

Article

An Efficient Time-Space Two-Grid Compact Difference Method for the Nonlinear Schrödinger Equation: Analysis and Simulation

Chelimuge Bai ¹, Siriguleng He ^{2,*}  and Eerdun Buhe ²¹ School of Computer and Information Technology, Hohhot Minzu College, Huhhot 010051, China; clmg@imnc.edu.cn² School of Mathematics and Big Data, Hohhot Minzu College, Huhhot 010051, China; eerdunbuhe@163.com

* Correspondence: hsr@imu.edu.cn

Abstract

This article proposes a novel time-space two-grid high-order compact difference scheme for the one-dimensional nonlinear Schrödinger equation subject to Dirichlet boundary conditions. In comparison with the fully nonlinear compact difference scheme, the proposed methodology combines a small-scale nonlinear fourth-order compact difference algorithm on a time-space coarse grid and a large-scale linearized correction compact difference algorithm on a fine grid. In contrast to the time two-grid compact difference method, the proposed scheme applies the two-grid technique in both the spatial and temporal domains, thereby further improving computational efficiency. Solutions from the coarse grid are projected onto the fine grid via a temporally linear and spatially cubic Lagrange interpolation operator. Unconditional stability and optimal convergence rates, which are fourth-order in space and second-order in time, are proven in both the discrete L^2 and L^∞ norms, without any constraints on the grid ratio. In addition to the standard techniques of the energy method, a discrete Sobolev inequality and an a priori error estimate are employed to demonstrate stability and high-order convergence. Finally, the theoretical results are validated through numerical experiments, which confirm the robustness and reliability of the proposed approach. A single-soliton experiment demonstrates that, compared with the fully nonlinear compact difference scheme, the proposed method achieves a significant reduction in CPU time while maintaining a comparable level of accuracy. Additional experiments further illustrate the algorithm's effectiveness in simulating two-soliton interactions and soliton birth. These findings establish the proposed scheme as a highly efficient alternative to conventional nonlinear approaches.

Keywords: time-space two-grid high-order method; compact difference scheme; spatial cubic lagrange interpolation; nonlinear Schrödinger equation; unconditional stability and high-order convergence

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1. Introduction

The idea of the two-grid finite element (FE) method was originally proposed by Xu [1] for the discretization of nonsymmetric and indefinite partial differential equations. The key idea is to first solve the nonlinear problem on a coarse grid and then use this solution to

linearize the problem on a finer grid. As a result, this approach significantly reduces the computational cost of solving large-scale nonlinear systems without compromising accuracy. Two-grid (mixed) finite (volume) element methods have been successfully applied to the nonlinear Schrödinger (NLS) equation, and its optimal-order error estimates have been established in a series of works [2–8]. Global H^1 -norm superconvergence for two-grid FE approximations of the NLS equation was studied by Wang et al. [9]. However, implementing two-grid FE methods often requires sophisticated mesh generation techniques, which increases computational complexity and poses practical implementation challenges. In contrast, the two-grid finite difference (FD) method simplifies both temporal and spatial discretization, making scheme construction more straightforward and computation more convenient. A spatial two-grid FD scheme for the NLS equation was proposed by Ignat et al. [10], in which the fine-grid problem is linearized but remains coupled. Subsequently, following the idea proposed in [11], a time two-grid (TTG) fourth-order compact difference scheme for the one-dimensional NLS equation was developed in [12], along with a detailed L^2 -norm error analysis. Later, Wang et al. [13] introduced a TTG fitted scheme for the one-dimensional time-fractional NLS equation with nonsmooth solutions and presented its L^2 -norm error estimate. In addition to two-grid methods, several other efficient approaches have been developed for the NLS equation, including evolutionary optimization-based Runge–Kutta methods [14] and parametric discretization methods [15]. Moreover, the study of analytical solutions to the NLS equation remains extensive. For results concerning the derivation, well-posedness, and dynamical properties of the NLS equation, we refer to [16–21] and the references therein.

Indeed, applying the two-grid technique simultaneously in both time and space can further enhance computational efficiency for solving nonlinear problems. Gao et al. [22] developed a space-time two-grid difference scheme for the symmetric regularized long wave equation and provided its second-order convergence analysis. Shi et al. [23] proposed a new time-space two-grid method for the generalized Burgers' equation, establishing its solvability, energy-preserving property, and convergence in the discrete L^2 -norm. Compared with the fully nonlinear scheme and the time two-grid method, the numerical results presented in both [22,23] demonstrate that the time-space two-grid approach achieves greater computational efficiency. However, both of the aforementioned studies focus on time-space two-grid schemes based on second-order finite difference methods. To the best of our knowledge, few studies have investigated time-space two-grid high-order compact difference methods for solving the NLS equations.

In this article, we develop a novel time-space two-grid compact difference (TSTG-CD) method by combining the two-grid technique with a high-order compact difference scheme. This method is applied to solve a model problem of the NLS equation, which arises in nonlinear optics, plasma physics, and quantum mechanics. The proposed algorithm consists of the following three steps:

- (i) **Obtain time-space coarse grid solutions.** A fully nonlinear fourth-order compact difference scheme is solved on a time-space coarse grid using a Newton-type iterative method.
- (ii) **Generate time-space fine grid rough solutions.** The coarse-grid solution is interpolated onto the time-space fine grid by means of a spatial cubic Lagrange interpolation operator and a temporal linear Lagrange interpolation operator.
- (iii) **Obtain time-space fine grid-corrected solutions.** A linearized correction compact scheme is solved on the time-space fine grid to correct the rough solutions.

Compared with the existing literature, the main contributions of this work are as follows:

1. A novel time-space two-grid high-order compact difference method is proposed for the NLS equation. This approach combines a compact difference discretization with a two-grid framework in which a spatial piecewise cubic Lagrange interpolation operator and a temporal piecewise linear Lagrange interpolation operator are employed to transfer the coarse-grid solution to the fine grid.
2. Unlike the two-grid difference methods for the NLS equation that apply the technique in only one direction (temporal or spatial) [2–9,12,13], the proposed method employs the two-grid technique simultaneously in both time and space to further enhance computational efficiency.
3. Compared with the existing space-time two-grid methods proposed in [22,23], the construction of the proposed method is based on a spatial fourth-order compact difference scheme.
4. A rigorous theoretical analysis of the proposed scheme is presented. By applying a discrete Sobolev inequality [24,25] together with the standard energy method, the discrete conservation laws of the nonlinear scheme are established. A priori estimates are then obtained. Subsequently, the unconditional stability and convergence of the scheme are proven based on these estimates.
5. Convergence results with an order of $\mathcal{O}(\tau_C^4 + \tau^2 + H^8 + h^4)$ are proven in both the discrete L^2 -norm and L^∞ -norm, which implies that coarse grids can be rougher than fine grids. Here, τ_C and H denote the step sizes on the coarse grid, while τ and h denote those on the fine grid, respectively.
6. Numerical experiments are conducted to validate the effectiveness and efficiency of the proposed method. Computational efficiency is further demonstrated through comparisons with the nonlinear compact difference method [26] and the time two-grid method [12]. The results show that our method is computationally more efficient than those in [12,26].

The remainder of this article is organized as follows: In Section 2, some notations are introduced and the time-space two-grid compact difference scheme is proposed. In Section 3, several basic lemmas are introduced or proven. In Section 4, the discrete conservation laws of the nonlinear scheme are discussed, and a priori estimates are obtained; then the stability and convergence of the proposed scheme are proven based on these estimates. The numerical results are reported in Section 5, and brief conclusions are given in the final section. For convenience, some definitions of notations and abbreviations used in this paper are summarized in Appendix A.

2. Notations and the New Time-Space Two-Grid Scheme

In this section, we use the following simple model problem of the NLS equation to illustrate our idea.

$$i \frac{\partial u}{\partial t} + \eta \frac{\partial^2 u}{\partial x^2} + q|u|^2 u = 0, \quad x \in (a, b), \quad t \in (0, T], \quad (1)$$

$$u(x, 0) = \phi(x), \quad x \in [a, b], \quad (2)$$

$$u(a, t) = u(b, t) = 0, \quad t \in (0, T], \quad (3)$$

where $i = \sqrt{-1}$, $u(x, t) \in \mathbb{C}([a, b] \times (0, T])$ is the complex wave function, and $\phi(x)$ is a given sufficiently smooth complex-valued initial function. The solution $u(x, t)$ is assumed to have compact support on a bounded interval $[a, b]$ during the time period $[0, T]$, and

the real constant parameters $\eta > 0$, $q \neq 0$. The parameter q in Equation (1) describes the strength of the local interactions between particles. Models (1)–(3) conserve the total mass and energy such that

$$\mathbf{Q}(u(\cdot, t)) := \int_{\mathbb{R}} |u(x, t)|^2 dx \equiv \mathbf{Q}(u(\cdot, 0)), \quad t > 0,$$

and

$$\mathbf{E}(u(\cdot, t)) := \int_{\mathbb{R}} \left(\eta \left| \frac{\partial u}{\partial x} \right|^2 - \frac{q}{2} |u|^4 \right) dx \equiv \mathbf{E}(u(\cdot, 0)), \quad t > 0,$$

respectively.

To design a time-space two-grid algorithm, we give the following notations: For two integers ω_1 and ω_2 , we denote a set $\mathcal{Z}[\omega_1, \omega_2] = \{j | \omega_1 \leq j \leq \omega_2, j \in \mathbb{Z}\}$. For the positive integers $L, M, R \geq 2, r \geq 2$, we define a spatial coarse grid $x_J = JH$ for $J \in \mathcal{Z}[0, L]$ on $[a, b]$ with $H = (b - a)/L$. We denote $\ell = LR$ and define a spatial fine grid $x_j = jh$ for $j \in \mathcal{Z}[0, \ell]$ with $h = H/R$. Similarly, we define a temporal coarse grid $t_Z = Z\tau_c$ for $Z \in \mathcal{Z}[0, M]$ on $[0, T]$ with $\tau_c = T/M$. Denote $m = Mr$ and define a temporal fine grid $t_z = z\tau$ for $z \in \mathcal{Z}[0, m]$ with $\tau = T/m$. Denote $\omega = H$ or h , $\kappa = \tau_c$ or τ , $N_\omega = L$ or ℓ , $M_\kappa = M$ or m . We introduce the mesh $\Omega_{\omega\kappa} = \Omega_\omega \times \Omega_\kappa$, where $\Omega_\omega = \{x_s | s \in \mathcal{Z}[0, N_\omega]\}$, $\Omega_\kappa = \{t_n | n \in \mathcal{Z}[0, M_\kappa]\}$. Define grid function spaces $\mathcal{J}_\omega = \{v | v = (v_0, v_1, \dots, v_{N_\omega})\}$ and $\mathcal{J}_\omega^0 = \{v | v \in \mathcal{J}_\omega, v_0 = v_{N_\omega} = 0\}$ with $v_s = v(x_s)$. Given a spatial grid function $v \in \mathcal{J}_\omega$, denote

$$\begin{aligned} \partial_\omega v_s &= \frac{v_{s+1} - v_s}{\omega}, \quad \delta_\omega^2 v_s = \frac{v_{s+1} - 2v_s + v_{s-1}}{\omega^2}, \\ \mathcal{A}_\omega v_s &= \left(1 + \frac{\omega^2}{12} \delta_\omega^2\right) v_s = \frac{1}{12} (v_{s-1} + 10v_s + v_{s+1}). \end{aligned}$$

Given a temporal grid function $w = (w^0, w^1, \dots, w^{M_\kappa})$ with $w^n = w(t_n)$, denote

$$\partial_\kappa w^n = \frac{w^{n+1} - w^n}{\kappa}, \quad w^{n+\frac{1}{2}} = \frac{w^{n+1} + w^n}{2}.$$

Let $v, w \in \mathcal{J}_\omega^0$, and define the discrete inner products and norms via

$$\begin{aligned} (v, w)_\omega &= \omega \sum_{s=1}^{N_\omega-1} v_s \bar{w}_s, \quad \|v\|_\omega = \sqrt{(v, v)}, \quad \|v\|_{\omega, \infty} = \max_{s \in \mathcal{Z}[0, N_\omega]} |v_s|, \\ \langle v, w \rangle_\omega &= \omega \sum_{s=0}^{N_\omega-1} v_s \bar{w}_s, \quad |v|_{\omega, 1} = \sqrt{\langle v, v \rangle_\omega}, \quad \|v\|_{\omega, 1}^2 = \|v\|_\omega^2 + |\partial_\omega v|_{\omega, 1}^2, \end{aligned}$$

where $\bar{w}_j$ denotes the complex conjugate of w_j . We also define the discrete L^p ($2 < p < \infty$) norm as

$$\|v\|_{\omega, p} = \left(\omega \sum_{s=1}^{N_\omega-1} |v_s|^p \right)^{1/p}.$$

For simplicity, we denote u_s^n and U_s^n as the exact value and the approximation of $u(x, t)$ at the point (x_s, t_n) , respectively. We also denote $(U_C)_s^n$, $(U\tau h)_s^n$, and $(U_F)_s^n$ as the coarse-grid approximate solution, the fine-grid rough solution, and the fine-grid corrected solution, respectively. Furthermore, note that C is a generic positive constant independent of the time and space steps, and it may take different values in different contexts.

To construct the proposed TSTG-CD algorithm, we introduce the following lemma and definitions.

Definition 1. For $v \in J_h$, the spatial cubic Lagrange interpolation operator Π_h is defined as

$$\Pi_h v(x) \Big|_{[x_{J-3}, x_J]} = \sum_{s=J-3}^J \psi_s(x) v_s, \quad J = 3k, \quad k \in \mathcal{Z} \left[1, \left\lfloor \frac{L}{3} \right\rfloor \right],$$

Furthermore, if $L \not\equiv 0 \pmod{3}$, then

$$\Pi_h v(x) \Big|_{[x_{L-2}, x_L]} = \sum_{s=L-3}^L \psi_s(x) v_s,$$

where $\left\lfloor \frac{L}{3} \right\rfloor$ is the floor of $\frac{L}{3}$, and $\psi_s(x)$ is the cubic Lagrange interpolation basis function defined as

$$\psi_s(x) = \prod_{J-3 \leq p \leq J, p \neq s} \frac{(x - x_p)}{(x_s - x_p)}.$$

Definition 2. For $w \in J_h$, the temporal linear Lagrange interpolation operator Π_τ is defined as

$$\Pi_\tau w(t) \Big|_{[t_{Z-1}, t_Z]} = \frac{t - t_Z}{t_{Z-1} - t_Z} w^{Z-1} + \frac{t - t_{Z-1}}{t_Z - t_{Z-1}} w^Z, \quad Z \in \mathcal{Z}[1, M].$$

Lemma 1. (Ref. [27]) Let function $g(x) \in C^6[0, 1]$ and $\zeta(\lambda) = 5(1 - \lambda)^3 - 3(1 - \lambda)^5$. Then it holds, for $s \in \mathcal{Z}[1, N_\omega - 1]$,

$$\mathcal{A}_\omega g''(x_s) = \delta_\omega^2 g(x_s) + \frac{\omega^4}{36} \int_0^1 \left[g^{(6)}(x_s - \lambda\omega) + g^{(6)}(x_s + \lambda\omega) \right] \zeta(\lambda) d\lambda.$$

Considering Equation (1) at the point $(x_s, t_{n+1/2})$ with $t_{n+1/2} = (t_{n+1} + t_n)/2$, we obtain

$$i\partial_\kappa u_s^n + \eta \left(\frac{\partial^2 u}{\partial x^2} \right)_s^{n+\frac{1}{2}} + q(|u|^2)_s^{n+\frac{1}{2}} u_s^{n+\frac{1}{2}} + r_{1,s}^n = 0, \quad (4)$$

where

$$\begin{aligned} r_{1,s}^n = & i \frac{\partial u}{\partial t}(x_s, t_{n+1/2}) - i\partial_\kappa u_s^n + \eta \frac{\partial^2 u}{\partial x^2}(x_s, t_{n+1/2}) - \eta \left(\frac{\partial^2 u}{\partial x^2} \right)_s^{n+\frac{1}{2}} \\ & + q \left[|u(x_s, t_{n+1/2})|^2 u(x_s, t_{n+1/2}) - (|u|^2)_s^{n+\frac{1}{2}} u_s^{n+\frac{1}{2}} \right]. \end{aligned}$$

Applying the operator $\mathcal{A}_\omega$ to both sides of Equation (4) and using Lemma 1, we have

$$i\mathcal{A}_\omega \partial_\kappa u_s^n + \eta \delta_\omega^2 u_s^{n+\frac{1}{2}} + q\mathcal{A}_\omega \left[(|u|^2)_s^{n+\frac{1}{2}} u_s^{n+\frac{1}{2}} \right] + r_s^n = 0, \quad (5)$$

where $r_s^n = \mathcal{A}_\omega r_{1,s}^n + \frac{\omega^4}{36} \int_0^1 \left[\frac{\partial^6 u}{\partial x^6}(x_s - \lambda\omega, t_{n+1/2}) + \frac{\partial^6 u}{\partial x^6}(x_s + \lambda\omega, t_{n+1/2}) \right] \zeta(\lambda) d\lambda$. Using Taylor expansion and assuming the solution is smooth, there exists a positive constant C such that

$$|r_{1,s}^n| \leq C\kappa^2, \quad |r_s^n| \leq C(\kappa^2 + \omega^4). \quad (6)$$

Omitting the small term r_s^n in (5), we obtain a fully nonlinear compact difference (FN-CD) scheme:

$$i\mathcal{A}_\omega \partial_\kappa U_s^n + \eta \delta_\omega^2 U_s^{n+\frac{1}{2}} + q\mathcal{A}_\omega \left[(|U|^2)_s^{n+\frac{1}{2}} U_s^{n+\frac{1}{2}} \right] = 0, \quad (7)$$

$$s \in \mathcal{Z}[1, N_\omega - 1], \quad n \in \mathcal{Z}[0, M_\kappa - 1],$$

$$U_s^0 = \phi(x_s), \quad s \in \mathcal{Z}[1, N_\omega - 1], \quad U_0^n = 0, \quad U_{N_\omega}^n = 0, \quad n \in \mathcal{Z}[0, M_\kappa]. \quad (8)$$

Remark 1. Generally, Newton's iteration method is used to solve nonlinear systems (7) and (8). Nevertheless, an excessively large number of grid subdivisions m, ℓ can severely degrade computational efficiency.

Based on (7) and (8), we construct a novel time-space two-grid compact difference (TSTG-CD) algorithm for (1)–(3), which consists of the following three steps:

Step 1. On the coarse grid $\Omega_{H\tau_C}$, we construct the following scheme for the numerical solutions $(U_C)_J^Z = (V_C)_J^Z + i(W_C)_J^Z$:

$$i\mathcal{A}_H \partial_{\tau_C} (U_C)_J^Z + \eta \delta_H^2 (U_C)_J^{Z+\frac{1}{2}} + q\mathcal{A}_H \left[(|U_C|^2)_J^{Z+\frac{1}{2}} (U_C)_J^{Z+\frac{1}{2}} \right] = 0, \quad (9)$$

$$J \in \mathcal{Z}[1, L-1], \quad Z \in \mathcal{Z}[0, M-1],$$

$$(U_C)_J^0 = \phi(x_J), \quad J \in \mathcal{Z}[1, L-1], \quad (U_C)_0^Z = 0, \quad (U_C)_L^Z = 0, \quad Z \in \mathcal{Z}[0, M], \quad (10)$$

where $(V_C)_J^Z$ and $(W_C)_J^Z$ denote the real part and the imaginary part of the complex value $(U_C)_J^Z$.

Step 2. On the time-space fine grid $\Omega_{h\tau}$, the interpolation operators Π_h and Π_τ are used to compute auxiliary variables $(U_{\tau h})_j^z := [\Pi_\tau(\Pi_h U_C)]_{(j-3)R+\alpha}^{(Z-1)r+\beta}$ on $\Omega_{h\tau}$ with $\alpha \in \mathcal{Z}[0, 3R]$ and $\beta \in \mathcal{Z}[0, r]$. To this end, in spatial direction for $Z \in \mathcal{Z}[0, M]$,

$$\left(\Pi_h U_C \right)_{(j-3)R+\alpha}^Z = \sum_{s=j-3}^j \phi_s(x_{(j-3)R+\alpha}) (U_C)_s^Z, \quad j = 3k, \quad k \in \mathcal{Z}\left[1, \left\lfloor \frac{L}{3} \right\rfloor\right], \quad (11)$$

and if $L \not\equiv 0 \pmod{3}$, then for $\alpha \in \mathcal{Z}[R, 3R]$,

$$\left(\Pi_h U_C \right)_{(L-3)R+\alpha}^Z = \sum_{s=L-3}^L \phi_s(x_{(L-3)R+\alpha}) (U_C)_s^Z.$$

Furthermore, the auxiliary variables $(U_{\tau h})_j^z$ are computed by

$$(U_{\tau h})_j^z := \left(\Pi_\tau(\Pi_h U_C) \right)_j^{(Z-1)r+\beta} = \left(1 - \frac{\beta}{r} \right) (U_C)_j^{Z-1} + \frac{\beta}{r} (U_C)_j^Z, \quad Z \in \mathcal{Z}[1, M]. \quad (12)$$

Step 3. On the time-space fine grid $\Omega_{h\tau}$, according to the solutions $(U_{\tau h})_j^z, (z \in \mathcal{Z}[0, m], j \in \mathcal{Z}[0, \ell])$ obtained in Step 2, we construct the following scheme to solve the numerical solutions $(U_F)_j^z, (z \in \mathcal{Z}[0, m], j \in \mathcal{Z}[0, \ell])$.

(i) For the case of $z = 0$, an iterative algorithm was implemented to compute $(U_F)_j^1$, such that

$$i\mathcal{A}_h (U_F)_j^{1(s+1)} + \frac{\eta}{2\tau} \delta_h^2 (U_F)_j^{1(s+1)} = i\mathcal{A}_h (U_F)_j^0 - \frac{\eta}{2\tau} \delta_h^2 (U_F)_j^0 - \frac{q}{2\tau} \mathcal{A}_h \left[(|U_F|^2)_j^{1(s)} (U_F)_j^{1(s)} + (|U_F|^2)_j^0 (U_F)_j^0 \right], \quad j \in \mathcal{Z}[1, \ell-1], \quad (13)$$

$$(U_F)_j^0 = \phi(x_j), \quad j \in \mathcal{Z}[1, \ell-1], \quad (U_F)_0^z = 0, \quad (U_F)_\ell^z = 0, \quad z \in \mathcal{Z}[0, m], \quad (14)$$

where s denotes the s th iteration, and set $(U_F)_j^{1(0)} = (U_F)_j^0$. If $\|(U_F)_j^{1(s+1)} - (U_F)_j^{1(s)}\|_{h,\infty} < \varepsilon$ (e.g., $\varepsilon = 10^{-6}$), we stop the iteration and set $(U_F)_j^1 = (U_F)_j^{1(s+1)}, j \in \mathcal{Z}[1, \ell-1]$.

(ii) For the case of $z \in \mathcal{Z}[1, m-1]$, solve $(U_F)_j^z$, such that

$$i\mathcal{A}_h \partial_\tau (U_F)_j^z + \eta \delta_h^2 (U_F)_j^{z+\frac{1}{2}} + q\mathcal{A}_h F_j^{z+\frac{1}{2}} = 0, \quad j \in \mathcal{Z}[1, \ell-1], \quad (15)$$

where $F_j^{n+\frac{1}{2}} = \tilde{F}_j^{n+\frac{1}{2}} + i\hat{F}_j^{n+\frac{1}{2}}$ with

$$\begin{aligned}\tilde{F}_j^{n+\frac{1}{2}} &= g_R^C + g_{R,v}^C \cdot \{E((V_F)_j^z) - (V_{th})_j^{z+\frac{1}{2}}\} + g_{R,w}^C \cdot \{E((W_F)_j^z) - (W_{th})_j^{z+\frac{1}{2}}\}, \\ \hat{F}_j^{n+\frac{1}{2}} &= g_I^C + g_{I,v}^C \cdot \{E((V_F)_j^z) - (V_{th})_j^{z+\frac{1}{2}}\} + g_{I,w}^C \cdot \{E((W_F)_j^z) - (W_{th})_j^{z+\frac{1}{2}}\},\end{aligned}$$

and $(V_{th})_j^z$ and $(W_{th})_j^z$ are the real part and the imaginary part of the complex value $(U_{th})_j^z$, respectively. $(V_F)_j^z$ and $(W_F)_j^z$ are the real part and the imaginary part of $(U_F)_j^z$, respectively. Here, the extrapolation operator $E(v_j^z)$ is defined by

$$E(v_j^z) = \frac{3v_j^z - v_j^{z-1}}{2}, \quad z \in \mathcal{Z}[1, m-1].$$

And the symbols are defined as follows:

$$\begin{aligned}g_R^C &= g_R((V_{th})_j^{z+\frac{1}{2}}, (W_{th})_j^{z+\frac{1}{2}}), \quad g_I^C = g_I((V_{th})_j^{z+\frac{1}{2}}, (W_{th})_j^{z+\frac{1}{2}}), \\ g_{R,v}^C &= g_{R,v}((V_{th})_j^{z+1/2}, (W_{th})_j^{z+\frac{1}{2}}) = \frac{\partial g_R}{\partial v}((V_{th})_j^{z+\frac{1}{2}}, (W_{th})_j^{z+\frac{1}{2}}), \\ g_{R,w}^C &= g_{R,w}((V_{th})_j^{z+\frac{1}{2}}, (W_{th})_j^{z+\frac{1}{2}}) = \frac{\partial g_R}{\partial w}((V_{th})_j^{z+\frac{1}{2}}, (W_{th})_j^{z+\frac{1}{2}}),\end{aligned}$$

with $g_R(v, w) = (v^2 + w^2)v$ and $g_I(v, w) = (v^2 + w^2)w$. The definitions of the symbols $g_{I,v}^C$ and $g_{I,w}^C$ are similar to those of $g_{R,v}^C$ and $g_{R,w}^C$.

3. Some Basic Lemmas

In this section, we will give some auxiliary lemmas which will be used later.

Lemma 2. (Ref. [26]) For any grid functions $v, w \in \mathcal{J}_\omega^0$, there is

$$(\delta_\omega^2 v, w)_\omega = -(\partial_\omega v, \partial_\omega w)_\omega.$$

Lemma 3. (Ref. [26]) For any grid function $v \in \mathcal{J}_\omega^0$, there are

$$|\partial_\omega v|_{\omega,1}^2 \leq \frac{4}{\omega^2} \|v\|_\omega^2, \quad \|v\|_{\omega,\infty} \leq \frac{\sqrt{b-a}}{2} |\partial_\omega v|_{\omega,1}.$$

Lemma 4. (Ref. [24]) For any function $v \in \mathcal{J}_\omega^0$, we have

$$\|v\|_{\omega,p} \leq C_\gamma |\partial_\omega v|_{\omega,1}^\gamma \cdot \|v\|_\omega^{1-\gamma}, \quad \gamma = \frac{d}{2} - \frac{d}{p}.$$

In general, for any function $v \in \mathcal{J}_\omega$, we have

$$\|v\|_{\omega,p} \leq C_\gamma (|\partial_\omega v|_{\omega,1} + \|v\|_\omega)^\gamma \cdot \|v\|_\omega^{1-\gamma}, \quad \gamma = \frac{d}{2} - \frac{d}{p},$$

where $p \in [2, 6]$ for $d = 3$ and $p \in [2, \infty)$ for $d = 1, 2$.

Proof. For the proof of Lemma 4, we refer to [25] for the case of $d = 1$, refs. [28,29] for $d = 2$, and [30] for $d = 3$. $\square$

In order to prove the convergence of the proposed scheme, some matrices and their related properties are provided below.

Let A_ω be a symmetric matrix defined by $A_\omega = \frac{1}{12} \text{trid}[1, 10, 1]_{N_\omega-1}$, where $\text{trid}[a_1, a_2, a_3]_{N_\omega-1}$ denotes the $(N_\omega - 1) \times (N_\omega - 1)$ tridiagonal matrix whose each row contains the values a_1, a_2 and a_3 on its subdiagonal, diagonal and superdiagonal, respectively. The eigenvalues of A_ω are in the form of

$$\lambda_j = \frac{5}{6} + \frac{1}{6} \cos \frac{k\pi}{J}, \quad k = 1, 2, \dots, N_\omega - 1, \quad (16)$$

which implies $\frac{2}{3} \leq \lambda_j \leq 1$. This indicates that the matrix A_ω is positive definite and invertible. Furthermore, we denote $M_\omega = A_\omega^{-1}$. Based on our knowledge of matrices, we know that M_ω is also real, positive definite and symmetric. Then, for any grid function $v = (v_1, v_2, \dots, v_{N_\omega-1})^T$, a discrete norm is introduced as follows:

$$\|v\|_\omega = (M_\omega v, v)_{\frac{1}{\omega}}. \quad (17)$$

Lemma 5. (Ref. [28]) For any grid function $v = (v_1, v_2, \dots, v_{N_\omega-1})^T$, there is

$$\|v\|_\omega \leq \|v\|_\omega \leq \frac{\sqrt{6}}{2} \|v\|_\omega.$$

Lemma 6. For any grid function $v = (v_1, v_2, \dots, v_{N_\omega-1})^T$, there are

$$M_\omega \partial_\omega v = \partial_\omega (M_\omega v), \quad M_\omega \delta_\omega^2 v = \delta_\omega^2 (M_\omega v).$$

Proof. Let $M_\omega = [m_{kj}]_{(N_\omega-1) \times (N_\omega-1)}$. For the k th entry of the vector $M_\omega v$, there is

$$(M_\omega v)_k = \sum_{j=1}^{N_\omega-1} m_{kj} v_j,$$

and

$$\partial_\omega (M_\omega v)_k = \sum_{j=1}^{N_\omega-1} m_{kj} \partial_\omega v_j. \quad (18)$$

On the other hand,

$$(M_\omega \partial_\omega v)_k = \sum_{j=1}^{N_\omega-1} m_{kj} \partial_\omega v_j. \quad (19)$$

Then $M_\omega \partial_\omega v = \partial_\omega (M_\omega v)$ is obtained from (18) and (19). Similarly, $M_\omega \delta_\omega^2 v = \delta_\omega^2 (M_\omega v)$ can be proven as well. $\square$

Lemma 7. For the grid functions $U^n \in \mathcal{J}_\omega^0$, $n \in \mathcal{Z}[0, M_\kappa - 1]$, there are

$$\begin{aligned} \text{Im}(M_\omega \delta_\omega^2 U^{n+\frac{1}{2}}, U^{n+\frac{1}{2}})_\omega &= 0, \\ \text{Re}(M_\omega \delta_\omega^2 U^{n+\frac{1}{2}}, \partial_\kappa U^n)_\omega &= -\frac{1}{2\kappa} \left(\|\partial_\omega U^{n+1}\|_\omega^2 - \|\partial_\omega U^n\|_\omega^2 \right)_\omega, \end{aligned}$$

where “Im(s)” and “Re(s)” refer to taking the imaginary part and the real part of a complex number s , respectively.

Proof. From Lemma 2, Lemma 6 and (17), we have

$$\begin{aligned}\operatorname{Im}\left(M_{\omega} \delta_{\omega}^2 U^{n+\frac{1}{2}}, U^{n+\frac{1}{2}}\right)_{\omega} &= \operatorname{Im}\left(\delta_{\omega}^2 M_{\omega} U^{n+\frac{1}{2}}, U^{n+\frac{1}{2}}\right)_{\omega} \\ &= -\operatorname{Im}\left\langle \partial_{\omega} M_{\omega} U^{n+\frac{1}{2}}, \partial_{\omega} U^{n+\frac{1}{2}} \right\rangle_{\omega} \\ &= -\operatorname{Im}\left\langle M_{\omega} \partial_{\omega} U^{n+\frac{1}{2}}, \partial_{\omega} U^{n+\frac{1}{2}} \right\rangle_{\omega} \\ &= -\operatorname{Im}\|\partial_{\omega} U^{n+\frac{1}{2}}\|_{\omega}^2 = 0.\end{aligned}$$

Using Lemma 2, Lemma 6 and (17), we also get

$$\begin{aligned}\operatorname{Re}\left(M_{\omega} \delta_{\omega}^2 U^{n+1/2}, \partial_{\kappa} U^n\right)_{\omega} &= -\operatorname{Re}\left\langle \partial_{\omega} M_{\omega} U^{n+1/2}, \partial_{\omega} \partial_{\kappa} U^n \right\rangle_{\omega} \\ &= -\operatorname{Re}\left\langle M_{\omega} \partial_{\omega} U^{n+1/2}, \partial_{\omega} \partial_{\kappa} U^n \right\rangle_{\omega} \\ &= -\frac{1}{2\kappa} \operatorname{Re}\left\langle M_{\omega} [\partial_{\omega} U^{n+1} + \partial_{\omega} U^n], [\partial_{\omega} U^{n+1} - \partial_{\omega} U^n] \right\rangle_{\omega} \\ &= -\frac{1}{2\kappa} \left(\|\partial_{\omega} U^{n+1}\|_{\omega}^2 - \|\partial_{\omega} U^n\|_{\omega}^2 \right).\end{aligned}$$

This completes the proof. $\square$

4. Numerical Analysis of the TSTG-CD Scheme

For simplicity, in numerical analysis, the subindex “s” is omitted and Equation (5) will be rewritten as

$$i\partial_{\kappa} u^n + \eta \mathcal{A}_{\omega}^{-1} \delta_{\omega}^2 u^{n+\frac{1}{2}} + q \left[(|u|^2)^{n+\frac{1}{2}} u^{n+\frac{1}{2}} \right] + \mathcal{A}_{\omega}^{-1} r^n = 0, \quad (20)$$

and Equation (7) can also be written as

$$i\partial_{\kappa} U^n + \eta \mathcal{A}_{\omega}^{-1} \delta_{\omega}^2 U^{n+\frac{1}{2}} + q \left[(|U|^2)^{n+\frac{1}{2}} U^{n+\frac{1}{2}} \right] = 0. \quad (21)$$

Then, the matrix form of Equation (20) can be written as

$$i\partial_{\kappa} u^n + \eta M_{\omega} \delta_{\omega}^2 u^{n+\frac{1}{2}} + q \left[(|u|^2)^{n+\frac{1}{2}} u^{n+\frac{1}{2}} \right] + M_{\omega} r^n = 0, \quad (22)$$

and the matrix form of Equation (21) is

$$i\partial_{\kappa} U^n + \eta M_{\omega} \delta_{\omega}^2 U^{n+\frac{1}{2}} + q \left[(|U|^2)^{n+\frac{1}{2}} U^{n+\frac{1}{2}} \right] = 0. \quad (23)$$

Similarly, the matrix form of Equation (15) is

$$i\partial_{\kappa} U_F^z + \eta M_{\omega} \delta_{\omega}^2 U_F^{z+\frac{1}{2}} + q F^{z+\frac{1}{2}} = 0, \quad z \in \mathcal{Z}[0, m-1]. \quad (24)$$

Remark 2. Equation (7) is suitable for computing in implementation. However, since the nonlinear terms in Equation (7) are distributed in different grids via a certain portion, it is not suitable for theoretical analysis. Thus, in numerical analysis, we will use the equivalent form (23).

4.1. Conservation

Theorem 1. The schemes (7) and (8) are conservative in the sense that

$$\mathcal{Q}^n := \|U^n\|_{\omega}^2 \equiv \mathcal{Q}^0, \quad n \in \mathcal{Z}[0, M_{\kappa}], \quad (25)$$

$$\mathcal{E}^n := \eta \|\partial_{\omega} U^n\|_{\omega}^2 - \frac{q}{2} \|U^n\|_{\omega, 4}^4 \equiv \mathcal{E}^0, \quad n \in \mathcal{Z}[0, M_{\kappa}]. \quad (26)$$

Proof. Doing the inner product of (23) with $2U^{n+\frac{1}{2}}$, and taking the imaginary part, we get

$$\partial_\kappa \|U^n\|_\omega^2 = 0, \quad n \in \mathcal{Z}[0, M_\kappa], \quad (27)$$

where Lemma 7 was used. This gives (25).

Doing the inner product of (23) with $\partial_\kappa U^n$, and taking the real part, we have

$$\frac{1}{2\kappa} \left[\left(\eta \| \partial_\omega U^{n+1} \|_\omega^2 - \frac{q}{2} \| U^{n+1} \|_{\omega,4}^4 \right) - \left(\eta \| \partial_\omega U^n \|_\omega^2 - \frac{q}{2} \| U^n \|_{\omega,4}^4 \right) \right] = 0, \quad (28)$$

where Lemma 7 was used. This gives (26) and completes the proof. $\square$

4.2. A Priori Estimate

Theorem 2. Let U^n be the solution obtained from schemes (7) and (8). Assume that one of the following two conditions is satisfied:

$$\begin{aligned} (a) \quad & \phi \in H_4^2(\Omega), \quad q < 0, \\ (b) \quad & \phi \in H_4^2(\Omega), \quad \|\phi\|_\omega^2 \leq \frac{2\eta}{qC_\ell}, \quad q > 0, \end{aligned}$$

then there exists a constant C such that

$$\|U^n\|_{\omega,\infty} \leq C, \quad n \in \mathcal{Z}[0, M_\kappa], \quad (29)$$

$$\|U_{\tau h}^z\|_{h,\infty} \leq C, \quad z \in \mathcal{Z}[0, m]. \quad (30)$$

Proof. Under the condition (a), applying Lemma 5 and (25), we get

$$\eta |\partial_\omega U^n|_{\omega,1}^2 \leq \eta \| \partial_\omega U^n \|_\omega^2 \leq \eta \| \partial_\omega \phi \|_\omega^2 - \frac{q}{2} \|\phi\|_{\omega,4}^4. \quad (31)$$

Then, (31), together with Lemma 3, gives (29).

Under condition (b), applying Lemma 4 and (25), we have

$$\|U^n\|_{\omega,4}^4 \leq C_\gamma |\partial_\omega U^n|_{\omega,1}^2 \|U^n\|_\omega^2 \leq C_\gamma \|\phi\|_\omega^2 |\partial_\omega U^n|_{\omega,1}^2. \quad (32)$$

This, together with (26) and Lemma 5, gives

$$\left(\eta - \frac{C_\gamma q}{2} \|\phi\|_\omega^2 \right) |\partial_\omega U^n|_{\omega,1}^2 \leq \eta \| \partial_\omega \phi \|_\omega^2 - \frac{q}{2} \|\phi\|_{\omega,4}^4 \leq \frac{3\eta}{2} \| \partial_\omega \phi \|_\omega^2. \quad (33)$$

Then, (33), together with Lemma 3, also gives (29).

Next, we prove (30). For any $z \in \mathcal{Z}[1, m]$, Definitions 1 and 2, combined with (11) and (12), imply that there exists a $Z \in \mathcal{Z}[1, M]$ such that $z \in \mathcal{Z}[(Z-1)r, Zr]$, and we have

$$(U_{\tau h})_j^z = [\Pi_\tau(\Pi_h U_c)]_j^z = [\Pi_\tau(\Pi_h U_c)]_j^{(Z-1)r+\beta} = \left(1 - \frac{\beta}{r}\right) [\Pi_h U_c]_j^{Z-1} + \frac{\beta}{r} [\Pi_h U_c]_j^Z. \quad (34)$$

Furthermore, for any $j \in \mathcal{Z}[1, \ell]$, there exists a $k \in \mathcal{Z}\left[1, \left\lfloor \frac{\ell}{3} \right\rfloor\right]$ and $J = 3k$ such that $j \in \mathcal{Z}[(J-3)R, JR]$, and we get

$$\left(\Pi_h U_c \right)_j^Z = \left(\Pi_h U_c \right)_{(J-3)R+\alpha}^Z = \sum_{s=J-3}^J \phi_s(x_{(J-3)R+\alpha}) (U_c)_s^Z. \quad (35)$$

Therefore, through (34) and (35) together with (29), we have

$$\begin{aligned} |(U_{\tau h})_j^z| &\leq (1 - \frac{\beta}{r}) \left| (\Pi_h U_C)_j^{Z-1} \right| + \frac{\beta}{r} \left| (\Pi_h U_C)_j^Z \right| \\ &\leq (1 - \frac{\beta}{r}) \sum_{s=j-3}^J \phi_s(x_{(J-3)R+\alpha}) \left| (U_C)_s^{Z-1} \right| + \frac{\beta}{r} \sum_{s=j-3}^J \phi_s(x_{(J-3)R+\alpha}) \left| (U_C)_s^Z \right| \\ &\leq \|U_C^{Z-1}\|_{H,\infty} + \|U_C^Z\|_{H,\infty} \leq 2C. \end{aligned} \quad (36)$$

This gives (30) and completes the proof. $\square$

4.3. Stability

Theorem 3. Suppose that $U_F^z, z \in \mathcal{Z}[0, m]$ is the solution of the fine grid scheme (13)–(15), and τ is sufficiently small. Then, there exists a positive constant C independent of h, τ such that

$$\|U_F^z\|_h \leq C\|\phi\|_h + \mathcal{O}(\tau^2), \quad z \in \mathcal{Z}[1, m]. \quad (37)$$

Proof. When $z = 1$, estimate $\|U_F^1\|_h \leq C\|\phi\|_h + \mathcal{O}(\tau^2)$ can be derived from (13) by following an argument analogous to that used in the proof of (25).

When $z \in \mathcal{Z}[1, m-1]$, doing the inner product of (15) with $2U_F^{z+\frac{1}{2}}$, and taking the imaginary part, we get

$$\frac{1}{\tau} \left(\|U_F^{z+1}\|_h^2 - \|U_F^z\|_h^2 \right) + 2\eta \operatorname{Im}(\mathbf{M}_h \delta_h^2 U_F^{z+\frac{1}{2}}, U_F^{z+\frac{1}{2}})_h + 2q \operatorname{Im}(F^{z+\frac{1}{2}}, U_F^{z+\frac{1}{2}})_h = 0. \quad (38)$$

By Lemma 7, we can see that

$$2\eta \operatorname{Im}(\mathbf{M}_h \delta_h^2 U_F^{z+\frac{1}{2}}, U_F^{z+\frac{1}{2}})_h = 0. \quad (39)$$

From the expressions $F_j^{z+\frac{1}{2}} = \tilde{F}_j^{z+\frac{1}{2}} + i\hat{F}_j^{z+\frac{1}{2}}$ and $U_{F,j}^{z+\frac{1}{2}} = V_{F,j}^{z+\frac{1}{2}} + iW_{F,j}^{z+\frac{1}{2}}$, we have

$$\operatorname{Im}(F^{z+\frac{1}{2}}, U_F^{z+\frac{1}{2}})_h = (\tilde{F}^{z+\frac{1}{2}}, W_F^{z+\frac{1}{2}})_h + (\hat{F}^{z+\frac{1}{2}}, V_F^{z+\frac{1}{2}})_h.$$

Furthermore, from the definitions of symbols $g_R^C, g_{R,v}^C, g_{R,w}^C$ and the extrapolation operator $E(v_j^z)$, we obtain

$$\begin{aligned} |(g_R^C, W_F^{z+\frac{1}{2}})_h| &= \left| \left((|U_{\tau h}|^2)^{z+\frac{1}{2}} V_{\tau h}^{z+\frac{1}{2}}, W_F^{z+\frac{1}{2}} \right)_h \right| \leq C \|V_{\tau h}^{z+\frac{1}{2}}\|_h \|W_F^{z+\frac{1}{2}}\|_h, \\ |(g_{R,v}^C \cdot \{E(V_F^z) - V_{\tau h}^{z+\frac{1}{2}}\}, W_F^{z+\frac{1}{2}})_h| &\leq C \left| \left((V_F^{z+\frac{1}{2}} + \mathcal{O}(\tau^2)), W_F^{z+\frac{1}{2}} \right)_h \right| + C \left| \left(V_{\tau h}^{z+\frac{1}{2}}, W_F^{z+\frac{1}{2}} \right)_h \right| \\ &\leq C \left(\|V_F^{z+\frac{1}{2}}\|_h + \|V_{\tau h}^{z+\frac{1}{2}}\|_h + \mathcal{O}(\tau^2) \right) \|W_F^{z+\frac{1}{2}}\|_h, \\ |(g_{R,w}^C \cdot \{E(W_F^z) - W_{\tau h}^{z+\frac{1}{2}}\}, W_F^{z+\frac{1}{2}})_h| &\leq C \left| \left((W_F^{z+\frac{1}{2}} + \mathcal{O}(\tau^2)), W_F^{z+\frac{1}{2}} \right)_h \right| + C \left| \left(W_{\tau h}^{z+\frac{1}{2}}, W_F^{z+\frac{1}{2}} \right)_h \right| \\ &\leq C \left(\|W_F^{z+\frac{1}{2}}\|_h + \|W_{\tau h}^{z+\frac{1}{2}}\|_h + \mathcal{O}(\tau^2) \right) \|W_F^{z+\frac{1}{2}}\|_h, \end{aligned} \quad (40)$$

where the boundedness condition (30) was used. Then, from (40), we have

$$|(\tilde{F}^{z+\frac{1}{2}}, W_F^{z+\frac{1}{2}})_h| \leq C \{ 2\|V_{\tau h}^{z+\frac{1}{2}}\|_h^2 + \|W_{\tau h}^{z+\frac{1}{2}}\|_h^2 + \|V_F^{z+\frac{1}{2}}\|_h^2 + 7\|W_F^{z+\frac{1}{2}}\|_h^2 + \mathcal{O}(\tau^4) \}.$$

By similar derivations, the term $|(\widehat{F}^{z+\frac{1}{2}}, V_F^{z+\frac{1}{2}})_h|$ can also be bounded. And then, combining (38)–(40), we get

$$\|U_F^{z+1}\|_h^2 - \|U_F^z\|_h^2 \leq 2\tau|q|C\{3\|U_{\tau h}^{z+\frac{1}{2}}\|_h^2 + 8\|U_F^{z+\frac{1}{2}}\|_h^2 + \mathcal{O}(\tau^4)\}. \quad (41)$$

In addition, according to (25), there is $\|U_c^Z\|_H^2 = \|U_c^0\|_H^2, Z \in \mathcal{Z}[1, M]$. Furthermore, an argument analogous to (34)–(36) is used to get

$$\|U_{\tau h}^z\|_h^2 \leq 2\|U_c^Z\|_H^2 = 2\|U_c^0\|_H^2, z \in \mathcal{Z}[1, m].$$

Then, summing up the inequality (41), and together with the above inequality, we have

$$\|U_F^z\|_h^2 - \|U_F^0\|_h^2 \leq 2|q|C\{16\tau \sum_{s=0}^{z-1} \|U_F^s\|_h^2 + 6T\|U_c^0\|_H^2 + 8\tau\|U_F^z\|_h^2 + \mathcal{O}(\tau^4)\}.$$

Then, taking sufficiently small τ and applying the discrete Gronwall's inequality, we have

$$\|U_F^z\|_h^2 \leq C\|U_c^0\|_H^2 + \|U_F^0\|_h^2 + \mathcal{O}(\tau^4) \leq C\|\phi\|_h^2 + \mathcal{O}(\tau^4), \quad z \in \mathcal{Z}[1, m].$$

This completes the proof. $\square$

4.4. Convergence

On the coarse grid, denoting $e_c^Z = u^Z - U_c^Z, Z \in \mathcal{Z}[0, M]$, and subtracting Equation (23) from Equation (22) yields the following error equation:

$$i\partial_{\tau_c} e_c^Z + \eta M_H \delta_H^2 e_c^{Z+\frac{1}{2}} + q\mathcal{G}^Z + M_H r^Z = 0, \quad Z \in \mathcal{Z}[0, M], \quad (42)$$

where

$$\begin{aligned} \mathcal{G}^Z &= (|u|^2)^{Z+\frac{1}{2}} u^{Z+\frac{1}{2}} - (|U_c|^2)^{Z+\frac{1}{2}} U_c^{Z+\frac{1}{2}} \\ &= \frac{1}{2}(|u^Z|^2 + |u^{Z+1}|^2 - |U_c^Z|^2 - |U_c^{Z+1}|^2) u^{Z+\frac{1}{2}} \\ &\quad + \frac{1}{2}(|U_c^Z|^2 + |U_c^{Z+1}|^2) e_c^{Z+\frac{1}{2}} = \theta^{Z+\frac{1}{2}} + \zeta^{Z+\frac{1}{2}}. \end{aligned}$$

In the following convergence analysis, we also suppose that the exact solution $u(x, t)$ to the problem (1) is sufficiently smooth, and there exist positive constants C_u, C_t, C_x such that

$$\max_{\substack{x \in [a, b], \\ 0 \leq t \leq T}} |u(x, t)| \leq C_u, \quad \max_{\substack{x \in [a, b], \\ 0 \leq t \leq T}} \left| \frac{\partial u^2(x, t)}{\partial t^2} \right| \leq C_t, \quad \max_{\substack{x \in [a, b], \\ 0 \leq t \leq T}} \left| \frac{\partial u^5(x, t)}{\partial x^5} \right| \leq C_x. \quad (43)$$

4.4.1. Error Estimate for Step 1

Theorem 4. Suppose that assumptions (a) and (b) are valid, and τ_c is sufficiently small. Then, for schemes (9) and (10), there exists a positive constant C independent of H, τ_c such that

$$\|u^Z - U_c^Z\|_H \leq C(\tau_c^2 + H^4), \quad Z \in \mathcal{Z}[0, M], \quad (44)$$

$$\|u^Z - U_c^Z\|_{H, \infty} \leq C(\tau_c^2 + H^4), \quad Z \in \mathcal{Z}[0, M]. \quad (45)$$

Proof. Computing the inner product of (42) with $2e_c^{Z+\frac{1}{2}}$, we have

$$i\partial_{\tau_c}\|e_c^Z\|_H^2 + 2\eta\|\partial_H e_c^{Z+\frac{1}{2}}\|_H^2 + 2q(\mathcal{G}^Z, e_c^{Z+\frac{1}{2}})_H + 2(M_H r^Z, e_c^{Z+\frac{1}{2}})_H = 0, \quad (46)$$

where Lemma 6 was used.

Taking the imaginary part of (46), we obtain

$$\partial_{\tau_c}\|e_c^Z\|_H^2 + 2q\text{Im}(\mathcal{G}^Z, e_c^{Z+\frac{1}{2}})_H + 2\text{Im}(M_H r^Z, e_c^{Z+\frac{1}{2}})_H = 0. \quad (47)$$

To obtain the bound for the second term of (47), using (29) and (43), we get

$$\begin{aligned} |2\theta^{Z+\frac{1}{2}}| &\leq \left\{ (|u^Z| + |U_c^Z|)|e_c^Z| + (|u^{Z+1}| + |U_c^{Z+1}|)|e_c^{Z+1}| \right\} |u^{Z+\frac{1}{2}}| \\ &\leq 2C_u(C_u + C)\{|e_c^Z| + |e_c^{Z+1}|\}. \end{aligned}$$

Furthermore, we obtain

$$\begin{aligned} |2q\text{Im}(\mathcal{G}^Z, e_c^{Z+\frac{1}{2}})_H| &= |2q\text{Im}(\theta^{Z+\frac{1}{2}}, e_c^{Z+1/2})_H| \\ &\leq 2|q|C_u(C_u + C)H \sum_{j=1}^{L-1} \{|(e_c)_j^Z| + |(e_c)_j^{Z+1}|\} |(e_c)_j^{Z+\frac{1}{2}}| \\ &\leq 2|q|C_u(C_u + C)\{\|e_c^Z\|_H^2 + \|e_c^{Z+1}\|_H^2\}, \end{aligned} \quad (48)$$

and

$$2|\text{Im}(M_H r^Z, e_c^{Z+\frac{1}{2}})_H| \leq \frac{9}{4}\|r^Z\|_H^2 + \|e_c^{Z+\frac{1}{2}}\|_H^2. \quad (49)$$

Then, from (47), together with (48) and (49), we have

$$\frac{1}{\tau_c}\{\|e_c^{Z+1}\|_H^2 - \|e_c^Z\|_H^2\} \leq \frac{9}{4}\|r^Z\|_H^2 + C\{\|e_c^Z\|_H^2 + \|e_c^{Z+1}\|_H^2\}. \quad (50)$$

Summing up the inequality (50), and noting that $e_c^0 = 0$, we get

$$\sigma\|e_c^{Z+1}\|_H^2 \leq \frac{9\tau_c}{4} \sum_{n=0}^Z \|r^n\|_H^2 + C\tau_c \sum_{n=0}^Z \|e_c^n\|_H^2,$$

where $\sigma = 1 - C\tau_c$. Then, taking sufficiently small τ_c and applying the discrete Gronwall's inequality, we have

$$\|u^Z - U_c^Z\|_H \leq C(\tau_c^2 + H^4), \quad Z \in \mathcal{Z}[0, M]. \quad (51)$$

For estimate (45), taking the real part of (46), we get

$$2\eta\|\partial_H e_c^{Z+\frac{1}{2}}\|_H^2 + 2q\text{Re}(\mathcal{G}^Z, e_c^{Z+\frac{1}{2}})_H + 2\text{Re}(M_H r^Z, e_c^{Z+\frac{1}{2}})_H = 0. \quad (52)$$

Analogously to the proof of (48) and (49), applying (29) and (43), we have

$$|2q\text{Re}(\theta^{Z+\frac{1}{2}}, e_c^{Z+\frac{1}{2}})_H| \leq 2|q|C_u(C_u + C)\{\|e_c^Z\|_H^2 + \|e_c^{Z+1}\|_H^2\}, \quad (53)$$

$$|2q\text{Re}(\xi^{Z+\frac{1}{2}}, e_c^{Z+\frac{1}{2}})_H| \leq |q|C\|e_c^{(p+\frac{1}{2})s}\|_H^2, \quad (54)$$

$$|2\text{Re}(M_H r^Z, e_c^{Z+\frac{1}{2}})_H| \leq \frac{9}{4}\|r^Z\|_H^2 + \|e_c^{Z+\frac{1}{2}}\|_H^2. \quad (55)$$

Then, from (52) together with (51) and (53)–(55), we obtain

$$\begin{aligned} 2\eta \|\partial_H e_c^{Z+\frac{1}{2}}\|_H^2 &\leq 2\eta \|\partial_H e_c^{Z+\frac{1}{2}}\|_H^2 \\ &\leq \left| 2q \operatorname{Re}(\mathcal{G}^n, e_c^{Z+\frac{1}{2}})_H \right| + \left| 2 \operatorname{Re}(M_H r^Z, e_c^{Z+\frac{1}{2}})_H \right| \\ &\leq C \{ \|e_c^Z\|_H^2 + \|e_c^{Z+1}\|_H^2 \} + C \|r^Z\|_H^2 \leq C(\tau_c^4 + H^8), \end{aligned} \quad (56)$$

where Lemma 5 was used. Then, (56), together with Lemma 3, gives

$$\|e_c^{Z+\frac{1}{2}}\|_{H,\infty} \leq C(\tau_c^2 + H^4), \quad Z \in \mathcal{Z}[0, M-1]. \quad (57)$$

And then applying (57) and noting that $e_c^0 = 0$, we obtain

$$\|\partial_H e_c^1\|_{H,\infty} = 2 \|\partial_H e_c^{\frac{1}{2}}\|_{H,\infty} \leq C(\tau_c^2 + H^4).$$

Furthermore, we assume that

$$\|\partial_H e_c^Z\|_{H,\infty} \leq C(\tau_c^2 + H^4), \quad Z \in \mathcal{Z}[0, M-1],$$

and then applying (57) and the triangle inequality, we have

$$\|\partial_H e_c^{Z+1}\|_{H,\infty} \leq 2 \|\partial_H e_c^{Z+\frac{1}{2}}\|_{H,\infty} + \|\partial_H e_c^Z\|_{H,\infty} \leq C(\tau_c^2 + H^4).$$

Therefore, according to mathematical induction, the error estimation (45) holds true. This completes the proof. $\square$

4.4.2. Error Estimate for the Step 2

Theorem 5. Suppose that assumptions (a) and (b) are valid, and τ_c is sufficiently small. Then there exists a positive constant C independent of H, τ_c such that

$$\|u^z - U_{\tau h}^z\|_h \leq C(\tau_c^2 + H^4), \quad z \in \mathcal{Z}[0, m], \quad (58)$$

$$\|u^z - U_{\tau h}^z\|_{h,\infty} \leq C(\tau_c^2 + H^4), \quad z \in \mathcal{Z}[0, m]. \quad (59)$$

where $(U_{\tau h})_j^z = [\Pi_\tau(\Pi_{\tau h} U_c)]_j^z$.

Proof. For any $z \in \mathcal{Z}[1, m]$, we have, cf. (34),

$$\begin{aligned} (U_{\tau h})_j^z &= \left(\Pi_\tau(\Pi_h U_c) \right)_j^z = \left(\Pi_\tau(\Pi_h U_c) \right)_j^{(Z-1)r+\beta} \\ &= \left(1 - \frac{\beta}{r} \right) \left(\Pi_h U_c \right)_j^{Z-1} + \frac{\beta}{r} \left(\Pi_h U_c \right)_j^Z. \end{aligned} \quad (60)$$

Additionally, based on the Lagrange's interpolation formula, we get

$$u_j^z = u_j^{(Z-1)r+\beta} = (\Pi_\tau u)_j^{(Z-1)r+\beta} + \frac{1}{2} \frac{\partial^2 u(\zeta_Z, x_j)}{\partial t^2} (t - t_{Z-1})(t - t_Z), \quad \zeta_Z \in (t_{Z-1}, t_Z). \quad (61)$$

Then, from (60) and (61) together with (43), we obtain

$$\begin{aligned} \|u^z - U_{\tau h}^z\|_h^2 &= h \sum_{j=1}^{\ell-1} |u_j^z - (U_{\tau h})_j^z|^2 \\ &= h \sum_{j=1}^{\ell-1} \left| \left(1 - \frac{\beta}{r}\right) \left(u_j^{Z-1} - (\Pi_h U_c)_j^{Z-1} + \frac{1}{2} \frac{\partial^2 u(\zeta_{Z-1}, x_j)}{\partial t^2} (t - t_{Z-1})(t - t_Z)\right) \right. \\ &\quad \left. + \frac{\beta}{r} \left(u_j^Z - (\Pi_h U_c)_j^Z + \frac{1}{2} \frac{\partial^2 u(\zeta_Z, x_j)}{\partial t^2} (t - t_{Z-1})(t - t_Z)\right) \right|^2 \\ &\leq 2 \|u^{Z-1} - (\Pi_h U_c)^{Z-1}\|_h^2 + 2 \|u^Z - (\Pi_h U_c)^Z\|_h^2 + C_t^2 (b-a) \tau_c^4. \end{aligned} \quad (62)$$

Analogously, for any $j \in \mathcal{Z}[1, \ell]$, we have, cf. (35),

$$(\Pi_h U_c)_j^Z = (\Pi_h U_c)_{(J-3)R+\alpha}^Z = \sum_{s=J-3}^J \phi_s(x_{(J-3)R+\alpha}) (U_c)_s^Z, \quad (63)$$

and there exists a $\varsigma_J \in (x_{J-3}, x_J)$ such that

$$u_j^Z = (\Pi_h u)_{(J-3)R+\alpha}^Z + \frac{1}{120} \frac{\partial^5 u(t_Z, \varsigma_J)}{\partial x^5} (x - x_{J-3})(x - x_{J-2})(x - x_{J-1})(x - x_J). \quad (64)$$

Then, from (63) and (64) together with (43) and (44), we get

$$\begin{aligned} \|u^Z - (\Pi_h U_c)^Z\|_h^2 &= h \sum_{j=1}^{\ell-1} |u_j^Z - (\Pi_h U_c)_j^Z|^2 \\ &\leq 6Rh \sum_{j=3}^{L-1} \left(|(e_c)_{J-3}^Z|^2 + |(e_c)_{J-2}^Z|^2 + |(e_c)_{J-1}^Z|^2 + |(e_c)_J^Z|^2 + \frac{H^8}{7200} \left| \frac{\partial^5 u(t_Z, \varsigma_J)}{\partial x^5} \right|^2 \right) \\ &\leq 24 \|e_c^Z\|_H^2 + \frac{C_x^2 (b-a)}{3600} H^8 \leq C(\tau_c^4 + H^8). \end{aligned} \quad (65)$$

Combining (62) and (65), we obtain the estimate (58). Using a similar argument together with (43) and (45), we can also prove the estimate (59), whose proof is omitted. $\square$

4.4.3. Error Estimate for the Step 3

On the fine mesh, denoting

$$\begin{aligned} e_F^z &= u^z - U_F^z, \quad e_{\tau h}^z = u^z - U_{\tau h}^z, \quad z \in \mathcal{Z}[0, m], \\ e_F^z &= e_{R,F}^z + i e_{I,F}^z, \quad e_{\tau h}^z = e_{R,\tau h}^z + i e_{I,\tau h}^z, \quad z \in \mathcal{Z}[0, m], \end{aligned}$$

and subtracting Equation (24) from Equation (22) yields the following error equation:

$$i \partial_\tau e_F^z + \eta \mathbf{M}_h \delta_h^2 e_F^{z+\frac{1}{2}} + q \mathcal{S}^z + \mathbf{M}_h r^z = 0, \quad (66)$$

where

$$\mathcal{S}^z = \left(g_R(v^{z+\frac{1}{2}}, w^{z+\frac{1}{2}}) - \tilde{F}^{z+\frac{1}{2}} \right) + i \left(g_I(v^{z+\frac{1}{2}}, w^{z+\frac{1}{2}}) - \hat{F}^{z+\frac{1}{2}} \right).$$

Additionally, applying the second-order Taylor expansion of functions $g_R(v, w)$ and $g_I(v, w)$ at a point $(V_{\tau h}^{z+\frac{1}{2}}, W_{\tau h}^{z+\frac{1}{2}})$, respectively, we have

$$\begin{aligned} g_R(v^{z+\frac{1}{2}}, w^{z+\frac{1}{2}}) - \tilde{F}^{z+\frac{1}{2}} &= g_{R,v}^C \cdot [v^{z+\frac{1}{2}} - E(V_F^z)] + g_{R,w}^C \cdot [w^{z+\frac{1}{2}} - E(W_F^z)] \\ &\quad + \frac{1}{2} g_{R,vv} \cdot [v^{z+\frac{1}{2}} - V_{\tau h}^{z+\frac{1}{2}}]^2 + \frac{1}{2} g_{R,ww} \cdot [w^{z+\frac{1}{2}} - W_{\tau h}^{z+\frac{1}{2}}]^2 \\ &\quad + g_{R,vw} \cdot [(v^{z+\frac{1}{2}} - V_{\tau h}^{z+\frac{1}{2}})(w^{z+\frac{1}{2}} - W_{\tau h}^{z+\frac{1}{2}})], \\ g_I(v^{z+\frac{1}{2}}, w^{z+\frac{1}{2}}) - \hat{F}^{z+\frac{1}{2}} &= g_{I,v}^C \cdot [v^{z+\frac{1}{2}} - E(V_F^z)] + g_{I,w}^C \cdot [w^{z+\frac{1}{2}} - E(W_F^z)] \\ &\quad + \frac{1}{2} g_{I,vv} \cdot [v^{z+\frac{1}{2}} - V_{\tau h}^{z+\frac{1}{2}}]^2 + \frac{1}{2} g_{I,ww} \cdot [w^{z+\frac{1}{2}} - W_{\tau h}^{z+\frac{1}{2}}]^2 \\ &\quad + g_{I,vw} \cdot [(v^{z+\frac{1}{2}} - V_{\tau h}^{z+\frac{1}{2}})(w^{z+\frac{1}{2}} - W_{\tau h}^{z+\frac{1}{2}})], \end{aligned}$$

where the definitions of the symbols $g_R, g_I, \tilde{F}^{z+\frac{1}{2}}, \hat{F}^{z+\frac{1}{2}}, g_R^C, g_I^C, g_{R,v}^C, g_{R,w}^C, g_{R,vv}^C, g_{R,ww}^C, g_{R,vw}^C$ and $E(v^n)$ have already been given in Step 3 of the TSTG-CD algorithm, respectively. The second-order derivative terms are $g_{R,vv} = 6\zeta_1, g_{R,ww} = 2\zeta_1, g_{R,vw} = 2\eta_1$ and $g_{I,vv} = 2\eta_2, g_{I,ww} = 6\eta_2, g_{I,vw} = 2\zeta_2$, in which the point (η_j, ζ_j) lies on the line segment between the points $(v^{z+\frac{1}{2}}, w^{z+\frac{1}{2}})$ and $(V_{\tau h}^{z+\frac{1}{2}}, W_{\tau h}^{z+\frac{1}{2}})$.

Theorem 6. Suppose that assumptions (a) and (b) are valid, and τ is sufficiently small. Then there exists a positive constant C independent of h, H, τ_c, τ such that

$$\|u^z - U_F^z\|_h \leq C(\tau^2 + \tau_c^4 + h^4 + H^8), \quad z \in \mathcal{Z}[0, m], \quad (67)$$

$$\|u^z - U_F^z\|_{h,\infty} \leq C(\tau^2 + \tau_c^4 + h^4 + H^8), \quad z \in \mathcal{Z}[0, m]. \quad (68)$$

Proof. For $z \in \mathcal{Z}[1, m-1]$, computing the inner product of (66) with $2e_F^{z+\frac{1}{2}}$ and taking the imaginary part, we obtain

$$\begin{aligned} \partial_\tau \|e_F^z\|_h^2 + 2\eta \| \partial_h e_F^z \|_h^2 + \left\{ \left(g_R(v^{z+\frac{1}{2}}, w^{z+\frac{1}{2}})_h - \tilde{F}^{z+\frac{1}{2}}, 2e_{I,F}^{z+\frac{1}{2}} \right) \right. \\ \left. + \left(g_I(v^{z+\frac{1}{2}}, w^{z+\frac{1}{2}}) - \hat{F}^{z+\frac{1}{2}}, 2e_{R,F}^{z+\frac{1}{2}} \right)_h \right\} + \text{Im}(\mathbb{M}_h r^z, 2e_F^{z+\frac{1}{2}})_h = 0, \end{aligned} \quad (69)$$

where $e_{R,F}^{z+\frac{1}{2}}$ and $e_{I,F}^{z+\frac{1}{2}}$ represent the real part and the imaginary part of $e_F^{z+\frac{1}{2}}$, respectively.

According to the smoothness assumption of the exact solution (43) and the a priori estimate (30), we have

$$\begin{aligned} \left| \left(g_{R,v}^C \cdot [v^{z+\frac{1}{2}} - E(V_F^z)], 2e_{I,F}^{z+\frac{1}{2}} \right)_h \right| &\leq C \left\{ \left\| \frac{3e_{R,F}^z - e_{R,F}^{z-1}}{2} \right\|_h^2 + \|2e_{I,F}^{z+\frac{1}{2}}\|_h^2 \right\} + \mathcal{O}(\tau^4), \\ \left| \left(g_{R,w}^C \cdot [w^{z+\frac{1}{2}} - E(W_F^z)], 2e_{I,F}^{z+\frac{1}{2}} \right)_h \right| &\leq C \left\{ \left\| \frac{3e_{I,F}^z - e_{I,F}^{z-1}}{2} \right\|_h^2 + \|2e_{I,F}^{z+\frac{1}{2}}\|_h^2 \right\} + \mathcal{O}(\tau^4), \\ \left| \left(\frac{1}{2} g_{R,vv} [e_{R,\tau h}^{z+\frac{1}{2}}]^2 + \frac{1}{2} g_{R,ww} [e_{I,\tau h}^{z+\frac{1}{2}}]^2 + g_{R,vw} [e_{R,\tau h}^{z+\frac{1}{2}} e_{I,\tau h}^{z+\frac{1}{2}}], 2e_{I,F}^{z+\frac{1}{2}} \right)_h \right| \\ &\leq C \left\{ \frac{1}{2} \| (e_{R,\tau h}^{z+\frac{1}{2}})^2 \|_h^2 + \| e_{R,\tau h}^{z+\frac{1}{2}} e_{I,\tau h}^{z+\frac{1}{2}} \|_h^2 + \frac{1}{2} \| (e_{I,\tau h}^{z+\frac{1}{2}})^2 \|_h^2 + \| 2e_{I,F}^{z+\frac{1}{2}} \|_h^2 \right\}. \end{aligned}$$

It follows

$$\begin{aligned} & \left| \left(g_R(v^{z+\frac{1}{2}}, w^{z+\frac{1}{2}}) - \tilde{F}^{z+\frac{1}{2}}, 2e_{I,F}^{z+\frac{1}{2}} \right)_h \right| \\ & \leq C \left\{ \left\| \frac{3e_F^z - e_F^{z-1}}{2} \right\|_h^2 + 3\|2e_{I,F}^{z+\frac{1}{2}}\|_h^2 + \|e_{\tau h}^{z+\frac{1}{2}}\|_{h,\infty}^2 \|e_{\tau h}^{z+\frac{1}{2}}\|_h^2 \right\} + \mathcal{O}(\tau^4). \end{aligned} \quad (70)$$

$$\begin{aligned} & \left| \left(g_I(v^{z+\frac{1}{2}}, w^{z+\frac{1}{2}}) - \hat{F}^{z+\frac{1}{2}}, 2e_{R,F}^{z+\frac{1}{2}} \right)_h \right| \\ & \leq C \left\{ \left\| \frac{3e_F^z - e_F^{z-1}}{2} \right\|_h^2 + 3\|2e_{R,F}^{z+\frac{1}{2}}\|_h^2 + \|e_{\tau h}^{z+\frac{1}{2}}\|_{h,\infty}^2 \|e_{\tau h}^{z+\frac{1}{2}}\|_h^2 \right\} + \mathcal{O}(\tau^4). \end{aligned} \quad (71)$$

Substituting (70) and (71) into (69), then summing up z from 1 to n and replacing n by z , we get

$$(1 - 4\tau)\|e_F^{z+1}\|_h^2 \leq \|e_F^1\|_h^2 + C\tau \left\{ \sum_{s=0}^z \|e_F^s\|_h^2 + \mathcal{O}(\tau^4 + \tau_c^8 + h^8 + H^{16}) \right\}. \quad (72)$$

Additionally, for the case of $z = 0$, $U_F^1 = U_F^{1(s+1)}$. Then from (13) and (22), we have

$$i\partial_\tau e_F^{1(s+1)} + \eta M_h \delta_h^2 e_F^{\frac{1}{2}(s+1)} + q\mathcal{G}^{1(s)} + M_h r^0 = 0, \quad (73)$$

in which

$$\begin{aligned} \partial_\tau e_F^{1(s+1)} &= \frac{e_F^{1(s+1)} - e_F^0}{\tau}, \quad e_F^{1(s+1)} = u^1 - U_F^{1(s+1)}, \\ e_F^{\frac{1}{2}(s+1)} &= u^{\frac{1}{2}} - U_F^{\frac{1}{2}(s+1)}, \quad U_F^{\frac{1}{2}(s+1)} = \frac{U_F^{1(s+1)} + U_F^0}{2}, \\ \mathcal{G}^{1(s)} &= (|u|^2)^{\frac{1}{2}} u^{\frac{1}{2}} - (|U_F|^2)^{\frac{1}{2}(s)} U_F^{\frac{1}{2}(s)} \\ &= \frac{1}{2} (|u^0|^2 + |u^1|^2 - |U_F^0|^2 - |U_F^{1(s)}|^2) u^{\frac{1}{2}} \\ &\quad + \frac{1}{2} (|U_F^0|^2 + |U_F^{1(s)}|^2) e_F^{\frac{1}{2}(s)} = \theta^{\frac{1}{2}(s)} + \zeta^{\frac{1}{2}(s)}. \end{aligned}$$

Computing the inner product of (73) with $2e_F^{\frac{1}{2}(s+1)}$ and taking the imaginary part, we obtain

$$\frac{1}{\tau} \left[\|e_F^{1(s+1)}\|_h^2 - \|e_F^0\|_h^2 \right] + 2q \operatorname{Im}(\mathcal{G}^{1(s)}, e_F^{\frac{1}{2}(s+1)})_h + 2\operatorname{Im}(M_h r^0, e_F^{\frac{1}{2}(s+1)})_h = 0. \quad (74)$$

Note that $|U_F^{1(s)}| \leq s\varepsilon + |U_F^0| \leq C$, where s denotes the s th iteration and $s\varepsilon$ is typically bounded (for example, in the numerical examples in this article, $s \leq 3$ and $\varepsilon = 10^{-6}$). Then, similarly to (48) and (49), we get

$$|2q \operatorname{Im}(\theta^{\frac{1}{2}(s)} + \zeta^{\frac{1}{2}(s)}, e_F^{\frac{1}{2}(s+1)})_h| \leq C|q| \{ \|e_F^{1(s)}\|_h^2 + \|e_F^0\|_h^2 + \|e_F^{1(s+1)}\|_h^2 \}.$$

Furthermore, noting that $e_F^0 = 0$ and combining (74) with the above estimate, we have

$$(1 - C\tau)\|e_F^{1(s+1)}\|_h^2 \leq C\tau \{ \|e_F^{1(s)}\|_h^2 + \mathcal{O}(\tau^4) \}.$$

Then, taking $\tau = \frac{1}{2C}$ and iterating over s , we get

$$\|e_F^1\|_h^2 = \|e_F^{1(s+1)}\|_h^2 \leq 4\mathcal{O}(\tau^4). \quad (75)$$

Finally, taking a sufficiently small τ and applying the discrete Gronwall inequality, we obtain (67) from (72) and (75). By a similar argument as in the proof of Theorem 4, we obtain (68) and complete the proof. $\square$

5. Numerical Examples

In this section, we evaluate the numerical accuracy and efficiency of the proposed TSTG-CD scheme through four numerical examples. The construction of the TSTG-CD method is based on the fully nonlinear compact (FN-CD) scheme proposed in [26], and it can also be viewed as an extension of the time two-grid compact difference (TTG-CD) method [12]. Therefore, we compare the performance of the TSTG-CD method with both the FN-CD method [26] and the TTG-CD method [12]. All experiments were performed using MATLAB R2019b on a computer equipped with an Intel Core i7 processor and 8 GB of RAM.

Let $e_{L_2}(h, \tau)$ and $e_{L_\infty}(h, \tau)$ denote errors $|e^z|/h$ and $|e^z|/h, \infty$, respectively, with the time step τ and the space step h . The spatial and temporal convergence orders in the discrete L^2 -norm and L^∞ -norm are computed by

$$\begin{aligned} \text{Ord}_{L_2}^S &= \frac{\log(e_{L_2}(h_1, \tau)/e_{L_2}(h_2, \tau))}{\log(h_1/h_2)}, & \text{Ord}_{L_\infty}^S &= \frac{\log(e_{L_\infty}(h_1, \tau)/e_{L_\infty}(h_2, \tau))}{\log(h_1/h_2)}, \\ \text{Ord}_{L_2}^T &= \frac{\log(e_{L_2}(h, \tau_1)/e_{L_2}(h, \tau_2))}{\log(\tau_1/\tau_2)}, & \text{Ord}_{L_\infty}^T &= \frac{\log(e_{L_\infty}(h, \tau_1)/e_{L_\infty}(h, \tau_2))}{\log(\tau_1/\tau_2)}, \end{aligned}$$

respectively. Furthermore, the numerical mass Q^z and energy $\mathcal{E}^z$ are computed by

$$Q^z = \sum_{j=1}^{\ell-1} |U_j^z|^2 h, \quad \mathcal{E}^z = \sum_{j=0}^{\ell-1} \left(\eta |\partial_h U_j^z|^2 - \frac{1}{2} q |U_j^z|^4 \right) h.$$

The flowcharts of the scheme of Equations (9) and (10) and the scheme of Equations (11)–(15) are shown in Figure 1 and Figure 2, respectively. In Figure 1, the matrices are defined as $A_H = \frac{1}{12} \text{trid}[1, 10, 1]_{L-1}$, $B_H = \frac{1}{H^2} \text{trid}[1, -2, 1]_{L-1}$ and $C_H(\mathbf{U}_g) = \frac{1}{12} \text{trid}[(|U_g|^2)_{J-1}, 10(|U_g|^2)_J, (|U_g|^2)_{J+1}]_{L-1}$. The Jacobian matrix $J(\mathbf{U}_g)$ and the right-hand side term $R(\mathbf{U}_g, \mathbf{U}_c(Z, :))$ are obtained from Equation (9). In Figure 2, the matrices A_h and B_h are analogous to A_H and B_H . The right-hand-side term $F(\mathbf{U}_f(z-1, :), \mathbf{U}_f(Z, :), \mathbf{U}_{th}(Z, :))$ is derived from Equation (15).

5.1. Propagation of a Soliton

As the first test problem, we consider the cubic NLS Equation (1) with $\eta = 1$ and $q = 2$, which admits the exact single-soliton solution [12,26,31,32]

$$u(x, t) = \text{sech}(x - 4t) \exp(2ix - 3it), \quad (t, x) \in [0, 1] \times [-20, 20].$$

The initial condition is $u(x, 0) = \text{sech}(x) \exp(2ix)$, and the Dirichlet boundary conditions are $u(-20, t) = u(20, t) = 0$.

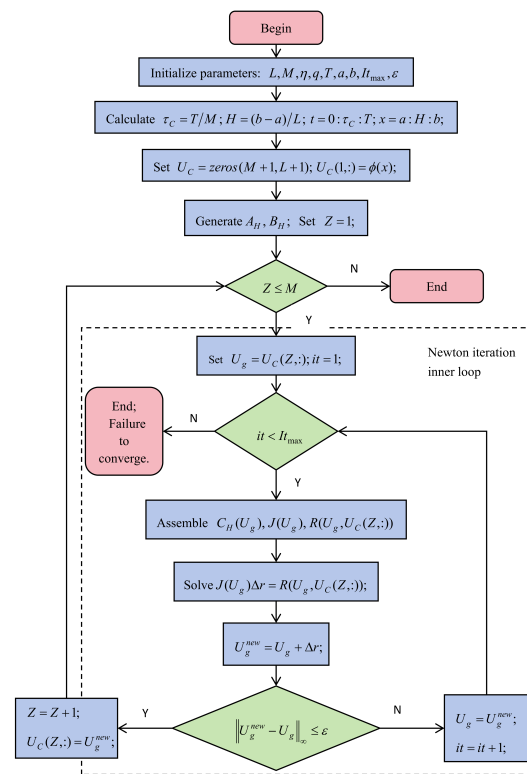


Figure 1. Flowchart of the scheme on the time-space coarse grid.

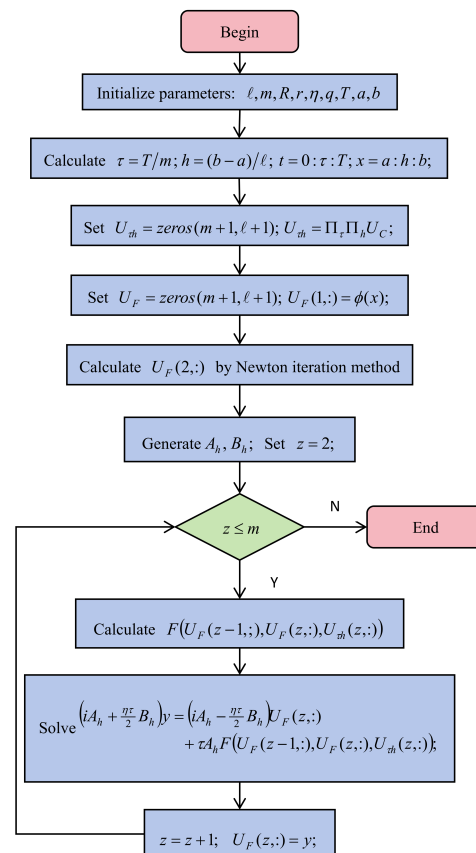


Figure 2. Flowchart of the scheme on the time-space fine grid.

To investigate the spatial convergence, we first set $\tau = h^2$ for all three methods. The TTG-CD method employs the time-step relation $\tau_C = \sqrt{\tau}$, while the TSTG-CD method

further uses a spatial step-size ratio of $R = 4$. The resulting errors, spatial convergence rates, and CPU times are listed in Table 1. For the temporal convergence tests, we fix $H = 4h = 0.025$ and adopt the same time-step ratio $\tau_C = \sqrt{\tau}$. The corresponding results are summarized in Table 2. Figure 3 shows the CPU time of the proposed method compared with the FN-CD method (left) and the TTG-CD method (right), respectively. For a time step $\tau = 1/625$ and a space step $h = 1/160$, Table 3 presents a comparison of the efficiency of the three numerical schemes. From these numerical results, we make the following observations.

- (i) All three methods exhibit $\text{Ord}L_2^S \approx 4$, $\text{Ord}L_\infty^S \approx 4$, $\text{Ord}L_2^T \approx 2$, and $\text{Ord}L_\infty^T \approx 2$, which verifies our estimates in Theorems 4 and 6.
- (ii) The proposed method significantly improves computational efficiency. For example, the FN-CD method requires more than 422.9 s when the time step is $\tau = 1/625$ and the spatial fine grid has $\ell = 6400$ points, while the TSTG-CD scheme, with a coarse time step $\tau_C = 1/25$ and a spatial coarse grid of $L = 1600$ points, consumes only about 62.9 s, reducing the CPU time by 85.1%. As the grid number ℓ increases, computational efficiency can be further improved, albeit with a corresponding increase in numerical error. Compared with the TTG-CD method, both the speedup ratio and the time reduction achieved by the TSTG-CD method are also improved. These gains are attributed to the application of the two-grid technique in both the temporal and spatial domains, which substantially reduces both the number of nonlinear iterations and the computational scale on the coarse mesh.

Table 1. Spatial errors and convergence rates for single-soliton at time $T = 1.0$ with $\tau = h^2$, $\tau_C = \sqrt{\tau}$ and space step-size ratio $R = 4$.

Method	h	$e_{L_2}(h, \tau)$	$\text{Ord}_{L_2}^S$	$e_{L_\infty}(h, \tau)$	$\text{Ord}_{L_\infty}^S$	CPU(s)
FN-CD [26]	1/16	3.7873×10^{-4}	*	2.2865×10^{-4}	*	35.6128
	1/25	6.3464×10^{-5}	4.0027	3.8355×10^{-5}	4.0003	177.7757
	1/36	1.4754×10^{-5}	4.0010	8.9178×10^{-6}	4.0007	709.0766
	1/64	1.4853×10^{-6}	3.9904	8.9248×10^{-7}	4.0006	6677.2448
TTG-CD [12]	1/16	3.8143×10^{-3}	*	2.6953×10^{-3}	*	6.1359
	1/25	6.4379×10^{-4}	3.9866	4.6038×10^{-4}	3.9598	29.5855
	1/36	1.4983×10^{-4}	3.9980	1.0790×10^{-4}	3.9788	122.1262
	1/64	1.4999×10^{-5}	4.0001	1.0874×10^{-5}	3.9884	1091.9605
TSTG-CD	1/16	3.9071×10^{-3}	*	2.7524×10^{-3}	*	4.0277
	1/25	6.5075×10^{-4}	4.0163	4.6470×10^{-4}	3.9859	23.0436
	1/36	1.5069×10^{-4}	4.0118	1.0840×10^{-4}	3.9917	97.2565
	1/64	1.5033×10^{-5}	4.0062	1.0892×10^{-5}	3.9937	983.8858

The symbol “*” indicates that the convergence order is not available at this position.

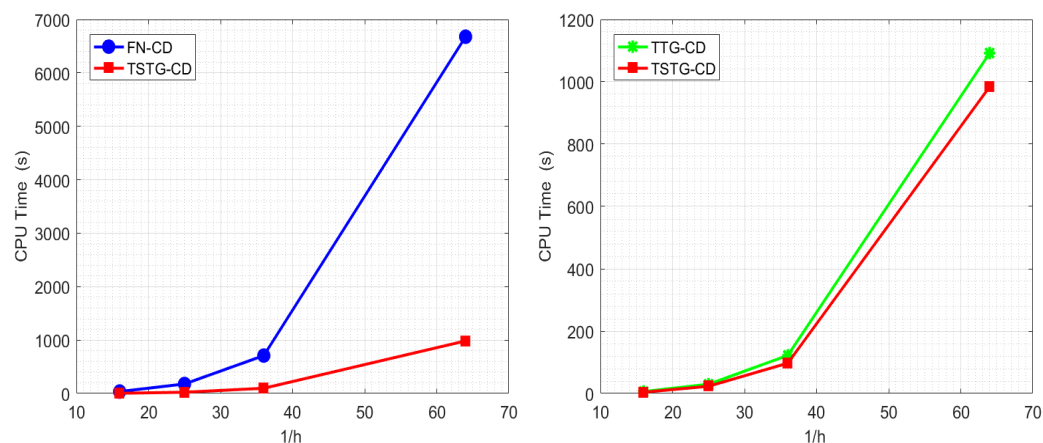
Table 2. Temporal errors and convergence rates for single-soliton at time $T = 1.0$ with the fixed space step $H = 4h = 0.025$ and time step-size ratio $r = 1/\tau_C = 1/\sqrt{\tau}$.

Method	τ	$e_{L_2}(h, \tau)$	$\text{Ord}_{L_2}^T$	$e_{L_\infty}(h, \tau)$	$\text{Ord}_{L_\infty}^T$	CPU(s)
FN-CD [26]	1/144	1.0719×10^{-3}	*	6.5085×10^{-4}	*	95.5828
	1/256	3.3934×10^{-4}	1.9991	2.0620×10^{-4}	1.9978	170.5751
	1/400	1.3950×10^{-4}	1.9918	8.4787×10^{-5}	1.9913	266.5152
	1/625	5.7722×10^{-5}	1.9773	3.5072×10^{-5}	1.9780	422.9237
TTG-CD [12]	1/144	1.1593×10^{-2}	*	7.8318×10^{-3}	*	20.6427
	1/256	3.8193×10^{-3}	1.9298	2.6929×10^{-3}	1.8555	32.4334
	1/400	1.5708×10^{-3}	1.9908	1.1134×10^{-3}	1.9790	45.6373
	1/625	6.4424×10^{-4}	1.9971	4.5954×10^{-4}	1.9829	67.2817
TSTG-CD	1/144	1.1597×10^{-2}	*	7.8341×10^{-3}	*	12.0784
	1/256	3.8214×10^{-3}	1.9294	2.6943×10^{-3}	1.8551	20.7736
	1/400	1.5722×10^{-3}	1.9901	1.1143×10^{-3}	1.9783	36.143
	1/625	6.4520×10^{-4}	1.9958	4.6017×10^{-4}	1.9817	62.875

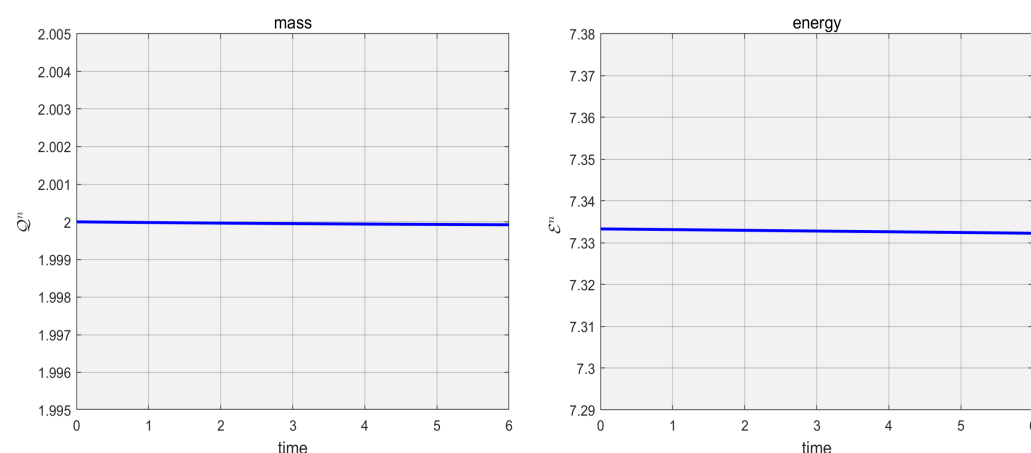
The symbol “*” indicates that the convergence order is not available at this position.

Table 3. Comparison of efficiency of different numerical schemes.

Method	L^∞ Error ($\tau = 1/625, h = 1/160$)	CPU Time (s)	Speedup Ratio	Time Reduction (%)
FN-CD [26]	3.5072×10^{-5}	422.9237	$1.00 \times$ (Reference)	
TTG-CD [12]	4.5954×10^{-4}	67.2817	$6.29 \times$	84.1%
TSTG-CD	4.6017×10^{-4}	62.875	$6.73 \times$	85.1%


Figure 3. CPU time of the single-soliton at time $T = 1.0$ with $\tau = h^2$, $\tau_c = \sqrt{\tau}$ and space step-size ratio $R = 4$: FN-CD vs. TSTG-CD (left) and TTG-CD vs. TSTG-CD (right), respectively.

To test the conservation laws, we present the numerical total mass and energy computed by the TSTG-CD scheme in Figure 4. It shows that the proposed TSTG-CD method preserves the two conservation laws in the discrete sense, which is consistent with the theoretical analysis. The corresponding absolute errors in mass and energy are presented in Figure 5, which are approximately $\mathcal{O}(10^{-4})$ and $\mathcal{O}(10^{-3})$, respectively. Consequently, the proposed method does not preserve these quantities to machine precision. This may be attributed to the inability of the TSTG-CD method on a time-space coarse uniform grid to adequately resolve rapid and sharp waveform variations, leading to increased local numerical errors. Finally, the properties of the three numerical schemes are summarized in Table 4.


Figure 4. Total mass (left) and energy (right) of the single-soliton under $h = 0.05, \tau = 0.005, r = 2, R = 2$.

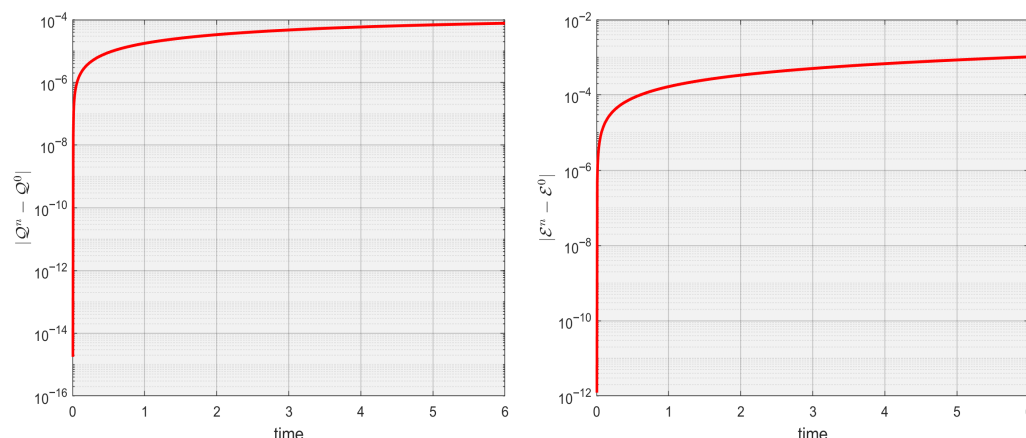


Figure 5. Absolute errors of total mass (**left**) and energy (**right**) of the single-soliton under $h = 0.05$, $\tau = 0.005$, $r = 2$, $R = 2$.

Table 4. Comparison of properties of different numerical schemes.

Property/Method	FN-CD [26]	TTG-CD [12]	TSTG-CD
Mass conservation	Yes	Yes	Yes
Energy conservation	Yes	Yes	Yes
Fully implicit	Yes	No	No
Linearly implicit	No	Yes	Yes
Temporal accuracy	2nd	2nd	2nd
Spatial accuracy	4th	4th	4th

5.2. Interaction of Two Solitons

This example aims to simulate the evolution of two solitons propagating in opposite directions using the TSTG-CD method. Therefore, we further consider the NLS Equation (1) with the initial condition

$$u(x, 0) = \text{sech}(x + 15)\exp(2ix) + 1.5\text{sech}(1.5x - 7.5)\exp(-2ix),$$

and boundary conditions $u(-30, t) = u(30, t) = 0$. This initial condition yields a two-soliton solution [12,33]. The first soliton with amplitude 1 is placed at $x = -15$ and moves to the right with speed 4, while the second soliton with amplitude 1.5 is placed on the other side at $x = 5$ and travels to the left with speed 4.

The evolution of the numerical solutions and their trajectories from $t = 0$ to $T = 6$ are presented in Figure 6 with $h = 0.025$, $\tau = 0.0025$, $r = 2$, and $R = 2$. As time progresses, the two solitons travel in opposite directions, collide at approximately $t = 2.5$, and then separate, recovering their shapes afterward despite a strongly nonlinear interaction. These results are in qualitative agreement with the behavior predicted by soliton theory [34]. Furthermore, the evolution of the two invariants and their absolute errors are shown in Figures 7 and 8. From Figure 7, we observe that the total mass Q^z and energy E^z remain constant except for a slight change between $2 < t < 3$, which occurs during the interaction of the two solitons. This fact can also be observed in Figure 8. Moreover, the proposed method does not preserve these quantities to machine precision. A possible reason for this phenomenon is the increase in local gradients caused by wave-packet collisions, which leads to insufficient spatial or temporal resolution in the numerical method, thereby inducing numerical errors and sudden changes in the conserved quantities. To address this issue, we will consider employing numerical methods with adaptive mesh refinement for solving the Schrödinger equation in future work.

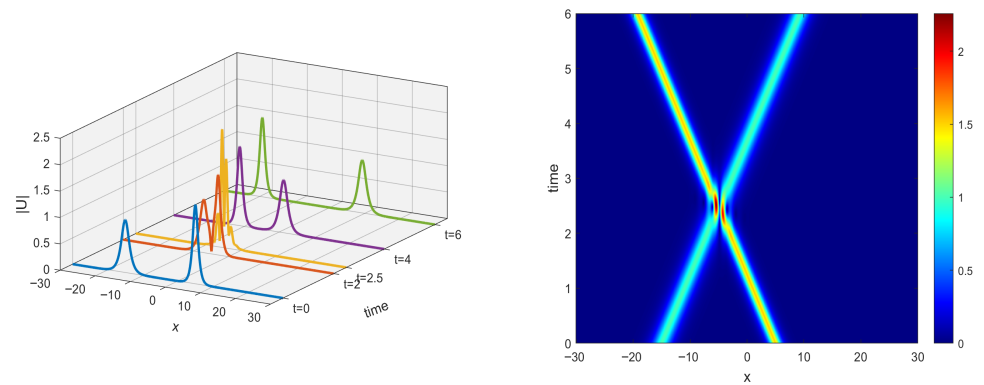


Figure 6. Collision of double solitons (left) and its trajectory (right) computed under $h = 0.025$, $\tau = 0.0025$, $r = 2$, and $R = 2$.

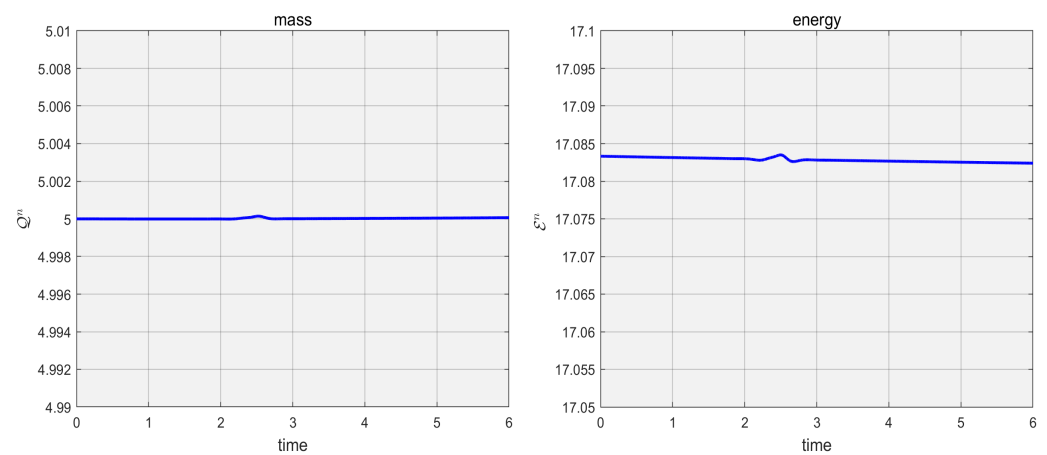


Figure 7. Total mass (left) and energy (right) of the two-soliton solution under $h = 0.025$, $\tau = 0.0025$, $r = 2$, and $R = 2$.

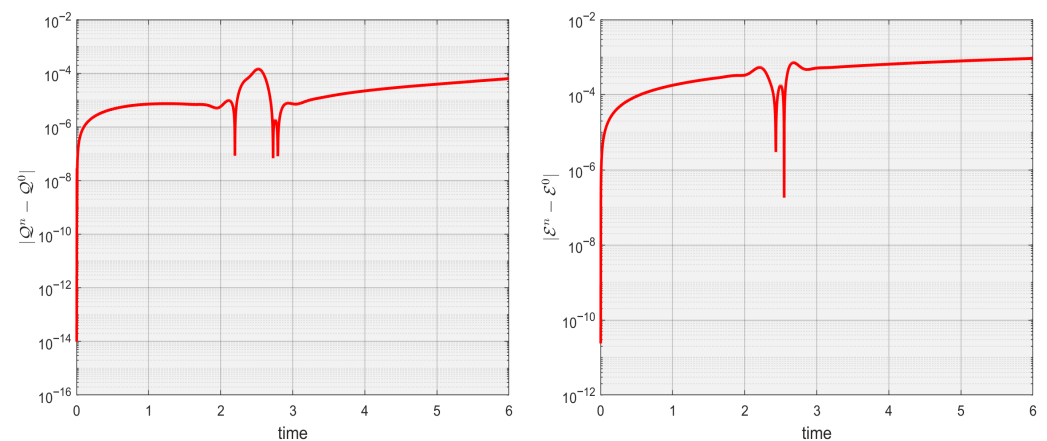


Figure 8. Absolute errors of total mass (left) and energy (right) of the two-soliton solution under $h = 0.025$, $\tau = 0.0025$, $r = 2$, and $R = 2$.

5.3. Birth of Soliton

With an initial condition $u(x, 0)$, the theory predicts that if

$$\int_{-\infty}^{\infty} u(x, 0) dx \geq \pi,$$

a soliton will appear over time; otherwise, the soliton decays away [12,35,36]. We investigate the birth of solitons computed by the TSTG-CD method using the following two initial conditions:

1. Case I. A birth of a standing soliton with the Maxwellian initial condition

$$u(x,0) = A \exp(-x^2), \quad -45 \leq x \leq 45.$$

2. Case II. A birth of a mobile soliton with the Maxwellian initial condition

$$u(x,0) = A \exp(-x^2 + 2ix), \quad -45 \leq x \leq 45.$$

Both cases produce a soliton when $A \geq \sqrt{\pi}$, whereas the initial condition fades out when $A < \sqrt{\pi}$.

Therefore, we select the parameters $A = 1$ and $A = 1.78$ to investigate the birth of solitons. The numerical simulations were performed with $h = 0.04$, $\tau = 0.00125$, $r = 2$, $R = 2$, and $q = 2$. For Case I, the formation and trajectories of standing solitons from $t = 0$ to $t = 6$ are illustrated in Figure 9 ($A = 1$) and Figure 10 ($A = 1.78$). Figure 11 shows the discrete conservation laws and the corresponding absolute conservation errors. For Case II, Figures 12 and 13 depict the behaviors and trajectories of mobile solitons for $A = 1$ and $A = 1.78$, respectively, over $t \in [0, 6]$. The associated conservation laws and absolute errors are presented in Figure 14. These figures reveal the following phenomena: (i) The approximate solution $|u|$ decays as time increases when the parameter $A = 1$ is taken in both cases. However, when $A = 1.78$ is taken, the soliton's amplitude, shape, and speed are preserved in both cases. (ii) The discrete total mass Q and energy $\mathcal{E}$ remain almost constant, which is consistent with the theoretical analysis. However, as in the simulations of single-soliton and two-soliton solutions, the corresponding absolute errors in mass and energy do not reach machine precision. Therefore, it is necessary to introduce adaptive mesh refinement techniques into the scheme to further enhance computational efficiency.

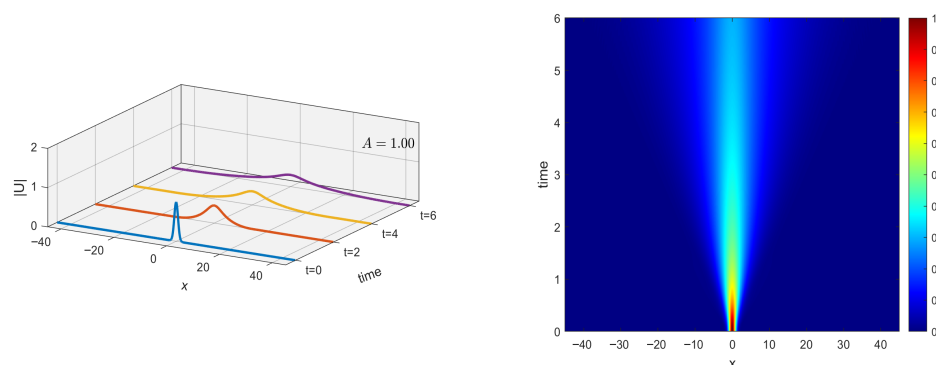


Figure 9. Formation of standing solitons (left) and its trajectory (right) computed for Case I ($A = 1$).

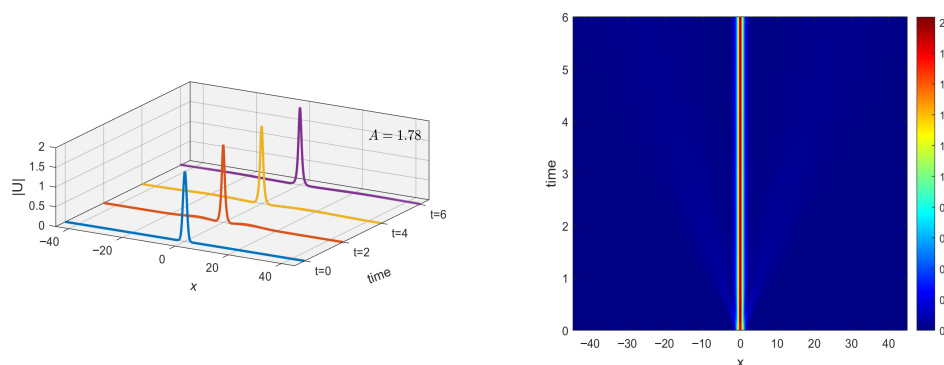


Figure 10. Formation of standing solitons (left) and its trajectory (right) computed for Case I ($A = 1.78$).

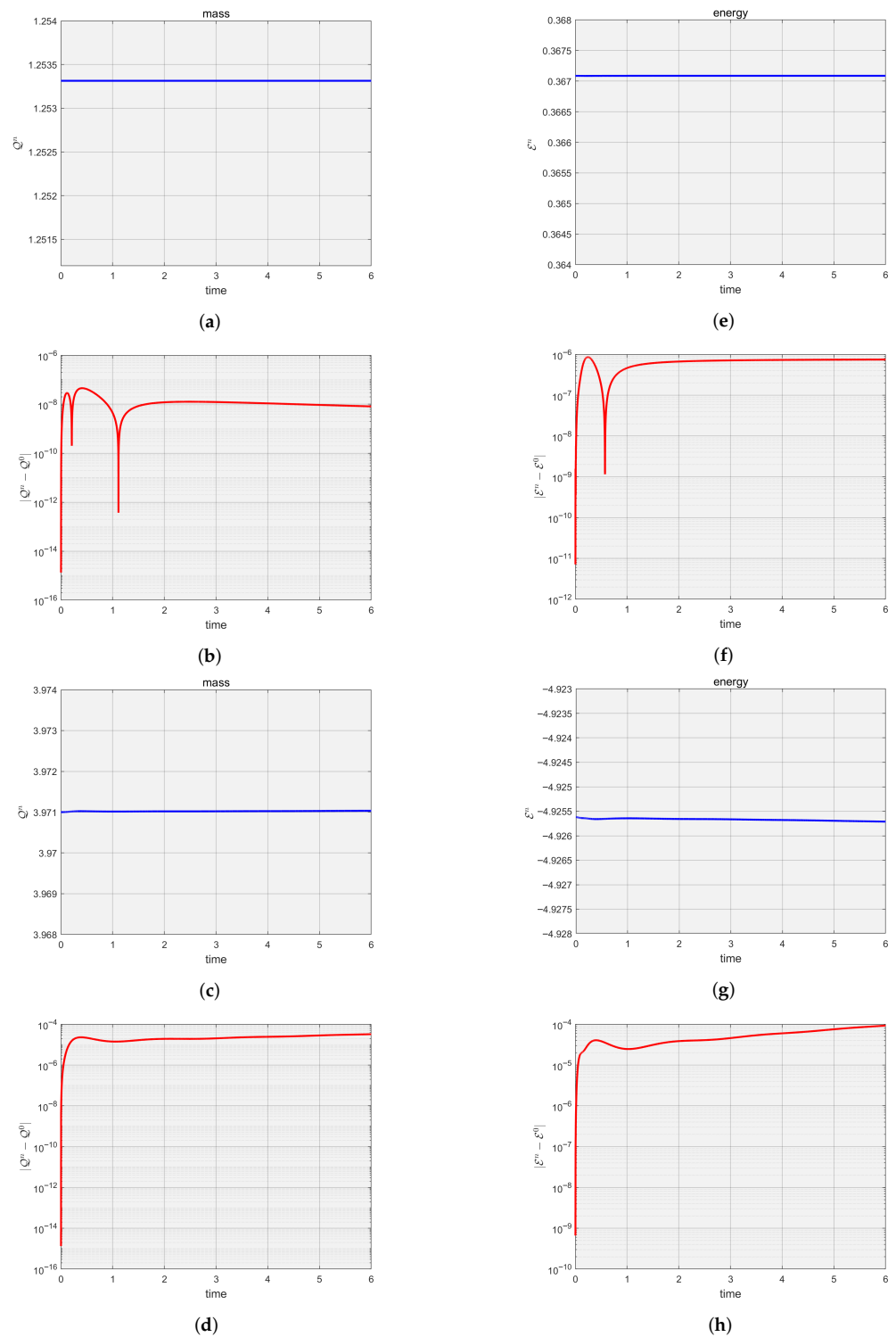


Figure 11. The discrete total mass (left), energy (right) and their absolute errors for Case I. (a) Total mass for $A = 1$. (b) Absolute error of mass $A = 1$. (c) Total mass for $A = 1.78$. (d) Absolute error of mass $A = 1.78$. (e) Total energy for $A = 1$. (f) Absolute error of energy $A = 1$. (g) Total energy for $A = 1.78$. (h) Absolute error of energy $A = 1.78$.

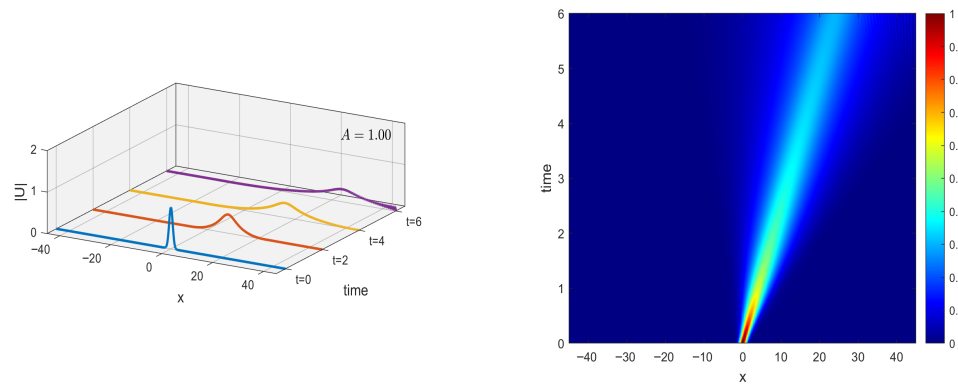


Figure 12. Formation of mobile solitons (left) and its trajectory (right) computed for Case II ($A = 1$).

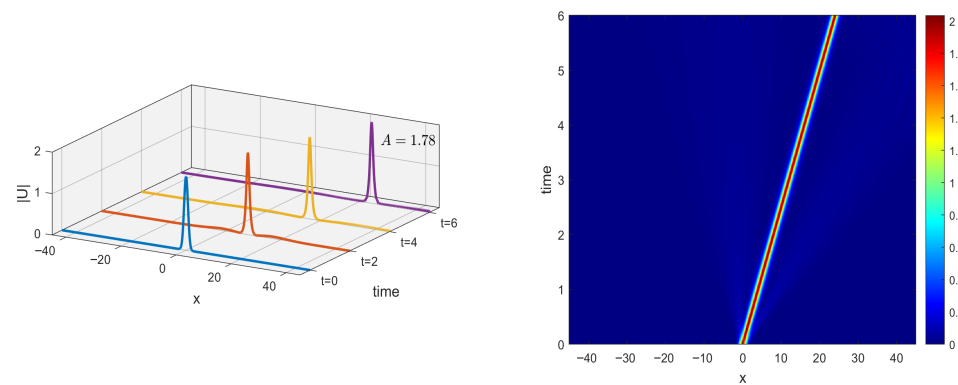


Figure 13. Formation of mobile solitons (left) and its trajectory (right) computed for Case II ($A = 1.78$).

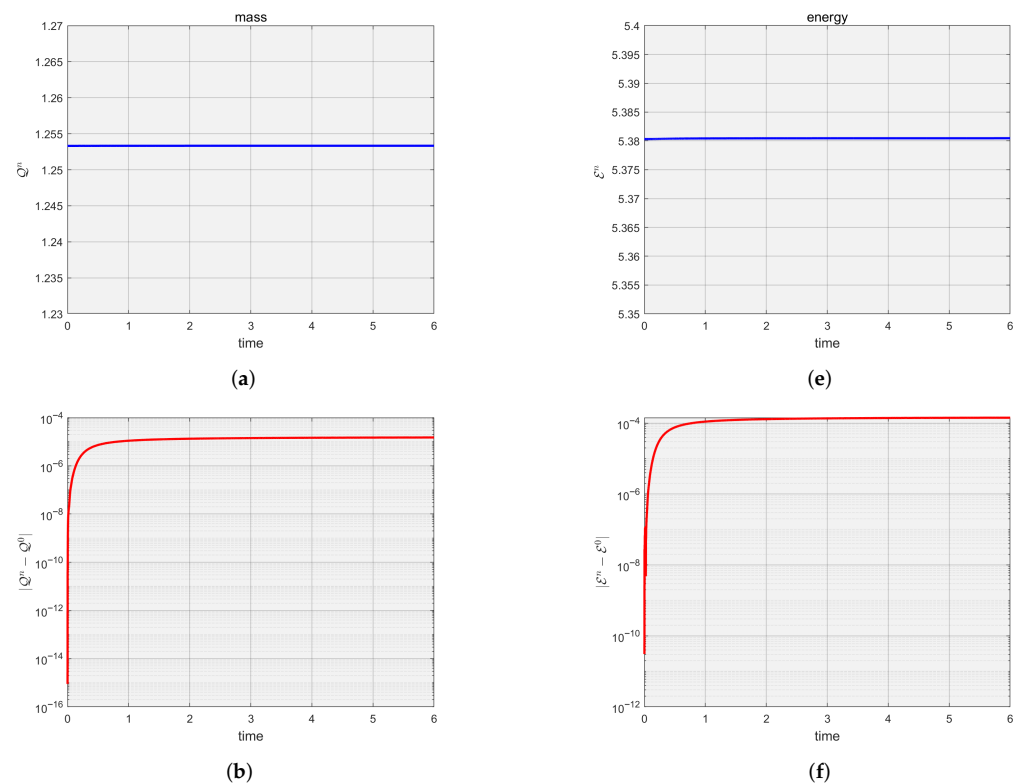


Figure 14. Cont.

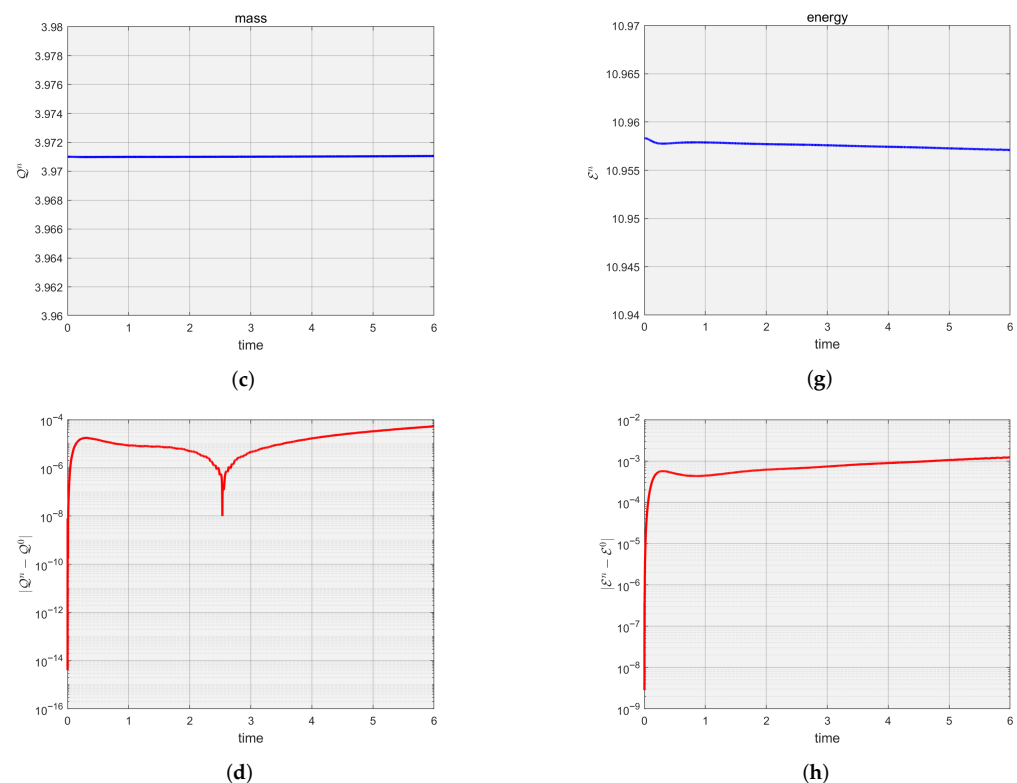


Figure 14. The discrete total mass (left), energy (right) and their absolute errors for Case II. (a) Total mass for $A = 1$. (b) Absolute error of mass $A = 1$. (c) Total mass for $A = 1.78$. (d) Absolute error of mass $A = 1.78$. (e) Total energy for $A = 1$. (f) Absolute error of energy $A = 1$. (g) Total energy for $A = 1.78$. (h) Absolute error of energy $A = 1.78$.

6. Conclusions

In this article, a novel time-space two-grid high-order compact difference scheme was proposed for the one-dimensional nonlinear Schrödinger equation. The core idea of the method is to combine a nonlinear compact scheme on a time-space coarse grid with a linearized correction compact scheme on a fine grid, thereby significantly reducing computational cost while preserving high-order accuracy. Unlike conventional time two-grid methods, the proposed approach applies the two-grid strategy simultaneously in both the spatial and temporal directions. The coarse-grid solutions are transferred to the fine grid via a temporally linear and spatially cubic Lagrange interpolation operator, ensuring accuracy and efficiency in the correction step.

Rigorous theoretical analysis demonstrates that the scheme is unconditionally stable and achieves optimal convergence rates of fourth-order in space and second-order in time in both the discrete L^2 and L^∞ norms, without any restriction on the grid ratio. The proof combines the energy method, a discrete Sobolev inequality, and an a priori error estimate. Numerical experiments validate the theoretical results. A single-soliton test shows that the proposed method significantly reduces CPU time compared with the fully nonlinear compact scheme while maintaining comparable accuracy. Additional simulations of two-soliton interactions and soliton birth phenomena further illustrate the algorithm's effectiveness in capturing complex nonlinear wave dynamics. These findings confirm the proposed scheme as a highly efficient and reliable alternative to conventional nonlinear approaches for solving the nonlinear Schrödinger equation. Future work may extend this methodology to higher-dimensional problems, variable-coefficient equations, or systems with more complex nonlinearities, as well as explore adaptive strategies to further enhance computational performance in regions with sharp gradients.

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Conflicts of Interest: The authors declare that they have no known competing financial interest or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A

A list of symbols is given in Tables A1–A3, and a list of abbreviations is provided in Table A4.

Table A1. List of symbols I.

Symbol	Description	Symbol	Description
L	Coarse grid number of in space	ℓ	Fine grid number of in space
M	Coarse grid number of in time	m	Fine grid number of in time
R	Ratio of coarse and fine grids in space	r	Ratio of coarse and fine grids in time
H	Space coarse grid step-size	h	Space fine grid step-size
τ_C	Time coarse grid step-size	τ	Time fine grid step-size
ω	Representative of space step-size H or h	κ	Representative of time step-size τ_C or τ
N_ω	Representative of L or ℓ	M_κ	Representative of M or m
$\Omega_{\omega\kappa}$	Time-space mesh	$\mathcal{J}_\omega$	grid function space
Π_h	Spatial cubic interpolation operator	Π_τ	Temporal linear interpolation operator

Table A2. List of symbols II.

Symbol	Description	Symbol	Description
U_C	Coarse grid solution	U_F	Fine grid corrected solution
V_C	Real part of U_C	W_C	Imaginary part of U_C
V_F	Real part of U_F	W_F	Imaginary part of U_F
$U_{\tau h}$	Fine grid rough solution	$E(v^z)$	extrapolation operator $(3v^z - v^{z-1})/2$
$V_{\tau h}$	Real part of $U_{\tau h}$	$W_{\tau h}$	Imaginary part of $U_{\tau h}$
$F_j^{z+\frac{1}{2}}$	Linearized complex terms	e_C	Coarse grid error term
$\tilde{F}_j^{n+\frac{1}{2}}$	Real part of $F_j^{z+\frac{1}{2}}$	$\hat{F}_j^{n+\frac{1}{2}}$	Imaginary part of $F_j^{z+\frac{1}{2}}$
$\mathcal{A}_\omega$	Compact difference operator	$\mathcal{A}_\omega$	Corresponding matrix of $\mathcal{A}_\omega$
M_ω	Inverse matrix of $\mathcal{A}_\omega$	e_C^Z	Coarse grid error
e_F^z	Fine grid error	$e_{\tau h}^z$	Auxiliary error
$e_{R,F}^z$	Real part of e_F^z	$e_{I,F}^z$	Imaginary part of e_F^z
$e_{R,\tau h}^z$	Real part of $e_{\tau h}^z$	$e_{I,\tau h}^z$	Imaginary part of $e_{\tau h}^z$
$g_R(v, w)$	Function $(v^2 + w^2)v$	$g_I(v, w)$	Function $(v^2 + w^2)w$
g_R^C	Value of g_R at $(V_{\tau h}, W_{\tau h})$	g_I^C	Value of g_I at $(V_{\tau h}, W_{\tau h})$

Table A3. List of symbols III.

Symbol	Description
$g_{R,v}^C$	Partial derivative of function g_R with respect to variable v at $(V_{\tau h}, W_{\tau h})$
$g_{R,w}^C$	Partial derivative of function g_R with respect to variable w at $(V_{\tau h}, W_{\tau h})$
$g_{I,v}^C$	Partial derivative of function g_I with respect to variable v at $(V_{\tau h}, W_{\tau h})$
$g_{I,w}^C$	Partial derivative of function g_I with respect to variable w at $(V_{\tau h}, W_{\tau h})$

Table A4. List of abbreviations.

Abbreviation	Meaning
FE	Finite element
FD	Finite difference
NLS	Nonlinear Schrödinger
FN-CD	Fully nonlinear compact difference
TTG-CD	Time two-grid compact difference
TSTG-CD	Temporal-spatial two-grid compact difference

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