

Article

Sharp Choice Number Thresholds for Complete Bipartite Graphs

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Abstract

Fix $m \geq 3$. The choice number $\text{ch}(K_{m,n})$ of the complete bipartite graph $K_{m,n}$ has two sharp thresholds as n grows. We give complete proofs of the Hoffman–Johnson values at levels $m + 1$ and m , and we pin down the extremal list assignments at the lower threshold

$$n_0 = (m - 1)^{m-1} - (m - 2)^{m-1}. \text{ Specifically, } \text{ch}(K_{m,n}) = \begin{cases} m + 1, & n \geq m^m, \\ m, & n_0 \leq n < m^m, \\ \leq m - 1, & n < n_0. \end{cases}$$

Our method centers on a transversal obstruction principle and a dichotomy for how the M -side lists can intersect when all lists have size $m - 1$: Case I, in which some $m - 1$ of the M -lists are pairwise disjoint, and Case II, in which three M -lists pairwise intersect with all remaining lists mutually disjoint. For $m \geq 5$ we show that the three-way intersection pattern (three pairwise intersecting M -lists) is strictly non-extremal, and we prove the uniqueness of extremal configurations: we classify all uniformly critical assignments at n_0 and show that, up to relabeling, there are exactly two extremal types for $m \geq 5$, while a third type appears for $m \in \{3, 4\}$. Finally, we propose a fixed- k block model for deeper levels $\text{ch}(K_{m,n}) = m - k + 1$ and contrast this unbalanced setting with the balanced case $m = n$, where $\text{ch}(K_{m,m}) \sim \log_2 m$, highlighting the shift from polynomial to logarithmic threshold growth.

Keywords: list coloring; choice number; complete bipartite graphs; transversals; extremal configurations; threshold phenomena

MSC: 05C15; 05C35; 05D15



Academic Editor: Cristina Flaut

Received: 17 February 2026

Revised: 17 March 2026

Accepted: 24 March 2026

Published: 27 March 2026

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1. Introduction

List coloring, introduced independently by Vizing [1] and by Erdős, Rubin, and Taylor [2], assigns to each vertex a prescribed set of admissible colors. Given a list assignment L , an L -coloring of a graph G is a proper coloring ϕ with $\phi(v) \in L(v)$ for every $v \in V(G)$. The *choice number* (or list chromatic number) of G , written $\text{ch}(G)$ throughout (Hoffman and Johnson [3] use $c(G)$), is the least integer k such that every assignment with $|L(v)| \geq k$ admits an L -coloring. A list assignment L is called *uniformly critical* if $|L(v)| = \text{ch}(G) - 1$ for all $v \in V(G)$ and G admits no proper L -coloring; the word “critical” refers exclusively to this uniform notion throughout the paper.

Complete bipartite graphs provide a natural test case. Although $\chi(K_{m,n}) = 2$, the choice number $\text{ch}(K_{m,n})$ grows without bound as $n \rightarrow \infty$ for fixed m . Erdős, Rubin, and Taylor [2] initiated the systematic study of this behavior. Hoffman and Johnson [3] identified the first two threshold ranges, proving that $\text{ch}(K_{m,n}) = m + 1$ if and only if $n \geq m^m$, and, for $m \geq 3$, that $\text{ch}(K_{m,n}) = m$ precisely when $(m - 1)^{m-1} - (m - 2)^{m-1} \leq n < m^m$, with $\text{ch}(K_{m,n}) \leq m - 1$ below that interval. They also described extremal list assignments at these thresholds, though several structural steps were only outlined. Further results on choosability of bipartite graphs appear in Hanson, MacGillivray, and Toft [4], where degree conditions yield general bounds; for the choice number of random bipartite graphs see [5,6].

Algebraic methods, notably those of Alon and Tarsi [7] and Alon’s Combinatorial Nullstellensatz [8], have proved powerful in list coloring through graph polynomial techniques. Galvin’s resolution [9] of the Dinitz conjecture, showing that the list chromatic index of a bipartite multigraph equals its chromatic index, highlights the strength of these tools in bipartite settings. Such algebraic approaches are effective in balanced or degree-regular settings, but they do not directly yield the combinatorial rigidity (in particular, the uniqueness of extremal assignments) that we establish here; for that purpose, explicit transversal enumeration is essential.

Hoffman and Johnson [3] established the threshold values m^m and $n_0 = (m - 1)^{m-1} - (m - 2)^{m-1}$ and outlined extremal constructions, but left several structural steps implicit. The present paper supplies rigorous proofs of all intermediate claims, namely Corollary 1 (transversal obstruction principle), Lemma 1 (minimum-cover bound), and the full AM–GM optimization in Proposition 1. Beyond completing those proofs, we contribute three new results. We prove that Case (II) configurations are strictly non-extremal for $m \geq 5$ (Lemma 6), which yields a complete classification of critical isomorphism types at n_0 into exactly two types for $m \geq 5$, with one additional type for $m \in \{3, 4\}$ (Theorem 3). We also introduce the fixed- k block construction (Construction 1, novel to this paper) for deeper levels $\text{ch}(K_{m,n}) = m - k + 1$, derive the necessary bound $n \geq N(\ell, k)$ (Theorem 4), and establish an asymptotic lower bound under structural hypotheses (Theorem 5). The transversal obstruction framework of Corollary 1, while implicit in the covering arguments of [2,3], has not previously appeared in this explicit form.

The setting also connects to applied combinatorics. In wireless frequency assignment, vertices on the M -side represent base stations with fixed channel constraints, vertices on the N -side represent mobile users, and lists encode locally available frequencies; the choice number gives the worst-case number of frequencies needed to guarantee conflict-free assignment regardless of local availability [10]. In scheduling under resource constraints, the two sides model jobs and machines with list sizes reflecting machine-specific feasibility windows, and timetabling feasibility reduces to graph colorability [11]. In index coding for broadcast channels, complete bipartite subgraphs capture interference patterns, and the choice number controls the minimum codeword length needed for universal decodability [12]. In each setting the unbalanced case (fixed m , large n) corresponds to a small number of constrained resources serving a large and variable demand, making the polynomial threshold behavior of $\text{ch}(K_{m,n})$ directly relevant.

Throughout, (M, N) denotes the bipartition of $K_{m,n}$ with $|M| = m$ and $|N| = n$. When $M = \{v_1, \dots, v_m\}$ we write $L_i := L(v_i)$. In Sections 3–7 we set $a := m - 1$, and in the fixed- k setting we put $\ell := m - k$. Several arguments depend only on uniform list size and non-colorability; these hypotheses are stated explicitly where invoked. We use the arithmetic–geometric mean inequality (AM–GM): for nonnegative reals $x_1, \dots, x_n$, $\frac{x_1 + \dots + x_n}{n} \geq (x_1 \cdots x_n)^{1/n}$, with equality if and only if $x_1 = \dots = x_n$ [13]. In particular, among positive reals with fixed sum, the product is maximized when all terms are equal.

The remainder of the paper is organized as follows. Section 2 introduces the transversal obstruction framework and the minimum-cover lemma. Section 3 establishes the upper threshold and derives the structural dichotomy for non-colorable assignments at list size $m - 1$. Section 4 analyzes Case I and completes the lower bound. Section 5 analyzes Case II. Section 6 proves Case II is non-extremal for $m \geq 5$. Section 7 handles $m \in \{3, 4\}$ and verifies Example 1. Section 8 assembles the main theorems, classifies critical assignments at n_0 , and develops the fixed- k block model. Section 9 treats the balanced case $m = n$, while Section 10 presents some open problems.

2. Transversal Obstructions

The threshold behavior of $\text{ch}(K_{m,n})$ admits a direct reformulation in terms of extremal set systems. We make this correspondence explicit via choice sets and extract the covering principle that drives the later extremal bounds.

Definition 1. Let $\mathcal{F} = \{S_1, \dots, S_t\}$ be a family of sets. A choice function is a map $f : [t] \rightarrow \bigcup_{i=1}^t S_i$ with $f(i) \in S_i$ for all i . Its choice set is $T = \{f(1), \dots, f(t)\}$; thus T meets every S_i and $|T| \leq t$. A transversal is a choice set of size exactly t , equivalently one arising from an injective choice function. In contrast to a transversal, a choice set need not select distinct elements: if $S_i \cap S_j \neq \emptyset$ for some $i \neq j$, the choice function may assign both i and j to a common element, yielding $|T| < t$. This distinction is critical in what follows, since the covering arguments apply precisely to choice sets that may be strictly smaller than transversals.

Remark 1. The existence of a transversal is equivalent to Hall's condition [14]: $(S_1, \dots, S_t)$ admits a transversal if and only if $|\bigcup_{i \in I} S_i| \geq |I|$ for every $I \subseteq [t]$. In what follows, transversals enter through the covering argument of Lemma 1, rather than via Hall's theorem directly.

We now connect this framework to list coloring. Let (M, N) be the bipartition of $K_{m,n}$. Given a proper L -coloring ϕ , the set $T = \{\phi(v) : v \in M\}$ is a choice set of $\{L(v) : v \in M\}$. Properness is equivalent to $\phi(u) \notin T$ for every $u \in N$, that is, $L(u) \not\subseteq T$. Conversely, if a choice set T satisfies $L(u) \not\subseteq T$ for all $u \in N$, then selecting colors from $T \cap L(v)$ for $v \in M$ and from $L(u) \setminus T$ for $u \in N$ produces a proper L -coloring. This yields the following characterization.

Corollary 1 (Transversal obstruction). A list assignment L on $K_{m,n}$ admits no proper L -coloring if and only if for every choice set T of $\{L(v) : v \in M\}$ there exists $u \in N$ with $L(u) \subseteq T$.

Proof. ($\Rightarrow$) Suppose L is non-colorable and that some choice set T satisfies $L(u) \not\subseteq T$ for all $u \in N$. Let f be a choice function on M with image T . For each $u \in N$, choose $\phi(u) \in L(u) \setminus T$, nonempty by assumption, and set $\phi(v) := f(v)$ for $v \in M$. Then ϕ is a proper L -coloring, since $\phi(M) = T$ and $\phi(u) \notin T$ for all $u \in N$, a contradiction.

($\Leftarrow$) Conversely, assume every choice set T contains some N -list. If ϕ were a proper L -coloring, then $T := \{\phi(v) : v \in M\}$ would be a choice set. By hypothesis there exists $u \in N$ with $L(u) \subseteq T$, so $\phi(u) \in L(u) \subseteq T = \phi(M)$, contradicting properness. $\square$

Lower bounds on n therefore reduce to determining how many $(m - 1)$ -subsets are required to meet all m -transversals arising from disjoint parts. The extremal configuration follows from the double-counting argument below.

Lemma 1 (Minimum cover). Let $S_1, \dots, S_m$ be finite pairwise disjoint sets with $|S_1| \leq \dots \leq |S_m|$ and $m \geq 2$. If $\mathcal{C}$ is a family of $(m - 1)$ -subsets of $\bigcup_{i=1}^m S_i$ such that every m -transversal of $(S_1, \dots, S_m)$ contains some $C \in \mathcal{C}$, then $|\mathcal{C}| \geq \prod_{i=1}^{m-1} |S_i|$.

Equality is attained by the family of $(m - 1)$ -transversals of $(S_1, \dots, S_{m-1})$, and if $|S_{m-1}| < |S_m|$ this family is the unique extremal configuration.

Proof. Let $\mathcal{T}$ denote the m -transversals; then $|\mathcal{T}| = \prod_{i=1}^m |S_i|$. For $T \in \mathcal{T}$ and any $(m - 1)$ -subset $C \subseteq T$, disjointness implies that C meets exactly $m - 1$ parts and misses a unique part $S_{j(C)}$. Hence $|\{T \in \mathcal{T} : C \subseteq T\}| = |S_{j(C)}| \leq |S_m|$.

Double-counting pairs (C, T) with $C \subseteq T$ gives $|\mathcal{T}| \leq \sum_{C \in \mathcal{C}} |\{T \in \mathcal{T} : C \subseteq T\}| \leq |C| |S_m|$, and therefore $|\mathcal{C}| \geq \frac{|\mathcal{T}|}{|S_m|} = \prod_{i=1}^{m-1} |S_i|$.

Let $\mathcal{T}'$ be the $(m - 1)$ -transversals of $(S_1, \dots, S_{m-1})$. Then $|\mathcal{T}'| = \prod_{i=1}^{m-1} |S_i|$, and each element of $\mathcal{T}'$ extends to exactly $|S_m|$ elements of $\mathcal{T}$; these extensions partition $\mathcal{T}$, so equality is achieved.

For uniqueness when $|S_{m-1}| < |S_m|$, suppose $|\mathcal{C}| = \prod_{i=1}^{m-1} |S_i|$. Then equality holds in both inequalities above. Thus every $T \in \mathcal{T}$ contains exactly one $C \in \mathcal{C}$, and for each $C \in \mathcal{C}$ we have $|\{T \in \mathcal{T} : C \subseteq T\}| = |S_m|$. Since $|\{T : C \subseteq T\}| = |S_{j(C)}| \leq |S_m|$ with equality only when $j(C) = m$, each C must omit S_m . Hence $C \subseteq \bigcup_{i=1}^{m-1} S_i$ and, because the parts are disjoint and $|C| = m - 1$, C contains exactly one element from each of $S_1, \dots, S_{m-1}$. Therefore $\mathcal{C}$ is precisely the family of $(m - 1)$ -transversals of $(S_1, \dots, S_{m-1})$. $\square$

Remark 2. Lemma 1 is a new derivation tailored to the bipartite list-coloring context. It can be viewed as a Turán-type statement for covering families: among all covering families of prescribed size, the $(m - 1)$ -transversals of the $m - 1$ smallest parts are uniquely optimal whenever the largest part is strictly larger than the second-largest. The pairwise-disjointness hypothesis is essential; when the sets S_i overlap, a common element can belong to multiple $(m - 1)$ -subsets, and the double-counting argument no longer applies directly, and this is precisely the source of the added complexity in Case II.

3. Extremal Structure at the First Two Thresholds

We determine the first two threshold values of $\text{ch}(K_{m,n})$ and identify the rigidity forced at each level. Throughout, (M, N) denotes the bipartition with $|M| = m$, and we write $L_i := L(v_i)$ for $v_i \in M$. The key input is Corollary 1, which translates non-colorability into a covering condition on M -side choice sets.

Theorem 1. For all integers $m, n \geq 1$, $\text{ch}(K_{m,n}) = m + 1 \iff n \geq m^m$.

Proof. Let $|M| = m$. Any choice set T of $\{L(v) : v \in M\}$ satisfies $|T| \leq m$. If $|L(x)| \geq m + 1$ for every vertex x , then for each $u \in N$ we have $|L(u)| > |T|$, hence $L(u) \not\subseteq T$. By Corollary 1, every such assignment is colorable, so $\text{ch}(K_{m,n}) \leq m + 1$.

For sharpness, take pairwise disjoint m -element lists $L_1, \dots, L_m$ on M . The number of transversals is m^m . Assign these transversals bijectively as the N -lists. Every choice set is then one of these transversals and hence equals an N -list; by Corollary 1, the assignment is non-colorable. Thus $\text{ch}(K_{m,m^m}) \geq m + 1$.

Conversely, assume $n < m^m$ and $|L(x)| \geq m$ for all vertices. Choose m -subsets $L'(x) \subseteq L(x)$. If L' were non-colorable, then by Corollary 1 every choice set T of $\{L'(v) : v \in M\}$ would contain some N -list. Since $|L'(u)| = m$ and $|T| \leq m$, this forces $|T| = m$ and $L'(u) = T$ for some $u \in N$.

If $L'_i \cap L'_j \neq \emptyset$ for some $i \neq j$, choosing a common element in these two lists and arbitrary elements elsewhere yields a choice set of size at most $m - 1$, contradicting $|T| = m$.

Hence the M -lists are pairwise disjoint. They therefore produce exactly m^m distinct transversals, each equal to an N -list, so $n \geq m^m$, contradicting the assumption. Thus $\text{ch}(K_{m,n}) \leq m$ whenever $n < m^m$. $\square$

The proof shows that non-colorability at list size m forces the M -lists to be pairwise disjoint. We now descend one level.

Assume $|L(x)| = m - 1$ for all vertices and that L is non-colorable. By Corollary 1, every choice set T of $\{L(v) : v \in M\}$ satisfies $|T| \geq m - 1$. This yields two immediate constraints:

- (i) No color lies in three distinct M -lists, for otherwise selecting that color in the three lists and arbitrary elements elsewhere produces $|T| \leq m - 2$.
- (ii) The intersection graph $\mathcal{I}$ on $\{L_1, \dots, L_m\}$ has matching number at most 1: two disjoint intersecting pairs would again give a choice set of size at most $m - 2$.

These restrictions lead to a complete structural dichotomy.

Lemma 2 (Structural dichotomy). *Let $|L(x)| = m - 1$ for all vertices of $K_{m,n}$ and suppose L is non-colorable. Then exactly one of the following holds:*

- (I) *Some $m - 1$ lists among $L_1, \dots, L_m$ are pairwise disjoint;*
- (II) *Three lists L_1, L_2, L_3 pairwise intersect, and every other pair of lists is disjoint; in particular, $L_4, \dots, L_m$ are pairwise disjoint and disjoint from $L_1 \cup L_2 \cup L_3$.*

Proof. Let $\mathcal{I}$ be the intersection graph on $\{L_1, \dots, L_m\}$. Since $\nu(\mathcal{I}) \leq 1$:

If $|E(\mathcal{I})| \leq 1$, removing one endpoint of the (possible) edge leaves $m - 1$ pairwise disjoint lists, giving Case (I).

If $|E(\mathcal{I})| \geq 2$, then any two edges share a vertex. Relabel so that $L_1 \cap L_2 \neq \emptyset$ and $L_1 \cap L_3 \neq \emptyset$. Then $L_3, \dots, L_m$ are pairwise disjoint, and likewise $L_2, L_4, \dots, L_m$ are pairwise disjoint; hence $L_4, \dots, L_m$ are pairwise disjoint.

If $L_2 \cap L_3 = \emptyset$, then $L_2, \dots, L_m$ are pairwise disjoint, so Case (I) holds. Otherwise L_1, L_2, L_3 pairwise intersect. Condition (i) implies that the intersections $L_1 \cap L_2$, $L_1 \cap L_3$, $L_2 \cap L_3$ are pairwise disjoint.

If some L_j with $j \geq 4$ meets $L_1 \cup L_2 \cup L_3$, then together with one intersecting pair among L_1, L_2, L_3 it forms two disjoint edges in $\mathcal{I}$, contradicting $\nu(\mathcal{I}) \leq 1$. Hence L_j is disjoint from $L_1 \cup L_2 \cup L_3$ for $j \geq 4$, establishing Case (II).

The two cases are mutually exclusive: in Case (II) every subfamily of $m - 1$ lists contains at least two of L_1, L_2, L_3 , which intersect. $\square$

Remark 3 (Forbidden subgraphs of $\mathcal{I}$). *The matching constraint $\nu(\mathcal{I}) \leq 1$ is equivalent to forbidding $2K_2$ (two vertex-disjoint edges) as a subgraph of $\mathcal{I}$. A graph has $\nu \leq 1$ if and only if every two edges share a vertex, which forces $\mathcal{I}$ to be a subgraph of either a star $K_{1,r}$ or a triangle K_3 . Case (I) arises when $\mathcal{I}$ is a subgraph of a star (at most one list meets all others in a pairwise-intersecting pair), so $m - 1$ lists are pairwise disjoint after removing at most one vertex. Case (II) arises when $\mathcal{I}$ contains K_3 as a component, with condition (i) additionally forcing the three pairwise intersections to be color-disjoint. The two cases are mutually exclusive since K_3 is not a subgraph of any star.*

When $m - 1$ lists are pairwise disjoint, every $(m - 1)$ -choice set is a transversal, allowing direct extremal enumeration.

4. Case I Analysis

Throughout this section we assume Case (I) of Lemma 2: the lists $L_2, \dots, L_m$ are pairwise disjoint. All M -lists have common size $a := m - 1$, and we distinguish L_1 . For $j = 2, \dots, m$ set $S_j := L_1 \cap L_j$, $k_j := |S_j|$, and $K := \sum_{j=2}^m k_j \leq a$.

Lemma 3 (AM–GM product bound). *Let $r \geq 1$ and let $u_1, \dots, u_r$ be nonnegative reals with $u_1 + \dots + u_r = C$. Then $\prod_{i=1}^r u_i \leq (C/r)^r$, with equality if and only if $u_1 = \dots = u_r = C/r$.*

Proof. This is AM–GM applied to $(u_1, \dots, u_r)$ [13]. $\square$

Proposition 1. *In Case (I), any list assignment L on $K_{m,n}$ with $|L(v)| = a = m - 1$ and no proper L -coloring satisfies*

$$n \geq a^a - (a - 1)^a = (m - 1)^{m-1} - (m - 2)^{m-1}.$$

Equality holds if and only if, up to relabeling, either $k_2 = \dots = k_m = 1$, or $k_2 = \dots = k_{m-1} = 1$ and $k_m = 0$. In both cases the N -lists are precisely the $(m - 1)$ -transversals of $L_2, \dots, L_m$ that intersect L_1 .

Proof. Since $L_2, \dots, L_m$ are pairwise disjoint, they admit $a^{m-1} = a^a (m - 1)$ -transversals. A transversal avoids L_1 precisely when it avoids each S_j , so the number avoiding L_1 equals $P := \prod_{j=2}^m (a - k_j)$. Hence $a^a - P$ transversals meet L_1 . Each such transversal T has $|T| = a$ and meets every M -list, so by Corollary 1 it must occur as an N -list. Thus $n \geq a^a - P$.

Case 1: $K < a$. Set $u_0 := a - K$ and $u_j := a - k_j$ for $2 \leq j \leq m$. Let $A_1 := L_1 \setminus \bigcup_{j=2}^m S_j$ and $A_j := L_j \setminus S_j$. Then $|A_1| = u_0$, $|A_j| = u_j$, and the sets $A_1, A_2, \dots, A_{m-1}, L_m$ are pairwise disjoint.

Consider the m -transversals of $(A_1, A_2, \dots, A_{m-1}, L_m)$. Each avoids every S_j , hence is disjoint from the previously counted $(m - 1)$ -transversals meeting L_1 . By Lemma 1, any family of $(m - 1)$ -subsets covering these transversals has size at least $Q := \prod_{i=1}^{m-1} \sigma_i$, where $\sigma_1 \leq \dots \leq \sigma_{m-1}$ are the $m - 1$ smallest values among $\{u_0, u_2, \dots, u_m\}$. Since Q is the product of the $m - 1$ smallest among m numbers, $Q = u_0 \prod_{j=2}^m u_j / \max\{u_0, u_2, \dots, u_m\}$. Each such covering set must appear as an N -list, so $n \geq a^a - P + Q$.

Subcase 1a: $K \leq a - 2$. Then $u_0 = a - K \geq 2$. If all $k_j \geq 1$, then $K \geq a$, contradicting $K \leq a - 2$; hence some $k_j = 0$, so some $u_j = a$. Let $u_{j^*} := \max\{u_0, u_2, \dots, u_m\}$. Since $u_{j^*} \leq a$, $Q = (u_0/u_{j^*})P \geq ((a - K)/a)P$, so $n \geq a^a - (K/a)P$.

As $\sum_{j=2}^m u_j = a^2 - K$, Lemma 3 with $r = a$ and $C = a^2 - K$ yields $P \leq (a - K/a)^a$. Write $\psi(K) := (K/a)(a - K/a)^a$. Then $n \geq a^a - \psi(K)$. A direct computation gives

$$\psi'(K) = \frac{1}{a} \left(a - \frac{K}{a} \right)^{a-1} \left(a - \frac{K(a+1)}{a} \right).$$

For $K \in [0, a - 2]$, the factor $a - K(a + 1)/a$ equals $(a^2 - K(a + 1))/a$; at $K = a - 2$ this is $(a + 2)/a > 0$, so $\psi'(K) > 0$ throughout $[0, a - 2]$. Hence $\psi(K) \leq \psi(a - 2)$. Now

$$\psi(a - 2) = (a - 1)^a \frac{a - 2}{a} \left(1 + \frac{2}{a(a - 1)} \right)^a \leq (a - 1)^a \frac{a - 2}{a} \exp\left(\frac{2}{a - 1} \right).$$

Since $(a - 1)^a > 0$, it suffices to show $(a - 2)/a \cdot \exp(2/(a - 1)) < 1$. Define $h(a) := -\log(1 - 2/a) - 2/(a - 1)$. Then $h'(a) = 2(1/(a - 1)^2 - 1/(a(a - 2))) < 0$ for $a > 2$, so h is strictly decreasing and $\lim_{a \rightarrow \infty} h(a) = 0$. Thus $h(a) > 0$ for all $a > 2$, and in particular for integers $a \geq 4$. Hence $\psi(a - 2) < (a - 1)^a$ and $n > a^a - (a - 1)^a$ whenever $K \leq a - 2$.

Subcase 1b: $K = a - 1$. Then $u_0 = 1$ and $\sum_{j=2}^m u_j = a^2 - a + 1$, so some $u_j = a$; relabel so $u_m = a$. Now $Q = \prod_{j=2}^{m-1} u_j = P/a$ and $n \geq a^a - P(1 - 1/a)$. Since $u_2 + \dots + u_{m-1} = (a - 1)^2$, Lemma 3 with $r = m - 2 = a - 1$ and $C = (a - 1)^2$ gives $\prod_{j=2}^{m-1} u_j \leq (a - 1)^{a-1}$, with equality iff $u_2 = \dots = u_{m-1} = a - 1$. Hence $P \leq a(a - 1)^{a-1}$ and $n \geq a^a - (a - 1)^a$, with equality exactly when $k_2 = \dots = k_{m-1} = 1$ and $k_m = 0$.

Case 2: $K = a$. Then $\sum_{j=2}^m u_j = a(a - 1)$, so Lemma 3 with $r = m - 1 = a$ and $C = a(a - 1)$ yields $P \leq (a - 1)^a$, with equality iff $k_2 = \dots = k_m = 1$. Thus $n \geq a^a - (a - 1)^a$, and equality holds precisely in this configuration. $\square$

Remark 4. In both extremal configurations exactly $m - 1$ of the M -lists share a uniform intersection pattern with L_1 . This symmetry forces equality in the AM–GM bounds and explains the rigidity at the threshold.

Lemma 4 (Existence at the lower threshold). Let $m \geq 3$ and $a = m - 1$. In each equality configuration of Proposition 1, define the N -lists to be exactly the $(m - 1)$ -transversals of $L_2, \dots, L_m$ that meet L_1 (and, in the second case, also the $(m - 1)$ -transversals of $A_1, \dots, A_{m-1}$). Then the resulting assignment on K_{m,n_0} with $n_0 = a^a - (a - 1)^a$ is non-colorable.

Proof. Let T be any choice set of $\{L(v) : v \in M\}$. If T is a transversal of $(L_2, \dots, L_m)$ meeting L_1 , then T is an N -list by construction.

If T avoids L_1 , then $T \subseteq \bigcup_{j=2}^m (L_j \setminus L_1)$. In the configuration $k_2 = \dots = k_m = 1$, we have $K = a$, so $A_1 = \emptyset$ and every transversal of $(L_2, \dots, L_m)$ meets L_1 ; this case cannot occur. In the configuration $k_2 = \dots = k_{m-1} = 1, k_m = 0$, the sets $A_1, \dots, A_{m-1}, L_m$ are pairwise disjoint, and any such T is exactly a transversal of $(A_1, \dots, A_{m-1})$; hence it is included among the N -lists.

Thus every choice set contains an N -list, and Corollary 1 implies non-colorability. $\square$

5. Case II Analysis

Assume Case (II) of Lemma 2. Thus L_1, L_2, L_3 pairwise intersect, while $L_4, \dots, L_m$ are pairwise disjoint and disjoint from $L_1 \cup L_2 \cup L_3$. Set $a := m - 1$ and define $S_{12} := L_1 \cap L_2$, $S_{13} := L_1 \cap L_3$, $S_{23} := L_2 \cap L_3$, with $x := |S_{12}|$, $y := |S_{13}|$, $z := |S_{23}|$. By the “no triple-color” restriction, S_{12}, S_{13}, S_{23} are pairwise disjoint.

Since $|L_i| = a$ for $i = 1, 2, 3$, we have $x, y, z \geq 1$ and $x + y \leq a$, $x + z \leq a$, $y + z \leq a$. Moreover, $|L_1 \setminus (S_{12} \cup S_{13})| = a - x - y$, $|L_2 \setminus (S_{12} \cup S_{23})| = a - x - z$, $|L_3 \setminus (S_{13} \cup S_{23})| = a - y - z$.

Lemma 5 (Case II lower bound). In Case (II), let L be a list assignment on $K_{m,n}$ with $|L(v)| = a = m - 1$ for all v and no proper L -coloring. Then

$$n \geq a^{m-3}(a(x + y + z) - (xy + xz + yz)) + a^{m-4}(a - x - y)(a - x - z)(a - y - z),$$

where the second term is omitted in the case $m = 3$.

Proof. Let T be an $(m - 1)$ -element choice set of $\{L(v) : v \in M\}$. Since $L_4, \dots, L_m$ are pairwise disjoint and disjoint from $L_1 \cup L_2 \cup L_3$, T contains exactly one element from each of $L_4, \dots, L_m$, contributing $m - 3$ elements, and exactly two elements from $L_1 \cup L_2 \cup L_3$.

We count the possible unordered pairs from $L_1 \cup L_2 \cup L_3$ that can occur in T . Such a pair must intersect each of L_1, L_2, L_3 . There are two types.

- (1) One element from each of two distinct intersections among S_{12}, S_{13}, S_{23} : $xy + xz + yz$ possibilities.

- (2) One element from some S_{ij} and one element from the portion of the third list disjoint from both intersections, summed cyclically: $x(a - y - z) + y(a - x - z) + z(a - x - y)$ possibilities.

Adding gives $a(x + y + z) - (xy + xz + yz)$ distinct unordered pairs. Each extends independently across $L_4, \dots, L_m$, contributing a factor a^{m-3} . Hence there are $a^{m-3}(a(x + y + z) - (xy + xz + yz))$ distinct $(m - 1)$ -choice sets, all of which must occur as N -lists by Corollary 1.

For $m \geq 4$, the disjoint sets $L_1 \setminus (S_{12} \cup S_{13}), L_2 \setminus (S_{12} \cup S_{23}), L_3 \setminus (S_{13} \cup S_{23}), L_4, \dots, L_m$ yield m -transversals avoiding all S_{ij} , disjoint from the previous family. By Lemma 1, covering them requires at least $(a - x - y)(a - x - z)(a - y - z) a^{m-4}$ additional $(m - 1)$ -subsets. Summing gives the bound. $\square$

6. Case II Cannot Be Extremal for $m \geq 5$

Lemma 6 (Optimization bound). *Let $m \geq 5$ and $a := m - 1 \geq 4$. For every feasible triple (x, y, z) arising in Case (II) (that is, $x, y, z \geq 1$ and $x + y, x + z, y + z \leq a$),*

$$a^{m-3}(a(x + y + z) - (xy + xz + yz)) + a^{m-4}(a - x - y)(a - x - z)(a - y - z) > a^a - (a - 1)^a.$$

Hence no Case (II) configuration attains $(m - 1)^{m-1} - (m - 2)^{m-1}$ when $m \geq 5$.

Proof. Write $u := a - x - y, v := a - x - z, w := a - y - z$. Feasibility gives $x, y, z \geq 1$ and $u, v, w \geq 0$. By Lemma 5, the left-hand side equals $a^{m-4}H(x, y, z)$, where $H = a(a(x + y + z) - (xy + xz + yz)) + uvw$.

Since $m = a + 1$, we have $a^{m-4} = a^{a-3}$ and $a^a - (a - 1)^a = a^{a-3}a^3(1 - (1 - 1/a)^a)$. Thus it suffices to prove

$$\frac{H(x, y, z)}{a^3} > 1 - \left(1 - \frac{1}{a}\right)^a.$$

Normalize $X := x/a, Y := y/a, Z := z/a$. Then $X, Y, Z \geq 1/a$ and $X + Y, X + Z, Y + Z \leq 1$. Let

$$\Delta := \{(X, Y, Z) \in \mathbb{R}_{\geq 0}^3 : X + Y \leq 1, X + Z \leq 1, Y + Z \leq 1\}.$$

A direct expansion gives $H(x, y, z)/a^3 = F(X, Y, Z)$, where

$$F = (X + Y + Z) - (XY + XZ + YZ) + (1 - X - Y)(1 - X - Z)(1 - Y - Z).$$

Since the discrete feasible triples $(x/a, y/a, z/a)$ lie in Δ , it suffices to minimize F over the continuous polytope Δ to obtain a lower bound for the discrete case. Since F is a polynomial, it is continuous on the compact polytope $\Delta \subset \mathbb{R}^3$, so F attains its minimum, either at an interior critical point (where $\nabla F = 0$) or on a boundary face; we locate both explicitly.

In the interior ($X + Y < 1, X + Z < 1, Y + Z < 1$), $\partial_X F = -(Y + Z - 1)(2X + Y + Z - 1)$ with cyclic permutations. Since $Y + Z - 1 \neq 0$ in the interior, $\nabla F = 0$ forces $2X + Y + Z = 1, X + 2Y + Z = 1, X + Y + 2Z = 1$, hence $X = Y = Z = \frac{1}{4}$. At this point $F(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}) = \frac{11}{16}$.

On the boundary, if $X + Y = 1$ then $F = 1 - X + X^2 \geq \frac{3}{4}$, with equality at $X = \frac{1}{2}$. The same bound holds when $X + Z = 1$ or $Y + Z = 1$.

If $Z = 0$, write $s := X + Y \in [0, 1]$. At fixed s , XY is maximized at $X = Y = s/2$, so $\min_{X+Y=s} F(X, Y, 0) = 1 - s + s^2 - s^3/4$. Differentiation gives a unique minimum at $s = 2/3$, where $F = \frac{19}{27} > \frac{11}{16}$. By symmetry the same holds on $X = 0$ and $Y = 0$. Hence the global minimum of F on Δ equals $\frac{11}{16}$, attained at $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$.

Finally, for $a \geq 4$ the function $(1 - 1/t)^t$ is increasing in t , so $(1 - 1/a)^a \geq (3/4)^4 = 81/256$. Therefore

$$1 - \left(1 - \frac{1}{a}\right)^a \leq \frac{175}{256} \approx 0.684 < \frac{11}{16} \approx 0.6875 \leq F(X, Y, Z) = \frac{H(x, y, z)}{a^3},$$

yielding the desired strict inequality. $\square$

Remark 5 (Base case for $m = 5$). For $m = 5$, $a = 4$, and $(x, y, z) = (1, 1, 1)$ (the minimizer within the feasible integers), Lemma 5 gives $n \geq 4^2(4 \cdot 3 - 3) + 4^1(4 - 2)^3 = 144 + 32 = 176 > 175 = n_0$, confirming that Case (II) exceeds n_0 already at the tightest integer point.

7. Small Values of m

The cases $m \in \{3, 4\}$ are not covered by Lemma 6 and must be handled directly. In both instances, the Case (II) lower bound from Lemma 5 coincides with the Case (I) threshold.

Proposition 2. For $m = 3$, Case (II) forces $(x, y, z) = (1, 1, 1)$ and yields $n = 3 = 2^2 - 1^2$. For $m = 4$, the minimum over all feasible integer triples is attained at $(x, y, z) = (1, 1, 1)$ with $n = 19 = 3^3 - 2^3$, while every other feasible triple satisfies $n \geq 21$. In both cases, the Case (II) bound equals the Case (I) threshold.

Proof. For $m = 3$, $a = 2$. The constraints $x, y, z \geq 1$ and $x + y, x + z, y + z \leq 2$ imply $x = y = z = 1$. Since Lemma 5 contributes only its first term when $m = 3$, we obtain $n \geq 2^0(2 \cdot 3 - 3) = 3 = 2^2 - 1^2$.

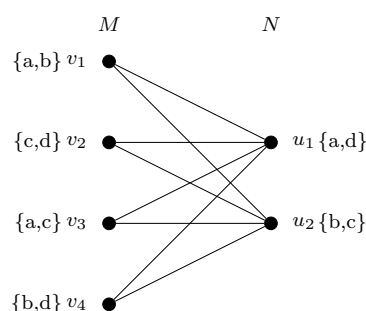
For $m = 4$, $a = 3$. The constraints allow only $(1, 1, 1)$ and permutations of $(2, 1, 1)$. For $(1, 1, 1)$: $n \geq 3^1(3 \cdot 3 - 3) + 3^0(3 - 1 - 1)^3 = 18 + 1 = 19 = 3^3 - 2^3$. For $(2, 1, 1)$: $x + y + z = 4$, $xy + xz + yz = 5$, so the first term alone yields $n \geq 3(3 \cdot 4 - 5) = 21$. $\square$

Remark 6 (Why $m \in \{3, 4\}$ are exceptions). For $m \in \{3, 4\}$ ($a \in \{2, 3\}$), the feasibility constraints $x + y, x + z, y + z \leq a$ force all intersection sizes to be at most 1 (when $a = 2$) or at most 2 (when $a = 3$), concentrating the feasible region near the interior minimizer of F . Once $a \geq 4$ (i.e., $m \geq 5$), the feasible region Δ is large enough and the normalized Case (I) threshold $1 - (1 - 1/a)^a$ small enough that the minimum $\frac{11}{16}$ of F on Δ strictly exceeds it for all $a \geq 4$.

The next example illustrates the threshold in the smallest case where $\text{ch}(K_{m,n}) > \chi(K_{m,n})$.

Example 1. We show that $\text{ch}(K_{4,2}) = 3$ by constructing a non-colorable 2-list assignment; see the graph of $K_{4,2}$ and its lists, below. Let (M, N) be the bipartition with $M = \{v_1, v_2, v_3, v_4\}$ and $N = \{u_1, u_2\}$, and define

$$\begin{aligned} L(v_1) &= \{a, b\}, & L(v_2) &= \{c, d\}, \\ L(v_3) &= \{a, c\}, & L(v_4) &= \{b, d\}, \\ L(u_1) &= \{a, d\}, & L(u_2) &= \{b, c\}. \end{aligned}$$



We verify non-colorability via Corollary 1. Since $L(v_1) \cap L(v_2) = \emptyset$, every choice set of $\{L(v_i) : 1 \leq i \leq 4\}$ contains at least one element from each of $\{a, b\}$ and $\{c, d\}$. A direct enumeration shows the distinct choice sets are $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$, $\{a, b, c, d\}$, $\{a, d\}$, and $\{b, c\}$. Each contains either $L(u_1) = \{a, d\}$ or $L(u_2) = \{b, c\}$. Thus every choice set contains an N -list, so the assignment admits no proper L -coloring. Hence $\text{ch}(K_{4,2}) \geq 3$. Since $2 < n_0 = 19$, Theorem 2 gives $\text{ch}(K_{4,2}) \leq 3$, and therefore $\text{ch}(K_{4,2}) = 3$.

8. Main Results

We now assemble the preceding analysis and determine the first two threshold ranges of $\text{ch}(K_{m,n})$ in the unbalanced setting. We also describe the rigidity of critical assignments at the lower threshold.

Theorem 2. Let $m \geq 3$ and set $n_0 := (m-1)^{m-1} - (m-2)^{m-1}$. Then

$$\text{ch}(K_{m,n}) = m \iff n_0 \leq n < m^m.$$

Proof. The proof has two parts. For the upper bound ($\text{ch}(K_{m,n}) \leq m$ when $n < m^m$) we use Theorem 1. For the existence of a non-colorable assignment at n_0 we combine Proposition 1 (lower bound in Case I) with Lemma 4 (explicit construction). The matching lower bound $n \geq n_0$ for any non-colorable assignment at list size $m-1$ follows from Lemma 2 (structural dichotomy), Proposition 1 (Case I bound), and Lemma 6/Proposition 2 (Case II bound for $m \geq 5$ and $m \in \{3, 4\}$ respectively).

By Theorem 1, $\text{ch}(K_{m,n}) \leq m$ for all $n < m^m$. Proposition 1 together with Lemma 4 produces a non-colorable assignment on K_{m,n_0} with $|L(v)| = m-1$ for all v ; hence $\text{ch}(K_{m,n_0}) \geq m$. Duplicating N -lists preserves non-colorability, so $\text{ch}(K_{m,n}) \geq m$ for every $n \geq n_0$. Thus $\text{ch}(K_{m,n}) = m$ whenever $n_0 \leq n < m^m$.

Conversely, suppose $\text{ch}(K_{m,n}) = m$. Then there exists a non-colorable assignment with uniform list size $m-1$. By Lemma 2 it lies in Case (I) or Case (II). In Case (I), Proposition 1 gives $n \geq n_0$. In Case (II), Lemma 6 (for $m \geq 5$) and Proposition 2 (for $m \in \{3, 4\}$) yield the same lower bound. Hence $n < n_0$ forces $\text{ch}(K_{m,n}) \leq m-1$. $\square$

Definition 2 (Isomorphism of list assignments). Two list assignments L and L' on $K_{m,n}$ are isomorphic if there exist bijections $\sigma : M \rightarrow M$, $\tau : N \rightarrow N$, and a bijection ρ on the ground color set such that $L'(\sigma(v)) = \rho(L(v))$ for every $v \in M$ and $L'(\tau(u)) = \rho(L(u))$ for every $u \in N$. The phrase “up to relabeling” in Theorems 2 and 3 refers to this isomorphism relation.

Definition 3 (Extremal types at n_0). Let $m \geq 5$ and $n_0 = (m-1)^{m-1} - (m-2)^{m-1}$. A critical list assignment on K_{m,n_0} with uniform list size $m-1$ is, up to relabeling of M and N , of one of the following forms.

Type A. For $2 \leq j \leq m$ we have $|L_1 \cap L_j| = 1$, and N consists precisely of the $(m-1)$ -transversals of $L_2, \dots, L_m$ that intersect L_1 .

Type B. For $2 \leq j \leq m-1$ we have $|L_1 \cap L_j| = 1$ and $L_1 \cap L_m = \emptyset$. Writing $S_j := L_1 \cap L_j$ ($2 \leq j \leq m-1$) and $A_1 := L_1 \setminus \bigcup_{j=2}^{m-1} S_j$, $A_j := L_j \setminus L_1$ ($2 \leq j \leq m-1$), the sets $A_1, \dots, A_{m-1}, L_m$ are pairwise disjoint with $|A_1| = 1$ and $|A_j| = m-2$ for $2 \leq j \leq m-1$. Moreover,

$$N = \{(m-1)\text{-transversals of } L_2, \dots, L_m \text{ meeting } L_1\} \cup \{(m-1)\text{-transversals of } A_1, \dots, A_{m-1}\}.$$

Theorem 3 (Critical assignments at n_0). Let $m \geq 5$ and $n_0 = (m-1)^{m-1} - (m-2)^{m-1}$. Every critical list assignment on K_{m,n_0} with list size $m-1$ belongs to Case (I) and is, up to relabeling

(in the sense of Definition 2), of Type A or Type B. For $m \in \{3, 4\}$, one additional configuration arises from Case (II).

Proof. For $m \geq 5$, Lemma 6 excludes Case (II) at n_0 . Hence every critical assignment lies in Case (I). Proposition 1 identifies the two extremal configurations, which are distinct up to relabeling. For $m \in \{3, 4\}$, Proposition 2 shows that Case (II) attains the same bound, producing one additional type. $\square$

For $r \geq 2$, let $n_r(m)$ denote the least integer n such that $\text{ch}(K_{m,n}) = r$ for fixed m . Since $\text{ch}(K_{m,n})$ is nondecreasing in n , the quantity $n_r(m)$ is well defined. In particular, $n_{m-k+1}(m)$ is the threshold at level $m - k + 1$. When the value is verified inside the block-disjoint model below, we write $n_{m-k+1}^{\text{block}}(m)$.

Beyond the first threshold we examine levels of the form $\text{ch}(K_{m,n}) = m - k + 1$ with $k \geq 2$. The guiding configuration is a block-disjoint model in which k distinguished lists select one marked element from each of $\ell = m - k$ disjoint blocks.

Construction 1 (k -special construction). Fix $k \geq 1$ and set $\ell = m - k$. Let $M = \{v_1, \dots, v_m\}$. Choose pairwise disjoint sets $B_{k+1}, \dots, B_m$ with $|B_j| = \ell$. For each $j \geq k + 1$, select distinct elements $b_{1,j}, \dots, b_{k,j} \in B_j$. Define

$$L(v_i) := \{b_{i,k+1}, \dots, b_{i,m}\} \quad (1 \leq i \leq k), \quad L(v_j) := B_j \quad (k+1 \leq j \leq m).$$

Then every M -list has size ℓ ; the blocks $B_{k+1}, \dots, B_m$ are disjoint; and each distinguished list $L(v_i)$ meets each block B_j in exactly one element.

In this model, the ℓ -element choice sets are precisely the block transversals. Among them, those meeting all k distinguished lists are the minimal obstructions.

Lemma 7 (Count of ℓ -choice sets). In Construction 1, the number of choice sets of $\{L(v) : v \in M\}$ of size exactly ℓ equals

$$N(\ell, k) = \sum_{t=0}^k (-1)^t \binom{k}{t} (\ell - t)^\ell. \quad (1)$$

Proof. A choice set of size ℓ selects one element from each block $B_{k+1}, \dots, B_m$. Such a set meets every M -list precisely when it meets each distinguished list. Fix $I \subseteq [k]$ with $|I| = t$. Avoiding all $L(v_i)$ for $i \in I$ removes exactly t marked elements from each block, leaving $(\ell - t)^\ell$ block transversals. Inclusion–exclusion over $I \subseteq [k]$ yields the formula. $\square$

Theorem 4 (Necessary condition for fixed k). Let $k \geq 1$ and set $\ell = m - k$. Assume the M -lists are given by Construction 1. If L is a list assignment on $K_{m,n}$ with $|L(v)| = \ell$ for all v and no proper L -coloring exists, then every ℓ -choice set counted in Lemma 7 occurs as an N -list. Consequently, $n \geq N(\ell, k)$. This bound applies within the block-disjoint structure of Construction 1; for general assignments outside this model, the lower bound on n may differ.

Proof. Let $\mathcal{T}$ be the family of ℓ -element choice sets of the M -lists. By Lemma 7, $|\mathcal{T}| = N(\ell, k)$. If $T \in \mathcal{T}$, then T meets every M -list and $|T| = \ell$. By Corollary 1, non-colorability implies there exists $u \in N$ with $L(u) \subseteq T$. Since $|L(u)| = \ell = |T|$, we have $L(u) = T$. Thus each $T \in \mathcal{T}$ must appear on N , giving $n \geq |\mathcal{T}|$. $\square$

Remark 7 (Sufficiency gap for $k \geq 2$). Theorem 4 provides a sharp bound when $k = 1$: every choice set contains an ℓ -choice set from $\mathcal{T}$, so taking $\mathcal{T}$ as the N -lists certifies non-colorability and recovers Proposition 1.

For $k \geq 2$, this set containment fails in general. A choice set may concentrate several elements inside a single block via different distinguished lists, obstructing extraction of a block transversal that meets all distinguished lists.

For instance, when $k = 2$ and $m = 5$, choose $f(v_1) = b_{1,3}$, $f(v_2) = b_{2,3}$, $f(v_3) = c$, $f(v_4) = d_4$, $f(v_5) = d_5$, where $c \in B_3 \setminus \{b_{1,3}, b_{2,3}\}$, $d_4 \in B_4$, $d_5 \in B_5$. Then $T = \{b_{1,3}, b_{2,3}, c, d_4, d_5\}$ is a choice set of size $\ell = 3$ meeting all M -lists. However, $T \cap B_3$ has size 3, so any ℓ -subset forming a block transversal must choose exactly one element from B_3 ; that element meets at most one distinguished list, so the resulting transversal does not lie in $\mathcal{T}$.

Whether the exact threshold satisfies $n_{m-k+1}(m) = N(\ell, k)$ for $k \geq 2$ remains open; see Conjecture 1 and Table 1.

Remark 8 (Possible approaches for $k \geq 2$). The sufficiency gap of Remark 7 reflects a structural obstacle: when multiple distinguished lists draw from the same block, the resulting choice set need not contain a block transversal meeting all k distinguished lists. Two general frameworks appear potentially applicable.

The container method [15] organizes independent sets (or, here, transversal families) into few structurally simple containers. Applied to the hypergraph whose vertices are block elements and whose edges are the obstructing choice sets, container techniques could in principle bound from below the number of N -lists forced by any non-colorable assignment, even outside the block-disjoint class.

Entropy-based counting arguments track the information content of a coloring across blocks; in the block-disjoint setting each block contributes essentially independently, and such methods might constrain which choice-set structures are compatible with global non-colorability for large m .

Proposition 3 (Asymptotics for fixed k). Fix $k \geq 1$. As $\ell \rightarrow \infty$,

$$N(\ell, k) = \ell^\ell (1 - e^{-1})^k \left(1 + O_k\left(\frac{1}{\ell}\right)\right).$$

In particular, the block-transversal obstruction in Construction 1 has asymptotic order $(1 - e^{-1})^k (m - k)^{m-k}$.

Proof. For fixed $t \leq k$, $(1 - t/\ell)^\ell = \exp(\ell \log(1 - t/\ell)) = e^{-t} (1 + O_k(1/\ell))$ uniformly in t . Hence $(\ell - t)^\ell = \ell^\ell e^{-t} (1 + O_k(1/\ell))$. Substituting into (1) yields $N(\ell, k) = \ell^\ell \sum_{t=0}^k (-1)^t \binom{k}{t} e^{-t} (1 + O_k(1/\ell))$. The binomial identity $\sum_{t=0}^k (-1)^t \binom{k}{t} e^{-t} = (1 - e^{-1})^k$ completes the proof. $\square$

Conjecture 1 (Sharp threshold for fixed k). Fix $k \geq 1$ and set $\ell = m - k$. For all sufficiently large m , $n_{m-k+1}(m) = N(\ell, k)$. Moreover, any list assignment attaining this value is, up to relabeling of vertices and colors, isomorphic to Construction 1.

The first unresolved instance is $k = 2$, where the conjecture predicts $n_{m-1}(m) = (m - 2)^{m-2} - 2(m - 3)^{m-2} + (m - 4)^{m-2}$. Table 1 records agreement with this value for $4 \leq m \leq 8$ within the block-disjoint class.

Theorem 5 (Conditional lower bound). Fix $k \geq 1$ and set $\ell = m - k$. Assume that every critical assignment with list size ℓ admits, after relabeling, disjoint blocks $B_{k+1}, \dots, B_m$ and distinguished lists $L_1, \dots, L_k$ such that $|L_i \cap B_j| \leq 1$ for all $i \in [k]$, $j \geq k + 1$, and each L_i meets all but $o(\ell)$ blocks as $m \rightarrow \infty$. Then

$$n_{m-k+1}(m) \geq (1 - e^{-1})^k (m - k)^{m-k} (1 - o(1)) \quad (m \rightarrow \infty).$$

Table 1. Verification for $k = 2$ within the block-disjoint model, with the $k = 1$ threshold n_0 shown for comparison. Here $\ell = m - 2$ and $N(\ell, 2) = \ell^\ell - 2(\ell - 1)^\ell + (\ell - 2)^\ell$.

m	ℓ	$N(\ell, 2)$	$n_{m-1}^{\text{block}}(m)$	$n_0 = (m - 1)^{m-1} - (m - 2)^{m-1}$
4	2	2	2	19
5	3	12	12	175
6	4	100	100	2101
7	5	1230	1230	31,031
8	6	19,926	19,926	543,607

Proof. Let L be critical and satisfy the hypothesis. For $j \geq k + 1$ define $I_j = \{i \in [k] : L_i \cap B_j \neq \emptyset\}$ and $r_j = |I_j|$. Let $\mathcal{T}$ denote the family of ℓ -element block transversals that meet every distinguished list $L_1, \dots, L_k$. Under the assumptions, each such transversal is a choice set of size ℓ . For $I \subseteq [k]$ with $|I| = t$, let E_I be the subfamily of $\mathcal{T}$ avoiding L_i for all $i \in I$. In block B_j , avoidance removes $r_{I,j} := |I \cap I_j|$ elements, so $|E_I| = \prod_{j=k+1}^m (\ell - r_{I,j})$. By hypothesis, $r_{I,j} = t$ for all but $o(\ell)$ indices j , and each exceptional factor differs from $\ell - t$ by at most k . Thus $|E_I| = (\ell - t)^\ell (1 + o(1))$ uniformly for fixed I .

Inclusion–exclusion gives $|\mathcal{T}| = \sum_{t=0}^k (-1)^t \binom{k}{t} (\ell - t)^\ell (1 + o(1))$. Using $(\ell - t)^\ell = \ell^\ell e^{-t} (1 + O_k(1/\ell))$, we obtain $|\mathcal{T}| = \ell^\ell (1 - e^{-1})^k (1 + o(1))$. Each $T \in \mathcal{T}$ must appear as an N -list by Corollary 1, hence $n \geq |\mathcal{T}|$. $\square$

Remark 9 (Computational methodology). *The entries in Tables 1 and 2 were obtained by exhaustive enumeration within the block-disjoint model. For each m in the given range, all list assignments of the prescribed uniform size were generated over a canonical color set, up to color permutation. Non-colorability was verified via Corollary 1: an assignment was classified as non-colorable if and only if every choice set of the M -lists contained some N -list. Isomorphism classes were identified by canonical labeling: two assignments were deemed isomorphic (in the sense of Definition 2) if one was obtainable from the other by permuting M -vertices, permuting N -vertices, and relabeling colors. The resulting isomorphism type counts are consistent with Theorem 3 in all cases verified.*

Table 2. Computational verification for $m \leq 7$ within the block model, with threshold $n_0 = (m - 1)^{m-1} - (m - 2)^{m-1}$ and the number of critical isomorphism types (Theorem 3).

m	n_0	m^m	Type A	Type B	Case II	Total
3	3	27	1	1	1	3
4	19	256	1	1	1	3
5	175	3125	1	1	0	2
6	2101	46,656	1	1	0	2
7	31,031	823,543	1	1	0	2

9. The Balanced Case

The preceding sections treat the strongly unbalanced setting with fixed m and $n \rightarrow \infty$, where explicit transversal obstructions produce polynomial thresholds. When the two parts grow on the same scale, the behavior changes fundamentally. In the symmetric case $m = n$, the choice number grows only logarithmically. We record the classical argument to emphasize this contrast.

Property B. A k -uniform hypergraph $\mathcal{H} \subseteq \binom{[r]}{k}$ has *Property B* if there exists a partition $[r] = R \sqcup B$ such that every edge of $\mathcal{H}$ meets both R and B . Let M_k denote the minimum size of a k -uniform hypergraph without Property B.

Erdős [16] proved $2^{k-1} < M_k$, and probabilistic constructions yield $M_k \leq (1 + o(1))(e \ln 2/2) k^2 2^k$ as $k \rightarrow \infty$ (see [15] (§1.3) and [17]).

Lemma 8. *If $m \geq M_k$, then $\text{ch}(K_{m,m}) \geq k + 1$.*

Proof. Let $\mathcal{H} = \{E_1, \dots, E_m\}$ be a k -uniform hypergraph with no Property B. Assign the k -set E_i to the i th vertex of M and to the i th vertex of N . Suppose φ is a proper L -coloring. Set $R := \varphi(M)$ and $B := \varphi(N)$. Since $K_{m,m}$ is complete bipartite, $R \cap B = \emptyset$. For each i , the vertex in M forces $E_i \cap R \neq \emptyset$ and the corresponding vertex in N forces $E_i \cap B \neq \emptyset$. Hence $\mathcal{H}$ has Property B, a contradiction. $\square$

Theorem 6. *As $m \rightarrow \infty$, $\frac{\text{ch}(K_{m,m})}{\log_2 m} \rightarrow 1$.*

Proof. Erdős, Rubin, and Taylor [2] proved $\text{ch}(K_{m,n}) \leq t$ whenever $m + n \leq 2^t$. With $m = n$ this gives $\text{ch}(K_{m,m}) \leq \lceil \log_2(2m) \rceil = \log_2 m + O(1)$, so $\limsup \text{ch}(K_{m,m}) / \log_2 m \leq 1$.

For the lower bound, let $k = k(m)$ be maximal with $M_k \leq m$. By Lemma 8, $\text{ch}(K_{m,m}) \geq k + 1$, while maximality implies $m < M_{k+1}$. Since $M_t \leq Ct^2 2^t$ for large t , we obtain $m < C(k+1)^2 2^{k+1}$. Taking $\log_2$ yields $\log_2 m \leq (k+1) + 2 \log_2(k+1) + O(1)$. Because $\log_2(k+1) = o(k)$, we have $k \geq \log_2 m - O(\log \log m)$. Hence $\liminf \text{ch}(K_{m,m}) / \log_2 m \geq 1$, and the limit follows. $\square$

Thus $\text{ch}(K_{m,m}) = (1 + o(1)) \log_2 m$, in sharp contrast with the polynomial thresholds $n \asymp m^m$ and, at level $m - k + 1$, $n \asymp (1 - e^{-1})^k m^{m-k}$ in the unbalanced setting.

Remark 10 (Convergence rate). *The rate of convergence in Theorem 6 is $O(\log \log m / \log m)$: the lower bound satisfies $k \geq \log_2 m - O(\log \log m)$ while the upper bound gives $\text{ch}(K_{m,m}) \leq \log_2 m + O(1)$. Whether the second-order term $\text{ch}(K_{m,m}) - \log_2 m$ can be determined exactly is an open problem; see [5] for related results in the random bipartite setting.*

10. Conclusions and Open Problems

The choice number $\text{ch}(K_{m,n})$ is fully determined at the first two thresholds in the unbalanced range, and the extremal structure at the lower threshold is now completely rigid. At $n_0 = (m-1)^{m-1} - (m-2)^{m-1}$ the critical list assignments are classified: for $m \geq 5$ exactly two isomorphism types exist, both arising from the disjoint configuration of Case (I), while for $m \in \{3, 4\}$ a third type appears from Case (II).

The two threshold settings differ in the obstruction mechanism that governs non-colorability. In the unbalanced setting, the obstruction is a covering condition: every choice set of the M -lists must contain some N -list, directly forcing $|N|$ to exceed the number of unavoidable choice sets (Corollary 1). In the balanced setting, the obstruction is a partition condition: a proper coloring would induce a partition of the color set into $R = \varphi(M)$ and $B = \varphi(N)$ such that every hyperedge of the associated hypergraph meets both parts, certifying Property B. The polynomial growth of thresholds in the unbalanced case and the logarithmic behavior $\text{ch}(K_{m,m}) \sim \log_2 m$ are direct consequences of this difference.

Whether the Type A/Type B dichotomy at $k = 1$ persists at deeper levels is a natural question. The fixed- k block model suggests an analogous two-type structure at each level, but confirming this requires resolving the sufficiency gap described in Remark 7.

Beyond n_0 , the fixed- k block construction suggests a hierarchy of deeper thresholds. We proved the necessary bound $n \geq N(\ell, k)$ (Theorem 4), which is sharp at $k = 1$. Whether it is also sufficient for $k \geq 2$, and whether the extremal assignments are unique up to relabeling, are the central unresolved questions.

Open Problem 1 (Exact threshold for $k = 2$). *Let $n_{m-1}(m)$ be the least n such that $\text{ch}(K_{m,n}) = m - 1$. Determine whether, for all sufficiently large m , $n_{m-1}(m) = (m-2)^{m-2} - 2(m-3)^{m-2} + (m-4)^{m-2}$, and classify the critical assignments at this value.*

Open Problem 2 (Fixed- k thresholds). Fix $k \geq 2$ and set $\ell = m - k$. Decide whether, for large m , $n_{m-k+1}(m) = \sum_{t=0}^k (-1)^t \binom{k}{t} (\ell - t)^\ell$, and whether every extremal assignment is, up to relabeling, isomorphic to Construction 1.

Open Problem 3 (Rigidity and asymptotics). Fix $k \geq 2$ and $\ell = m - k$. Identify structural conditions on critical assignments of list size ℓ that force $n_{m-k+1}(m) \geq (1 - e^{-1})^k \ell^\ell (1 - o(1))$, and determine whether a comparable structure is unavoidable.

Author Contributions: Conceptualization, J.A.; Methodology, J.A., B.B. and S.G.; Software, J.S., W.G. and S.G.; Validation, B.B., J.S., W.G. and S.G.; Formal analysis, B.B.; Investigation, J.A., J.S. and W.G.; Writing—original draft, J.A.; Writing—review & editing, B.B., J.S., W.G. and S.G.; Funding acquisition, J.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by US department of Education grant number P382G240006.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Conflicts of Interest: The authors declare no conflicts of interest.

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