame vector along $\boldsymbol{p}$ *is* $\mathbf{\Xi}_h(\boldsymbol{p}(s)) = (\sqrt{s^2-1},0,s)$*. Differentiating, we obtain* $\sigma'(s) = 1$*,* $\boldsymbol{F}_h'(s) = (1,0,s/\sqrt{s^2-1})$*, and* $(\boldsymbol{F}_h^{\ell})'(s) = (1,0,s/\sqrt{s^2-1})$*. Both derivatives are clearly parallel to* $\mathbf{\Xi}_h(\boldsymbol{p}(s))$ *for all* $s > 1$*. (When* $s \to 1^+$*,* $\boldsymbol{F}_h(s)$ *approaches the lightlike point* $(1,0,0)$ *and the derivative diverges, so the envelope is defined only on the open interval* $(1,\infty)$*.) Thus, conditions* ***(H1)***, ***(H2)***, *and* ***(H3)*** *are satisfied. Consequently,* $\boldsymbol{p}$ *is a pre-envelope,* $\boldsymbol{F}_h$ *is a hyperbolic envelope (Figure 1), and* $\boldsymbol{F}_h^{\ell}$ *is a lightcone envelope (Figure 2). The* Δ_2*-duality between the two families is illustrated in Figure 3 for a fixed* $\sigma = 1.5$*, and Figure 4 shows several members of the families for* $\sigma = 1, 1.5, 2, 2.5$ *along with their envelopes.*

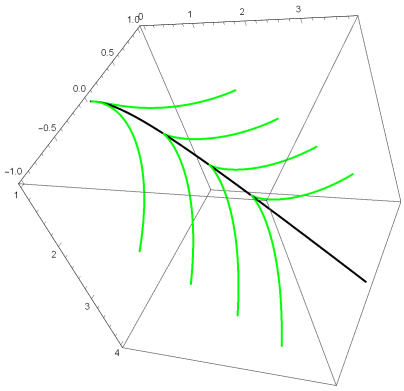


Figure 1. The images of $\gamma_h(t,\sigma)$ and its envelope (in black), where $\sigma = 1, 1.5, 2, 2.5$.

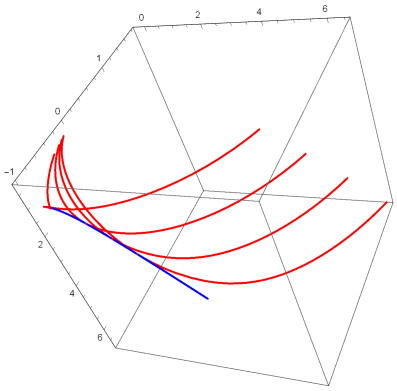


Figure 2. The images of $\gamma_h(t,\sigma)$ and its envelope (in blue), where $\sigma = 1, 1.5, 2, 2.5$.

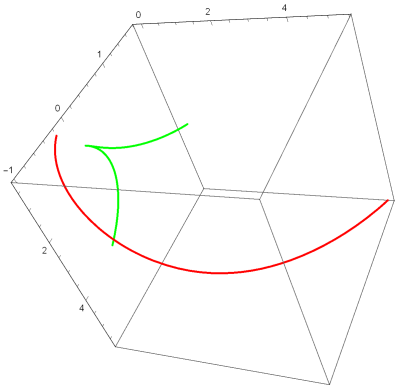


Figure 3. The Δ_2-duality of $\gamma_h(t,\sigma)$ (in green) and $\gamma_h^\ell(t,\sigma)$ (in red) when $\sigma = 1.5$.

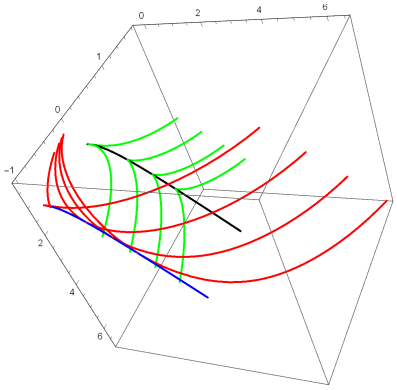


Figure 4. The Δ_2-duality of $\gamma_h(t,\sigma)$ and $\gamma_h^\ell(t,\sigma)$ when $\sigma = 1, 1.5, 2, 2.5$ and their envelopes.

Example 2 (A Δ_3-dual family and its envelopes). *Let $\mathcal{U} \subset \mathbb{R}^2$ be the open set defined by*

$$\mathcal{U} = \{(t,\sigma) \in \mathbb{R}^2 \mid V(t,\sigma) > 0,\ W(t,\sigma) > 0\},$$

where V and W are given below. Consider the maps $\mathbf{\Gamma}_d, \mathbf{\Gamma}_d^L : \mathcal{U} \to \mathbb{E}_1^3$ defined by

$$\mathbf{\Gamma}_d(t,\sigma) = \Big(t^3,\ \sqrt{V},\ t^2+\sigma\Big),$$
$$\mathbf{\Gamma}_d^L(t,\sigma) = \Big(t^3 + \frac{-2+t^6+3t^4\sigma}{\sqrt{W}},\ \sqrt{V} + \frac{t(t^2+3\sigma)\sqrt{V}}{\sqrt{W}},\ t^2+\sigma+\frac{t(-3+t^4+4t^2\sigma+3\sigma^2)}{\sqrt{W}}\Big),$$

with the auxiliary functions

$$V = 1 - t^4 + t^6 - 2t^2\sigma - \sigma^2, \qquad W = 4 + t^6 + 6t^4\sigma + 9t^2(-1+\sigma^2).$$

(Note that W takes the same form as in Example 1; it is linear in σ, not quadratic.) One can check that $(\mathbf{\Gamma}_d, \mathbf{\Gamma}_d^L)$ maps into Δ_3 and satisfies the Legendrian condition $\langle \partial_t \mathbf{\Gamma}_d, \mathbf{\Gamma}_d^L \rangle = 0$, making it a parameter-dependent family of spacelike Legendrian curves. The third frame vector is

$$\Xi_d(t,\sigma) = \Big(\frac{2\sqrt{V}}{\sqrt{W}},\ \frac{3t\sqrt{V}}{\sqrt{W}},\ \frac{2t^2-3t^4+2\sigma}{\sqrt{W}}\Big) \in \mathbb{S}_1^2.$$

The spacelike de Sitter–Legendrian curvature functions $(\alpha_d, \beta_d, \gamma_d, \delta_d, \varepsilon_d)$ are given by

$$\alpha_d(t,\sigma) = \frac{t\sqrt{W}}{\sqrt{V}},$$
$$\beta_d(t,\sigma) = \frac{t^{10}+24t^2\sigma+9t^8\sigma-6+6\sigma^2-15t^6+27t^6\sigma^2+10t^4-27t^4\sigma}{W\sqrt{V}} + \frac{27t^4\sigma^3+(4t+t^7+6t^5\sigma-9t^3+9t^3\sigma^2)\sqrt{W}}{W\sqrt{V}},$$
$$\gamma_d(t,\sigma) = \frac{-3t}{\sqrt{W}},$$
$$\delta_d(t,\sigma) = \frac{2-t^6-3t^4\sigma}{\sqrt{V}\sqrt{W}},$$
$$\varepsilon_d(t,\sigma) = -\frac{(-2+t^6+3t^4\sigma)(t^3+3t\sigma+\sqrt{W})}{W\sqrt{V}}.$$

Now, choose the pre-envelope candidate $\boldsymbol{p} : \mathcal{V} \to \mathcal{U}$ given by $\boldsymbol{p}(s) = (0,s)$, where we take $\mathcal{V} = (-1,1)$. For $s \in (-1,1)$, we have $t = 0$, $\sigma = s$, so $V = 1 - s^2 > 0$ and $W = 4 > 0$, hence $\boldsymbol{p}(s) \in \mathcal{U}$. The restricted curves are

$$\boldsymbol{F}_d(s) = \mathbf{\Gamma}_d(0,s) = (0, \sqrt{1-s^2}, s),$$
$$\boldsymbol{F}_d^L(s) = \mathbf{\Gamma}_d^L(0,s) = (-1, \sqrt{1-s^2}, s),$$

*and $\Xi_d(\boldsymbol{p}(s)) = (0, -s, \sqrt{1-s^2})$. Differentiation yields $\sigma'(s) = 1$, $\boldsymbol{F}_d'(s) = (0, -s/\sqrt{1-s^2}, 1)$, and $(\boldsymbol{F}_d^L)'(s) = (0, -s/\sqrt{1-s^2}, 1)$. Both derivatives are parallel to $\Xi_d(\boldsymbol{p}(s))$ for all $s \in (-1,1)$. (When $s \to \pm 1$, $\boldsymbol{F}_d(s)$ approaches the lightlike point $(0,0,\pm 1)$ and the derivative diverges, so the envelope is defined only on the open interval $(-1,1)$.) Hence, conditions (**H**1), ($\widetilde{\mathbf{H}}$2), and ($\widetilde{\mathbf{H}}$3) are met. Therefore, $\boldsymbol{p}$ is a pre-envelope, $\boldsymbol{F}_d$ is a de Sitter envelope (Figure 5), and $\boldsymbol{F}_d^L$ is a lightcone envelope (Figure 6). The Δ_3-duality is depicted in Figure 7 for $\sigma = -1$, and Figure 8 shows the families for various σ values $(0, \pm 0.1, \pm 0.5, \pm 0.7, -1)$ together with their envelopes.*

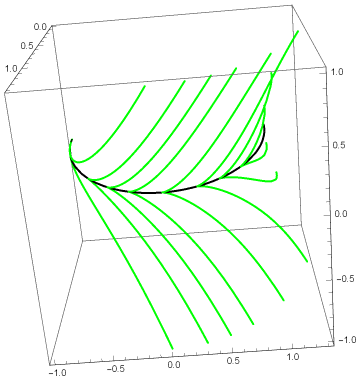


Figure 5. The images of $\gamma_d(t,\sigma)$ and its envelope (in black), where $\sigma = 0, \pm 0.1, \pm 0.5, \pm 0.7, -1$.

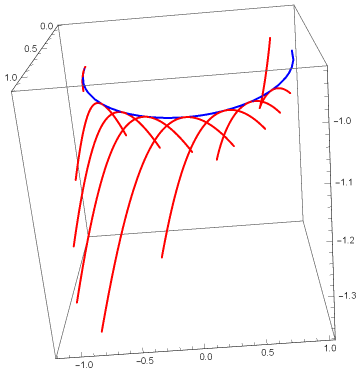


Figure 6. The images of $\gamma_d(t,\sigma)$ and its envelope (in blue), where $\sigma = 0, \pm 0.1, \pm 0.5, \pm 0.7, -1$.

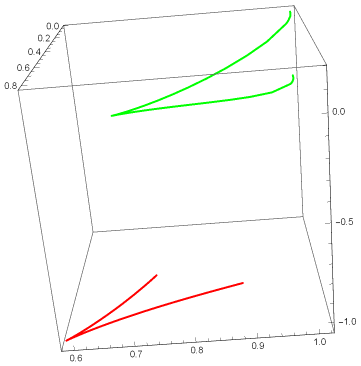


Figure 7. The Δ_3-duality of $\gamma_d(t,\sigma)$ and $\gamma_d^L(t,\sigma)$ when $\sigma = -1$.

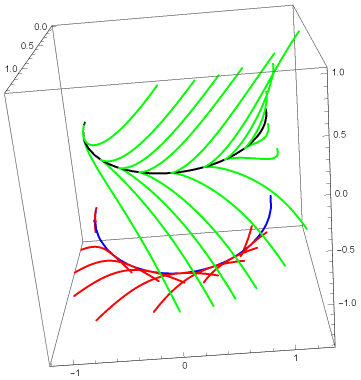


Figure 8. The Δ_3-duality of $\gamma_d(t,\sigma)$ and $\gamma_d^L(t,\sigma)$ and their envelopes, where $\sigma = 0, \pm 0.1, \pm 0.5, \pm 0.7, -1$.

6. Concluding Remarks

This work has established a comprehensive framework for understanding envelopes of parameter-dependent families of frontals that are governed by Δ_2 and Δ_3 dualities within the three pseudo-spheres. Through the formulation of Envelope Theorems, we have provided clear criteria for the existence of these envelopes and characterized them as frontals themselves in the respective spaces (hyperbolic, de Sitter, or lightcone). A pivotal result is the demonstration that Δ_2-dual families, as well as Δ_3-dual families, invariably share the same pre-envelope, underscoring the profound geometric link imposed by Legendrian duality. Furthermore, we have shown that the operation of taking a hyperbolic or de Sitter parallel commutes with the process of finding an envelope.

The primary methodological contribution of this paper lies in presenting an effective strategy for determining envelopes of frontal families depending on a parameter, particularly those residing in the degenerate light cone. Two complementary pathways are now available: a direct computation based on the definition involving variability and tangency conditions, and an indirect, often simpler, computation that leverages the duality relations. By transforming the problem to a dual curve family, one can bypass some of the complexities inherent in the light cone geometry.

Looking ahead, several avenues for future research present themselves. A natural extension is to investigate envelope problems for two parameter-dependent families of frontals that exhibit Δ_4-duality, also within the light cone. The highly degenerate nature of this setting will necessitate the construction of entirely new types of frames adapted to light cone frontals and the formulation of appropriate tangency conditions to define light cone envelopes for families of light cone Legendrian curves indexed by a parameter. Another promising direction involves extending this study to families of curves parametrized by a single variable lying on more general surfaces—spacelike, timelike, or lightlike surfaces with non-constant Gaussian curvature. Such an investigation promises to yield more general and powerful results in the field of singularity theory and differential geometry. Inspired by Arnold's classical work on Legendrian submanifolds in optics, the dualities developed here may also find applications in relativistic optics, a direction we leave for future investigation.

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