

Article

Unifying Bipolar and Cubic Set Theories: Ideals in Sheffer Stroke Hilbert Algebras

Amal S. Alali ¹, Hashem Bordbar ², Ravikumar Bandaru ^{3,*}, Rajesh Neelamegarajan ⁴, Tahsin Oner ^{5,*}

¹ Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia; asalali@pnu.edu.sa

² Centre for Information Technologies and Applied Mathematics, University of Nova Gorica, 5000 Nova Gorica, Slovenia; hashem.bordbar@ung.si

³ Department of Mathematics, School of Advanced Sciences, VIT-AP University, Amaravati 522241, Andhra Pradesh, India

⁴ Department of Mathematics, Rajah Serfoji Government College, Thanjavur 613005, Tamil Nadu, India; nrjesh_topology@yahoo.co.in

⁵ Department of Mathematics, Faculty of Science, Ege University, Izmir 35100, Turkey

* Correspondence: ravimaths83@gmail.com (R.B.); tahsin.oner@ege.edu.tr (T.O.)

Abstract

In recent years, the study of generalized fuzzy structures in algebraic systems has attracted considerable attention due to their ability to represent uncertainty and bipolar information. In this paper, we introduce the notion of cubic bipolar ideals in the framework of Sheffer stroke Hilbert algebras. This concept integrates the descriptive capability of cubic sets with the dual representation of bipolar information, providing a broader perspective for investigating algebraic structures associated with the Sheffer stroke operation. We establish the definition of cubic bipolar ideals and investigate several of their fundamental properties. In particular, the structural behavior of these ideals is examined within Sheffer stroke Hilbert algebras. Furthermore, the preservation of cubic bipolar ideals under algebraic homomorphisms is analyzed through the study of images and preimages. The Cartesian product of cubic bipolar ideals is also discussed, and conditions ensuring the stability of the resulting structures are obtained. The results presented here contribute to the development of fuzzy algebraic theory and extend existing approaches to Sheffer stroke-based algebraic systems.

Keywords: Sheffer stroke Hilbert algebra; fuzzy subalgebra; fuzzy ideal; (cubic bipolar) ideal

MSC: 06F35; 03G25; 08A72



Academic Editor: Cristina Flaut

Received: 24 March 2026

Revised: 16 April 2026

Accepted: 20 April 2026

Published: 22 April 2026

Copyright: © 2026 by the authors.

Licensee MDPI, Basel, Switzerland.

This article is an open access article distributed under the terms and conditions of the [Creative Commons Attribution \(CC BY\)](https://creativecommons.org/licenses/by/4.0/) license.

1. Introduction

The Sheffer stroke operation, also known as the NAND operation, was introduced by H. M. Sheffer [1] and has since played a significant role in both logic and algebra. Owing to its functional completeness in Boolean logic, the Sheffer stroke provides a powerful tool for constructing and analyzing various algebraic systems. In particular, its integration into Hilbert algebras has led to the development of several generalized structures that enable a deeper investigation of logical operations from an algebraic perspective. Within this framework, the study of ideals has become an important topic due to its relevance in describing internal algebraic behavior and its applications in logic, computation, and information processing.

Bipolar structures constitute another important direction in the study of generalized algebraic models. Such structures typically involve the simultaneous consideration of positive and negative components, reflecting the dual aspects of information and logical interpretation. From an algebraic viewpoint, the interaction of these dual components provides a useful mechanism for examining complex relationships between operations. In particular, the analysis of cubic bipolar configurations offers a richer framework for investigating how these dual aspects interact within algebraic environments.

The importance of the Sheffer stroke stems from its ability to express all classical logical operations within a single connective. Because of this property, it serves as a convenient foundation for constructing diverse algebraic systems. Hilbert algebras, which provide an algebraic representation of implication-based logical systems, form a suitable setting for studying such operations. When the Sheffer stroke is incorporated into these structures, new algebraic behaviors emerge, especially in relation to ideals and their associated properties.

The Sheffer stroke has been investigated in various algebraic and fuzzy environments (see [2–4]). In particular, Oner et al. [5] introduced the notion of Sheffer stroke Hilbert algebras and examined several of their fundamental properties. Later, in [6], they studied deductive systems and filters in these algebras, including their fuzzy extensions, and provided a detailed analysis of ideals in Sheffer stroke Hilbert algebras.

The motivation for the present study is to further explore the structural aspects of these algebras and to extend existing results to more generalized frameworks. Although previous studies have examined several properties of Sheffer stroke algebras (see [7,8]) and their possible generalizations (see [9]), the investigation of cubic bipolar structures related to ideals has not yet been sufficiently explored. This observation provides the primary motivation for the work presented in this paper.

In this study, we introduce the notion of cubic bipolar ideals in Sheffer stroke Hilbert algebras and investigate several of their fundamental properties. Furthermore, we examine relationships between cubic bipolar ideals and study the behavior of these structures under algebraic mappings. In particular, the concepts of the images and preimages of cubic bipolar ideals are introduced, and conditions ensuring the preservation of these structures are established. Finally, the Cartesian products of cubic bipolar ideals are considered within product algebras.

A list of acronyms is given in Table 1.

Table 1. List of acronyms.

Acronyms	Representation
SSH-algebra	Sheffer stroke Hilbert algebra
BF-subalgebra	bipolar fuzzy subalgebra
BIVF-subalgebra	bipolar interval-valued fuzzy subalgebra
IVF-set	interval-valued fuzzy set
CB-set	cubic bipolar set
CB-subalgebra	cubic bipolar subalgebra
CB-ideal	cubic bipolar ideal

2. Preliminaries

In this section, we review fundamental concepts and results regarding Sheffer stroke Hilbert algebras that will be used throughout this paper.

Definition 1 ([1]). Let $\mathcal{H} = (H, |)$ be a groupoid. The binary operation $|$ is called a Sheffer stroke operation on H if, for all $v, q, v \in H$, the following identities hold:

$$(S1) \quad v | q = q | v.$$

$$(S2) \quad (\nu \mid \nu) \mid (\nu \mid \varrho) = \nu.$$

(S3)

$$\nu \mid ((\varrho \mid \nu) \mid (\varrho \mid \nu)) = ((\nu \mid \varrho) \mid (\nu \mid \varrho)) \mid \nu.$$

(S4)

$$(\nu \mid ((\nu \mid \nu) \mid (\varrho \mid \varrho))) \mid (\nu \mid ((\nu \mid \nu) \mid (\varrho \mid \varrho))) = \nu.$$

To simplify notation, we adopt the convention

$$\nu \mid (\varrho \mid \varrho) = \nu^\varrho.$$

Definition 2 ([5]). A Sheffer stroke Hilbert algebra (briefly, SSH-algebra) is an algebraic structure

$$\mathcal{H} = (H, \mid),$$

where H is a nonempty set, and $\mid$ is a Sheffer stroke operation on H such that for all $\nu, \varrho, v \in H$, the following conditions hold:

1.

$$(\nu \mid (\varrho^\nu \mid \varrho^\nu)) \mid ((\nu^\varrho)^{(\nu^\nu)} \mid (\nu^\varrho)^{(\nu^\nu)}) = \nu^\nu.$$

2. If $\nu^\varrho = \nu^\nu = \varrho^\nu$, then $\nu = \varrho$.

Proposition 1 ([5]). Let $\mathcal{H}$ be an SSH-algebra. Then the relation

$$\nu \leq \varrho \iff \nu^\varrho = 1$$

defines a partial order on H .

Definition 3 ([5]). Let $\mathcal{H}$ be an SSH-algebra. A nonempty subset $\mathcal{G} \subseteq H$ is called a subalgebra of $\mathcal{H}$ if

$$\nu^\varrho \mid \nu^\varrho \in \mathcal{G}, \quad \text{for all } \nu, \varrho \in \mathcal{G}.$$

Definition 4 ([5]). Let $\mathcal{H}$ be an SSH-algebra. A nonempty subset $\mathcal{G} \subseteq H$ is called an ideal of $\mathcal{H}$ if, for all $\nu, \varrho \in H$, the following conditions hold:

1. $0 \in \mathcal{G}$.

2. If $\nu^\varrho \mid \nu^\varrho \in \mathcal{G}$ and $\varrho \in \mathcal{G}$, then $\nu \in \mathcal{G}$.

Theorem 1 ([5]). Let $(A, \mid_A)$ and $(B, \mid_B)$ be SSH-algebras. Then the structure

$$(A \times B, \mid_{A \times B})$$

defined by

$$(a_1, b_1) \mid_{A \times B} (a_2, b_2) = (a_1 \mid_A a_2, b_1 \mid_B b_2)$$

for all $(a_1, b_1), (a_2, b_2) \in A \times B$ is an SSH-algebra with zero element $(0_A, 0_B)$.

Definition 5 ([5]). Let $(A, \mid_A)$ and $(B, \mid_B)$ be SSH-algebras. A mapping $f : A \rightarrow B$ is called a homomorphism if

$$f(\nu \mid_A \varrho) = f(\nu) \mid_B f(\varrho), \quad \text{for all } \nu, \varrho \in A,$$

and $f(0_A) = 0_B$.

Definition 6 ([10]). Let X be a nonempty set. A function

$$\tilde{\mu}_{\mathcal{F}} : X \rightarrow D[0, 1],$$

which assigns to each $x \in X$ an interval

$$\tilde{\mu}_{\mathcal{F}}(x) = [\tilde{\mu}_A^-(x), \tilde{\mu}_A^+(x)],$$

is called an interval-valued fuzzy set on X . Here, $D[0, 1]$ denotes the set of all closed subintervals of $[0, 1]$.

The functions $\tilde{\mu}_A^-$ and $\tilde{\mu}_A^+$ are mappings

$$\tilde{\mu}_A^- : X \rightarrow [0, 1], \quad x \mapsto \tilde{\mu}_A^-(x),$$

$$\tilde{\mu}_A^+ : X \rightarrow [0, 1], \quad x \mapsto \tilde{\mu}_A^+(x),$$

such that $\tilde{\mu}_A^-(x) \leq \tilde{\mu}_A^+(x)$ for all $x \in X$.

Definition 7 ([11]). Let X be a nonempty set. A cubic set in X is an ordered pair

$$A = (\vartheta^+, \psi^+),$$

where ϑ^+ is an interval-valued fuzzy set on X , and ψ^+ is a fuzzy set on X .

We now introduce the concept of interval-valued fuzzy sets (IVF-sets). Let $D[0, 1]$ denote the set of all closed subintervals of $[0, 1]$; that is,

$$D[0, 1] = \{\tilde{h} = [h^-, h^+] \mid 0 \leq h^- \leq h^+ \leq 1\}.$$

Each element of $D[0, 1]$ is called an *interval-valued number*. For any $\tilde{h}, \tilde{k} \in D[0, 1]$, we define the relations $\ll, \gg, =$ and the operations $\bar{\wedge}$ (minimum) and $\bar{\vee}$ (maximum) as follows:

- $\tilde{h} \ll \tilde{k}$ if and only if $h^- \leq k^-$ and $h^+ \leq k^+$.
- $\tilde{h} \gg \tilde{k}$ if and only if $h^- \geq k^-$ and $h^+ \geq k^+$.
- $\tilde{h} = \tilde{k}$ if and only if $h^- = k^-$ and $h^+ = k^+$.
- $\tilde{h} \bar{\wedge} \tilde{k} = [h^- \wedge k^-, h^+ \wedge k^+]$.
- $\tilde{h} \bar{\vee} \tilde{k} = [h^- \vee k^-, h^+ \vee k^+]$.

Let $D_1 = [D_1^-, D_1^+]$ and $D_2 = [D_2^-, D_2^+]$ be elements of $D[0, 1]$. The refined minimum and refined maximum are defined, respectively, by

$$r \min\{D_1, D_2\} = [\min\{D_1^-, D_2^-\}, \min\{D_1^+, D_2^+\}],$$

$$r \max\{D_1, D_2\} = [\max\{D_1^-, D_2^-\}, \max\{D_1^+, D_2^+\}].$$

If $D_i = [h_i, k_i] \in D[0, 1]$ for $i \in I$, then the refined supremum and infimum are defined by

$$r \sup_i(D_i) = \left[\sup_i(h_i), \sup_i(k_i) \right], \quad r \inf_i(D_i) = \left[\inf_i(h_i), \inf_i(k_i) \right].$$

Moreover, for $D_1 = [h_1, k_1]$ and $D_2 = [h_2, k_2]$, we write

$$D_1 \gg D_2 \iff h_1 \geq h_2 \text{ and } k_1 \geq k_2,$$

and similarly define $D_1 \ll D_2$ and $D_1 = D_2$.

Definition 8 ([10]). Let X be a nonempty set. A function

$$\vartheta^+ : X \rightarrow D[0, 1], \quad x \mapsto [\alpha_A^-(x), \alpha_A^+(x)],$$

is called an interval-valued fuzzy set on X , where $D[0, 1]$ denotes the set of all closed subintervals of $[0, 1]$.

The functions α_A^- and α_A^+ are defined by

$$\alpha_A^- : X \rightarrow [0, 1], \quad x \mapsto \alpha_A^-(x),$$

$$\alpha_A^+ : X \rightarrow [0, 1], \quad x \mapsto \alpha_A^+(x),$$

such that $\alpha_A^-(x) \leq \alpha_A^+(x)$ for all $x \in X$.

3. Cubic Bipolar Subalgebra of SSH-Algebras

Throughout this section, X will denote an SSH-algebra unless explicitly stated otherwise. SSH-algebras form an important class of algebraic systems exhibiting remarkable symmetry properties and attracting interest due to their theoretical relevance and possible applications. In this section, we study cubic bipolar subalgebras in the context of SSH-algebras and examine their main structural properties. This investigation contributes to a better understanding of the internal structure of SSH-algebras and may lead to further developments and applications in related algebraic frameworks.

Definition 9. A CB-set $\mathcal{A} = (\vartheta^+, \psi^+)$ of X is called a cubic bipolar subalgebra (CB-subalgebra) of X if, for all $v, q \in X$,

$$\begin{aligned} \vartheta^+(v^q \mid v^q) &\gg \vartheta^+(v) \bar{\wedge} \vartheta^+(q), \\ \vartheta^-(v^q \mid v^q) &\ll \vartheta^-(v) \vee \vartheta^-(q), \\ \psi^+(v^q \mid v^q) &\geq \psi^+(v) \wedge \psi^+(q), \\ \psi^-(v^q \mid v^q) &\leq \psi^-(v) \vee \psi^-(q). \end{aligned} \tag{1}$$

Example 1. Consider the set $X = \{1, 2, 3, 0\}$, and define a Sheffer stroke operation “ $\mid$ ” by Table 2.

Table 2. A Cayley table for the Sheffer stroke “ $\mid$ ”.

$\mid$	1	2	3	0
1	0	3	2	1
2	3	3	1	1
3	2	1	2	1
0	1	1	1	1

Then X is an SSH-algebra (see [5]).

Define the functions ϑ^+ , ϑ^- , ψ^+ , and ψ^- on X by

$$\vartheta^+(v) = v, \quad \vartheta^-(v) = 4 - v, \quad \psi^+(v) = v, \quad \psi^-(v) = 4 - v,$$

for all $v \in X$.

For $v = 1$ and $q = 2$, we have $1 \mid 2 = 3$. Then,

- $\vartheta^+(1^2 \mid 1^2) = 3 \gg \vartheta^+(1) \bar{\wedge} \vartheta^+(2) = 1$.
- $\vartheta^-(1^2 \mid 1^2) = 1 \ll \vartheta^-(1) \vee \vartheta^-(2) = 3$.
- $\psi^+(1^2 \mid 1^2) = 3 \geq \psi^+(1) \wedge \psi^+(2) = 1$.
- $\psi^-(1^2 \mid 1^2) = 1 \leq \psi^-(1) \vee \psi^-(2) = 3$.

Thus, all required conditions are satisfied, and $\mathcal{A}$ is a CB-subalgebra of X .

Proposition 2. If $\mathcal{A} = (\vartheta^+, \psi^+)$ is a CB-subalgebra of X , then for all $v \in X$,

$$\begin{aligned}\vartheta^+(0) &\gg \vartheta^+(v), & \vartheta^-(0) &\ll \vartheta^-(v), \\ \psi^+(0) &\geq \psi^+(v), & \psi^-(0) &\leq \psi^-(v).\end{aligned}\tag{2}$$

Proof. For any $v \in X$, we have

$$\begin{aligned}\vartheta^+(0) &= \vartheta^+(v^\nu \mid v^\nu) \gg \vartheta^+(v) \bar{\wedge} \vartheta^+(v) = \vartheta^+(v), \\ \vartheta^-(0) &= \vartheta^-(v^\nu \mid v^\nu) \ll \vartheta^-(v) \vee \vartheta^-(v) = \vartheta^-(v), \\ \psi^+(0) &= \psi^+(v^\nu \mid v^\nu) \geq \psi^+(v) \wedge \psi^+(v) = \psi^+(v), \\ \psi^-(0) &= \psi^-(v^\nu \mid v^\nu) \leq \psi^-(v) \vee \psi^-(v) = \psi^-(v).\end{aligned}$$

□

Definition 10. Let X be a nonempty set, and let $\mathcal{A} = (\vartheta^+, \psi^+)$ be a CB-set of X . Let $\tilde{\mathfrak{h}}$ be a cubic bipolar number. Then the subset

$$[\mathcal{A}]_{\tilde{\mathfrak{h}}} = \{v \in X : \vartheta^+(v) \gg \tilde{\mathfrak{h}}^+, \vartheta^-(v) \ll \tilde{\mathfrak{h}}^-, \psi^+(v) \geq \mathfrak{h}^+, \psi^-(v) \leq \mathfrak{h}^-\}$$

is called the $\tilde{\mathfrak{h}}$ -level set of $\mathcal{A}$, where $\tilde{\mathfrak{h}}^+ = [\mathfrak{h}^{+, -}, \mathfrak{h}^{+, +}] \in D[0, 1]$, and $\tilde{\mathfrak{h}}^- = [\mathfrak{h}^{-, -}, \mathfrak{h}^{-, +}] \in D[-1, 0]$.

Example 2. Consider the SSH-algebra X from Example 1. Let $\mathcal{A} = (\vartheta^+, \psi^+)$ be a CB-set of X , where

$$\vartheta^+(v) = v, \quad \vartheta^-(v) = 4 - v, \quad \psi^+(v) = v, \quad \psi^-(v) = 4 - v,$$

for all $v \in X$.

Let $\tilde{\mathfrak{h}}^+ = [0.5, 1] \in D[0, 1]$ and $\tilde{\mathfrak{h}}^- = [-0.5, 0] \in D[-1, 0]$. Then the $\tilde{\mathfrak{h}}$ -level set is

$$[\mathcal{A}]_{\tilde{\mathfrak{h}}} = \{v \in X : \vartheta^+(v) \gg \tilde{\mathfrak{h}}^+, \vartheta^-(v) \ll \tilde{\mathfrak{h}}^-, \psi^+(v) \geq \mathfrak{h}^+, \psi^-(v) \leq \mathfrak{h}^-\}.$$

For $v = 1$, we have

$$\vartheta^+(1) = 1, \vartheta^-(1) = 3, \psi^+(1) = 1, \psi^-(1) = 3.$$

Thus, $\vartheta^+(1) \gg 0.5$, $\vartheta^-(1) \ll -0.5$, $\psi^+(1) \geq 0.5$, and $\psi^-(1) \leq 0$, so $v = 1$ satisfies all conditions.

For $v = 2$, we have

$$\vartheta^+(2) = 2, \vartheta^-(2) = 2, \psi^+(2) = 2, \psi^-(2) = 2.$$

Again, all required conditions are satisfied, so $v = 2$ belongs to the level set.

Therefore,

$$[\mathcal{A}]_{\tilde{\mathfrak{h}}} = \{1, 2\}.$$

Theorem 2. Let $\mathcal{A} = (\vartheta^+, \psi^+)$ be a CB-set of X , and let $\tilde{\mathfrak{h}}$ be any cubic bipolar number. Then $\mathcal{A}$ is a CB-subalgebra of X if and only if the level set $[\mathcal{A}]_{\tilde{\mathfrak{h}}}$ is a subalgebra of X , provided that $[\mathcal{A}]_{\tilde{\mathfrak{h}}} \neq \emptyset$.

Proof. Suppose that $\mathcal{A}$ is a CB-subalgebra of X and that $[\mathcal{A}]_{\tilde{\mathfrak{h}}} \neq \emptyset$. Let $\tilde{\mathfrak{h}}^+ = [\mathfrak{h}^{+, -}, \mathfrak{h}^{+, +}] \in D[0, 1]$ and $\tilde{\mathfrak{h}}^- = [\mathfrak{h}^{-, -}, \mathfrak{h}^{-, +}] \in D[-1, 0]$.

Take any $\nu, \varrho \in [\mathcal{A}]_{\tilde{\mathfrak{h}}}$. Then

$$\vartheta^+(\nu) \gg \tilde{\mathfrak{h}}^+, \quad \vartheta^-(\nu) \ll \tilde{\mathfrak{h}}^-, \quad \psi^+(\nu) \geq \mathfrak{h}^+, \quad \psi^-(\nu) \leq \mathfrak{h}^-,$$

and similarly for ϱ .

Since $\mathcal{A}$ is a CB-subalgebra of X , we have

$$\vartheta^+(\nu^{\varrho} | \nu^{\varrho}) \gg \vartheta^+(\nu) \bar{\wedge} \vartheta^+(\varrho) \gg \tilde{\mathfrak{h}}^+ \bar{\wedge} \tilde{\mathfrak{h}}^+ = \tilde{\mathfrak{h}}^+,$$

$$\vartheta^-(\nu^{\varrho} | \nu^{\varrho}) \ll \vartheta^-(\nu) \vee \vartheta^-(\varrho) \ll \tilde{\mathfrak{h}}^- \vee \tilde{\mathfrak{h}}^- = \tilde{\mathfrak{h}}^-,$$

$$\psi^+(\nu^{\varrho} | \nu^{\varrho}) \geq \psi^+(\nu) \wedge \psi^+(\varrho) \geq \mathfrak{h}^+,$$

$$\psi^-(\nu^{\varrho} | \nu^{\varrho}) \leq \psi^-(\nu) \vee \psi^-(\varrho) \leq \mathfrak{h}^-.$$

Thus, $\nu^{\varrho} | \nu^{\varrho} \in [\mathcal{A}]_{\tilde{\mathfrak{h}}}$, showing that $[\mathcal{A}]_{\tilde{\mathfrak{h}}}$ is closed under the operation. Hence it is a subalgebra of X .

Conversely, assume that $[\mathcal{A}]_{\tilde{\mathfrak{h}}} \neq \emptyset$ is a subalgebra of X . We prove that $\mathcal{A}$ is a CB-subalgebra by contradiction.

Suppose that

$$\vartheta^+(\nu^{\varrho} | \nu^{\varrho}) \gg \vartheta^+(\nu) \bar{\wedge} \vartheta^+(\varrho)$$

does not hold. Then there exist $\nu_0, \varrho_0 \in X$ such that

$$\vartheta^+(\nu_0^{\varrho_0} | \nu_0^{\varrho_0}) \ll \vartheta^+(\nu_0) \bar{\wedge} \vartheta^+(\varrho_0).$$

Let

$$\vartheta^+(\nu_0) = [\mathfrak{k}^{+, -}, \mathfrak{k}^{+, +}], \quad \vartheta^+(\varrho_0) = [\mathfrak{l}^{+, -}, \mathfrak{l}^{+, +}], \quad \vartheta^+(\nu_0^{\varrho_0} | \nu_0^{\varrho_0}) = [\mathfrak{h}^{+, -}, \mathfrak{h}^{+, +}].$$

Then

$$[\mathfrak{h}^{+, -}, \mathfrak{h}^{+, +}] \ll [\mathfrak{k}^{+, -} \wedge \mathfrak{l}^{+, -}, \mathfrak{k}^{+, +} \wedge \mathfrak{l}^{+, +}].$$

Choose an interval

$$[\mathfrak{f}^{+, -}, \mathfrak{f}^{+, +}] = \left[\frac{1}{2}(\mathfrak{h}^{+, -} + \mathfrak{k}^{+, -} \wedge \mathfrak{l}^{+, -}), \frac{1}{2}(\mathfrak{h}^{+, +} + \mathfrak{k}^{+, +} \wedge \mathfrak{l}^{+, +}) \right].$$

Then

$$[\mathfrak{k}^{+, -} \wedge \mathfrak{l}^{+, -}, \mathfrak{k}^{+, +} \wedge \mathfrak{l}^{+, +}] \gg [\mathfrak{f}^{+, -}, \mathfrak{f}^{+, +}] \gg [\mathfrak{h}^{+, -}, \mathfrak{h}^{+, +}].$$

Hence $\nu_0, \varrho_0 \in [\mathcal{A}]_{\tilde{\mathfrak{h}}}$, but $\nu_0^{\varrho_0} | \nu_0^{\varrho_0} \notin [\mathcal{A}]_{\tilde{\mathfrak{h}}}$, contradicting the assumption that it is a subalgebra.

Similarly, if any of

$$\vartheta^-(\nu^{\varrho} | \nu^{\varrho}) \ll \vartheta^-(\nu) \vee \vartheta^-(\varrho), \quad \psi^+(\nu^{\varrho} | \nu^{\varrho}) \geq \psi^+(\nu) \wedge \psi^+(\varrho), \quad \psi^-(\nu^{\varrho} | \nu^{\varrho}) \leq \psi^-(\nu) \vee \psi^-(\varrho),$$

fails, then by constructing appropriate intermediate values (as above), we obtain elements in $[\mathcal{A}]_{\tilde{\mathfrak{h}}}$ whose operation does not remain in the set, again leading to a contradiction.

Therefore, all defining conditions hold, and $\mathcal{A}$ is a CB-subalgebra of X . $\square$

The next result shows that a cubic bipolar set is a subalgebra if and only if its components satisfy the corresponding fuzzy subalgebra conditions.

Theorem 3. Let $\mathcal{A} = (\vartheta^+, \psi^+)$ be a CB-set of X . Then $\mathcal{A}$ is a CB-subalgebra of X if and only if ϑ^+ is a BIVF-subalgebra and ψ^+ is a BF-subalgebra of X .

Proof. Suppose that $\mathcal{A}$ is a CB-subalgebra of X . Let $\nu, \varrho \in X$. Then

$$\begin{aligned}\vartheta^+(\nu^e|\nu^e) &\gg \vartheta^+(\nu) \wedge \vartheta^+(\varrho) \\ &= [\vartheta^{+, -}(\nu) \wedge \vartheta^{+, -}(\varrho), \vartheta^{+, +}(\nu) \wedge \vartheta^{+, +}(\varrho)],\end{aligned}$$

and

$$\begin{aligned}\vartheta^-(\nu^e|\nu^e) &\ll \vartheta^-(\nu) \vee \vartheta^-(\varrho) \\ &= [\vartheta^{-, -}(\nu) \vee \vartheta^{-, -}(\varrho), \vartheta^{-, +}(\nu) \vee \vartheta^{-, +}(\varrho)].\end{aligned}$$

Thus, the components satisfy

$$\begin{aligned}\vartheta^{+, -}(\nu^e|\nu^e) &\geq \vartheta^{+, -}(\nu) \wedge \vartheta^{+, -}(\varrho), \\ \vartheta^{+, +}(\nu^e|\nu^e) &\geq \vartheta^{+, +}(\nu) \wedge \vartheta^{+, +}(\varrho), \\ \vartheta^{-, -}(\nu^e|\nu^e) &\leq \vartheta^{-, -}(\nu) \vee \vartheta^{-, -}(\varrho), \\ \vartheta^{-, +}(\nu^e|\nu^e) &\leq \vartheta^{-, +}(\nu) \vee \vartheta^{-, +}(\varrho).\end{aligned}$$

Similarly, from the definition of CB-subalgebra, we obtain

$$\begin{aligned}\psi^+(\nu^e|\nu^e) &\geq \psi^+(\nu) \wedge \psi^+(\varrho), \\ \psi^-(\nu^e|\nu^e) &\leq \psi^-(\nu) \vee \psi^-(\varrho).\end{aligned}$$

Hence, ϑ^+ is a BIVF-subalgebra and ψ^+ is a BF-subalgebra of X .

Conversely, assume that ϑ^+ is a BIVF-subalgebra and ψ^+ is a BF-subalgebra of X . Let $\nu, \varrho \in X$. Then

$$\begin{aligned}\vartheta^+(\nu^e|\nu^e) &= [\vartheta^{+, -}(\nu^e|\nu^e), \vartheta^{+, +}(\nu^e|\nu^e)] \\ &\gg [\vartheta^{+, -}(\nu) \wedge \vartheta^{+, -}(\varrho), \vartheta^{+, +}(\nu) \wedge \vartheta^{+, +}(\varrho)] \\ &= \vartheta^+(\nu) \wedge \vartheta^+(\varrho).\end{aligned}$$

Similarly,

$$\begin{aligned}\vartheta^-(\nu^e|\nu^e) &\ll \vartheta^-(\nu) \vee \vartheta^-(\varrho), \\ \psi^+(\nu^e|\nu^e) &\geq \psi^+(\nu) \wedge \psi^+(\varrho), \\ \psi^-(\nu^e|\nu^e) &\leq \psi^-(\nu) \vee \psi^-(\varrho).\end{aligned}$$

Therefore, $\mathcal{A}$ is a CB-subalgebra of X . $\square$

Definition 11. Let $(X, |_A, 0_X)$ and $(Y, |_B, 0_Y)$ be two SSH-algebras, and let $\mathcal{A} = (\Phi_A, \Psi_A)$ and $\mathcal{B} = (\Phi_B, \Psi_B)$ be CB-subsets of X and Y , respectively.

(i) The image of $\mathcal{A}$ under $\mathfrak{f}$, denoted by $\mathfrak{f}(\mathcal{A})$, is a CB-set in Y defined as follows: for each $\varrho \in Y$,

$$\mathfrak{f}(\mathcal{A})(\varrho) = \begin{cases} \bigwedge_{\nu \in \mathfrak{f}^{-1}(\varrho)} \mathcal{A}(\nu), & \text{if } \mathfrak{f}^{-1}(\varrho) \neq \emptyset, \\ \langle ([0, 0], [0, 0], (0, 0)) \rangle, & \text{otherwise.} \end{cases}$$

Here,

$$\bigwedge_{\nu \in \mathfrak{f}^{-1}(\varrho)} \mathcal{A}(\nu) = \left\langle \bigwedge_{\nu \in \mathfrak{f}^{-1}(\varrho)} \Phi_A(\nu), \bigwedge_{\nu \in \mathfrak{f}^{-1}(\varrho)} \Psi_A(\nu) \right\rangle,$$

where

$$\bigwedge_{v \in f^{-1}(q)} \Phi_A(v) = \left(\left[\bigvee_{v \in f^{-1}(q)} \vartheta^{-,-}(v), \bigvee_{v \in f^{-1}(q)} \vartheta^{-,+}(v) \right], \left[\bigwedge_{v \in f^{-1}(q)} \vartheta^{+,-}(v), \bigwedge_{v \in f^{-1}(q)} \vartheta^{+,+}(v) \right] \right),$$

and

$$\bigwedge_{v \in f^{-1}(q)} \Psi_A(v) = \left(\bigvee_{v \in f^{-1}(q)} \psi^-(v), \bigwedge_{v \in f^{-1}(q)} \psi^+(v) \right).$$

(ii) The preimage of $\mathcal{B}$ under f , denoted by $f^{-1}(\mathcal{B})$, is a CB-set in X defined by

$$[f^{-1}(\mathcal{B})](v) = \mathcal{B}(f(v)) = \langle \Phi_B(f(v)), \Psi_B(f(v)) \rangle = \langle [f^{-1}(\Phi_B)](v), [f^{-1}(\Psi_B)](v) \rangle,$$

for all $v \in X$.

To improve readability in the context of Sheffer stroke Hilbert algebras, we adopt the following notation:

$$x|_P(y|_P y) = \zeta_P(x^y),$$

and

$$(x|_P(y|_P y))|_P(x|_P(y|_P y)) = \zeta_{PP}(x^y).$$

Theorem 4. Let $f : X \rightarrow Y$ be a homomorphism of SSH-algebras. If $\mathcal{B} = (\Phi_B, \Psi_B)$ is a CB-ideal of Y , then the preimage $f^{-1}(\mathcal{B})$ is a CB-ideal of X .

Proof. Suppose that $\mathcal{B} = (\Phi_B, \Psi_B)$ is a CB-ideal of Y . Then Φ_B is a BIVF-ideal and Ψ_B is a BF-ideal of Y .

For all $v \in X$, we have

$$\begin{aligned} f^{-1}(\vartheta_B^+)(v) &= \vartheta_B^+(f(v)) \leq \vartheta_B^+(0_Y) = \vartheta_B^+(f(0_X)) = f^{-1}(\vartheta_B^+)(0_X), \\ f^{-1}(\vartheta_B^-)(v) &= \vartheta_B^-(f(v)) \geq \vartheta_B^-(0_Y) = \vartheta_B^-(f(0_X)) = f^{-1}(\vartheta_B^-)(0_X), \\ f^{-1}(\psi_B^+)(v) &= \psi_B^+(f(v)) \geq \psi_B^+(0_Y) = \psi_B^+(f(0_X)) = f^{-1}(\psi_B^+)(0_X), \\ f^{-1}(\psi_B^-)(v) &= \psi_B^-(f(v)) \leq \psi_B^-(0_Y) = \psi_B^-(f(0_X)) = f^{-1}(\psi_B^-)(0_X). \end{aligned}$$

Let $v, q \in X$. Using the fact that f is a homomorphism, we obtain

$$f(\zeta_{AA}(v^q)) = \zeta_{BB}(f(v)^{f(q)}).$$

Then

$$\begin{aligned} f^{-1}(\vartheta_B^+)(v) &= \vartheta_B^+(f(v)) \\ &\geq \vartheta_B^+(\zeta_{BB}(f(v)^{f(q)})) \bar{\wedge} \vartheta_B^+(f(q)) \\ &= \vartheta_B^+(f(\zeta_{AA}(v^q))) \bar{\wedge} \vartheta_B^+(f(q)) \\ &= f^{-1}(\vartheta_B^+)(\zeta_{AA}(v^q)) \bar{\wedge} f^{-1}(\vartheta_B^+)(q), \\ f^{-1}(\vartheta_B^-)(v) &= \vartheta_B^-(f(v)) \\ &\leq \vartheta_B^-(\zeta_{BB}(f(v)^{f(q)})) \vee \vartheta_B^-(f(q)) \\ &= \vartheta_B^-(f(\zeta_{AA}(v^q))) \vee \vartheta_B^-(f(q)) \\ &= f^{-1}(\vartheta_B^-)(\zeta_{AA}(v^q)) \vee f^{-1}(\vartheta_B^-)(q), \end{aligned}$$

$$\begin{aligned}
 f^{-1}(\psi_B^+)(\nu) &= \psi_B^+(f(\nu)) \\
 &\geq \psi_B^+(\xi_{BB}(f(\nu)^{f(q)})) \wedge \psi_B^+(f(q)) \\
 &= \psi_B^+(f(\xi_{AA}(\nu^q))) \wedge \psi_B^+(f(q)) \\
 &= f^{-1}(\psi_B^+)(\xi_{AA}(\nu^q)) \wedge f^{-1}(\psi_B^+)(q), \\
 f^{-1}(\psi_B^-)(\nu) &= \psi_B^-(f(\nu)) \\
 &\leq \psi_B^-(\xi_{BB}(f(\nu)^{f(q)})) \vee \psi_B^-(f(q)) \\
 &= \psi_B^-(f(\xi_{AA}(\nu^q))) \vee \psi_B^-(f(q)) \\
 &= f^{-1}(\psi_B^-)(\xi_{AA}(\nu^q)) \vee f^{-1}(\psi_B^-)(q).
 \end{aligned}$$

Therefore, $f^{-1}(\mathcal{B})$ satisfies all the conditions of a CB-ideal of X . $\square$

Theorem 5. Let $f : X \rightarrow Y$ be a homomorphism of SSH-algebras. If $\mathcal{B} = (\Phi_B, \Psi_B)$ is a CB-subalgebra of Y , then the preimage $f^{-1}(\mathcal{B})$ is a CB-subalgebra of X .

Proof. Suppose that $\mathcal{B} = (\Phi_B, \Psi_B)$ is a CB-subalgebra of Y . Then Φ_B is a BIVF-subalgebra and Ψ_B is a BF-subalgebra of Y .

Let $\nu, q \in X$. Since f is a homomorphism, we have

$$f(\xi_{AA}(\nu^q)) = \xi_{BB}(f(\nu)^{f(q)}).$$

Now,

$$\begin{aligned}
 f^{-1}(\vartheta_B^+)(\xi_{AA}(\nu^q)) &= \vartheta_B^+(f(\xi_{AA}(\nu^q))) \\
 &= \vartheta_B^+(\xi_{BB}(f(\nu)^{f(q)})) \\
 &\gg \vartheta_B^+(f(\nu)) \bar{\wedge} \vartheta_B^+(f(q)) \\
 &= f^{-1}(\vartheta_B^+)(\nu) \bar{\wedge} f^{-1}(\vartheta_B^+)(q), \\
 f^{-1}(\vartheta_B^-)(\xi_{AA}(\nu^q)) &= \vartheta_B^-(f(\xi_{AA}(\nu^q))) \\
 &= \vartheta_B^-(\xi_{BB}(f(\nu)^{f(q)})) \\
 &\ll \vartheta_B^-(f(\nu)) \vee \vartheta_B^-(f(q)) \\
 &= f^{-1}(\vartheta_B^-)(\nu) \vee f^{-1}(\vartheta_B^-)(q), \\
 f^{-1}(\psi_B^+)(\xi_{AA}(\nu^q)) &= \psi_B^+(f(\xi_{AA}(\nu^q))) \\
 &= \psi_B^+(\xi_{BB}(f(\nu)^{f(q)})) \\
 &\geq \psi_B^+(f(\nu)) \wedge \psi_B^+(f(q)) \\
 &= f^{-1}(\psi_B^+)(\nu) \wedge f^{-1}(\psi_B^+)(q), \\
 f^{-1}(\psi_B^-)(\xi_{AA}(\nu^q)) &= \psi_B^-(f(\xi_{AA}(\nu^q))) \\
 &= \psi_B^-(\xi_{BB}(f(\nu)^{f(q)})) \\
 &\leq \psi_B^-(f(\nu)) \vee \psi_B^-(f(q)) \\
 &= f^{-1}(\psi_B^-)(\nu) \vee f^{-1}(\psi_B^-)(q).
 \end{aligned}$$

Therefore, $f^{-1}(\mathcal{B})$ is a CB-subalgebra of X . $\square$

4. Cubic Bipolar Ideals of SSH-Algebra

In this section, we study cubic bipolar ideals in the framework of Sheffer stroke Hilbert algebras. A cubic bipolar ideal is a special type of CB-set satisfying certain inequalities related to the operations of a Sheffer stroke algebra. These structures play an important role in understanding the algebraic properties of SSH-algebras, particularly in connection with logical operations and their algebraic interpretations.

Definition 12. A CB-set $A = (\vartheta^+, \psi^+)$ of X is called a cubic bipolar ideal (CB-ideal) of X if it satisfies the following conditions:

$$(\forall v \in X) \begin{pmatrix} \vartheta^+(0) \gg \vartheta^+(v), & \vartheta^-(0) \ll \vartheta^-(v) \\ \psi^+(0) \geq \psi^+(v), & \psi^-(0) \leq \psi^-(v) \end{pmatrix}, \quad (3)$$

$$(\forall v, q \in X) \begin{pmatrix} \vartheta^+(v) \gg \vartheta^+(v^q | v^q) \bar{\wedge} \vartheta^+(q) \\ \vartheta^-(v) \ll \vartheta^-(v^q | v^q) \vee \vartheta^-(q) \\ \psi^+(v) \geq \psi^+(v^q | v^q) \wedge \psi^+(q) \\ \psi^-(v) \leq \psi^-(v^q | v^q) \vee \psi^-(q) \end{pmatrix}. \quad (4)$$

Example 3. Consider the SSH-algebra X in Example 1. Let $A = (\vartheta^+, \psi^+)$ be a CB-set of X , where the functions are defined by

$$\vartheta^+(v) = v, \quad \psi^+(v) = 4 - v, \quad \text{for all } v \in X.$$

We check whether A satisfies the conditions of a CB-ideal.

Since

$$\vartheta^+(0) = 0, \quad \vartheta^-(0) = 4, \quad \psi^+(0) = 4, \quad \psi^-(0) = 0,$$

the inequalities in (3) hold for all $v \in X$. Moreover, the inequalities in (4) are also satisfied for all $v, q \in X$. Hence, $A = (\vartheta^+, \psi^+)$ is a CB-ideal of X .

Lemma 1. If $A = (\vartheta^+, \psi^+)$ is a CB-ideal of X , then for all $v, q \in X$,

$$v \leq q \Rightarrow \begin{cases} \vartheta^+(q) \ll \vartheta^+(v), & \vartheta^-(q) \gg \vartheta^-(v) \\ \psi^+(q) \geq \psi^+(v), & \psi^-(q) \leq \psi^-(v) \end{cases}$$

Proof. Let $v, q \in X$ such that $v \leq q$. Then $v^q | v^q = 0$, and hence

$$\vartheta^+(v) \gg \vartheta^+(v^q | v^q) \bar{\wedge} \vartheta^+(q) = \vartheta^+(0) \bar{\wedge} \vartheta^+(q) = \vartheta^+(q),$$

$$\vartheta^-(v) \ll \vartheta^-(v^q | v^q) \vee \vartheta^-(q) = \vartheta^-(0) \vee \vartheta^-(q) = \vartheta^-(q),$$

$$\psi^+(v) \geq \psi^+(v^q | v^q) \wedge \psi^+(q) = \psi^+(0) \wedge \psi^+(q) = \psi^+(q),$$

$$\psi^-(v) \leq \psi^-(v^q | v^q) \vee \psi^-(q) = \psi^-(0) \vee \psi^-(q) = \psi^-(q).$$

□

The following theorem demonstrates that every cubic bipolar ideal inherently possesses the structure of a cubic bipolar subalgebra.

Theorem 6. Every CB-ideal of X is a CB-subalgebra of X .

Proof. Let $\mathcal{A} = (\vartheta^+, \psi^+)$ be a CB-ideal of X .

$$\begin{aligned}\vartheta^+(v^e|v^e) &\gg \vartheta^+(((v^e|v^e)|(v|v))|(((v^e|v^e)|(v|v))) \bar{\wedge} \vartheta^+(v) \\ &= \vartheta^+(((q|q)^{v^v|v^v})|(((q|q)^{v^v|v^v}))) \bar{\wedge} \vartheta^+(v) \\ &= \vartheta^+((q|q)^1|(q|q)^1) \bar{\wedge} \vartheta^+(q) \\ &= \vartheta^+(1|1) \bar{\wedge} \vartheta^+(v) \\ &= \vartheta^+(0) \bar{\wedge} \vartheta^+(q) \\ &= \vartheta^+(0) \\ &\gg \vartheta^+(v) \bar{\wedge} \vartheta^+(q),\end{aligned}$$

$$\begin{aligned}\vartheta^-(v^e|v^e) &\ll \vartheta^-(((v^e|v^e)|(v|v))|(((v^e|v^e)|(v|v))) \vee \vartheta^-(x) \\ &= \vartheta^-(((q|q)^{v^v|v^v})|(((q|q)^{v^v|v^v}))) \vee \vartheta^-(x) \\ &= \vartheta^-((q|q)^1|(q|q)^1) \vee \vartheta^-(v) \\ &= \vartheta^-(1|1) \vee \vartheta^-(v) \\ &= \vartheta^-(0) \vee \vartheta^-(q) \\ &= \vartheta^-(0) \\ &\ll \vartheta^-(v) \vee \vartheta^-(q),\end{aligned}$$

$$\begin{aligned}\psi^+(v^e|v^e) &\geq \psi^+(((v^e|v^e)|(v|v))|(((v^e|v^e)|(v|v))) \wedge \psi^+(v) \\ &= \psi^+(((q|q)^{v^v|v^v})|(((q|q)^{v^v|v^v}))) \wedge \psi^+(v) \\ &= \psi^+((q|q)^1|(q|q)^1) \wedge \psi^+(v) \\ &= \psi^+(1|1) \wedge \psi^+(v) \\ &= \psi^+(0) \wedge \psi^+(q) \\ &= \psi^+(0) \\ &\geq \psi^+(v) \wedge \psi^+(q),\end{aligned}$$

$$\begin{aligned}\psi^-(v^e|v^e) &\leq \psi^-(((v^e|v^e)|(v|v))|(((v^e|v^e)|(v|v))) \vee \psi^-(v) \\ &= \psi^-(((q|q)^{v^v|v^v})|(((q|q)^{v^v|v^v}))) \vee \psi^-(v) \\ &= \psi^-((q|q)^1|(q|q)^1) \vee \psi^-(v) \\ &= \psi^-(1|1) \vee \psi^-(v) \\ &= \psi^-(0) \vee \psi^-(q) \\ &= \psi^-(0) \\ &\leq \psi^-(v) \vee \psi^-(q),\end{aligned}$$

Hence A is a CB-subalgebra of X . $\square$

The following example shows that the converse of Theorem 6 does not hold in general.

Example 4. Let $X = \{0, 1, 2, 3\}$ be the SSH-algebra defined in Example 1 with the operation $|$ given by Table 2.

Define a cubic bipolar set $A = (\vartheta^+, \psi^+)$ on X as follows:

$$\vartheta^+(v) = \begin{cases} [0.7, 0.9], & \text{if } v = 0, 1, \\ [0.2, 0.4], & \text{if } v = 2, 3, \end{cases} \quad \psi^+(v) = \begin{cases} 0.8, & \text{if } v = 0, 1, \\ 0.3, & \text{if } v = 2, 3. \end{cases}$$

Then, for all $v, q \in X$, one can verify that

$$\vartheta^+(v|q) \gg \vartheta^+(v) \bar{\wedge} \vartheta^+(q), \quad \psi^+(v|q) \geq \psi^+(v) \wedge \psi^+(q),$$

and hence A is a CB-subalgebra of X .

However, A is not a CB-ideal of X . Indeed, take $v = 2$ and $q = 1$. Then

$$\vartheta^+(2) = [0.2, 0.4], \quad \vartheta^+(2|1) = \vartheta^+(3) = [0.2, 0.4],$$

and

$$r \min\{\vartheta^+(2|1), \vartheta^+(1)\} = r \min\{[0.2, 0.4], [0.7, 0.9]\} = [0.2, 0.4].$$

Thus, the required dominance condition for CB-ideals is not satisfied.

Hence, A is a CB-subalgebra but not a CB-ideal of X . Therefore, the converse of Theorem 6 does not hold.

Theorem 7. Let $\tilde{a}$ be any cubic bipolar number and $[\mathcal{A}]_{\tilde{a}}$ be a CB-set of X . Then $\mathcal{A}$ is a CB-ideal of X if and only if $[\mathcal{A}]_{\tilde{a}}$ is an ideal of X , where $[\mathcal{A}]_{\tilde{a}} \neq \emptyset$.

Proof. Suppose $\mathcal{A}$ is a CB-ideal of X and $[\mathcal{A}]_{\tilde{a}} \neq \emptyset$. Let $\nu \in [\mathcal{A}]_{\tilde{a}}$, and let $\tilde{h}^+ = [h^{+, -}, h^{+, +}] \in D[0, 1]$ and $\tilde{h}^- = [h^{-, -}, h^{-, +}]$. Then

$$\vartheta^+(\nu) \gg \tilde{h}^+, \vartheta^-(\nu) \ll \tilde{h}^-, \psi^+(\nu) \geq h^+, \psi^-(\nu) \leq h^-.$$

Since $\mathcal{A}$ is a cubic bipolar ideal of X ,

$$\vartheta^+(0) \gg \vartheta^+(\nu) \gg \tilde{h}^+, \vartheta^+(0) \ll \vartheta^-(\nu) \ll \tilde{h}^-,$$

$$\psi^+(0) \geq \psi^+(\nu) \geq \tilde{h}^+, \psi^-(0) \leq \psi^-(\nu) \leq \tilde{h}^-.$$

Hence $0 \in [\mathcal{A}]_{\tilde{a}}$. For any $(\nu^q | \nu^q), q \in [\mathcal{A}]_{\tilde{a}}$. Then

$$\vartheta^+(\nu^q | \nu^q) \gg \tilde{h}^+, \vartheta^-(\nu^q | \nu^q) \ll \tilde{h}^-,$$

$$\psi^+(\nu^q | \nu^q) \geq h^+, \psi^-(\nu^q | \nu^q) \leq h^-,$$

$$\vartheta^+(y) \gg \tilde{h}^+, \vartheta^-(q) \ll \tilde{h}^-, \psi^+(q) \geq h^+, \psi^-(q) \leq h^-.$$

Since $\mathcal{A}$ is a CB-ideal of X ,

$$\vartheta^+(\nu) \gg \vartheta^+(\nu^q | \nu^q) \wedge \vartheta^+(q) \gg \tilde{h}^+ \wedge \tilde{h}^+ = \tilde{h}^+,$$

$$\vartheta^-(\nu) \ll \vartheta^-(\nu^q | \nu^q) \vee \vartheta^-(q) \ll \tilde{h}^- \vee \tilde{h}^- = \tilde{h}^-,$$

$$\psi^+(\nu) \geq \psi^+(\nu^q | \nu^q) \wedge \psi^+(q) \geq \tilde{h}^+ \wedge \tilde{h}^+ = \tilde{h}^+,$$

$$\psi^-(\nu) \leq \psi^-(\nu^q | \nu^q) \vee \psi^-(q) \leq \tilde{h}^- \vee \tilde{h}^- = \tilde{h}^-.$$

Hence $\nu \in [\mathcal{A}]_{\tilde{a}}$. Therefore, $\mathcal{A}$ is a CB-ideal of X .

Conversely, suppose $[\mathcal{A}]_{\tilde{a}} \neq \emptyset$ is an ideal of X . We prove that $\mathcal{A}$ satisfies the conditions of Definition 12. Assume that $\vartheta^+(0) \gg \vartheta^+(\nu)$ is not true. Then there exists $\nu_0 \in X$ such that $\vartheta^+(0) < \vartheta^+(\nu_0)$. Let $\vartheta^+(\nu_0) = [\mathfrak{t}^{+, -}, \mathfrak{t}^{+, +}]$, $\vartheta^+(0) = [h^{+, -}, h^{+, +}]$, where $[h^{+, -}, h^{+, +}], [\mathfrak{t}^{+, -}, \mathfrak{t}^{+, +}] \in D[0, 1]$. Then clearly, $[h^{+, -}, h^{+, +}] < [\mathfrak{t}^{+, -}, \mathfrak{t}^{+, +}]$. Let $[\mathfrak{l}^{+, -}, \mathfrak{l}^{+, +}] = [\frac{1}{2}(h^{+, -} + \mathfrak{t}^{+, -}), \frac{1}{2}(h^{+, +} + \mathfrak{t}^{+, +})]$. Then we can easily obtain the following inequalities:

$$\mathfrak{t}^{+, -} > \mathfrak{l}^{+, -} = \frac{1}{2}(h^{+, -} + \mathfrak{t}^{+, -}) > h^{+, -},$$

$$\mathfrak{t}^{+, +} > \mathfrak{l}^{+, +} = \frac{1}{2}(h^{+, +} + \mathfrak{t}^{+, +}) > h^{+, +}.$$

Thus $[\mathfrak{t}^{+, -}, \mathfrak{t}^{+, +}] > [\mathfrak{l}^{+, -}, \mathfrak{l}^{+, +}] > [h^{+, -}, h^{+, +}] = \vartheta^+(0)$. Hence $0 \notin [\mathcal{A}]_{\tilde{a}}$. Assume that $\vartheta^-(0) \ll \vartheta^-(x)$ is not true. Then there exists $\nu_0 \in X$ such that $\vartheta^-(0) > \vartheta^-(\nu_0)$. Let $\vartheta^-(\nu_0) = [\mathfrak{t}^{-, -}, \mathfrak{t}^{-, +}]$, $\vartheta^-(0) = [h^{-, -}, h^{-, +}]$, where $[h^{-, -}, h^{-, +}], [\mathfrak{t}^{-, -}, \mathfrak{t}^{-, +}] \in D[0, 1]$. Then

clearly, $[\mathfrak{h}^{-,-}, \mathfrak{h}^{-,+}] < [\mathfrak{k}^{-,-}, \mathfrak{k}^{-,+}]$. Let $[\mathfrak{l}^{-,-}, \mathfrak{l}^{-,+}] = [\frac{1}{2}(\mathfrak{h}^{-,-} + \mathfrak{k}^{-,-}), \frac{1}{2}(\mathfrak{h}^{-,+} + \mathfrak{k}^{-,+})]$. Then we can easily obtain the following inequalities:

$$\mathfrak{k}^{-,-} < \mathfrak{c}^{-,-} = \frac{1}{2}(\mathfrak{h}^{-,-} + \mathfrak{k}^{-,-}) < \mathfrak{h}^{-,-},$$

$$\mathfrak{k}^{-,+} > \mathfrak{c}^{-,+} = \frac{1}{2}(\mathfrak{h}^{-,+} + \mathfrak{k}^{-,+}) < \mathfrak{h}^{-,+}.$$

Thus $[\mathfrak{k}^{-,-}, \mathfrak{k}^{-,+}] < [\mathfrak{l}^{-,-}, \mathfrak{l}^{-,+}] < [\mathfrak{h}^{-,-}, \mathfrak{h}^{-,+}] = \vartheta^-(0)$. Hence $0 \notin [\mathcal{A}]_{\tilde{\mathfrak{h}}}^-$. Assume that $\psi^+(0) \geq \psi^+(\nu)$ is not true. Then there exists $\nu_0 \in X$ such that $\psi^+(0) < \psi^+(\nu_0)$. Let $\psi^+(\nu_0) = \mathfrak{k}^+, \psi^+(0) = \mathfrak{h}^+$, where $\mathfrak{h}^+, \mathfrak{k}^+ \in [0, 1]$. Then clearly, $\mathfrak{h}^+ < \mathfrak{k}^+$. Let $\mathfrak{l}^+ = \frac{1}{2}(\mathfrak{h}^+ + \mathfrak{k}^+)$. Then we can easily obtain the following inequalities:

$$\mathfrak{k}^+ > \mathfrak{l}^+ = \frac{1}{2}(\mathfrak{h}^+ + \mathfrak{k}^+) > \mathfrak{h}^+,$$

$$\mathfrak{k}^+ > \mathfrak{h}^+ = \frac{1}{2}(\mathfrak{h}^+ + \mathfrak{k}^+) > \mathfrak{h}^+.$$

Thus $\mathfrak{k}^+ > \mathfrak{l}^+ > \mathfrak{h}^+ = \psi^+(0)$. Hence $0 \notin [\mathcal{A}]_{\tilde{\mathfrak{h}}}^+$. Assume that $\psi^-(0) \leq \psi^-(\nu)$ is not true. Then there exists $\nu_0 \in X$ such that $\psi^-(0) > \psi^-(\nu_0)$. Let $\psi^-(\nu_0) = \mathfrak{b}^-, \psi^-(0) = \mathfrak{h}^-$, where $\mathfrak{h}^-, \mathfrak{k}^- \in [0, 1]$. Then clearly, $\mathfrak{h}^- > \mathfrak{k}^-$. Let $\mathfrak{l}^- = \frac{1}{2}(\mathfrak{h}^- + \mathfrak{k}^-)$. Then we can easily obtain the following inequalities:

$$\mathfrak{k}^- < \mathfrak{l}^- = \frac{1}{2}(\mathfrak{h}^- + \mathfrak{k}^-) < \mathfrak{h}^-,$$

$$\mathfrak{k}^- < \mathfrak{l}^- = \frac{1}{2}(\mathfrak{h}^- + \mathfrak{k}^-) < \mathfrak{h}^-.$$

Thus $\mathfrak{k}^- < \mathfrak{l}^- < \mathfrak{h}^- = \psi^-(0)$. Hence $0 \notin [\mathcal{A}]_{\tilde{\mathfrak{h}}}^-$. Assume that $\vartheta^+(\nu) \gg \vartheta^+(\nu^e | \nu^e) \wedge \vartheta^+(\varrho)$ is not true. Then there exist $\nu_0, \varrho_0 \in X$ such that $\vartheta^+(\nu_0) < \vartheta^+(\nu_0^{e_0} | \nu_0^{e_0}) \wedge \vartheta^+(\varrho_0)$. Let $\vartheta^+(\nu_0^{e_0} | \nu_0^{e_0}) = [\mathfrak{k}^{+,-}, \mathfrak{k}^{+,+}]$, $\vartheta^+(\varrho_0) = [\mathfrak{l}^{+,-}, \mathfrak{l}^{+,+}]$, and $\vartheta^+(\nu_0) = [\mathfrak{h}^{+,-}, \mathfrak{h}^{+,+}]$, where $[\mathfrak{h}^{+,-}, \mathfrak{h}^{+,+}], [\mathfrak{k}^{+,-}, \mathfrak{k}^{+,+}], [\mathfrak{l}^{+,-}, \mathfrak{l}^{+,+}] \in D[0, 1]$. Then $[\mathfrak{h}^{+,-}, \mathfrak{h}^{+,+}] < [\mathfrak{k}^{+,-}, \mathfrak{k}^{+,+}] \wedge [\mathfrak{l}^{+,-}, \mathfrak{l}^{+,+}] = [\mathfrak{k}^{+,-} \wedge \mathfrak{l}^{+,-}, \mathfrak{k}^{+,+} \wedge \mathfrak{l}^{+,+}]$. Let $[\mathfrak{f}^{+,-}, \mathfrak{f}^{+,+}] = [\frac{1}{2}(\mathfrak{h}^{+,-} + \mathfrak{k}^{+,-} \wedge \mathfrak{l}^{+,-}), \frac{1}{2}(\mathfrak{h}^{+,+} + \mathfrak{k}^{+,+} \wedge \mathfrak{l}^{+,+})]$. Then

$$\mathfrak{k}^{+,-} \wedge \mathfrak{l}^{+,-} > \mathfrak{f}^{+,-} = \frac{1}{2}(\mathfrak{h}^{+,-} + \mathfrak{k}^{+,-} \wedge \mathfrak{l}^{+,-}) > \mathfrak{h}^{+,-},$$

$$\mathfrak{k}^{+,+} \wedge \mathfrak{l}^{+,+} > \mathfrak{f}^{+,+} = \frac{1}{2}(\mathfrak{h}^{+,+} + \mathfrak{k}^{+,+} \wedge \mathfrak{l}^{+,+}) > \mathfrak{h}^{+,+}.$$

Thus $[\mathfrak{k}^{+,-} \wedge \mathfrak{l}^{+,-}, \mathfrak{k}^{+,+} \wedge \mathfrak{l}^{+,+}] > [\mathfrak{f}^{+,-}, \mathfrak{f}^{+,+}] > [\mathfrak{h}^{+,-}, \mathfrak{h}^{+,+}] = \vartheta^+(\nu_0)$. So $(\nu_0^{e_0} | \nu_0^{e_0}), \varrho_0 \in [\mathcal{A}]_{\tilde{\mathfrak{h}}}^+$, but $\nu_0 \notin [\mathcal{A}]_{\tilde{\mathfrak{h}}}^+$. Assume that $\vartheta^-(\nu) \ll \vartheta^-(\nu^e | \nu^e) \vee \vartheta^-(\varrho)$ is not true. Then there exist $\nu_0, \varrho_0 \in X$ such that $\vartheta^-(\nu_0) > \vartheta^-(\nu_0^{e_0} | \nu_0^{e_0}) \vee \vartheta^-(\varrho_0)$. Let $\vartheta^-(\nu_0^{e_0} | \nu_0^{e_0}) = [\mathfrak{k}^{+,-}, \mathfrak{k}^{+,+}]$, $\vartheta^-(\varrho_0) = [\mathfrak{l}^{+,-}, \mathfrak{l}^{+,+}]$, $\vartheta^-(\nu_0) = [\mathfrak{h}^{+,-}, \mathfrak{h}^{+,+}]$, where $[\mathfrak{h}^{+,-}, \mathfrak{h}^{+,+}], [\mathfrak{k}^{+,-}, \mathfrak{k}^{+,+}], [\mathfrak{l}^{+,-}, \mathfrak{l}^{+,+}] \in D[0, 1]$. Then clearly, $[\mathfrak{h}^{+,-}, \mathfrak{h}^{+,+}] < [\mathfrak{k}^{+,-}, \mathfrak{k}^{+,+}] \wedge [\mathfrak{l}^{+,-}, \mathfrak{l}^{+,+}] = [\mathfrak{k}^{+,-} \vee \mathfrak{l}^{+,-}, \mathfrak{k}^{+,+} \vee \mathfrak{l}^{+,+}]$. Let $[\mathfrak{f}^{+,-}, \mathfrak{f}^{+,+}] = [\frac{1}{2}(\mathfrak{h}^{+,-} + \mathfrak{k}^{+,-} \vee \mathfrak{l}^{+,-}), \frac{1}{2}(\mathfrak{h}^{+,+} + \mathfrak{k}^{+,+} \vee \mathfrak{l}^{+,+})]$. Then

$$\mathfrak{k}^{+,-} \vee \mathfrak{l}^{+,-} < \mathfrak{f}^{+,-} = \frac{1}{2}(\mathfrak{h}^{+,-} + \mathfrak{k}^{+,-} \vee \mathfrak{l}^{+,-}) < \mathfrak{h}^{+,-},$$

$$\mathfrak{k}^{+,+} \vee \mathfrak{l}^{+,+} < \mathfrak{f}^{+,+} = \frac{1}{2}(\mathfrak{h}^{+,+} + \mathfrak{k}^{+,+} \vee \mathfrak{l}^{+,+}) < \mathfrak{h}^{+,+}.$$

Thus $[\mathfrak{k}^{+,-} \vee \mathfrak{l}^{+,-}, \mathfrak{k}^{+,+} \vee \mathfrak{l}^{+,+}] < [\mathfrak{f}^{+,-}, \mathfrak{f}^{+,+}] < [\mathfrak{h}^{+,-}, \mathfrak{h}^{+,+}] = \vartheta^-(\nu_0)$. So $(\nu_0^{e_0} | \nu_0^{e_0}), \varrho_0 \in [\mathcal{A}]_{\tilde{\mathfrak{h}}}^-$, but $\nu_0 \notin [\mathcal{A}]_{\tilde{\mathfrak{h}}}^-$. Assume that $\psi^+(\nu) \geq \psi^+(\nu^e | \nu^e) \wedge \psi^+(\varrho)$ is not true. Then there exist $\nu_0, \varrho_0 \in X$ such that $\psi^+(\nu_0) < \psi^+(\nu_0^{e_0} | \nu_0^{e_0}) \wedge \psi^+(\varrho_0)$. Let $\psi^+(\nu_0^{e_0} | \nu_0^{e_0}) = \mathfrak{k}^+, \psi^+(\varrho_0) =$

$\mathfrak{l}^+, \psi^+(\nu_0) = \mathfrak{h}^+$, where $\mathfrak{h}^+, \mathfrak{k}^+, \mathfrak{l}^+ \in [0, 1]$. Then clearly, $\mathfrak{h}^+ < \mathfrak{k}^+ \wedge \mathfrak{l}^+$. Let $\mathfrak{f}^+ = \frac{1}{2}(\mathfrak{h}^+ + \mathfrak{k}^+ \wedge \mathfrak{l}^+)$. Then

$$\mathfrak{k}^+ \wedge \mathfrak{l}^+ > \mathfrak{f}^+ = \frac{1}{2}(\mathfrak{h}^+ + \mathfrak{k}^+ \wedge \mathfrak{l}^+) > \mathfrak{h}^+,$$

$$\mathfrak{k}^+ \wedge \mathfrak{l}^+ > \mathfrak{f}^+ = \frac{1}{2}(\mathfrak{h}^+ + \mathfrak{k}^+ \wedge \mathfrak{l}^+) > \mathfrak{h}^+.$$

Thus $\mathfrak{k}^+ \wedge \mathfrak{l}^+ > \mathfrak{f}^+ > \mathfrak{h}^+ = \psi^+(\nu_0)$. So $(\nu_0^{e_0} | \nu_0^{e_0}), \varrho_0 \in [\mathcal{A}]_{\tilde{\mathfrak{h}}}$, but $\nu_0 \notin [\mathcal{A}]_{\tilde{\mathfrak{h}}}$. Assume that $\psi^-(\nu) \leq \psi^-(\nu^e | \nu^e) \wedge \psi^-(\varrho)$ is not true. Then there exist $\nu_0, \varrho_0 \in X$ such that $\psi^-(\nu_0) > \psi^-(\nu_0^{e_0} | \nu_0^{e_0}) \vee \psi^-(\varrho_0)$. Let $\psi^-(\nu_0^{e_0} | \nu_0^{e_0}) = \mathfrak{k}^-, \psi^-(\varrho_0) = \mathfrak{l}^-, \psi^-(\nu_0) = \mathfrak{h}^-$, where $\mathfrak{h}^-, \mathfrak{k}^-, \mathfrak{l}^- \in [0, 1]$. Then clearly, $\mathfrak{h}^- < \mathfrak{k}^- \vee \mathfrak{l}^-$. Let $\mathfrak{f}^- = \frac{1}{2}(\mathfrak{h}^- + \mathfrak{k}^- \vee \mathfrak{l}^-)$. Then

$$\mathfrak{k}^- \vee \mathfrak{l}^- > \mathfrak{f}^- = \frac{1}{2}(\mathfrak{h}^- + \mathfrak{k}^- \vee \mathfrak{l}^-) > \mathfrak{h}^-,$$

$$\mathfrak{k}^- \vee \mathfrak{l}^- > \mathfrak{f}^- = \frac{1}{2}(\mathfrak{h}^- + \mathfrak{k}^- \vee \mathfrak{l}^-) > \mathfrak{h}^-.$$

Thus $\mathfrak{k}^- \vee \mathfrak{l}^- > \mathfrak{f}^- > \mathfrak{h}^- = \psi^-(\nu_0)$. So $(\nu_0^{e_0} | \nu_0^{e_0}), \varrho_0 \in [\mathcal{A}]_{\tilde{\mathfrak{h}}}$, but $\nu_0 \notin [\mathcal{A}]_{\tilde{\mathfrak{h}}}$. $\square$

The following theorem provides a characterization of cubic bipolar ideals in terms of their interval-valued fuzzy and bipolar fuzzy components.

Theorem 8. Let $\mathcal{A} = (\vartheta^+, \psi^+)$ be any CB-set of X . Then $\mathcal{A}$ is a CB-ideal of X if and only if ϑ^+ is a BIVF-ideal and ψ^+ is a BF-ideal of X .

Proof. Suppose that $\mathcal{A}$ is a CB-ideal of X . Then, for each $\nu \in X$, we have

$$\left(\begin{array}{ll} \vartheta^{+,-}(0) \gg \vartheta^{+,-}(\nu), & \vartheta^{+,+}(0) \gg \vartheta^{+,+}(\nu), \\ \psi^{-,-}(0) \leq \psi^{-,-}(\nu), & \psi^{-,+}(0) \leq \psi^{-,+}(\nu) \end{array} \right),$$

and equivalently,

$$\vartheta^+(0) \gg \vartheta^+(\nu), \quad \vartheta^-(0) \ll \vartheta^-(\nu).$$

Let $\nu, \varrho \in X$. Then

$$\begin{aligned} \vartheta^+(\nu) &= [\vartheta^{+,-}(\nu), \vartheta^{+,+}(\nu)] \\ &\gg \vartheta^+(\nu^e | \nu^e) \wedge \vartheta^+(\varrho) \\ &= [\vartheta^{+,-}(\nu^e | \nu^e) \wedge \vartheta^{+,-}(\varrho), \vartheta^{+,+}(\nu^e | \nu^e) \wedge \vartheta^{+,+}(\varrho)], \end{aligned}$$

and

$$\begin{aligned} \vartheta^-(\nu) &= [\vartheta^{-,-}(\nu), \vartheta^{-,+}(\nu)] \\ &\ll \vartheta^-(\nu^e | \nu^e) \vee \vartheta^-(\varrho) \\ &= [\vartheta^{-,-}(\nu^e | \nu^e) \vee \vartheta^{-,-}(\varrho), \vartheta^{-,+}(\nu^e | \nu^e) \vee \vartheta^{-,+}(\varrho)]. \end{aligned}$$

Thus we obtain

$$\left(\begin{array}{l} \vartheta^{+,-}(\nu) \geq \vartheta^{+,-}(\nu^e | \nu^e) \wedge \vartheta^{+,-}(\varrho), \\ \vartheta^{+,+}(\nu) \geq \vartheta^{+,+}(\nu^e | \nu^e) \wedge \vartheta^{+,+}(\varrho), \\ \psi^{-,-}(\nu) \leq \psi^{-,-}(\nu^e | \nu^e) \vee \psi^{-,-}(\varrho), \\ \psi^{-,+}(\nu) \leq \psi^{-,+}(\nu^e | \nu^e) \vee \psi^{-,+}(\varrho) \end{array} \right).$$

Also, we easily see that

$$\begin{pmatrix} \vartheta^+(v) \gg \vartheta^+(v^e|v^e) \wedge \vartheta^+(q), \\ \vartheta^-(v) \ll \vartheta^-(v^e|v^e) \vee \vartheta^-(q) \end{pmatrix}.$$

Hence, ϑ^+ is a BIVF-ideal and ψ^+ is a BF-ideal of X .

Conversely, suppose that ϑ^+ is a BIVF-ideal and ψ^+ is a BF-ideal of X . Then for all $v \in X$,

$$\begin{pmatrix} \vartheta^+(0) \gg \vartheta^+(v), & \vartheta^-(0) \ll \vartheta^-(v), \\ \psi^+(0) \geq \psi^+(v), & \psi^-(0) \leq \psi^-(v) \end{pmatrix}.$$

Let $v, q \in X$. Then

$$\begin{aligned} \vartheta^+(v) &= [\vartheta^{+,-}(v), \vartheta^{+,+}(v)] \\ &\gg [\vartheta^{+,-}(v^e|v^e) \wedge \vartheta^{+,-}(q), \vartheta^{+,+}(v^e|v^e) \wedge \vartheta^{+,+}(q)] \\ &= \vartheta^+(v^e|v^e) \wedge \vartheta^+(q). \end{aligned}$$

Similarly,

$$\begin{aligned} \vartheta^-(v) &\ll \vartheta^-(v^e|v^e) \vee \vartheta^-(q), & \psi^+(v) &\geq \psi^+(v^e|v^e) \wedge \psi^+(q), \\ \psi^-(v) &\leq \psi^-(v^e|v^e) \vee \psi^-(q). \end{aligned}$$

Hence, $\mathcal{A}$ is a cubic bipolar ideal of X . $\square$

Example 5. Let $X = \{0, 1, 2, 3\}$ and $Y = \{0, 1\}$ be SSH-algebras. The operation on X is given in Example 1, and on Y , it is defined by

$$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1 & 1 \\ 1 & 1 & 0 \end{array}$$

Define a mapping $f : X \rightarrow Y$ by

$$f(0) = 0, \quad f(1) = 1, \quad f(2) = 1, \quad f(3) = 0.$$

Then f is a homomorphism, since

$$f(v|q) = f(v)f(q), \quad \forall v, q \in X.$$

Now define a cubic bipolar set $A = (\vartheta^+, \psi^+)$ on X by

$$\vartheta^+(v) = \begin{cases} [0.6, 0.9], & \text{if } v = 0, 1, \\ [0.2, 0.5], & \text{if } v = 2, 3, \end{cases} \quad \psi^+(v) = \begin{cases} 0.7, & \text{if } v = 0, 1, \\ 0.3, & \text{if } v = 2, 3. \end{cases}$$

Define the image $f(A) = (f(\vartheta^+), f(\psi^+))$ on Y by

$$f(\vartheta^+)(y) = \sup\{\vartheta^+(v) : f(v) = y\}, \quad f(\psi^+)(y) = \sup\{\psi^+(v) : f(v) = y\}.$$

Then,

$$f^{-1}(0) = \{0, 3\}, \quad f^{-1}(1) = \{1, 2\}.$$

Hence,

$$f(\vartheta^+)(0) = \sup\{[0.6, 0.9], [0.2, 0.5]\} = [0.6, 0.9],$$

$$f(\vartheta^+)(1) = \sup\{[0.6, 0.9], [0.2, 0.5]\} = [0.6, 0.9],$$

and

$$f(\psi^+)(0) = 0.7, \quad f(\psi^+)(1) = 0.7.$$

One can verify that $f(A)$ satisfies the defining conditions of a CB-ideal in Y .

Thus, this example illustrates the behavior of images of CB-ideals under homomorphisms.

Proposition 3. Let $f : X \rightarrow Y$ be a homomorphism of SSH-algebras. If $\mathcal{A} = (\vartheta^+, \psi^+)$ is a CB-ideal of Y , then the preimage $f^{-1}(\mathcal{A})$ of $\mathcal{A}$ under f is a CB-ideal of X .

Proof. Let $\mathcal{A} = (\vartheta^+, \psi^+)$ be a CB-ideal of Y . For all $v \in X$, we have

$$f^{-1}(\vartheta^+)(v) = \vartheta^+(f(v)) \gg \vartheta^+(0_Y) = \vartheta^+(f(0_X)) = f^{-1}(\vartheta^+)(0_X),$$

$$f^{-1}(\vartheta^-)(v) = \vartheta^-(f(v)) \ll \vartheta^-(0_Y) = \vartheta^-(f(0_X)) = f^{-1}(\vartheta^-)(0_X),$$

$$f^{-1}(\psi^+)(v) = \psi^+(f(v)) \geq \psi^+(0_Y) = \psi^+(f(0_X)) = f^{-1}(\psi^+)(0_X),$$

$$f^{-1}(\psi^-)(v) = \psi^-(f(v)) \leq \psi^-(0_Y) = \psi^-(f(0_X)) = f^{-1}(\psi^-)(0_X).$$

Now let $v, q \in X$. Since f is a homomorphism, we have

$$f(\xi_{AA}(v^q)) = \xi_{BB}(f(v)f(q)).$$

Then,

$$\begin{aligned} f^{-1}(\vartheta^+)(v) &= \vartheta^+(f(v)) \\ &\gg r \min\{\vartheta^+(\xi_{BB}(f(v)f(q))), \vartheta^+(f(q))\} \\ &= r \min\{\vartheta^+(f(\xi_{AA}(v^q))), \vartheta^+(f(q))\} \\ &= r \min\{f^{-1}(\vartheta^+)(\xi_{AA}(v^q)), f^{-1}(\vartheta^+)(q)\}, \end{aligned}$$

$$\begin{aligned} f^{-1}(\vartheta^-)(v) &= \vartheta^-(f(v)) \\ &\ll r \max\{\vartheta^-(\xi_{BB}(f(v)f(q))), \vartheta^-(f(q))\} \\ &= r \max\{\vartheta^-(f(\xi_{AA}(v^q))), \vartheta^-(f(q))\} \\ &= r \max\{f^{-1}(\vartheta^-)(\xi_{AA}(v^q)), f^{-1}(\vartheta^-)(q)\}, \end{aligned}$$

$$\begin{aligned} f^{-1}(\psi^+)(v) &= \psi^+(f(v)) \\ &\geq \max\{\psi^+(\xi_{BB}(f(v)f(q))), \psi^+(f(q))\} \\ &= \max\{\psi^+(f(\xi_{AA}(v^q))), \psi^+(f(q))\} \\ &= \max\{f^{-1}(\psi^+)(\xi_{AA}(v^q)), f^{-1}(\psi^+)(q)\}, \end{aligned}$$

$$\begin{aligned} f^{-1}(\psi^-)(v) &= \psi^-(f(v)) \\ &\leq \min\{\psi^-(\xi_{BB}(f(v)f(q))), \psi^-(f(q))\} \\ &= \min\{\psi^-(f(\xi_{AA}(v^q))), \psi^-(f(q))\} \\ &= \min\{f^{-1}(\psi^-)(\xi_{AA}(v^q)), f^{-1}(\psi^-)(q)\}. \end{aligned}$$

Hence $f^{-1}(\mathcal{A})$ is a CB-ideal of X . $\square$

5. Cartesian Product of CB-Structures in SSH-Algebras

In this section, we introduce the Cartesian product of CB-sets, CB-subalgebras, and CB-ideals in SSH-algebras.

Definition 13. Let $\mathcal{A} = (\Phi_A, \Psi_A)$ and $\mathcal{B} = (\Phi_B, \Psi_B)$ be cubic sets of X and Y , respectively. Then the Cartesian product $\mathcal{A} \times \mathcal{B}$ is defined as the cubic set of $X \times Y$ given by

$$\mathcal{A} \times \mathcal{B} = \{ \langle (x, y), \Phi_{A \times B}^+(x, y), \Psi_{A \times B}^+(x, y) \rangle : x \in X, y \in Y \},$$

where

$$\Phi_{A \times B}^+(x, y) = r \min\{\vartheta_A^+(x), \alpha_B(y)\}, \quad \Psi_{A \times B}^+(x, y) = \max\{\psi_A^+(x), \mu_B(y)\},$$

and

$$\Phi_{A \times B}^+ : X \times Y \rightarrow D[0, 1], \quad \Psi_{A \times B}^+ : X \times Y \rightarrow [0, 1].$$

For clarity of notation in SSH-algebras, we write

$$\xi_{A \times B}((x_1, y_1)^{(x_2, y_2)}) := (x_1 \mid_A (x_2 \mid_A x_2), y_1 \mid_B (y_2 \mid_B y_2)).$$

Definition 14. A cubic set

$$\mathcal{A} \times \mathcal{B} = (\Phi_{A \times B}, \Psi_{A \times B})$$

of $X \times Y$ is called a CB-subalgebra of $X \times Y$ if for all $(x_1, y_1), (x_2, y_2) \in X \times Y$, the following conditions hold:

$$\begin{aligned} \vartheta_{A \times B}^+(\xi_{A \times B}((x_1, y_1)^{(x_2, y_2)})) &\gg \vartheta_{A \times B}^+(x_1, y_1) \bar{\wedge} \vartheta_{A \times B}^+(x_2, y_2), \\ \vartheta_{A \times B}^-(\xi_{A \times B}((x_1, y_1)^{(x_2, y_2)})) &\ll \vartheta_{A \times B}^-(x_1, y_1) \vee \vartheta_{A \times B}^-(x_2, y_2), \\ \psi_{A \times B}^+(\xi_{A \times B}((x_1, y_1)^{(x_2, y_2)})) &\geq \psi_{A \times B}^+(x_1, y_1) \wedge \psi_{A \times B}^+(x_2, y_2), \\ \psi_{A \times B}^-(\xi_{A \times B}((x_1, y_1)^{(x_2, y_2)})) &\leq \psi_{A \times B}^-(x_1, y_1) \vee \psi_{A \times B}^-(x_2, y_2). \end{aligned}$$

Definition 15. A cubic set $\mathcal{A} = (\vartheta_{A \times B}, \psi_{A \times B})$ of $X \times Y$ is called a CB-ideal of $X \times Y$ if the following conditions hold:

For all $(x, y), (x_1, y_1), (x_2, y_2) \in X \times Y$, we have

$$\begin{aligned} \vartheta_{A \times B}^+(0, 0) &\gg \vartheta_{A \times B}^+(x, y), \\ \vartheta_{A \times B}^-(0, 0) &\ll \vartheta_{A \times B}^-(x, y), \\ \psi_{A \times B}^+(0, 0) &\geq \psi_{A \times B}^+(x, y), \\ \psi_{A \times B}^-(0, 0) &\leq \psi_{A \times B}^-(x, y), \end{aligned}$$

and

$$\begin{aligned} \vartheta_{A \times B}^+(x_1, y_1) &\gg \vartheta_{A \times B}^+(\xi_{A \times B}((x_1, y_1)^{(x_2, y_2)})) \bar{\wedge} \vartheta_{A \times B}^+(x_2, y_2), \\ \vartheta_{A \times B}^-(x_1, y_1) &\ll \vartheta_{A \times B}^-(\xi_{A \times B}((x_1, y_1)^{(x_2, y_2)})) \vee \vartheta_{A \times B}^-(x_2, y_2), \\ \psi_{A \times B}^+(x_1, y_1) &\geq \psi_{A \times B}^+(\xi_{A \times B}((x_1, y_1)^{(x_2, y_2)})) \wedge \psi_{A \times B}^+(x_2, y_2), \\ \psi_{A \times B}^-(x_1, y_1) &\leq \psi_{A \times B}^-(\xi_{A \times B}((x_1, y_1)^{(x_2, y_2)})) \vee \psi_{A \times B}^-(x_2, y_2). \end{aligned}$$

Proposition 4. Let $\mathcal{A} = (\Phi_A, \Psi_A)$ and $\mathcal{B} = (\Phi_B, \Psi_B)$ be CB-subalgebras of X and Y , respectively. Then the Cartesian product $\mathcal{A} \times \mathcal{B}$ is a CB-subalgebra of $X \times Y$.

Proof. Let $(x_1, y_1), (x_2, y_2) \in X \times Y$. Then, by the definition of the product structure, we obtain

$$\begin{aligned}
 \vartheta_{A \times B}^+ \left(\xi_{A \times B}((x_1, y_1)^{(x_2, y_2)}) \right) &= \vartheta_A^+(\xi_{AA}(x_1^{x_2})) \bar{\wedge} \vartheta_B^+(\xi_{BB}(y_1^{y_2})) \\
 &\gg (\vartheta_A^+(x_1) \bar{\wedge} \vartheta_A^+(x_2)) \bar{\wedge} (\vartheta_B^+(y_1) \bar{\wedge} \vartheta_B^+(y_2)) \\
 &= (\vartheta_A^+(x_1) \bar{\wedge} \vartheta_B^+(y_1)) \bar{\wedge} (\vartheta_A^+(x_2) \bar{\wedge} \vartheta_B^+(y_2)) \\
 &= \vartheta_{A \times B}^+(x_1, y_1) \bar{\wedge} \vartheta_{A \times B}^+(x_2, y_2),
 \end{aligned}$$

$$\begin{aligned}
 \vartheta_{A \times B}^- \left(\xi_{A \times B}((x_1, y_1)^{(x_2, y_2)}) \right) &= \vartheta_A^-(\xi_{AA}(x_1^{x_2})) \vee \vartheta_B^-(\xi_{BB}(y_1^{y_2})) \\
 &\ll (\vartheta_A^-(x_1) \vee \vartheta_A^-(x_2)) \vee (\vartheta_B^-(y_1) \vee \vartheta_B^-(y_2)) \\
 &= (\vartheta_A^-(x_1) \vee \vartheta_B^-(y_1)) \vee (\vartheta_A^-(x_2) \vee \vartheta_B^-(y_2)) \\
 &= \vartheta_{A \times B}^-(x_1, y_1) \vee \vartheta_{A \times B}^-(x_2, y_2),
 \end{aligned}$$

$$\begin{aligned}
 \psi_{A \times B}^+ \left(\xi_{A \times B}((x_1, y_1)^{(x_2, y_2)}) \right) &= \psi_A^+(\xi_{AA}(x_1^{x_2})) \wedge \psi_B^+(\xi_{BB}(y_1^{y_2})) \\
 &\geq (\psi_A^+(x_1) \wedge \psi_A^+(x_2)) \wedge (\psi_B^+(y_1) \wedge \psi_B^+(y_2)) \\
 &= (\psi_A^+(x_1) \wedge \psi_B^+(y_1)) \wedge (\psi_A^+(x_2) \wedge \psi_B^+(y_2)) \\
 &= \psi_{A \times B}^+(x_1, y_1) \wedge \psi_{A \times B}^+(x_2, y_2),
 \end{aligned}$$

$$\begin{aligned}
 \psi_{A \times B}^- \left(\xi_{A \times B}((x_1, y_1)^{(x_2, y_2)}) \right) &= \psi_A^-(\xi_{AA}(x_1^{x_2})) \vee \psi_B^-(\xi_{BB}(y_1^{y_2})) \\
 &\leq (\psi_A^-(x_1) \vee \psi_A^-(x_2)) \vee (\psi_B^-(y_1) \vee \psi_B^-(y_2)) \\
 &= (\psi_A^-(x_1) \vee \psi_B^-(y_1)) \vee (\psi_A^-(x_2) \vee \psi_B^-(y_2)) \\
 &= \psi_{A \times B}^-(x_1, y_1) \vee \psi_{A \times B}^-(x_2, y_2).
 \end{aligned}$$

Hence, $\mathcal{A} \times \mathcal{B}$ is a CB-subalgebra of $X \times Y$. $\square$

Proposition 5. If $\mathcal{A} = (\Phi_A, \Psi_A)$ and $\mathcal{B} = (\Phi_B, \Psi_B)$ are CB-ideals of X , then the Cartesian product $\mathcal{A} \times \mathcal{B}$ is a CB-ideal of $X \times Y$.

Proof. Let $(x, y) \in X \times Y$. Then we have

$$\begin{aligned}
 \vartheta_{A \times B}^+(0, 0) &= \vartheta_A^+(0) \bar{\wedge} \vartheta_B^+(0) \gg \vartheta_A^+(x) \bar{\wedge} \vartheta_B^+(y) = \vartheta_{A \times B}^+(x, y), \\
 \vartheta_{A \times B}^-(0, 0) &= \vartheta_A^-(0) \vee \vartheta_B^-(0) \ll \vartheta_A^-(x) \vee \vartheta_B^-(y) = \vartheta_{A \times B}^-(x, y), \\
 \psi_{A \times B}^+(0, 0) &= \psi_A^+(0) \wedge \psi_B^+(0) \geq \psi_A^+(x) \wedge \psi_B^+(y) = \psi_{A \times B}^+(x, y), \\
 \psi_{A \times B}^-(0, 0) &= \psi_A^-(0) \vee \psi_B^-(0) \leq \psi_A^-(x) \vee \psi_B^-(y) = \psi_{A \times B}^-(x, y).
 \end{aligned}$$

Now let $(x_1, y_1), (x_2, y_2) \in X \times Y$. Then

$$\begin{aligned}
 \vartheta_{A \times B}^+(x_1, y_1) &= \vartheta_A^+(x_1) \bar{\wedge} \vartheta_B^+(y_1) \\
 &\gg (\vartheta_A^+(\xi_{AA}(x_1^{x_2})) \bar{\wedge} \vartheta_A^+(x_2)) \bar{\wedge} (\vartheta_B^+(\xi_{BB}(y_1^{y_2})) \bar{\wedge} \vartheta_B^+(y_2)) \\
 &= \vartheta_{A \times B}^+(\xi_{A \times B}((x_1, y_1)^{(x_2, y_2)})) \bar{\wedge} \vartheta_{A \times B}^+(x_2, y_2), \\
 \vartheta_{A \times B}^-(x_1, y_1) &= \vartheta_A^-(x_1) \vee \vartheta_B^-(y_1) \\
 &\ll (\vartheta_A^-(\xi_{AA}(x_1^{x_2})) \vee \vartheta_A^-(x_2)) \vee (\vartheta_B^-(\xi_{BB}(y_1^{y_2})) \vee \vartheta_B^-(y_2)) \\
 &= \vartheta_{A \times B}^-(\xi_{A \times B}((x_1, y_1)^{(x_2, y_2)})) \vee \vartheta_{A \times B}^-(x_2, y_2),
 \end{aligned}$$

$$\begin{aligned}\psi_{A \times B}^+(x_1, y_1) &= \psi_A^+(x_1) \wedge \psi_B^+(y_1) \\ &\geq (\psi_A^+(\xi_{AA}(x_1^{x_2})) \wedge \psi_A^+(x_2)) \wedge (\psi_B^+(\xi_{BB}(y_1^{y_2})) \wedge \psi_B^+(y_2)) \\ &= \psi_{A \times B}^+(\xi_{A \times B}((x_1, y_1)^{(x_2, y_2)})) \wedge \psi_{A \times B}^+(x_2, y_2),\end{aligned}$$

$$\begin{aligned}\psi_{A \times B}^-(x_1, y_1) &= \psi_A^-(x_1) \vee \psi_B^-(y_1) \\ &\leq (\psi_A^-(\xi_{AA}(x_1^{x_2})) \vee \psi_A^-(x_2)) \vee (\psi_B^-(\xi_{BB}(y_1^{y_2})) \vee \psi_B^-(y_2)) \\ &= \psi_{A \times B}^-(\xi_{A \times B}((x_1, y_1)^{(x_2, y_2)})) \vee \psi_{A \times B}^-(x_2, y_2).\end{aligned}$$

Hence, $\mathcal{A} \times \mathcal{B}$ is a CB-ideal of $X \times Y$. $\square$

6. Conclusions

This study introduced the notion of cubic bipolar ideals in Sheffer stroke Hilbert algebras by combining cubic set theory with bipolar fuzzy information, thereby providing a richer framework for analyzing algebraic structures. Fundamental properties and structural characteristics were established to clarify their role within this algebraic setting.

Moreover, the behavior of cubic bipolar ideals under algebraic mappings was examined through the notions of images and preimages. It was shown that these ideals were preserved under homomorphic mappings. In addition, the study of Cartesian products described how such ideals behaved in product algebras and identified conditions under which their structure was maintained.

Overall, the results enhanced the theory of Sheffer stroke Hilbert algebras and provided a foundation for further research. Future work may focus on cubic bipolar filters, homomorphism theorems, and extensions to other algebraic systems such as BCK- and EQ-algebras, along with potential applications in uncertainty modeling and decision-making.

Author Contributions: The current paper resulted from a long-term collaboration and some short communications. Conceptualization, A.S.A., T.O., R.N., R.B. and H.B.; supervision, T.O., R.N., R.B. and H.B.; writing—original draft preparation, A.S.A., T.O., R.N., R.B., and H.B.; resources, T.O., R.N., H.B. and A.S.A.; writing—review and editing, T.O., R.N., R.B., H.B. and A.S.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research is funded by Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia, under Researchers Supporting Project Number PNURSP2026R231.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author

Acknowledgments: The authors extend their appreciation to Princess Nourah bint Abdulrahman University for funding this research under Researchers Supporting Project number PNURSP2026R231, Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Conflicts of Interest: The authors declare that they have no conflicts of interests concerning the publication of this article.

References

1. Sheffer, H.M. A set of five independent postulates for Boolean algebras, with application to logical constants. *Trans. Am. Math. Soc.* **1913**, *14*, 481–488. [CrossRef]
2. Chajad, I. Sheffer operation in ortholattices. *Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math.* **2005**, *44*, 19–23.
3. Kozarkiewicz, V.; Grabowski, A. Axiomatization of Boolean algebras based on Sheffer stroke. *Formaliz. Math.* **2004**, *12*, 355–361.
4. Oner, T.; Katican, T.; Borumand Saeid, A. Class of Sheffer stroke BCK-algebras. *An. Şt. Univ. Ovidius Constanţa* **2022**, *30*, 247–269. [CrossRef]

5. Oner, T.; Katican, T.; Borumand Saeid, A. Relation between Sheffer Stroke and Hilbert Algebras. *Categ. Gen. Algebr. Struct. Appl.* **2021**, *14*, 245–268. [[CrossRef](#)]
6. Oner, T.; Katican, T.; Borumand Saeid, A. Fuzzy filters of Sheffer stroke Hilbert algebras. *J. Intell. Fuzzy Syst.* **2021**, *40*, 759–772. [[CrossRef](#)]
7. Kalmbach, G. *Algebraic Approaches to Logic*; North-Holland: Amsterdam, The Netherlands, 1983.
8. Priest, G. *An Introduction to Non-Classical Logic: From If to Is*; Cambridge University Press: Cambridge, UK, 2008.
9. Dummett, M. *Elements of Intuitionistic Logic*; Oxford University Press: Oxford, UK, 1978.
10. Atanassov, K.T.; Gargov, G. Interval-valued intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1989**, *31*, 343–349. [[CrossRef](#)]
11. Jun, Y.B.; Kim, C.S.; Yang, K.O. Cubic sets. *Ann. Fuzzy Math. Inform.* **2012**, *4*, 83–98.

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.