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Jordan Ideals of Vector-Valued Beurling Algebras and θ-Lau Product of Banach Algebras

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Abstract

In this paper, we give a characterization of the closed Jordan ideals of vector-valued Beurling algebras $L^1(G, \mathcal{A}, \omega)$ and the direct sum of Banach algebras. We also investigate Jordan ideals of the θ-Lau product of Banach algebras and give some results about Banach algebras $\mathcal{A}^\sharp$ and $\mathcal{A} \oplus \mathcal{B}$. We also prove that every Jordan ideal of $\mathcal{A} \oplus \mathcal{B}$ is an ideal if and only if every Jordan ideal of $\mathcal{A}$ and $\mathcal{B}$ is an ideal of $\mathcal{A}$ and $\mathcal{B}$, respectively, where $\mathcal{A}$ is a unital Banach algebra and $\mathcal{B}$ is any Banach algebra. Finally, we apply our results to some Banach algebras related to a locally compact group.

Keywords: Jordan ideal; vector valued Beurling algebra; θ-Lau product Banach algebra

MSC: 17C65; 43A20; 22D15

1. Introduction and Preliminaries

Let $(\mathcal{A}, \| \ \|_{\mathcal{A}})$ be a Banach algebra, G be a locally compact group with a left Haar measure μ and the modular function Δ. For any Borel measurable function $f\colon G \to \mathcal{A}$, let

$$\|f\|_{\mathcal{A}} = \int_G \|f(x)\|_{\mathcal{A}} d\mu(x).$$

We denote by $L^1(G, \mathcal{A})$ the Banach space of all Borel measurable functions $f\colon G \to \mathcal{A}$ such that $\|f\|_{\mathcal{A}} < \infty$. For each $f \in L^1(G, \mathcal{A})$, define $\overline{f}(x) = \|f(x)\|_{\mathcal{A}}$. Then $f \in L^1(G, \mathcal{A})$ if and only if $\overline{f} \in L^1(G)$. Let us recall that the convolution on $L^1(G, \mathcal{A})$ can be defined by

$$(f * g)(x) = \int_G f(xy) g(y^{-1}) d\mu(y) = \int_G f(y) g(y^{-1}x) d\mu(y)$$

for all $f, g \in L^1(G, \mathcal{A})$. With convolution as multiplication, $L^1(G, \mathcal{A})$ becomes a Banach algebra under pointwise addition and scalar multiplication; this algebra is called the *vector-valued group algebra* [1]. Let us recall that a *weight function* on G is a Borel measurable function $\omega\colon G \to [1, \infty)$ such that $\omega(e) = 1$ and

$$\omega(xy) \leq \omega(x)\omega(y) \quad (x, y \in G),$$

where e is the identity element of G. Thus we can define the *vector-valued Beurling algebra*

$$L^1(G, \mathcal{A}, \omega) = \{f \in L^1(G, \mathcal{A}) : \int_G \|f(x)\|_{\mathcal{A}} \omega(x) d\mu(x) < \infty\}.$$

Check for updates

Academic Editor: Deguang Han

Received: 22 March 2026
Revised: 18 April 2026
Accepted: 21 April 2026
Published: 23 April 2026

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In the case where $\mathcal{A} = \mathbb{C}$, we set $L^1(G, \omega)$ for $L^1(G, \mathcal{A}, \omega)$. If also, $\omega = 1$, then we write $L^1(G)$ for $L^1(G, \mathcal{A}, \omega)$.

Let $\mathcal{B}$ be also a Banach algebra. Then the Cartesian product space $\mathcal{A} \times \mathcal{B}$ is a space with the usual operations. If we choose a linear functional θ: $\mathcal{B} \to \mathbb{C}$ and equip $\mathcal{A} \times \mathcal{B}$ with the product

$$(a, b) \cdot_\theta (x, y) = (ax + \theta(b)x + \theta(y)a, by), \quad \big((a, b), (x, y) \in \mathcal{A} \times \mathcal{B}\big).$$

and the norm

$$\|(a, b)\|_{\mathcal{A}\times\mathcal{B}} = \|a\|_{\mathcal{A}} + \|b\|_{\mathcal{B}},$$

then $\mathcal{A} \times \mathcal{B}$ is a Banach algebra. This Banach algebra is called the *θ-Lau product* of $\mathcal{A}$ and $\mathcal{B}$, denoted it by $\mathcal{A} \times_\theta \mathcal{B}$. The θ-Lau products $\mathcal{A} \times_\theta \mathcal{B}$ were first introduced by Lau [2], for certain Banach algebras. Sangani Monfared [3] extended this product to arbitrary Banach algebras $\mathcal{A}$ and $\mathcal{B}$; see also [4,5].

A linear subspace $\mathcal{J}$ of $\mathcal{A}$ is called a *Jordan ideal* if for every $a \in \mathcal{A}$ and $b \in \mathcal{J}$, we have

$$a \circ b := ab + ba \in \mathcal{J}.$$

Recall that a standard two-sided ideal requires both $ab \in \mathcal{J}$ and $ba \in \mathcal{J}$ for all $a \in \mathcal{A}$ and $b \in \mathcal{J}$. It is obvious that every such ideal is also a Jordan ideal. Note that the converse does not remain valid, in general.

Example 1. *Let $\mathbb{F}$ be a field with* $\mathrm{char}(\mathbb{F}) \neq 2$. *Let $M_4(\mathbb{F})$ be the associative algebra of 4×4 matrices over $\mathbb{F}$. Denote by E_{ij} the standard matrix units, so that $E_{ij}E_{kl} = \delta_{jk}E_{il}$. Define three matrices in $M_4(\mathbb{F})$ by*

$$e := E_{21} + E_{43}, \qquad f := E_{31} - E_{42}, \qquad g := E_{41}.$$

Let

$$\mathcal{A} := \mathrm{span}_{\mathbb{F}}\{e, f, g\} \subseteq M_4(\mathbb{F}), \qquad \mathcal{J} := \mathrm{span}_{\mathbb{F}}\{e, f\} \subseteq \mathcal{A}.$$

Take arbitrary

$$a = \alpha e + \beta f + \gamma g \in \mathcal{A}, \qquad b = ue + vf \in \mathcal{J}.$$

Since $ef = g$ and $fe = -g$, we have $a \circ b = 0 \in \mathcal{J}$. Thus $\mathcal{A} \circ J \subseteq J$, and $\mathcal{J}$ is a Jordan ideal of $\mathcal{A}$. But $ef = g \notin \mathcal{J}$, this shows $\mathcal{J}$ is not an two-sided ideal of $\mathcal{A}$.

Many authors have been interested in finding conditions under which a non-zero Jordan ideal is an associative ideal [6–12]. It was proved that the Jordan ideals of all simple algebras [9], all full matrix algebras $M_n(\mathcal{A})$, where $\mathcal{A}$ is a unital algebra [10], all W^*-algebras [8] are associative ideals. Li and Lu [11] showed that a weakly closed Jordan ideal in a nest algebra on a Banach space, which satisfies certain conditions, is an associative ideal. In ref. [13], the authors recently characterized the closed Lie ideals of vector-valued group algebras.

In this paper, we investigate Jordan ideals of Banach algebras. In Section 2, we give a characterization of the closed Jordan ideal of vector-valued Beurling algebras $L^1(G, \mathcal{A}, \omega)$ and show that a closed subspace $\mathcal{J}$ of $L^1(G, \mathcal{A}, \omega)$ is a Jordan ideal if and only if

$$\Delta(x^{-1})(f_{x^{-1}} \cdot a) + a \cdot {}_x f \in \mathcal{J}$$

for all $f \in \mathcal{J}, x \in G$ and $a \in \mathcal{A}$. In Section 3, we study Jordan ideals of θ-Lau products $\mathcal{A} \times_\theta \mathcal{B}$. In particular, we present a necessary and sufficient condition under which a

subspace of the direct sum of Banach algebras is a Jordan ideal. For a unital Banach algebra $\mathcal{A}$, we establish that every Jordan ideal of $\mathcal{A} \oplus \mathcal{B}$ is an ideal if and only if every Jordan ideal of $\mathcal{A}$ and $\mathcal{B}$ is an ideal of $\mathcal{A}$ and $\mathcal{B}$, respectively. Finally, we apply our results to finite group algebra $\mathbb{C}[G]$, as well as to the Banach algebras $\mathrm{LUC}(G)^*$ and $M_*(G)^*$.

2. Jordan Ideals of Vector-Valued Beurling Algebras

We denote by $C_c(G, \mathcal{A})$ the space of all continuous functions from G into $\mathcal{A}$ with compact support. If $\mathcal{A} = \mathbb{C}$, then we put $C_c(G) := C_c(G, \mathbb{C})$. It is well known that $C_c(G)$ is norm-dense in $L^1(G)$. To establish the density of this space in the vector-valued and weighted setting, we employ a decoupling approximation strategy. Specifically, we first treat $f\omega$ as an unweighted vector-valued function and approximate it by a continuous, compactly supported vector-valued function h. Subsequently, we locally approximate the weight ω by a continuous positive scalar function η. By constructing the quotient function $g = h/\eta$, we can effectively control the Bochner norm. This reasoning naturally leads to the following result.

Lemma 1. *Let G be a locally compact group and ω be a weight function on G. Then $C_c(G, \mathcal{A})$ is a norm dense subspace of $L^1(G, \mathcal{A}, \omega)$.*

By [1] (Lemma 1.3.6), the maps $x \mapsto {}_xf$ and $x \mapsto f_x$ from G into $L^1(G, \mathcal{A})$ are continuous, where ${}_xf(y) = f(x^{-1}y)$ and $f_x(y) = f(yx)$ for all $x, y \in G$. Similarly, since ω is locally bounded [1] (Lemma 1.3.3), with the necessary modifications in the proof, the result can also be established for $L^1(G, \mathcal{A}, \omega)$ in place of $L^1(G, \mathcal{A})$.

Our next goal is to characterize the closed Jordan ideals of vector-valued Beurling algebras. For every $a \in \mathcal{A}$ and $f \in L^1(G, \mathcal{A}, \omega)$, consider the mappings $a \cdot f, f \cdot a \colon G \to \mathcal{A}$ given by

$$(a \cdot f)(x) = af(x) \quad \text{and} \quad (f \cdot a)(x) = f(x)a$$

for all $x \in G$. It is easy seen that $a \cdot f$ and $f \cdot a \in L^1(G, \mathcal{A}, \omega)$ for all $a \in \mathcal{A}$ and $f \in L^1(G, \mathcal{A}, \omega)$. Laursen [14] established that, for a Banach algebra $\mathcal{A}$ with an approximate identity, a closed subspace $\mathcal{J} \subseteq L^1(G, \mathcal{A})$ is an ideal precisely when it is $\mathcal{A}$-translation invariant, i.e., ${}_xf, f_x, a \cdot f, f \cdot a \in \mathcal{J}$ for all $f \in \mathcal{J}, a \in \mathcal{A}, x \in G$. This assertion also holds for vector-valued Beurling algebra $L^1(G, \mathcal{A}, \omega)$; see [15]. We now state the main result of this section.

Theorem 1. *Let G be a locally compact group, ω a weight function on G, and $\mathcal{A}$ a Banach algebra. A closed subspace $\mathcal{J}$ of $L^1(G, \mathcal{A}, \omega)$ is a Jordan ideal if and only if*

$$\Delta(x^{-1})(f_{x^{-1}} \cdot a) + a \cdot {}_xf \in \mathcal{J}$$

for all $f \in \mathcal{J}, x \in G$, and $a \in \mathcal{A}$.

Proof. First, we denote by χ_K the characteristic function of K on G, where K is a subset of G. Let $f \in \mathcal{J}, x \in G$ and $a \in \mathcal{A}$. Then for every $y \in G$, we have

$$\begin{aligned}
(\mu(K^{-1})f_{x^{-1}} - f * \chi_K)(y) &= \mu(K^{-1})f_{x^{-1}}(y) - \int f(yz)\chi_K(z^{-1})dz \\
&= \int_{K^{-1}} f_{x^{-1}}(y)dz - \int f(yz)\chi_{K^{-1}}(z)dz \\
&= \int_{K^{-1}} (f_{x^{-1}}(y) - f(yz))dz \\
&= \int_{K^{-1}} (f_{x^{-1}} - f_z)(y)dz.
\end{aligned}$$

https://doi.org/10.3390/axioms15050306

This implies that

$$\|\mu(K^{-1})f_{x^{-1}} - f * \chi_K\|_\omega \leq \int_{K^{-1}} \|f_{x^{-1}} - f_z\|_\omega dz. \tag{1}$$

For every $f \in L^1(G, \mathcal{A}, \omega)$, the maps $z \mapsto {}_zf, f_z$ are continuous. Hence for every $\epsilon > 0$, we can choose a compact symmetric neighborhood V of e such that

$$\sup_{z\in V} \|f_{x^{-1}} - f_{z^{-1}x^{-1}}\|_\omega < \frac{\epsilon}{2\|a\|\Delta(x^{-1})} \quad \text{and} \quad \sup_{z\in V} \|{}_xf - {}_{xz}f\|_\omega < \frac{\epsilon}{2\|a\|}.$$

Taking $K = xV$ in (1), we have

$$\begin{aligned} \|\mu(Vx^{-1})f_{x^{-1}} \cdot a - f * (a \cdot \chi_{xV})\|_\omega &\leq \mu(Vx^{-1})\|a\| \sup_{z\in Vx^{-1}} \|f_{x^{-1}} - f_z\|_\omega \\ &= \Delta(x^{-1})\mu(V)\|a\| \sup_{z\in V} \|f_{x^{-1}} - f_{z^{-1}x^{-1}}\|_\omega \\ &< \epsilon/2\mu(V). \end{aligned}$$

This together with the fact $\mu(Vx^{-1}) = \Delta(x^{-1})\mu(V)$ shows that

$$\|\Delta(x^{-1})f_{x^{-1}} \cdot a - \frac{1}{\mu(V)} f * (a \cdot \chi_{xV})\|_\omega < \epsilon/2. \tag{2}$$

Similarly, one can prove that

$$(\mu(V){}_xf - \chi_{xV} * f)(y) = \int_V ({}_xf - {}_{xz}f)(y)dz$$

for all $y \in G$. Hence we have

$$\|\mu(V)(a \cdot {}_x f) - (a \cdot \chi_{xV}) * f\|_\omega \leq \|a\| \int_V \|{}_xf - {}_{xz}f\|dz < \mu(V)\epsilon/2.$$

But,

$$\Delta(x^{-1})\|\mu(V)(a \cdot {}_x f) - (a \cdot \chi_{xV}) * f\|_\omega = \mu(Vx^{-1})\|a \cdot {}_xf - \frac{1}{\mu(V)} a_{xV} * f\|_\omega.$$

Thus

$$\|a \cdot {}_xf - \frac{1}{\mu(V)} a_{xV} * f\|_\omega < \epsilon/2. \tag{3}$$

From (2) and (3) we see that

$$\|\Delta(x^{-1})(f_{x^{-1}} \cdot a) + a \cdot {}_xf - \frac{1}{\mu(V)} \big(f * (a \cdot \chi_{xV}) + (a \cdot \chi_{xV}) * f\big)\|_\omega < \epsilon.$$

This implies that if $\mathcal{J}$ is a closed Jordan ideal of $L^1(G, \mathcal{A}, \omega)$, then

$$\Delta(x^{-1})(f_{x^{-1}} \cdot a) + a \cdot {}_xf \in \mathcal{J}.$$

Conversely, assume that $\phi \in C_c(G, \mathcal{A})$ and $K = \text{supp}(\phi) \cup \text{supp}(\phi)^{-1}$, where $\text{supp}(\phi)^{-1} = \{y^{-1} : y \in \text{supp}(\phi)\}$. Since the inversion map in the topological group G is continuous and $\text{supp}(\phi)$ is compact, the set K is also a compact subset of G. To approximate the integral, we partition the compact set K into a finite number of mutually

disjoint Borel measurable sets. Let $K = \cup_{i=1}^{n} K_i$, where $K_i \cap K_j = \emptyset$ for $i \neq j$. Then for every $f \in \mathcal{J}, y_i \in K_i$ and $x \in G$, we have

$$\begin{aligned}(\phi * f - \sum_{i=1}^{n} \mu(K_i)\phi(y_i)_{y_i} f)(x) &= \int_K \phi(y) f(y^{-1}x)dy - \sum_{i=1}^{n} \int_{K_i} \phi(y_i)_{y_i} f(x)dy \\ &= \sum_{i=1}^{n} \int_{K_i} (\phi(y)_y f(x) - \phi(y_i)_{y_i} f(x))dy.\end{aligned}$$

A similar argument shows that

$$\begin{aligned}(f * \phi - \sum_{i=1}^{n} \mu(K_i)\Delta(y_i^{-1}) f_{y_i^{-1}} \phi(y_i))(x) &= \textstyle\sum_{i=1}^{n} \int_{K_i} \Delta(y^{-1}) f_{y^{-1}} \phi(y) \\ &\quad -\Delta(y_i^{-1}) f_{y_i}(x)\phi(y_i).\end{aligned}$$

Hence

$$\begin{aligned}\|\phi * f + f * \phi &- \sum_{i=1}^{n} \mu(K_i)\big(\phi(y_i)\ {}_{y_i} f + \Delta(y_i^{-1}) f_{y_i^{-1}} \phi(y_i)\big)\|_\omega \\ &\leq \sum_{i=1}^{n} \int_{K_i} \|\phi(y)\ {}_y f - \phi(y_i)\ {}_{y_i} f + \Delta(y^{-1}) f_{y^{-1}} \phi(y) \\ &- \Delta(y_i^{-1}) f_{y_i^{-1}}(x)\phi(y_i)dy\big)\|_\omega dy.\end{aligned}$$

Since left and right translations, inverse function and module function are continuous, for any $\epsilon > 0$ we can choose finitely many measurable disjoint subsets $K_i (i = 1, \cdots n)$ of K such that

$$\|\phi(y)\ {}_y f - \Delta(y^{-1}) f_{y^{-1}} \phi(y) + \phi(z)\ {}_z f - \Delta(z^{-1}) f_{z^{-1}} \phi(z)\|_\omega \leq \frac{\epsilon}{\mu(K)}$$

for all $y, z \in K_i$, $1 \leq i \leq n$. The proof will be complete if we only recall that $C_c(G, \mathcal{A})$ is dense in $L^1(G, \mathcal{A}, \omega)$. □

As an immediate consequence of the preceding theorem, we obtain the following corollary.

Corollary 1. *Let G be a locally compact group and ω a weight function on G. A closed subspace $\mathcal{J}$ of Beurling algebra $L^1(G, \omega)$ is a Jordan ideal if and only if*

$$\Delta(x^{-1}) f_{x^{-1}} + {}_x f \in \mathcal{J}$$

for all $f \in \mathcal{J}, x \in G$.

The following example shows that one-sided invariance does not imply the Jordan ideal property; the symmetric condition in Corollary 1 is essential.

Example 2. *Let $G = \{e\}$ be the trivial group and let $\omega \equiv 1$. Then, in the isometric isomorphism sense,*

$$L^1(G, A, \omega) \cong A.$$

Take

$$A = M_2(\mathbb{C}), \qquad \mathcal{J} = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} : a, b \in \mathbb{C} \right\}.$$

For any

$$x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in A, \qquad y = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \in \mathcal{J},$$

we have

$$xy = \begin{pmatrix} x_{11}a + x_{12}b & 0 \\ x_{21}a + x_{22}b & 0 \end{pmatrix} \in \mathcal{J}.$$

Hence $\mathcal{J}$ is a closed left ideal of A. However, J is not a Jordan ideal. Let E_{ij} be the matrix units in $M_2(\mathbb{C})$; we have

$$E_{21}E_{12} + E_{12}E_{21} = E_{22} + E_{11} = I,$$

but $I \notin \mathcal{J}$.

3. Jordan Ideals of θ-Lau Product Banach Algebras

Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras. For linear functionals θ and γ on $\mathcal{B}$, a subspace $\mathcal{J}$ of $\mathcal{A} \times \mathcal{B}$ is called a (θ, γ)-*Jordan ideal* if for every $(a, b) \in \mathcal{A} \times \mathcal{B}$ and $(x, y) \in \mathcal{J}$, we have

$$\frac{1}{2}[(a, b) \cdot_\theta (x, y) + (x, y) \cdot_\gamma (a, b)] \in \mathcal{J}.$$

In the following of this section, we assume that $\mathcal{J}$ is a (θ, γ)-Jordan ideal and put

$$\mathcal{J}_1 = \{a \in \mathcal{A} : (a, y) \in \mathcal{J} \text{ for some } y \in \mathcal{B}\}$$

and

$$\mathcal{J}_2 = \{b \in \mathcal{B} : (x, b) \in \mathcal{J} \text{ for some } x \in \mathcal{A}\}.$$

Note that if $a \in \mathcal{J}_1$, then there is $y \in \mathcal{J}_2$ such that $(a, y) \in \mathcal{J}$. This is true for elements of $\mathcal{J}_2$. Hence $\mathcal{J} \subseteq \mathcal{J}_1 \times \mathcal{J}_2$. The following example shows that there are (θ, γ)-Jordan ideals $\mathcal{J} \neq \mathcal{J}_1 \times \mathcal{J}_2$.

Example 3. *Let $\mathcal{A} = \mathcal{B} = C_0(\mathbb{R})$, the space of all continuous functions on real numbers filed $\mathbb{R}$ vanishes at infinity. Define $\theta : \mathcal{B} \to \mathbb{C}$ by $\theta(f) = f(0)$. Then*

$$\mathcal{J} = \{(f, g) \in \mathcal{A} \times \mathcal{B} : f(0) = -g(0)\}$$

is a (θ, θ)-Jordan ideal and $\mathcal{J}_1 = \mathcal{J}_2 = C_0(\mathbb{R})$. Clearly, $\mathcal{J} \neq \mathcal{J}_1 \times \mathcal{J}_2$.

Proposition 1. *Let θ and γ be linear functionals on $\mathcal{B}$ such that $\mathcal{J}_2 \subseteq ker(\theta + \gamma)$. Then the following statements hold.*

(i) *If $\mathcal{A}$ has an approximate identity and $\mathcal{J}$ is closed, then $\mathcal{J} = \mathcal{J}_1 \times \mathcal{J}_2$.*
(ii) *If $\mathcal{A}$ has an identity, then $\mathcal{J} = \mathcal{J}_1 \times \mathcal{J}_2$.*

Proof. (i) Let $\{e_\alpha\}_\alpha$ be an approximate identity for $\mathcal{A}$. Let $x \in \mathcal{J}_1$ and $y \in \mathcal{J}_2$. Then $(x, z), (u, y) \in \mathcal{J}$ for some $u \in \mathcal{J}_1$ and $z \in \mathcal{J}_2$. Since $\mathcal{J}_2$ is the subspace of $\ker(\theta + \gamma)$, it follows that

$$\begin{aligned} \frac{1}{2}[(e_\alpha, 0) \cdot_\theta (x, z) + (x, z) \cdot_\gamma (e_\alpha, 0)] &= \frac{1}{2}[(e_\alpha x + x e_\alpha + e_\alpha(\theta + \gamma)(z), 0)] \\ &= \frac{1}{2}(e_\alpha x + x e_\alpha, 0) \to (x, 0). \end{aligned}$$

Therefore, $(x, 0) \in \mathcal{J}$. Similarly, $(u, 0) \in \mathcal{J}$. Consequently, $(u, y) - (u, 0) \in \mathcal{J}$. This shows that $(x, y) = (x, 0) + (0, y) \in \mathcal{J}$.

(ii) The proof is similar to (i). □

https://doi.org/10.3390/axioms15050306

It is easy to see that $\mathcal{J}_2$ is a Jordan ideal of $\mathcal{B}$. It is natural to ask: Is $\mathcal{J}_1$ a Jordan ideal of $\mathcal{A}$?

Proposition 2. *Let θ and γ be linear functionals on $\mathcal{B}$. Then $\mathcal{J}_1$ is a Jordan ideal if and only if $\mathcal{J}_1 = \mathcal{A}$ or $\mathcal{J}_2 \subseteq ker(\theta + \gamma)$.*

Proof. Let $\mathcal{J}_2 \subseteq \ker(\theta + \gamma)$. If $x \in \mathcal{J}_1$, then $(x, y) \in \mathcal{J}$ for some $y \in \mathcal{J}_2$. So, for every $a \in \mathcal{A}$, we have

$$\begin{aligned}(a,0) \cdot_\theta (x,y) + (x,y) \cdot_\gamma (a,0) &= (ax + xa + a(\theta+\gamma)(y), 0) \\ &= (ax + xa, 0) \in \mathcal{J}.\end{aligned}$$

Thus $ax + xa \in \mathcal{J}_1$. That is, $\mathcal{J}_1$ is a Jordan ideal.

Conversely, let $\mathcal{J}_1$ be a Jordan ideal and $\mathcal{J}_2 \not\subseteq \ker(\theta + \gamma)$. Choose $y \in \mathcal{J}_2$ with $(\theta + \gamma)(y) \neq 0$. Then there exists $x \in \mathcal{J}_1$ such that $(x, y) \in \mathcal{J}$. If $a \in \mathcal{A}$, then

$$(x,y) \cdot_\theta (a,0) + (a,0) \cdot_\gamma (x,y) = (xa + ax + a(\theta+\gamma)(y), 0) \in \mathcal{J}.$$

Hence $xa + ax + a(\theta + \gamma)(y) \in \mathcal{J}_1$. This shows that $a(\theta + \gamma)(y) \in \mathcal{J}_1$. Therefore, $a \in \mathcal{J}_1$. □

In the following, let $\mathcal{A}^\sharp$ be the unitization of $\mathcal{A}$.

Corollary 2. *Let $\mathcal{J}$ be a Jordan ideal of $\mathcal{A}^\sharp$ and $\mathcal{J}_2 \neq 0$. If $\mathcal{J}_1$ is a Jordan ideal of $\mathcal{A}$, then $\mathcal{J}_1 = \mathcal{A}$ and $\mathcal{J}_2 = \mathbb{C}$.*

Proof. Let $\theta = \gamma = \mathrm{id}_\mathcal{B}$, the identity map on $\mathcal{B}$. Then $\mathcal{A}^\sharp = \mathcal{A} \times_\theta \mathbb{C}$ and $\mathcal{J}$ is a (θ, γ)-Jordan ideal. Since $\mathcal{J}_2$ is a Jordan ideal of $\mathbb{C}$, it follows that it is an ideal of $\mathbb{C}$ and so $\mathcal{J}_2 = 0$ or $\mathbb{C}$. Thus $\mathcal{J}_2 = \mathbb{C}$. Hence $\mathcal{J}_2 \not\subseteq \ker(\theta + \gamma)$. Now, if $\mathcal{J}_1$ is a Jordan ideal of $\mathcal{A}$, then by Proposition 2, $\mathcal{J}_1 = \mathcal{A}$. □

Lemma 2. *Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be Jordan ideals of $\mathcal{A}$ and $\mathcal{B}$, respectively, and let θ and γ be linear functionals on $\mathcal{B}$. Then $\mathcal{I}_1 \times \mathcal{I}_2$ is a (θ, γ)-Jordan ideal if and only if $\mathcal{I}_1 = \mathcal{A}$ or $\mathcal{I}_2 \subseteq ker(\theta + \gamma)$.*

Proof. Let $\mathcal{J} = \mathcal{I}_1 \times \mathcal{I}_2$ be a (θ, γ)-Jordan ideal. Then $\mathcal{J}_1 = \mathcal{I}_1$ and $\mathcal{J}_2 = \mathcal{I}_2$. Since $\mathcal{J}_1$ is a Jordan ideal, by Proposition 2, $\mathcal{J}_1 = \mathcal{A}$ or $\mathcal{J}_2 \subseteq \ker(\theta + \gamma)$. The converse is trivial. □

Theorem 2. *The following statements are valid.*

(i) *If $\mathcal{J}$ is a Jordan ideal of $\mathcal{A} \oplus \mathcal{B}$, then $\mathcal{J}_1$ and $\mathcal{J}_2$ are Jordan ideals of $\mathcal{A}$ and $\mathcal{B}$, respectively.*

(ii) *Let $\mathcal{A}$ be unital. Then $\mathcal{J}$ is a Jordan ideal of $\mathcal{A} \oplus \mathcal{B}$ if and only if $\mathcal{J}_1$ and $\mathcal{J}_2$ are Jordan ideals of $\mathcal{A}$ and $\mathcal{B}$, respectively, and $\mathcal{J} = \mathcal{J}_1 \times \mathcal{J}_2$.*

(iii) *Let $\mathcal{A}$ be unital. If $\mathcal{A}$ and $\mathcal{B}$ are both simple, then the Jordan ideals of $\mathcal{A} \oplus \mathcal{B}$ are 0, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{A} \oplus \mathcal{B}$.*

Proof. Let $\theta = \gamma = 0$, and let $\mathcal{J}$ be a Jordan ideal of $\mathcal{A} \oplus \mathcal{B} = \mathcal{A} \times_\theta \mathcal{B}$. Then $\mathcal{J}$ is a (θ, γ)-Jordan ideal. So, $\mathcal{J}_2$ is a Jordan ideal of $\mathcal{B}$ and $\mathcal{J} \subseteq \ker(\theta + \gamma)$. Hence, by Proposition 2, $\mathcal{J}_1$ is a Jordan ideal. That is, (i) holds. Assume now that $\mathcal{A}$ has identity. By Proposition 1 and (i), $\mathcal{J}_1$ and $\mathcal{J}_2$ are Jordan ideals of $\mathcal{A}$ and $\mathcal{B}$, respectively, and $\mathcal{J} = \mathcal{J}_1 \times \mathcal{J}_2$. The converse follows from Lemma 2. Therefore, (ii) holds. The statement (iii) follows from (ii). □

Theorem 3. *Let $\mathcal{A}$ be unital. Then every Jordan ideal of $\mathcal{A} \oplus \mathcal{B}$ is an ideal if and only if every Jordan ideal of $\mathcal{A}$ and $\mathcal{B}$ is an ideal of $\mathcal{A}$ and $\mathcal{B}$, respectively.*

Proof. Let $\mathcal{J}$ be a Jordan ideal of $\mathcal{A} \oplus \mathcal{B}$. Then by Theorem 2 (ii), $\mathcal{J}_1$ and $\mathcal{J}_2$ are Jordan ideals of $\mathcal{A}$ and $\mathcal{B}$, respectively, and $\mathcal{J} = \mathcal{J}_1 \times \mathcal{J}_2$. If every Jordan ideal of $\mathcal{A}$ and $\mathcal{B}$ is an ideal of $\mathcal{A}$ and $\mathcal{B}$, respectively, then $\mathcal{J}_1$ and $\mathcal{J}_2$ are ideals. Hence, $\mathcal{J}$ is an ideal of $\mathcal{A} \oplus \mathcal{B}$. Conversely, let $\mathcal{I}_1$ and $\mathcal{I}_2$ be Jordan ideals of $\mathcal{A}$ and $\mathcal{B}$, respectively. Then $\mathcal{J} = \mathcal{I}_1 \times \mathcal{I}_2$ is a Jordan ideal of $\mathcal{A} \oplus \mathcal{B}$. Hence J is an ideal of $\mathcal{A} \oplus \mathcal{B}$. Therefore $\mathcal{I}_1$ and $\mathcal{I}_2$ are ideals of $\mathcal{A}$ and $\mathcal{B}$, respectively. □

Let us recall that for a finite group G, the complex group algebra $\mathbb{C}[G] := \{\sum_{s\in G} a_s s : a_s \in \mathbb{C}\}$ is a $*$-algebra equipped with the following operations:

$$\left(\sum_{s\in G} a_s s\right)^* = \sum_{s\in G} \overline{a}_s s^{-1}, \quad \sum_{s\in G} a_s s + \sum_{s\in G} b_s s = \sum_{s\in G} (a_s + b_s)s$$

and

$$\left(\sum_{s\in G} a_s s\right)\left(\sum_{s\in G} b_s s\right) = \sum_{s\in G}\left(\sum_{t\in G} a_t b_{t^{-1}s}\right)s.$$

Consider the left regular representation $\lambda : G \to \mathcal{U}(\ell^2(G))$, defined by $\lambda_x(\xi)(y) = \xi(x^{-1}y)$, where $x, y \in G$ and $\xi \in \ell^2(G)$, extends to an injective $*$-representation λ: $\mathbb{C}[G] \to B(\ell^2(G))$. This endows $\mathbb{C}[G]$ with a C^*-norm defined by $\|u\| = \|\lambda_u\|$ for $u \in \mathbb{C}[G]$. As a finite-dimensional C^*-algebra, $\mathbb{C}[G]$ admits a $*$-isomorphism ψ: $\mathbb{C}[G] \to \bigoplus_{j=1}^{k} M_{n_j}$ by the standard structure theorem, where M_{n_j} is the algebra of $n_j \times n_j$ complex matrices and n_j is the dimension of the corresponding irreducible $*$-representation. This fact, together with Theorem 3, yields the following result.

Corollary 3. *Let G be a finite group. Then every Jordan ideal of $\mathbb{C}[G]$ is an ideal.*

Let G be a locally compact group. Denote by $C_b(G)$ the space of all bounded continuous functions on G, and by $\mathrm{LUC}(G)$ the subspace of all $f \in C_b(G)$ for which the map $x \mapsto {}_x f$ from G into $C_b(G)$ is continuous. It is well known that $\mathrm{LUC}(G)^*$ is a Banach algebra with respect to the following product.

$$\langle m \cdot n\rangle = \langle m, nf\rangle,$$

where $\langle nf, x\rangle = \langle n, {}_x f\rangle$ for all $m, n \in \mathrm{LUC}(G)^*$, $f \in \mathrm{LUC}(G)$ and $x \in G$. One can prove that $\mathrm{LUC}(G)^*$ is unital and

$$\mathrm{LUC}(G)^* = M(G) \oplus C_0(G)^{\perp},$$

where $M(G)$ is the Banach algebra of all regular Borel measure on G and

$$C_0(G)^{\perp} = \{m \in \mathrm{LUC}(G)^* : \langle m, f\rangle = 0 \text{ for all } f \in C_0(G)\};$$

see for example [16].

Corollary 4. *Let G be an abelian locally compact group. Then every Jordan ideal of $LUC(G)^*$ is an ideal if and only if every Jordan ideal of $C_0(G)^{\perp}$ is an ideal.*

Proof. Let G be abelian. Then $M(G)$ is commutative. So every Jordan ideal of $M(G)$ is an ideal of $M(G)$. Now apply Theorem 3. □

https://doi.org/10.3390/axioms15050306

Remark 1. *Let $M_*(G)$ be the subspace of $M(G)^*$ consisting of all linear functionals that vanish at infinity. It is proved that $C_0(G)$ is a subspace of $M_*(G)$. Also, $M_*(G)^*$ is a unital Banach algebra; for more details see [17–20]. So*

$$M_*(G)^* = M(G) \oplus M_0(G),$$

where

$$M_0(G) = \{m \in M_*(G)^* : \langle m, f \rangle = 0 \text{ for all } f \in C_0(G)\}.$$

By an argument similar to that given in the proof of Corollary 4, we prove that for an abelian locally compact group G, every Jordan ideal of $M_*(G)^*$ is an ideal if and only if every Jordan ideal of $M_0(G)$ is an ideal.

Author Contributions: L.C.: conceptualization, writing—original draft, formal analysis. M.J.M.: conceptualization, formal analysis. Z.Q.: writing—review and editing. All authors have read and agreed to the published version of the manuscript.

Funding: The first author is supported by the National Natural Science Foundation of China (Grant No. 12061018). The third author is supported by the National Natural Science Foundation of China (Grant No. 12501163) and the Natural Science Foundation of the Jiangsu Higher Education Institutions of China (Grant No. 24KJB110001).

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Acknowledgments: The authors wish to thank anonymous reviewers for their constructive and valuable suggestions which have considerably improved the presentation of the paper.

Conflicts of Interest: The authors declare that they have no conflicts of interest.

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