axioms

Article

On Sawyer Duality in One and Higher Dimensions

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Abstract

We develop the Sawyer duality principle for non-increasing functions where, in the denominator, the function g is replaced by $\int_x^\infty \frac{g(t)}{t}dt$. This result complements Stepanov's result in which g was replaced by $\frac{1}{x}\int_0^x g(t)dt$. We use our result and obtain Stepanov's result for non-decreasing functions and similarly use Stepanov's result in proving our result for non-decreasing functions. These results have also been obtained in $\mathbb{R}^n$.

Keywords: sawyer duality principle; hardy averaging operator; adjoint operator; monotone functions; radially monotone functions

MSC: 26D10; 26D15

1. Introduction

Characterizing the boundedness of integral operators between various function spaces is an old problem. To this end, let us mention the following celebrity theorem, which gives the boundedness of the Hardy operator and its adjoint between weighted Lebesgue spaces [1]:

Theorem 1. *Let $1 < p \leqslant q < \infty$ and u, v be weight functions on $(0,\infty)$, i.e., the functions which are measurable, positive and finite almost everywhere on $(0,\infty)$.*

(a) *The inequality*

$$\left(\int_0^\infty u(x)\left(\int_0^x f(t)dt\right)^q dx\right)^{1/q} \leqslant c\left(\int_0^\infty v(x)f^p(x)dx\right)^{1/p}$$

holds for all $f \geqslant 0$ if and only if

$$\sup_{x>0}\left(\int_x^\infty u(t)dt\right)^{1/q}\left(\int_0^x v^{1-p'}(t)dt\right)^{1/p'} < \infty.$$

(b) *The inequality*

$$\left(\int_0^\infty u(x)\left(\int_x^\infty f(t)dt\right)^p dx\right)^{1/q} \leqslant c\left(\int_0^\infty v(x)f^p(x)dx\right)^{1/p}$$

holds for all $f \geqslant 0$ if and only if

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$$\sup_{x>0}\left(\int_0^x u(t)dt\right)^{1/q}\left(\int_x^\infty v^{1-p'}(t)dt\right)^{1/p'}<\infty.$$

From the application point of view, many times it is required to consider Theorem 1 for functions that are non-negative and non-increasing ($\downarrow$), see [2–5]. In order to address this problem, among other possible strategies, a much-used one is via the duality principle of Sawyer [3] (see also [1]), which concerns the duality in the weighted Lebesgue spaces for $\downarrow$ functions: Let $1<p<\infty$, $f\geqslant 0$, and v be a positive, measurable, and locally integrable function on $(0,\infty)$. Then

$$\begin{aligned} I(f) &:= \sup_{0\leqslant g\downarrow}\frac{\int_0^\infty f(x)g(x)dx}{\left(\int_0^\infty g^p(x)v(x)dx\right)^{1/p}} \\ &\simeq \left(\int_0^\infty f\right)\left(\int_0^\infty v\right)^{-1/p}+\left(\int_0^\infty\left(\int_0^x f(t)dt\right)^{p'}\left(\int_0^x v(t)dt\right)^{-p'}v(x)dx\right)^{1/p'}. \end{aligned} \tag{1}$$

Let us point out that in [1], the equivalence (1) has also been studied for the case when g is non-decreasing ($\uparrow$) and also when $0<p\leqslant 1$. In fact, it was proved that when g is $\downarrow$ and $0<p\leqslant 1$,

$$I(f)=\sup_{r\geqslant 0}\left(\int_0^r f\right)\left(\int_0^r v\right)^{-1/p}.$$

Recently, in [6], the duality principle (1) has been used to find necessary and sufficient conditions for the boundedness of the Hausdorff operator between weighted Lebesgue spaces for non-increasing functions. The duality principle in the framework of Orlicz spaces has been developed in [7].

In [4], Stepanov considered the LHS of (1), where in the denominator, g is replaced by $\frac{1}{x}\int_0^x g$, and obtained the lower as well as upper estimates of the corresponding expression for the case $1<p<\infty$. Precisely, the following was proved in [4]:

Theorem 2. *Let $1<p<\infty$, f and v be measurable non-negative functions on $[0,\infty)$ with v locally integrable. Consider the expression*

$$J^\downarrow(f):=\sup_{0\leqslant g\downarrow}\frac{\int_0^\infty fg}{\left(\int_0^\infty\left(\frac{1}{x}\int_0^x g(t)dt\right)^p v(x)dx\right)^{1/p}}.$$

Then

$$J_L^\downarrow(f)\ll J^\downarrow(f)\ll J_R^\downarrow(f),$$

where

$$J_L^\downarrow(f)=\frac{\int_0^\infty f}{\left(\int_0^\infty v\right)^{1/p}}+\left(\int_0^\infty\left(\int_0^x f\right)^{p'}\left(\int_0^x v+x^p\int_x^\infty t^{-p}v(t)dt\right)^{-p'}v(x)dx\right)^{1/p'}$$

and

$$\begin{aligned} J_R^\downarrow(f) &= \frac{\int_0^\infty f}{\left(\int_0^\infty v\right)^{1/p}} \\ &+\left(\int_0^\infty\left(\int_0^x f\right)^{p'}\left(\int_0^x v+x^p\int_x^\infty t^{-p}v(t)dt\right)^{-p'}x^{p-1}\int_x^\infty t^{-p}v(t)dtdx\right)^{1/p'}. \end{aligned}$$

The first goal of this paper is to complement Stepanov's result by considering the LHS of (1), where in the denominator, g is replaced by $\int_x^\infty \frac{g(t)}{t}dt$, i.e., we consider the expression

$$I^{\downarrow}(f) := \sup_{0 \leqslant g\downarrow} \frac{\int_0^\infty fg}{\left(\int_0^\infty \left(\int_x^\infty \frac{g(t)}{t}dt\right)^p v(x)dx\right)^{1/p}} \tag{2}$$

and obtain its lower and upper estimates. As an application of our result, we prove Theorem 2 for non-decreasing functions. Moreover, we apply Theorem 2 to prove our result for non-decreasing functions. By convention, we assign the value zero to products and quotients of the types $0 \times \infty$, $\frac{\infty}{\infty}$, and $\frac{0}{0}$.

Finally, we also consider the n-dimensional analogue of the expression (2). A function $\mathcal{G} : \mathbb{R}^n \to \mathbb{R}$ is said to be radially decreasing, written $\mathcal{G} \downarrow r$, if $\mathcal{G}(x) = \mathcal{G}(y)$ for $|x| = |y|$ and $\mathcal{G}(x) \geqslant \mathcal{G}(y)$ for $|x| < |y|$. The radially increasing functions are defined similarly. By $\mathbb{B}(x)$, we shall denote the ball in $\mathbb{R}^n$ centered at the origin and with radius $|x|$. We consider

$$\mathbb{I}^{\downarrow}(\mathcal{F}) := \sup_{0 \leqslant \mathcal{G}\downarrow \mathrm{r}} \frac{\int_{\mathbb{R}^n} \mathcal{F}(x)\mathcal{G}(x)dx}{\left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n \setminus \mathbb{B}(x)} \frac{\mathcal{G}(t)}{|\mathbb{B}(t)|}dt\right)^p \mathcal{V}(x)dx\right)^{1/p}}$$

and obtain its two-sided estimate for radially decreasing function $\mathcal{G}$. The equivalence (1) for radially monotone functions has been developed by Barza, Johansson, and Persson [8]. In [9], another n-dimensional analogue of (1) has been studied where the functions have been considered coordinate-wise. Also, in [9], the discrete version of (1) has been investigated. Some related multidimensional imbeddings in the setting of monotone functions have been studied in [10].

If there is no confusion, we have used the symbols x, y, and t to be the members of $\mathbb{R}$ as well as $\mathbb{R}^n$, as it will be clear from the usage. The symbol $A \ll B$ will mean $A \leqslant cB$ for some constant $c > 0$. The symbol p' will denote the conjugate index to p, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$.

2. Sawyer Type Duality Result Involving Adjoint Operator

In this section, we shall obtain the upper and lower estimates of the expression (2). We shall be using the following notations:

$$V(x) = \int_0^x v(t)dt, \quad w(x) = x^{-p}V^p(x)v^{1-p}(x),$$

$$W(x) = \int_0^x w(t)dt, \quad \gamma(x) = \int_x^\infty t^{-p}w(t)dt, \quad W_0(x) = W(x) + x^p\gamma(x).$$

Our first main theorem is the following:

Theorem 3. *Let $1 < p < \infty$, $f \geqslant 0$ and v be a function such that $v(t) = \int_0^t \xi(\tau)\,d\tau$, where ξ is positive, measurable and locally integrable function on $[0,\infty)$. Let $I^{\downarrow}(f)$ be as defined by (2). Then*

$$I_L^{\downarrow}(f) \ll I^{\downarrow}(f) \ll I_R^{\downarrow}(f),$$

where

$$I_L^{\downarrow}(f) = \left(\int_0^\infty \left(\int_0^x f(t)dt\right)^{p'} W^{-p'}(x)w(x)dx\right)^{1/p'} \tag{3}$$

and

$$I_R^{\downarrow}(f) = p\left(\int_0^\infty x^{-p}V^p(x)\left(\int_x^\infty f(t)W_0^{-1}(t)dt\right)^{p'} v^{1-p}(x)dx\right)^{1/p'}$$
$$+\left(\int_0^\infty \left(x^p\gamma(x)V^{-1}(x)\right)^{p'}\left(\int_x^\infty f(t)W_0^{-1}(t)dt\right)^{p'} v(x)dx\right)^{1/p'}$$
$$+\,p\left(\int_0^\infty x^p\gamma^{p'}(x)\left(\int_x^\infty f(t)W_0^{-1}(t)dt\right)^{p'} v(x)dx\right)^{1/p'}. \tag{4}$$

Proof. Let us obtain the lower bound first. We consider the function $g(x) = \int_x^\infty h$ with $h \geqslant 0$. We find that g is $\downarrow$ and

$$I^{\downarrow}(f) \geqslant \sup_{g=\int_x^\infty h, h\geqslant 0} \frac{\int_0^\infty fg}{\left(\int_0^\infty\left(\int_x^\infty \frac{g(t)}{t}dt\right)^p v(x)dx\right)^{1/p}}. \tag{5}$$

It can be checked that

$$\left(\int_0^x v(t)dt\right)^{1/p}\left(\int_x^\infty V^{-p'}(t)v(t)dt\right)^{1/p'} \ll 1$$

so that in view of Theorem 1(b), the inequality

$$\left(\int_0^\infty\left(\int_x^\infty \frac{g(t)}{t}dt\right)^p v(x)dx\right)^{1/p} \ll \left(\int_0^\infty g^p(x)w(x)dx\right)^{1/p} \tag{6}$$

holds. It can also be checked that

$$\left(\int_0^x w(t)dt\right)^{1/p}\left(\int_x^\infty W^{-p'}(t)w(t)dt\right)^{1/p'} \ll 1$$

so that, again, in view of Theorem 1(b), the inequality

$$\left(\int_0^\infty\left(\int_x^\infty h(t)dt\right)^p w(x)dx\right)^{1/p} \ll \left(\int_0^\infty h^p(x)W^p(x)w^{-p/p'}(x)dx\right)^{1/p} \tag{7}$$

holds. Consequently, by (6) and (7) and using the well known dual characterization of the norm of weighted Lebesgue spaces (see, e.g., [11] proc. 4.1, p. 225), we obtain

$$I^{\downarrow}(f) \geqslant \sup_{g=\int_x^\infty h, h\geqslant 0} \frac{\int_0^\infty fg}{\left(\int_0^\infty\left(\int_x^\infty \frac{g(t)}{t}dt\right)^p v(x)dx\right)^{1/p}}$$
$$= \sup_{h\geqslant 0}\left(\int_0^\infty\left(\int_0^x f\right)h(x)dx\right)\left(\int_0^\infty\left(\int_x^\infty \frac{g(t)}{t}dt\right)^p v(x)dx\right)^{-1/p}$$
$$\approx \left(\int_0^\infty\left(\int_0^x f(t)dt\right)^{p'} W^{-p'}(x)w(x)dx\right)^{1/p'},$$

which combined with (5) gives the lower bound (3) for $I^{\downarrow}(f)$.

We now proceed to prove the upper bound. Let $g \geqslant 0$ be a non-increasing function. We write

$$\int_0^\infty fg = \int_0^\infty fgW_0W_0^{-1} = \int_0^\infty fgW_0^{-1}(W + x^p\gamma(x)) =: I_1 + I_2.$$

First we estimate I_1. Let us write $G(t) := \int_t^\infty f(x)W_0^{-1}(x)dx$ and $\sigma(t) = tw(t)V^{-1}(t)G(t)$. Then by Fubini's theorem, we get

$$\begin{aligned}
I_1 &= \int_0^\infty f(x)g(x)W_0^{-1}(x)W(x)dx \\
&= \int_0^\infty f(x)g(x)W_0^{-1}(x)\left(\int_0^x w(t)dt\right)dx \\
&= \int_0^\infty w(t)\left(\int_t^\infty f(x)g(x)W_0^{-1}(x)dx\right)dt \\
&\le \int_0^\infty w(t)g(t)\left(\int_t^\infty f(x)W_0^{-1}(x)dx\right)dt \\
&= \int_0^\infty w(t)g(t)G(t)dt \\
&= \int_0^\infty w(t)g(t)G(t)V^{-1}(t)\left(\int_0^t v(x)dx\right)dt \\
&= \int_0^\infty v(x)\left(\int_x^\infty w(t)g(t)G(t)V^{-1}(t)dt\right)dx \\
&= \int_0^\infty v(x)\left(\int_x^\infty \sigma(t)\frac{g(t)}{t}dt\right)dx. \qquad (8)
\end{aligned}$$

Without loss of generality, we may assume that $g(y)/y$ is integrable so that integration by parts gives

$$\begin{aligned}
\int_x^\infty \sigma(t)\frac{g(t)}{t}dt &= -\int_x^\infty \sigma(t)\frac{d}{dt}\left(\int_t^\infty \frac{g(y)}{y}dy\right)dt \\
&= \sigma(x)\int_x^\infty \frac{g(y)}{y}dy + \int_x^\infty \sigma'(t)\left(\int_t^\infty \frac{g(y)}{y}dy\right)dt. \qquad (9)
\end{aligned}$$

Note that $\sigma(t) = \left(\frac{V(t)}{tv(t)}\right)^{p-1}\int_t^\infty fW_0^{-1}$. Without loss of generality, we may assume that fW_0^{-1} is integrable, otherwise the upper estimate is trivial. Further, the assumption on v makes it uniformly continuous and non-decreasing so that $\frac{V(t)}{tv(t)}$ exists and moreover bounded as well. These arguments give that

$$\begin{aligned}
\sigma'(t) &= \left(\int_t^\infty fW_0^{-1}\right)\frac{d}{dt}\left(\frac{V(t)}{tv(t)}\right)^{p-1} - \left(\frac{V(t)}{tv(t)}\right)^{p-1} fW_0^{-1} \\
&\le \left(\int_t^\infty fW_0^{-1}\right)\frac{d}{dt}\left(\frac{V(t)}{tv(t)}\right)^{p-1} \\
&\le (p-1)G(t)\left(\frac{V(t)}{tv(t)}\right)^{p-2}\left(\frac{1}{t}\right)
\end{aligned}$$

using which in (9) gives

$$\int_x^\infty \sigma(t)\frac{g(t)}{t}dt \le \sigma(x)\int_x^\infty \frac{g(y)}{y}dy + (p-1)\int_x^\infty G(t)\left(\frac{V(t)}{tv(t)}\right)^{p-2}\left(\frac{1}{t}\right)\left(\int_t^\infty \frac{g(y)}{y}dy\right)dt$$

which together with (8) and Fubini's theorem gives

$$\begin{aligned} I_1 \leq & \int_0^\infty v(x)\sigma(x)\left(\int_x^\infty \frac{g(y)}{y}dy\right)dx \\ & + (p-1)\int_0^\infty v(x)\left\{\int_x^\infty G(t)\left(\frac{V(t)}{tv(t)}\right)^{p-2}\left(\frac{1}{t}\right)\left(\int_t^\infty \frac{g(y)}{y}dy\right)dt\right\}dx \\ = & \int_0^\infty v(x)\sigma(x)\int_x^\infty \frac{g(y)}{y}dy \\ & + (p-1)\int_0^\infty G(t)\left(\frac{V(t)}{tv(t)}\right)^{p-2}\left(\frac{1}{t}\right)\left(\int_t^\infty \frac{g(y)}{y}dy\right)\left(\int_0^t v(x)dx\right)dt \\ = & \int_0^\infty v(x)\sigma(x)\int_x^\infty \frac{g(y)}{y}dy \\ & + (p-1)\int_0^\infty v(t)\sigma(t)\left(\int_t^\infty \frac{g(y)}{y}dy\right)dt \\ = & \, p\int_0^\infty v(x)\sigma(x)\left(\int_x^\infty \frac{g(y)}{y}dy\right)dx \\ = & \, p\int_0^\infty v(x)x^{1-p}V^{p-1}(x)v^{1-p}(x)G(x)\left(\int_x^\infty \frac{g(t)}{t}dt\right)dx. \end{aligned}$$

An application of Hölder's inequality now gives

$$I_1 \leq p\left(\int_0^\infty \left(\int_x^\infty \frac{g(t)}{t}dt\right)^p v(x)dx\right)^{1/p}\left(\int_0^\infty x^{-p}V^p(x)v^{1-p}(x)G^{p'}(x)dx\right)^{1/p'}. \tag{10}$$

Now, we proceed to estimate I_2. By Fubini's theorem and Hölder's inequality, we obtain

$$\begin{aligned} I_2 &= \int_0^\infty f(x)g(x)W_0^{-1}(x)x^p\gamma(x)dx \\ &= p\int_0^\infty g(x)f(x)W_0^{-1}(x)\gamma(x)\left(\int_0^x t^{p-1}dt\right)dx \\ &= p\int_0^\infty t^{p-1}\left(\int_t^\infty g(x)f(x)W_0^{-1}(x)\gamma(x)dx\right)dt \\ &\leqslant p\int_0^\infty t^{p-1}g(t)\gamma(t)\left(\int_t^\infty f(x)W_0^{-1}(x)dx\right)dt \\ &= p\int_0^\infty t^{p-1}g(t)G(t)\gamma(t)dt \\ &= p\int_0^\infty t^{p-1}g(t)G(t)\gamma(t)V^{-1}(t)\int_0^t v(x)dxdt \\ &= p\int_0^\infty v(x)\left(\int_x^\infty t^{p-1}g(t)G(t)\gamma(t)V^{-1}(t)dt\right)dx \\ &= p\int_0^\infty v(x)\left(\int_x^\infty \frac{g(t)}{t}H(t)dt\right)dx. \end{aligned} \tag{11}$$

Note that $H(t) = \frac{t^pG(t)\gamma(t)}{V(t)}$ which is the ratio of products of absolutely continuous functions so that integration by parts gives that

$$\begin{aligned} \int_x^\infty H(t)\frac{g(t)}{t}dt &= -\int_x^\infty H(t)\frac{d}{dt}\left(\int_t^\infty \frac{g(y)}{y}dy\right)dt \\ &= H(x)\int_x^\infty \frac{g(y)}{y}dy + \int_x^\infty H'(t)\left(\int_t^\infty \frac{g(y)}{y}dy\right)dt. \end{aligned} \tag{12}$$

Further, $G(t)\gamma(t)$ is decreasing being the product of two positive decreasing functions so that

$$H'(t) \leqslant \frac{pt^{p-1}G(t)\gamma(t)}{V(t)}$$

using which in (12) gives

$$\int_x^\infty \frac{g(t)}{t} H(t)dt \leqslant H(x) \int_x^\infty \frac{g(t)}{t} dt + p \int_x^\infty \frac{t^{p-1}G(t)\gamma(t)}{V(t)} \left(\int_t^\infty \frac{g(y)}{y} dy \right) dt$$

which together with (11) and Fubini's theorem gives

$$\begin{aligned} I_2 &\leqslant p \int_0^\infty v(x)H(x) \left(\int_x^\infty \frac{g(t)}{t} dt \right) dx + p^2 \int_0^\infty v(x) \int_x^\infty \frac{t^{p-1}G(t)\gamma(t)}{V(t)} \left(\int_t^\infty \frac{g(y)}{y} dy \right) dt \\ &\leqslant p \int_0^\infty v(x)H(x) \left(\int_x^\infty \frac{g(t)}{t} dt \right) dx + p^2 \int_0^\infty t^{p-1}G(t)\gamma(t) \left(\int_t^\infty \frac{g(y)}{y} dy \right) dt \\ &=: I_3 + I_4. \end{aligned}$$

Now applying Hölder's inequality we get

$$\begin{aligned} I_3 &= p \int_0^\infty v(x)H(x) \left(\int_x^\infty \frac{g(t)}{t} dt \right) dx \\ &\leqslant p \left(\int_0^\infty \left(\int_x^\infty \frac{g(t)}{t} dt \right)^p v(x)dx \right)^{1/p} \left(\int_0^\infty H^{p'}(x)v(x)dx \right)^{1/p'} \end{aligned} \tag{13}$$

Again, applying hölder's inequality to calculate I_4, we get

$$I_4 \leqslant p^2 \left(\int_0^\infty \left(\int_x^\infty \frac{g(t)}{t} dt \right)^p v(x)dx \right)^{1/p} \left(\int_0^\infty \left(t^{p-1}G(t)\gamma(t) \right)^{p'} (x)v(x)dx \right)^{1/p'}. \tag{14}$$

Now, combining the estimates (13) and (14), we get

$$\begin{aligned} I_2 \leqslant p \left(\int_0^\infty \left(\int_x^\infty \frac{g(t)}{t} dt \right)^p v(x)dx \right)^{1/p} \Bigg[&\left(\int_0^\infty H^{p'}(x)v(x)dx \right)^{1/p'} \\ &+ p \left(\int_0^\infty \left(t^{p-1}G(t)\gamma(t) \right)^{p'} (x)v(x)dx \right)^{1/p'} \Bigg]. \end{aligned} \tag{15}$$

Finally, combining the estimates (10) and (15), we get the required upper bound (4). □

As a corollary, we deduce the power weight case of Theorem 3:

Corollary 1. *Let $1 < p < \infty$, $f \geqslant 0$, $\alpha \in \mathbb{R}$ and $I_\alpha^\downarrow(f)$ be as defined by*

$$I_\alpha^\downarrow(f) := \sup_{0 \leqslant g \downarrow} \frac{\int_0^\infty fg}{\left(\int_0^\infty \left(\int_x^\infty \frac{g(t)}{t} dt \right)^p x^\alpha dx \right)^{1/p}}.$$

Then

$$I_{\alpha,L}^\downarrow(f) \ll I_\alpha^\downarrow(f) \ll I_{\alpha,R}^\downarrow(f),$$

where

$$I^{\downarrow}_{\alpha,L}(f) = \left(\int_0^\infty \left(\int_0^x f(t)dt\right)^{p'} x^{p'(1+\alpha)+\alpha} dx\right)^{1/p'}$$

and

$$I^{\downarrow}_{\alpha,R}(f) = p \times \Bigg[\left(\int_0^\infty \left(\int_x^\infty f(t)t^{-1-\alpha}(t)dt\right)^{p'} x^{\alpha} dx\right)^{1/p'} + p\left(\int_0^\infty \left(\int_x^\infty f(t)t^{-1-\alpha}(t)dt\right)^{p'} x^{(1+p')\alpha} dx\right)^{1/p'}\Bigg].$$

3. The Case of Non-Decreasing Functions

We begin this section by proving Theorem 2 for $g\uparrow$, as an application of Theorem 3.

Theorem 4. *Let $1 < p < \infty$, $f \geqslant 0$ and v be a function such that $v(t) = \int_0^t \xi(\tau)\,d\tau$, where ξ is positive, measurable and locally integrable function on $[0,\infty)$. Consider the expression*

$$J^{\uparrow}(f) := \sup_{0\leqslant g\uparrow} \frac{\int_0^\infty fg}{\left(\int_0^\infty \left(\frac{1}{x}\int_0^x g(t)dt\right)^p v(x)dx\right)^{1/p}}.$$

Then

$$J^{\uparrow}_L(f) \ll J^{\uparrow}(f) \ll J^{\uparrow}_R(f),$$

where

$$J^{\uparrow}_L(f) = \left(\int_0^\infty \left(\int_x^\infty tf(t)dt\right)^{p'} \left(\int_x^\infty \left(\int_t^\infty y^{-p}v(y)dy\right)^p t^{p(p-2)}v^{1-p}(t)dt\right)^{-p'} \times \left(\int_x^\infty t^{-p}v(t)dt\right)^p x^{p(p-2)}v^{1-p}(x)dx\right)^{1/p'}$$

and

$$\begin{aligned}
J^{\uparrow}_R(f) = p\Bigg[&\left(\int_0^\infty x^{p^2-2p+4}\left(\int_x^\infty y^{-p}v(y)dy\right)^p\right.\\
&\times\left(\int_0^x tf(t)\left(\int_t^\infty y^{p^2-2p+4}\left(\int_y^\infty z^{-p}v(z)dz\right)^p v^{1-p}(t)dt\right.\right.\\
&\left.\left.+t^{-p}\int_0^t y^{p^2-2p+4}\left(\int_y^\infty z^{-p}v(z)dz\right)^p v^{1-p}(y)dy\right)^{-1}dt\right)^p v^{1-p}(x)dx\right)^{1/p'}\\
&+\left(\int_0^\infty \left(x^{-p}\int_0^x t^p\left(\int_t^\infty y^{-p}v(y)dy\right)^p v(t)dt\left(\int_x^\infty t^{-p}v(t)dt\right)^{-1}\right)^{p'}\right.\\
&\times\left(\int_0^x tf(t)\left(\int_t^\infty y^{p^2-2p+4}\left(\int_y^\infty z^{-p}v(z)dz\right)^p v^{1-p}(y)dy\right.\right.\\
&\left.\left.+t^{-p}\left(\int_0^t y^{p^2-2p+4}\left(\int_y^\infty z^{-p}v(z)dz\right)^p v^{1-p}(y)dy\right)^{-1}dt\right)^p x^{-p}v(x)dx\right)^{1/p'}\\
&+p\left(\int_0^\infty x^{-2p}\left(\int_0^x t^p\left(\int_t^\infty y^{-p}v(y)dy\right)^p v(t)dt\right)^{p'}\right.\\
&\times\left(\int_0^x tf(t)\left(\int_t^\infty y^{p^2-2p+4}\left(\int_y^\infty z^{-p}v(z)dz\right)^p v^{1-p}(y)dy\right.\right.\\
&\left.\left.+t^{-p}\int_0^t y^{p^2-2p+4}\left(\int_y^\infty z^{-p}v(z)dz\right)^p v^{1-p}(y)dy\right)^{-1}dt\right)^{p'} v(x)dx\right)^{1/p'}\Bigg].
\end{aligned}$$

Proof. A change of variable shows that

$$J^{\uparrow}(f) := \sup_{0\leqslant g\uparrow} \frac{\int_0^\infty fg}{\left(\int_0^\infty \left(\frac{1}{x}\int_0^x g(t)dt\right)^p v(x)dx\right)^{1/p}} = \sup_{0\leqslant g\uparrow} \frac{\int_0^\infty f(\frac{1}{x})g(\frac{1}{x})\frac{dx}{x^2}}{\left(\int_0^\infty \left(\frac{1}{x}\int_{1/x}^\infty g(\frac{1}{t})\frac{dt}{t^2}\right)^p v(x)dx\right)^{1/p}}$$
$$= \sup_{0\leqslant g\uparrow} \frac{\int_0^\infty f(\frac{1}{x})g(\frac{1}{x})\frac{dx}{x^2}}{\left(\int_0^\infty \left(x\int_x^\infty g(\frac{1}{t})\frac{dt}{t^2}\right)^p v(\frac{1}{x})\frac{dx}{x^2}\right)^{1/p}}$$
$$= \sup_{0\leqslant \hat{g}\downarrow} \frac{\int_0^\infty \hat{f}(x)\hat{g}(x)dx}{\left(\int_0^\infty \left(\int_x^\infty \frac{\hat{g}(t)}{t}dt\right)^p \hat{v}(x)dx\right)^{1/p}},$$

where $\hat{f}(t) = \frac{f(\frac{1}{t})}{t}$, $\hat{g}(t) = \frac{g(\frac{1}{t})}{t}$ and $\hat{v}(t) = v(\frac{1}{t})t^{p-2}$. Then $\hat{g}\downarrow$ and the assertion follows by applying Theorem 3. □

Remark 1. *A corollary to Theorem 4 for power weight similar to Corollary 1 can be formulated.*

In turn, we can apply Theorem 2 to prove the version of Theorem 3 for $g\uparrow$. Precisely, we prove the following

Theorem 5. *Let $1 < p < \infty$, $f \geq 0$ and v be a positive, measurable and locally integrable function on $[0,\infty)$. Further, let $I^{\uparrow}(f)$ be defined by*

$$I^{\uparrow}(f) = \sup_{0\leqslant g\uparrow} \frac{\int_0^\infty fg}{\left(\int_0^\infty \left(\int_x^\infty \frac{g(t)}{t}dt\right)^p v(x)dx\right)^{1/p}}.$$

Then

$$I_L^{\uparrow}(f) \ll I^{\uparrow}(f) \ll I_R^{\uparrow}(f),$$

where

$$I_L^{\uparrow}(f) = \left(\int_0^\infty xf(x)dx\right)\left(\int_0^\infty x^{-p}v(x)dx\right)^{-1/p}$$
$$+ \left(\int_0^\infty \left(\int_x^\infty tf(t)dt\right)^p \left(\int_x^\infty t^{-p}v(t)dt + x^{-p}\int_0^x t^{-p}v(t)dt\right)^{-p'} x^{-p}v(x)dx\right)^{1/p'}$$

and

$$I_R^{\uparrow}(f)$$
$$= \left(\int_0^\infty \left(\int_x^\infty tf(t)dt\right)^p \left(\int_x^\infty t^{-p}v(t)dt + x^{-p}\int_0^x t^{-p}v(t)dt\right)^{-p'} x^{3-p}\int_0^x t^{-p}v(t)dtdx\right)^{1/p'}$$
$$+ \left(\int_o^\infty xf(x)dx\right)\left(\int_o^\infty x^{-p}v(x)dx\right)^{-1/p}.$$

Proof. As in the proof of Theorem 4, a change of variable gives that

$$
\begin{aligned}
J^{\downarrow}(f) &:= \sup_{0 \leqslant g \uparrow} \frac{\int_0^\infty fg}{\left(\int_0^\infty \left(\int_x^\infty \frac{g(t)}{t} dt\right)^p v(x)dx\right)^{1/p}} \\
&= \sup_{0 \leqslant g \uparrow} \frac{\int_0^\infty f(\frac{1}{x})g(\frac{1}{x})\frac{dx}{x^2}}{\left(\int_0^\infty \left(\int_0^{1/x} g(\frac{1}{t})\frac{dt}{t}\right)^p v(x)dx\right)^{1/p}} \\
&= \sup_{0 \leqslant g \uparrow} \frac{\int_0^\infty f(\frac{1}{x})g(\frac{1}{x})\frac{dx}{x^2}}{\left(\int_0^\infty \left(\int_0^{x} g(\frac{1}{t})\frac{dt}{t}\right)^p v(\frac{1}{x})\frac{dx}{x^2}\right)^{1/p}} \\
&= \sup_{0 \leqslant \hat{g} \downarrow} \frac{\int_0^\infty \hat{f}(x)\hat{g}(x)dx}{\left(\int_0^\infty \left(\frac{1}{x}\int_0^{x} \frac{\hat{g}(t)}{t} dt\right)^p \hat{v}(x)dx\right)^{1/p}},
\end{aligned}
$$

where $\hat{f}(t) = \frac{f(\frac{1}{t})}{t}$, $\hat{g}(t) = \frac{g(\frac{1}{t})}{t}$ and $\hat{v}(t) = v(\frac{1}{t})t^{p-2}$. Then $\hat{g} \downarrow$ and the assertion follows by applying Theorem 2. □

Finally, in this section, we prove Theorem 2 for the case $0 < p \leqslant 1$.

Theorem 6. *Let $0 < p \leqslant 1$, f and v be non-negative measurable functions on $[0, \infty)$ with v locally integrable. Then*

$$
J_L^{\downarrow}(f) \ll J^{\downarrow}(f) \ll J_R^{\downarrow}(f),
$$

where

$$
J_L^{\downarrow}(f) = \sup_{r \geqslant 0} \left(\int_0^r g(x)dx\right)\left(\int_0^r x^{-p}v(x)dx\right)^{-1/p}
$$

and

$$
J_R^{\downarrow}(f) = \sup_{r \geqslant 0} \left(\int_0^r f(x)dx\right)\left(\int_0^r v(x)dx\right)^{-1/p}.
$$

Proof. The lower bound is obtained by using the test function

$$
\chi_{(0,r)}f(x) = \begin{cases} 0 & \text{if } x > r \\ 1 & \text{if } x \leqslant r. \end{cases}
$$

Next, note that for $f \downarrow$

$$
f(x) \leqslant \frac{1}{x}\int_0^x f(t)dt
$$

and the upper bound follows by using [1] (Theorem 6.5). □

Remark 2. *It is of interest if Theorems 3–5 can be proved for $0 < p \leqslant 1$ and also if Theorem 5 can be proved for $g \uparrow$.*

4. Radially Monotone Functions

Recall that by $\mathbb{B}(x)$, we shall denote the ball in $\mathbb{R}^n$ centered at the origin and with radius $|x|$. We shall denote by $\sum_{n-1}$ the surface of the unit ball in $\mathbb{R}^n$ and by $|\sum_{n-1}|$ its area. By using the polar coordinates, we shall be writing

$$\int_{\mathbb{R}^n} \mathcal{F}(x)dx = \int_0^\infty \int_{\sum_{n-1}} t^{n-1}\mathcal{F}(t\tau)dtd\tau$$

and similarly other integrals in $\mathbb{R}^n$.

We mention below the n-dimensional version of Theorem 1 proved by Drábek, Heinig and Kufner [12]:

Theorem 7. *Let $1 < p \leqslant q < \infty$ and $\mathcal{W}, \mathcal{U}$ be weight functions on $\mathbb{R}^n$, i.e., the functions which are measurable, positive and finite almost everywhere on $\mathbb{R}^n$.*

(a) *The inequality*

$$\left(\int_{\mathbb{R}^n} \mathcal{W}(x)\left(\int_{\mathbb{B}(x)} \mathcal{F}(y)dy\right)^q dx\right)^{1/q} \leqslant c\left\{\int_{\mathbb{R}^n} \mathcal{U}(x)\mathcal{F}^p(x)dx\right\}^{1/p}$$

holds for $\mathcal{F} \geqslant 0$ if and only if

$$\sup_{\alpha>0}\left(\int_{|x|\geqslant\alpha} \mathcal{W}(x)dx\right)^{1/q}\left(\int_{|x|\leqslant\alpha} \mathcal{U}^{1-p'}(x)dx\right)^{1/p'} < \infty.$$

(b) *The inequality*

$$\left(\int_{\mathbb{R}^n} \mathcal{W}(x)\left(\int_{\mathbb{R}^n\setminus\mathbb{B}(x)} \mathcal{F}(y)dy\right)^q dx\right)^{1/q} \leqslant c\left(\int_{\mathbb{R}^n} \mathcal{U}(x)\mathcal{F}^p(x)dx\right)^{1/p}$$

holds for $\mathcal{F} \geqslant 0$ if and only if

$$\sup_{\alpha>0}\left(\int_{|x|\leqslant\alpha} \mathcal{W}(x)dx\right)^{1/q}\left(\int_{|x|\geqslant\alpha} \mathcal{U}^{1-p'}(x)dx\right)^{1/p'} < \infty.$$

Below we prove the n-dimensional version of Theorem 4.

Theorem 8. *Let $1 < p < \infty$, $f \geqslant 0$ and v be a function such that $v(t) = \int_{\mathbb{B}(t)} \xi(\tau)\,d\tau$, where ξ is positive, measurable and locally integrable function on $\mathbb{R}^n$. Let $\mathbb{I}^{\downarrow}(\mathcal{F})$ be defined by*

$$\mathbb{I}^{\downarrow}(\mathcal{F}) = \sup_{0\leqslant\mathcal{G}\downarrow r} \frac{\int_{\mathbb{R}^n} \mathcal{F}(x)\mathcal{G}(x)dx}{\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n\setminus\mathbb{B}(x)} \frac{\mathcal{G}(t)}{|\mathbb{B}(t)|}dt\right)^p \mathcal{V}(x)dx\right)^{1/p}}.$$

Then

$$\mathbb{I}_L^{\downarrow}(\mathcal{F}) \ll \mathbb{I}^{\downarrow}(\mathcal{F}) \ll \mathbb{I}_R^{\downarrow}(\mathcal{F}),$$

where

$$\mathbb{I}_L^{\downarrow}(\mathcal{F}) = \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{B}(x)} \mathcal{F}(t)dt\right)^p \tilde{\mathcal{W}}^{-p'}(x)\mathcal{W}(x)dx\right)^{1/p'} \tag{16}$$

and

$$\mathbb{I}_R^{\downarrow}(\mathcal{F}) = p\Bigg[\Bigg(\int_{\mathbb{R}^n} x^{-p}\tilde{\mathcal{V}}^p(x)\mathcal{V}^{1-p}(x)\Bigg(\int_{\mathbb{R}^n\setminus\mathbb{B}(x)}\mathcal{F}\tilde{\mathcal{W}}_0^{-1}\Bigg)^{p'}(x)dx\Bigg)^{1/p'}$$
$$+\Bigg(\int_{\mathbb{R}^n}\Big(x^p\tilde{\gamma}(x)\tilde{\mathcal{V}}^{-1}(x)\Big)^{p'}\Bigg(\int_{\mathbb{R}^n\setminus\mathbb{B}(x)}\mathcal{F}\tilde{\mathcal{W}}_0^{-1}\Bigg)^{p'}\mathcal{V}(x)dx\Bigg)^{1/p'}$$
$$+p\Bigg(\int_{\mathbb{R}^n} x^p\Bigg(\int_{\mathbb{R}^n\setminus\mathbb{B}(x)}\mathcal{F}\tilde{\mathcal{W}}_0^{-1}\Bigg)^{p'}\tilde{\gamma}^{p'}(x)\mathcal{V}(x)dx\Bigg)^{1/p'}\Bigg], \tag{17}$$

where $\mathcal{W}(x) = |\mathbb{B}(x)|^{-p}\tilde{\mathcal{V}}^p(x)\mathcal{V}^{1-p}(x)$ *and* $\tilde{\mathcal{W}}(x) = \int_{\mathbb{B}(x)}\mathcal{W}(t)dt.$

Proof. Make the following changes of variables.

Let $t = s\sigma$ and $x = y\tau$, where $s, y \in (0,\infty)$ and $\sigma, \tau \in S^{n-1}$. By using the fact that $g(s\sigma) = g(s)$ since g is radial, we get:

$$\Bigg(\int_{\mathbb{R}^n}\mathcal{F}(x)\mathcal{G}(x)dx\Bigg)\Bigg(\int_{\mathbb{R}^n}\Bigg(\int_{\mathbb{R}^n\setminus\mathbb{B}(x)}\frac{\mathcal{G}(t)}{|\mathbb{B}(t)|}dt\Bigg)^p\mathcal{V}(x)dx\Bigg)^{-1/p}$$
$$= \frac{1}{n}\Bigg(\int_0^\infty\hat{\mathcal{F}}(y)\mathcal{G}(y)dy\Bigg)\Bigg(\int_0^\infty\Bigg(\int_x^\infty\frac{\mathcal{G}(t)}{t}dt\Bigg)^p\tilde{\mathcal{V}}(x)dx\Bigg)^{-1/p},$$

where $\hat{\mathcal{F}}(y) = y^{n-1}\int_{\sum_{n-1}}\hat{\mathcal{F}}(y\sigma)d\sigma$ and $\hat{v}(y) = y^{n-1}\int_{\sum_{n-1}}\hat{v}(y\sigma)d\sigma$.

and hence, by using the one-dimensional result (3) and using the fact $|B(x)| = \frac{|x|^n|\sum_{n-1}|}{n}$, we find that

$$\begin{aligned}
\mathbb{I}_L^{\downarrow}(\hat{\mathcal{F}}) &= \frac{1}{n}\Bigg(\int_0^\infty\Bigg(\int_0^x\hat{\mathcal{F}}(t)dt\Bigg)^{p'}\hat{\mathcal{W}}^{-p'}(x)\mathcal{W}_1(x)dx\Bigg)^{1/p'}\\
&= \frac{1}{n}\Bigg(\int_0^\infty\Bigg(\int_0^x y^{n-1}\int_{\sum_{n-1}}\hat{\mathcal{F}}(y\sigma)d\sigma dt\Bigg)^{p'}\\
&\times\Bigg(\int_0^x t^{-p}\Bigg(\int_0^t y^{n-1}\int_{\sum_{n-1}}\hat{\mathcal{V}}(y\sigma)d\sigma dy\Bigg)^p\Bigg(t^{n-1}\int_{\sum_{n-1}}\hat{\mathcal{V}}(t\sigma)d\sigma\Bigg)^{1-p}dt\Bigg)^{-p'}\\
&\times x^{-p}\Bigg(\int_0^x y^{n-1}\int_{\sum_{n-1}}\hat{\mathcal{V}}(y\sigma)d\sigma dy\Bigg)^p\Bigg(x^{n-1}\int_{\sum_{n-1}}\hat{\mathcal{V}}(x\sigma)d\sigma\Bigg)^{1-p}dx\Bigg)^{1/p'}\\
&= \Bigg(\int_0^\infty x^{n-1}\int_{\sum_{n-1}}\Bigg(\int_0^x y^{n-1}\int_{\sum_{n-1}}\hat{\mathcal{F}}(y\sigma)d\sigma dt\Bigg)^{p'}\\
&\times\Bigg(\int_0^x t^{n-1}\int_{\sum_{n-1}}\Bigg(\int_0^t y^{n-1}\int_{\sum_{n-1}}\hat{\mathcal{V}}(y\sigma)d\sigma dy\Bigg)^p|\mathbb{B}(t)|^{-p}\hat{\mathcal{V}}^{1-p}(t\sigma)d\sigma dt\Bigg)^{-p'}\\
&\times\Bigg(\int_0^x y^{n-1}\int_{\sum_{n-1}}\hat{\mathcal{V}}(y\sigma)d\sigma dy\Bigg)^p|\mathbb{B}(x)|^{-p}\hat{\mathcal{V}}^{1-p}(x\tau)d\tau dx\Bigg)^{1/p'}\\
&= \Bigg(\int_{\mathbb{R}^n}\Bigg(\int_{\mathbb{B}(x)}\mathcal{F}(t)dt\Bigg)^p\mathcal{W}^{-p'}(x)\tilde{\mathcal{W}}(x)dx\Bigg)^{1/p'}
\end{aligned} \tag{18}$$

and

$$\mathbb{I}_R^{\downarrow}(\hat{\mathcal{F}}) = p\Bigg[\bigg(\int_{\mathbb{R}^n} x^{-p}\tilde{\mathcal{V}}^p(x)\mathcal{V}^{1-p}(x)\Big(\int_{\mathbb{R}^n\setminus\mathbb{B}(x)} \mathcal{F}\tilde{\mathcal{W}}_0^{-1}\Big)^{p'}(x)dx\bigg)^{1/p'}$$
$$+\bigg(\int_{\mathbb{R}^n}\Big(x^p\tilde{\gamma}(x)\tilde{\mathcal{V}}^{-1}(x)\Big)^{p'}\Big(\int_{\mathbb{R}^n\setminus\mathbb{B}(x)} \mathcal{F}\tilde{\mathcal{W}}_0^{-1}\Big)^{p'}\mathcal{V}(x)dx\bigg)^{1/p'}$$
$$+p\bigg(\int_{\mathbb{R}^n} x^p\Big(\int_{\mathbb{R}^n\setminus\mathbb{B}(x)} \mathcal{F}\tilde{\mathcal{W}}_0^{-1}\Big)^{p'}\tilde{\gamma}^{p'}(x)\mathcal{V}(x)dx\bigg)^{1/p'}\Bigg]. \tag{19}$$

Now combining (18) and (19) we will have the upper and lower bound. □

Using the similar arguments as in the previous theorem, the following theorem can be proved which is the n-dimensional analogue of Theorem 2.

Theorem 9. *Let $1 < p < \infty$, $\mathcal{F} \geqslant 0$ and $\mathcal{V}$ be a positive, measurable and locally integrable function on $\mathbb{R}^n$ and let $\mathbb{I}^{\downarrow}(\mathcal{F})$ be defined by*

$$\mathbb{I}^{\downarrow}(\mathcal{F}) = \sup_{0\leqslant\mathcal{G}\downarrow r} \frac{\int_{\mathbb{R}^n}\mathcal{F}(x)\mathcal{G}(x)dx}{\Big(\int_{\mathbb{R}^n}(\frac{1}{|\mathbb{B}(x)|}\int_{\mathbb{B}(x)}\mathcal{G}(t)dt)^p(x)\mathcal{V}(x)dx\Big)^{1/p}}.$$

Then

$$\mathbb{I}_L^{\downarrow}(\mathcal{F}) \ll \mathbb{I}^{\downarrow}(\mathcal{F}) \ll \mathbb{I}_R^{\downarrow}(\mathcal{F}),$$

where

$$\mathbb{I}^{\downarrow}(\mathcal{F}) \approx \|v\|_1^{-1/p}\|\mathcal{F}\|_1$$
$$+\bigg(\int_{\mathbb{R}^n}\Big(\int_{\mathbb{B}(x)}\mathcal{F}(t)dt\Big)^{p'}\Big(\int_{\mathbb{B}(x)}\mathcal{V}(t)dt+|\mathbb{B}(x)|^p\int_{\mathbb{R}^n\setminus\mathbb{B}(x)}|\mathbb{B}(t)|^{-p}\mathcal{V}(t)dt\Big)^{-p'}\mathcal{V}(x)dx\bigg)^{1/p'}$$

and

$$\mathbb{I}^{\downarrow}(\mathcal{F}) \approx \|v\|_1^{-1/p}\|\mathcal{F}\|_1$$
$$+\bigg(\int_{\mathbb{R}^n}\Big(\int_{\mathbb{B}(x)}\mathcal{F}(t)dt\Big)^{p'}\Big(\int_{\mathbb{B}(x)}\mathcal{V}(t)dt+|\mathbb{B}(x)|^p\int_{\mathbb{R}^n\setminus\mathbb{B}(x)}|\mathbb{B}(t)|^{-p}\mathcal{V}(t)dt\Big)^{-p'}$$
$$\times|\mathbb{B}(x)|^{p-1}\int_{\mathbb{R}^n\setminus\mathbb{B}(x)}|\mathbb{B}(t)|^{-p}\mathcal{V}(t)dtdx\bigg)^{1/p'}.$$

Remark 3. *Writing $x = t\sigma$ and making change of variable $t = 1/s$, we get that*

$$\begin{aligned}\int_{\mathbb{R}^n}\mathcal{F}(x)\mathcal{G}(x)dx &= \int_0^\infty\int_{\sum_{n-1}} t^{n-1}\mathcal{F}(t\sigma)g(t\sigma)dtd\sigma\\ &= \int_0^\infty\int_{\sum_{n-1}} s^{n-1}\tilde{\mathcal{F}}(s\sigma)\tilde{\mathcal{G}}(s\sigma)dsd\sigma\\ &= \int_{\mathbb{R}^n}\tilde{\mathcal{F}}(y)\tilde{\mathcal{G}}(y)dy,\end{aligned}$$

where $\tilde{\mathcal{F}}(s\sigma) = \frac{1}{s^{2n-1}}\mathcal{F}(\frac{\sigma}{s})$ and $\tilde{\mathcal{G}}(s\sigma) = \frac{1}{s^{2n-1}}\mathcal{G}(\frac{\sigma}{s})$. Clearly, if $\mathcal{F}\uparrow r$ then $\tilde{\mathcal{F}}\downarrow r$ and similarly, $\mathcal{G}$ and $\tilde{\mathcal{G}}$ are related.

Remark 4. *In view of Remark 3, the n-dimensional analogue of Theorem 2 can be formulated and proved. We skip the details for conciseness.*

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