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The New Bipolar Intuitionistic Fuzzy Metric Space (NBIFM-Space) with Applications

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Abstract

This paper introduces the *New Bipolar Intuitionistic Fuzzy Metric Space* (*NBIFM*-space)—a mathematical framework that extends intuitionistic and previously proposed bipolar intuitionistic structures by providing a complete three-component formulation based on positive similarity, negative similarity, and indeterminacy. Unlike earlier bipolar intuitionistic models, the NBIFM-space employs normalized metric components and coordinated triangular norms denoted by t-norm/t-conorm interactions, yielding a fully consistent topological and analytic setting. We have developed the basic properties of this structure and have demonstrated its effectiveness in image processing, where the explicit separation of attraction, repulsion, and uncertainty leads to robust edge-preserving filtering. Furthermore, a Banach-type fixed point theorem is established in the full NBIFM framework.

Keywords: bipolar intuitionistic fuzzy metric space; fuzzy metric space; fixed point theorem; Banach contraction principle; image processing

MSC: 47H10; 54H25; 94A08

1. Introduction and Preliminaries

The study of uncertainty-driven metric structures has evolved significantly since Zadeh introduced fuzzy sets in the mid-1960s [1]. A decade later, Kramosil and Michálek [2] proposed fuzzy metric (FM) spaces, later refined by George and Veeramani [3,4], providing the foundations for analysis in environments where distances are not crisp. Their framework has since been extended in several directions, particularly in fixed point theory and convergence analysis [5–11]. Atanassov [12] achieved a further generalization through intuitionistic fuzzy sets that explicitly separate membership, non-membership, and hesitation. Intuitionistic fuzzy metric spaces (IFMSs) by Park [13] incorporate this idea into the metric structure and have been studied in various analytical contexts [14–17].

Bipolar fuzzy sets, introduced by Lee in [18], distinguish between positive and negative evidence, allowing the modeling of dual relations such as agreement/disagreement, or attraction/repulsion. A formal metric version of this concept was introduced more recently by Zararsız and Riaz [19], where proximity is described simultaneously through positive and negative components. However, these bipolar fuzzy metrics do not handle intuitionistic

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uncertainty, and they use a negative component defined in the closed interval $[-1, 0]$, which is not fully compatible with the standard fuzzy and intuitionistic frameworks.

Existing bipolar fuzzy metric models therefore capture only the interaction between positive and negative information, without incorporating a separate intuitionistic hesitation component or normalizing all metric quantities to the closed interval $[0, 1]$.

Novelty of the present work: The New Bipolar Intuitionistic Fuzzy Metric Space (NBIFM-space) introduced in this paper unifies and extends all previously mentioned models—both classical and modern. Its main contributions are as follows:

- simultaneous inclusion of positive similarity, negative similarity, and indeterminacy through the triplet (μ_P, μ_N, ν);
- triangle inequalities involving both a triangular norm denoted by t-norm and a t-conorm ([20]);
- a complete topological framework (open balls, convergence, completeness, etc.);
- a Banach-type fixed point theorem formulated in the full three-component setting;
- applications to iterative algorithms, including recommended systems and edge–preserving image filtering.

Thus, the proposed NBIFM-space provides a more comprehensive and more flexible structure than classical fuzzy metric spaces (FMSs), intuitionistic FMSs, or bipolar FMSs.

We briefly review the fundamental classical notions underlying BIFM-spaces, which will be essential for the subsequent analysis.

Definition 1 ([20]). *A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is called a t-norm if it satisfies for all a, b, $c \in [0, 1]$:*

(t1) $a * b = b * a$,
(t2) $(a * b) * c = a * (b * c)$,
(t3) *if* $a \leq a'$ *and* $b \leq b'$ *then* $a * b \leq a' * b'$,
(t4) $a * 1 = a$ *for all* $a \in [0, 1]$.

Example 1. *The following are standard examples of t-norms on* $[0, 1]$*:*

$$a \star b = \min\{a, b\},$$

$$a \star b = a \cdot b,$$

$$a \star b = \max\{a + b - 1, 0\}.$$

Definition 2 ([9]). *Let $\star : [0, 1] \times [0, 1] \to [0, 1]$ be a t-norm. For $x \in [0, 1]$ and $n \in \mathbb{N}$, define*

$$T_n(x) = \underbrace{x \star x \star \cdots \star x}_{n \text{ times}},$$

i.e., recursively,

$$T_1(x) = x, \qquad T_{n+1}(x) = T_n(x) \star x.$$

The t-norm $\star$ is said to be of H-type if the family $\{T_n(x)\}_{n\in\mathbb{N}}$ is equicontinuous at the point $x = 1$; that is, for every $\lambda \in (0, 1)$ there exists $\delta(\lambda) \in (0, 1)$ such that

$$x > 1 - \delta(\lambda) \;\Rightarrow\; T_n(x) > 1 - \lambda, \quad \text{for every } n \in \mathbb{N}.$$

Definition 3 ([20]). *A binary operation $\diamond : [0, 1] \times [0, 1] \to [0, 1]$ is called a t-conorm if it satisfies for all $a, b, c \in [0, 1]$:*

(c1) $a \diamond b = b \diamond a$,

(c2) $(a \diamond b) \diamond c = a \diamond (b \diamond c)$,
(c3) *if* $a \leq a'$ *and* $b \leq b'$ *then* $a \diamond b \leq a' \diamond b'$,
(c4) $a \diamond 0 = a$ *for all* $a \in [0,1]$.

Example 2. *Basic examples of t-conorms are:*

$$x \diamond y = \max\{x, y\},$$

$$x \diamond y = x + y - xy,$$

$$x \diamond y = \min\{1, x + y\}.$$

Definition 4 ([21]). *Let* $\star : [0,1] \times [0,1] \to [0,1]$ *be a t-norm, and let* $\diamond : [0,1] \times [0,1] \to [0,1]$ *be a t-conorm. For* $x_i \in [0,1]$, $i = 1, 2, \ldots, n, n+1$ *and* $n \in \mathbb{N}$, *define*

$$\star_{i=1}^{n} x_i = x_1 \star x_2 \star \cdots \star x_n, \quad \diamond_{i=1}^{n} x_i = x_1 \diamond x_2 \diamond \cdots \diamond x_n,$$

i.e., recursively,

$$\star_{i=1}^{1} x_i = x_1, \quad \star_{i=1}^{n+1} x_i = (\star_{i=1}^{n} x_i) \star x_{n+1} \quad \text{and} \quad \diamond_{i=1}^{1} x_i = x_1, \quad \diamond_{i=1}^{n+1} x_i = (\diamond_{i=1}^{n} x_i) \diamond x_{n+1}.$$

Definition 5 ([2,3]). *A triple* $(X, M, *)$ *is a fuzzy metric space if* X *is a non-empty set,* $*$ *is a continuous t-norm, and* $M : X \times X \times (0, \infty) \to [0,1]$ *satisfies, for all* $x, y, z \in X$ *and* $s, t > 0$:

(F1) $M(x, y, t) = 1 \iff x = y$,
(F2) $M(x, y, t) = M(y, x, t)$,
(F3) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$,
(F4) $M(x, y, \cdot) : (0, \infty) \to [0,1]$ *is continuous.*

Definition 6 ([13]). *A quintuple* $(X, \mu_P, \nu, *, \diamond)$ *is an intuitionistic fuzzy metric space if* $\mu_P, \nu : X \times X \times (0, \infty) \to [0,1]$ *satisfies, for all* $x, y, z \in X$ *and* $s, t > 0$:

(I1) $\mu_P(x, y, t) + \nu(x, y, t) \leq 1$,
(I2) $\mu_P(x, y, t) > 0$,
(I3) $\nu(x, y, t) < 1$,
(I4) $\mu_P(x, y, t) = 1$ *and* $\nu(x, y, t) = 0 \iff x = y$,
(I5) $\mu_P(x, y, t) = \mu_P(y, x, t)$, $\nu(x, y, t) = \nu(y, x, t)$,
(I6) $\mu_P(x, y, t) \leq \mu_P(x, y, t + s)$, $\nu(x, y, t) \geq \nu(x, y, t + s)$,
(I7) $\mu_P(x, z, t + s) \geq \mu_P(x, y, t) * \mu_P(y, z, s)$, $\nu(x, z, t + s) \leq \nu(x, y, t) \diamond \nu(y, z, s)$,
(I8) $\mu_P(x, y, \cdot) : (0, \infty) \to [0,1]$ *and* $\nu(x, y, \cdot) : (0, \infty) \to [0,1]$ *are continuous.*

Definition 7. *A doubled arithmetic operation*

$$\nabla : [-1, 0] \times [-1, 0] \to [-1, 1]$$

is called a bipolar continuous symmetry t_s*-conorm if it satisfies the following conditions:*

(S1) ∇ *commutative and associative,*
(S2) ∇ *is continuous,*
(S3) $a \nabla 0 = a$ *for all* $a \in [-1, 0]$,
(S4) $a \nabla b \leq c \nabla d$, *where* $a \leq c$ *and* $b \leq d$ *for all* $a, b, c, d \in [-1, 0]$.

Definition 8 ([19]). *Let* A *be a non-empty set. Let* $\mu_A^r : A \times A \times (0, \infty) \to [0,1]$, $\mu_A^l : A \times A \times (0, \infty) \to [-1, 0]$, $B : A \times A \times (0, \infty) \to [-1, 1]$, $\alpha, \gamma > 0$. *Assume that* $\{\langle u, \mu_A^r(u), \mu_A^l(u)\rangle : u \in A\}$ *is a bipolar fuzzy set such that* μ_A^r *and* μ_A^l *are fuzzy sets on* $A \times A \times (0, \infty)$. *Suppose that*

$*$ is a continuous t-norm and ∇ is a bipolar continuous symmetry t_s-conorm. The quadruple set $(A, B, *, \nabla)$ is called a bipolar fuzzy metric space (BFMS) if for all $u, v, z \in A$ and all $\gamma, \alpha > 0$ the following conditions hold:

(B1) $0 \leq \mu_A^r(u, v, \gamma) \leq 1, -1 \leq \mu_A^l(u, v, \gamma) \leq 0,$
(B2) $\mu_A^r(u, v, \gamma) + \mu_A^l(u, v, \gamma) \leq 1,$
(B3) $\mu_A^r(u, v, \gamma) = 1 \iff u = v,$
(B4) $\mu_A^r(u, v, \gamma) = \mu_A^r(v, u, \gamma),$
(B5) $\mu_A^r(u, v, \gamma) * \mu_A^r(v, z, \alpha) \leq \mu_A^r(u, z, \gamma + \alpha),$
(B6) $\mu_A^r(u, v, \cdot) : [0, \infty) \to [0, 1]$ *is continuous,*
(B7) $\lim_{\gamma \to \infty} \mu_A^r(u, v, \gamma) = 1,$
(B8) $\mu_A^l(u, v, \gamma) = -1 \iff u = v,$
(B9) $\mu_A^l(u, v, \gamma) = \mu_A^l(v, u, \gamma),$
(B10) $\mu_A^l(u, v, \gamma) \nabla \mu_A^l(v, z, \alpha) \geq \mu_A^l(u, z, \gamma + \alpha).$
(B11) $\mu_A^l(u, v, \cdot) : [0, \infty) \to [-1, 0]$ *is continuous,*
(B12) $\lim_{\gamma \to \infty} \mu_A^l(u, v, \gamma) = -1.$

The functions $\mu_A^r(u, v, \gamma)$ and $\mu_A^l(u, v, \gamma)$ are the value of closeness and the value of a distance between u and v with respect to γ, respectively.

2. The Definition of an NBIFM-Space

In this section, we introduce a new definition of a bipolar intuitionistic fuzzy metric space. This formulation extends the earlier bipolar fuzzy metric structure by providing a clearer and more balanced representation of positive similarity, negative dissimilarity, and uncertainty. Unlike the previous approach, where the negative component was defined over the interval $[-1, 0]$, the new framework normalizes all quantities to the standard fuzzy interval $[0, 1]$. This makes the model easier to interpret, fully compatible with modern fuzzy and intuitionistic methods, and more suitable for practical computations. This improvement is especially important in image processing, where similarity and dissimilarity between pixels must be measured consistently, allowing smoother iterative behavior and more stable filtering operators.

Definition 9. *Let X be a non-empty set. Define the functions:*

$$\mu_P : X \times X \times (0, \infty) \to [0, 1], \quad \mu_N : X \times X \times (0, \infty) \to [0, 1], \quad \nu : X \times X \times (0, \infty) \to [0, 1],$$

where μ_P is the positive fuzzy measure of closeness, μ_N is the negative fuzzy measure of closeness, and ν is the measure of indeterminacy.

The ordered 6-tuple $(X, \mu_P, \mu_N, \nu, *, \diamond)$ is called a New Bipolar Intuitionistic Fuzzy Metric space (NBIFM-space) if for all $x, y, z \in X$ and all $t, s > 0$ the following conditions hold:

(A0) If for all $t > 0$ we have

$$\mu_P(x, y, t) = 1, \quad \mu_N(x, y, t) = 0, \quad \nu(x, y, t) = 0$$

then $x = y$.

(A1) $\mu_P(x, y, t) = \mu_P(y, x, t), \quad \mu_N(x, y, t) = \mu_N(y, x, t), \quad \nu(x, y, t) = \nu(y, x, t).$

(A2) $\mu_P(x, y, t) + \mu_N(x, y, t) + \nu(x, y, t) \leq 1.$

(A3) $\mu_P(x, x, t) = 1, \quad \mu_N(x, x, t) = 0, \quad \nu(x, x, t) = 0.$

(A4) There exist a continuous t-norm $*$ and t-conorm $\diamond$ such that

$$\mu_P(x, z, t + s) \geq \mu_P(x, y, t) * \mu_P(y, z, s),$$

$$\mu_N(x,z,t+s) \le \mu_N(x,y,t) \diamond \mu_N(y,z,s),$$
$$\nu(x,z,t+s) \le \nu(x,y,t) \diamond \nu(y,z,s).$$

(A5) If $t_1 < t_2$ then

$$\mu_P(x,y,t_1) \le \mu_P(x,y,t_2), \quad \mu_N(x,y,t_1) \ge \mu_N(x,y,t_2), \quad \nu(x,y,t_1) \ge \nu(x,y,t_2).$$

(A6) For all $x, y \in X$, the mappings

$$\mu_P(x,y,\cdot), \quad \mu_N(x,y,\cdot), \quad \nu(x,y,\cdot) : (0,\infty) \to [0,1]$$

are continuous.

Now, let us observe the following.

Example 3. *Typical choices are:*

$$a * b = a \cdot b, \quad a \diamond b = a + b - ab.$$

Remark 1. *Some special values determine certain structures as follows:*

- *Fuzzy metric space: if $\mu_P = M$, $\mu_N = \nu = 0$;*
- *Intuitionistic fuzzy metric space: if $\mu_N = 0$;*
- *Bipolar fuzzy metric space (existing models): if only μ_P and a negative component $\mu_A \in [-1,0]$ are used (typically normalized as $\mu_N = -\mu_A \in [0,1]$), without an indeterminacy term;*
- *NBIFM-space: the most general case with (μ_P, μ_N, ν) active.*

Remark 2. *The three components (μ_P, μ_N, ν) describe different aspects of the relationship between two points in an NBIFM-space.*

- *$\mu_P(x,y,t)$ is the positive degree of closeness between x and y. A higher value indicates that the two points are strongly similar or compatible in a positive sense.*
- *$\mu_N(x,y,t)$ measures negative closeness; that is, how strongly x and y are related in an opposite or conflicting way. This component captures the bipolar nature of the model and does not appear in classical fuzzy metrics or in intuitionistic fuzzy metrics.*
- *$\nu(x,y,t)$ quantifies the level of indeterminacy. It reflects how much uncertainty or lack of information is present in the relation between x and y. When ν is small, the relationship is well-understood; when it is large, the interaction is vague or unclear.*

Together, these three functions provide a more comprehensive and more flexible description of similarity, opposition, and uncertainty than the classical fuzzy or intuitionistic fuzzy settings.

3. The Topology of NBIFM-Space

We now consider an NBIFM-space $(X, \mu_P, \mu_N, \nu, *, \diamond)$.

Definition 10. *For $x \in X$, $r \in (0,1)$, and $t > 0$, define*

$$B_{\mathrm{NBIFM}}(x,r,t) := \{y \in X \mid \mu_P(x,y,t) > 1-r,\ \mu_N(x,y,t) < r,\ \nu(x,y,t) < r\}.$$

Lemma 1. *The family*

$$\mathcal{B} = \{B_{\mathrm{NBIFM}}(x,r,t) \mid x \in X,\ r \in (0,1),\ t > 0\}$$

forms a base for a topology on X.

Proof. (B1) Let $x \in X$. By axiom (A3),

$$\mu_P(x,x,t) = 1, \quad \mu_N(x,x,t) = 0, \quad \nu(x,x,t) = 0$$

for every $t > 0$. Hence, for every $r \in (0,1)$,

$$x \in B_{\mathrm{NBIFM}}(x,r,t).$$

Thus, each point of X belongs to some element of $\mathcal{B}$.

(B2) Let

$$z \in B(x_1,r_1,t_1) \cap B(x_2,r_2,t_2).$$

Fix $i \in \{1,2\}$. Since

$$z \in B(x_i,r_i,t_i),$$

we have

$$\mu_P(x_i,z,t_i) > 1-r_i, \quad \mu_N(x_i,z,t_i) < r_i, \quad \nu(x_i,z,t_i) < r_i.$$

By continuity in the third variable, there exists t_i^0, $0 < t_i^0 < t_i$ such that

$$\mu_P(x_i,z,t_i^0) > 1-r_i, \quad \mu_N(x_i,z,t_i^0) < r_i, \quad \nu(x_i,z,t_i^0) < r_i.$$

Let

$$a_i = \mu_P(x_i,z,t_i^0), \quad b_i = \mu_N(x_i,z,t_i^0), \quad c_i = \nu(x_i,z,t_i^0).$$

Then

$$a_i > 1-r_i, \quad b_i < r_i, \quad c_i < r_i.$$

Given that the t-norm $*$ is continuous and increasing, and $a_i * 1 = a_i$, there exists $\delta_i^1 > 0$ such that

$$u > 1-\delta_i^1 \quad \Longrightarrow \quad a_i * u > 1-r_i. \tag{1}$$

Since the t-conorm $\diamond$ is continuous and $b_i \diamond 0 = b_i$, $c_i \diamond 0 = c_i$, there exists $\delta_i^2 > 0$ such that

$$v < \delta_i^2 \quad \Longrightarrow \quad b_i \diamond v < r_i, \quad c_i \diamond v < r_i.$$

Let

$$\delta_i = \min\{\delta_i^1, \delta_i^2\}.$$

We show that

$$B(z,\delta_i,t_i - t_i^0) \subset B(x_i,r_i,t_i).$$

For every

$$y \in B(z,\delta_i,t_i - t_i^0),$$

the inequalities give

$$\mu_P(y,z,t_i - t_i^0) > 1-\delta_i, \quad \mu_N(y,z,t_i - t_i^0) < \delta_i, \quad \nu(y,z,t_i - t_i^0) < \delta_i.$$

From the triangle inequality and (1), it follows that

$$\mu_P(y,x_i,t_i) > \mu_P(y,z,t_i - t_i^0) * \mu_P(z,x_i,t_i^0) > 1-r_i.$$

It is proven similarly that

$$\mu_N(y,x_i,t_i) < r_i, \quad \nu(y,x_i,t_i) < r_i.$$

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So,

$$B(z, \delta_1, t_1 - t_1^0) \subset B(x_1, r_1, t_1),$$
$$B(z, \delta_2, t_2 - t_2^0) \subset B(x_2, r_2, t_2).$$

Let

$$\delta = \min\{\delta_1, \delta_2\}, \qquad s = \min\{t_1 - t_1^0,\ t_2 - t_2^0\}.$$

Then

$$B(z, \delta, s) \subset B(x_1, r_1, t_1) \cap B(x_2, r_2, t_2).$$

Thus, the base condition (B2) is satisfied. □

Corollary 1. *The family*

$$\mathcal{T}_{\mathrm{NBIFM}} = \left\{ U \subset X \mid \forall x \in U,\ \exists r \in (0,1),\ \exists t > 0 \text{ such that } B_{\mathrm{NBIFM}}(x, r, t) \subset U \right\}$$

is a topology on X, called the topology induced by the NBIFM.

Proof. Considering that $\mathcal{B}$ is a base, the family of all unions of members of $\mathcal{B}$ forms a topology on X. □

Theorem 1. *The topology $\mathcal{T}_{\mathrm{NBIFM}}$ induced by an NBIFM-space $(X, \mu_P, \mu_N, \nu, *, \diamond)$ is Hausdorff.*

Proof. Let $x \neq y$.

By (A0) there exists $t > 0$ such that at least one of the following holds:

$$\mu_P(x, y, t) < 1, \quad \mu_N(x, y, t) > 0, \quad \nu(x, y, t) > 0.$$

We distinguish three cases:

Case 1. $\mu_P(x, y, t) < 1$.

Set $r = \mu_P(x, y, t)$ and take r_0 such that

$$r < r_0 < 1.$$

Since $*$ is continuous and increasing, there exists $r_1 \in (0, 1)$ such that

$$r_1 * r_1 > r_0.$$

Consider the balls

$$B(x, 1 - r_1, t/2) \quad \text{and} \quad B(y, 1 - r_1, t/2).$$

Suppose

$$z \in B(x, 1 - r_1, t/2) \cap B(y, 1 - r_1, t/2).$$

Then

$$\mu_P(x, z, t/2) > r_1, \qquad \mu_P(y, z, t/2) > r_1.$$

By the triangle inequality,

$$\mu_P(x, y, t) = \mu_P(x, y, t/2 + t/2) \geq \mu_P(x, z, t/2) * \mu_P(z, y, t/2) > r_1 * r_1 > r_0 > r,$$

which contradicts $r = \mu_P(x, y, t)$. Hence, the balls are disjoint.

Case 2. $\mu_N(x, y, t) > 0$.

Set $s = \mu_N(x, y, t)$ and take s_0 such that

$$0 < s_0 < s.$$

Since $\diamond$ is continuous and increasing, there exists $s_1 \in (0, 1)$ such that

$$s_1 \diamond s_1 < s_0.$$

Consider the balls

$$B(x, s_1, t/2) \quad \text{and} \quad B(y, s_1, t/2).$$

Suppose

$$z \in B(x, s_1, t/2) \cap B(y, s_1, t/2).$$

Then

$$\mu_N(x, z, t/2) < s_1, \qquad \mu_N(y, z, t/2) < s_1.$$

By the triangle inequality,

$$\mu_N(x, y, t) = \mu_N(x, y, t/2 + t/2) \leq \mu_N(x, z, t/2) \diamond \mu_N(z, y, t/2) < s_1 \diamond s_1 < s_0 < s,$$

which contradicts $s = \mu_N(x, y, t)$. Hence, the balls are disjoint.
Case 3. $\nu(x, y, t) > 0$.

The argument is identical to Case 2, using the triangle inequality for ν. We again obtain a contradiction.

In all the cases, we obtain disjoint NBIFM-balls around x and y. Therefore, $\mathcal{T}_{\text{NBIFM}}$ is Hausdorff. □

Definition 11. *A sequence (x_n) in X converges to $x \in X$ if for every $\varepsilon > 0$ and $\lambda \in (0, 1)$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$:*

$$\mu_P(x_n, x, \varepsilon) > 1 - \lambda, \quad \mu_N(x_n, x, \varepsilon) < \lambda, \quad \nu(x_n, x, \varepsilon) < \lambda.$$

Theorem 2. *Let $(X, \mu_P, \mu_N, \nu, *, \diamond)$ be an NBIFM-space. If a sequence (x_n) converges to both x and y then $x = y$.*

Proof. Let $\varepsilon > 0$ and $\eta > 0$ be arbitrary. Since $*$ is continuous, we have

$$\lim_{\lambda \to 0^+} (1 - \lambda) * (1 - \lambda) = 1 * 1 = 1.$$

Hence, there exists $\lambda \in (0, 1)$ such that

$$(1 - \lambda) * (1 - \lambda) > 1 - \eta.$$

Since $x_n \to x$ and $x_n \to y$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\mu_P(x_n, x, \varepsilon) > 1 - \lambda, \qquad \mu_P(x_n, y, \varepsilon) > 1 - \lambda.$$

Using the triangle inequality for μ_P and the monotonicity of the t-norm $*$, we obtain for every $n \geq N$

$$\mu_P(x, y, 2\varepsilon) \geq \mu_P(x, x_n, \varepsilon) * \mu_P(x_n, y, \varepsilon) > (1 - \lambda) * (1 - \lambda) > 1 - \eta.$$

Since $\eta > 0$ was arbitrary and $\mu_P(x,y,2\varepsilon) \in [0,1]$, it follows that $\mu_P(x,y,2\varepsilon) = 1$ for all $\varepsilon > 0$. Hence, $\mu_P(x,y,t) = 1$ for all $t > 0$. By axiom (A2) we obtain $\mu_N(x,y,t) = 0$ and $\nu(x,y,t) = 0$ for all $t > 0$. Therefore, by axiom (A0) we conclude that $x = y$, as desired. □

Definition 12. *A sequence $(x_n) \subset X$ is a Cauchy sequence in an NBIFM-space if for every $\varepsilon \in (0,1)$ and $t > 0$ there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$:*

$$\mu_P(x_m, x_n, t) > 1 - \varepsilon, \quad \mu_N(x_m, x_n, t) < \varepsilon, \quad \nu(x_m, x_n, t) < \varepsilon.$$

Definition 13. *An NBIFM-space is complete if every Cauchy sequence converges to a point in X with respect to the induced topology.*

4. Banach Fixed Point Theorem in NBIFM-Spaces

The Banach contraction principle [22] has been extensively generalized in fuzzy, intuitionistic fuzzy, probabilistic, statistical, and bipolar metric frameworks (see, e.g., [2,3,8,10,13,14,17,19,23–26]); here, we establish a Banach-type fixed point theorem in NBIFM-spaces.

Theorem 3. *Let ordered 6-tuple $(X, \mu_P, \mu_N, \nu, *, \diamond)$ be a complete NBIFM-space. Assume that for all $x, y \in X$ we have*

$$\lim_{t\to+\infty} \mu_P(x,y,t) = 1, \qquad \lim_{t\to+\infty} \mu_N(x,y,t) = 0, \qquad \lim_{t\to+\infty} \nu(x,y,t) = 0.$$

Suppose that there exist $\mu \in (0,1)$ and $x_0, x_1 \in X$ such that for the arbitrarily $t > 0$ one has

$$\lim_{n\to\infty} *_{i=1}^{n} \mu_P\Big(x_0, x_1, \frac{t}{\mu^i}\Big) = 1, \quad \lim_{n\to\infty} \diamond_{i=1}^{n} \mu_N\Big(x_0, x_1, \frac{t}{\mu^i}\Big) = 0, \quad \lim_{n\to\infty} \diamond_{i=1}^{n} \nu\Big(x_0, x_1, \frac{t}{\mu^i}\Big) = 0. \tag{2}$$

Let $f : X \to X$ be a mapping for which there exists a constant $c > 1$ such that for all $x, y \in X$ and all $t > 0$ the following hold:

$$\mu_P(fx, fy, t) \geq \mu_P(x, y, c\,t), \qquad \mu_N(fx, fy, t) \leq \mu_N(x, y, c\,t), \qquad \nu(fx, fy, t) \leq \nu(x, y, c\,t). \tag{3}$$

Then mapping f has a unique fixed point in X.

Proof. First, fix $x_0 \in X$ and define the Picard iteration $x_{n+1} = f(x_n)$ for $n \geq 0$. If for some n we have $x_{n+1} = x_n$ then x_n is a fixed point and we are done. Hence, assume $x_{n+1} \neq x_n$ for all n.

From (3) by the mathematical induction principle we get, for every $n \geq 0$ and every $t > 0$,

$$\mu_P(x_{n+1}, x_n, t) = \mu_P(fx_n, fx_{n-1}, t) \geq \mu_P(x_n, x_{n-1}, ct) \geq \cdots \geq \mu_P(x_1, x_0, c^n t).$$

In a similar fashion,

$$\mu_N(x_{n+1}, x_n, t) \leq \mu_N(x_1, x_0, c^n t), \qquad \nu(x_{n+1}, x_n, t) \leq \nu(x_1, x_0, c^n t).$$

Since $c > 1$, we have $c^n t \to +\infty$ as $n \to \infty$, and by the limit assumptions it follows that

$$\lim_{n\to\infty} \mu_P(x_{n+1}, x_n, t) = 1, \qquad \lim_{n\to\infty} \mu_N(x_{n+1}, x_n, t) = 0, \qquad \lim_{n\to\infty} \nu(x_{n+1}, x_n, t) = 0.$$

We now prove that the sequence (x_n) is a Cauchy one in the NBIFM sense. Choose $\sigma \in \left(\frac{1}{c}, 1\right)$. Then $\sigma c > 1$; therefore, the number $\mu' := \dfrac{1}{\sigma c}$ satisfies $0 < \mu' < 1$. By the

assumption (2) there exists N_0 such that for all $n \geq N_0$ and for every $p \in \mathbb{N}$ the finite products and conorm-iterates satisfy that

$$*_{i=n}^{n+p-1} \mu_P\Big(x_0, x_1, \frac{t}{\mu'^i}\Big) \quad \text{is arbitrarily close to 1,}$$

and that

$$\diamond_{i=n}^{n+p-1} \mu_N\Big(x_0, x_1, \frac{t}{\mu'^i}\Big), \quad \diamond_{i=n}^{n+p-1} \nu\Big(x_0, x_1, \frac{t}{\mu'^i}\Big) \quad \text{are arbitrarily small.}$$

Now, fix $n \geq N_0$ and $m \geq 1$. Using the monotonicity of μ_P in the third argument and the triangle inequality (A4) in the NBIFM-space (applied m times), we obtain the following:

$$\begin{aligned}
\mu_P(x_n, x_{n+m}, t) &\geq \mu_P\Big(x_n, x_{n+m}, t \sum_{i=n}^{n+m-1} \sigma^i\Big) \\
&\geq *_{i=n}^{n+m-1} \mu_P\Big(x_{i+1}, x_i, t\sigma^i\Big) \geq *_{i=n}^{n+m-1} \mu_P\Big(x_1, x_0, c^i t\sigma^i\Big) \\
&= *_{i=n}^{n+m-1} \mu_P\Big(x_0, x_1, \frac{t}{(1/(\sigma c))^i}\Big) = *_{i=n}^{n+m-1} \mu_P\Big(x_0, x_1, \frac{t}{\mu'^i}\Big).
\end{aligned}$$

During the displayed proof chain we used the following: monotonicity of μ_P (so that replacing t by the smaller scalar $t\sum\sigma^i$ gives a smaller value, whence the first inequality), the triangle inequality that produces the product of adjacent values μ_P (second step), the iterated contractive estimate to relate $\mu_P(x_{i+1}, x_i, t\sigma^i)$ to $\mu_P(x_1, x_0, c^i t\sigma^i)$, and finally the definition $\mu' = (\sigma c)^{-1}$. Note that μ appearing in the assumptions of the theorem is actually μ'; also, it is clear that x_1 from the assumptions is equal to $f(x_0)$.

Analogously, for the negative and indeterminacy components we get

$$\mu_N(x_n, x_{n+m}, t) \leq \diamond_{i=n}^{n+m-1} \mu_N\Big(x_0, x_1, \frac{t}{\mu'^i}\Big), \qquad \nu(x_n, x_{n+m}, t) \leq \diamond_{i=n}^{n+m-1} \nu\Big(x_0, x_1, \frac{t}{\mu'^i}\Big).$$

Hence, for every fixed $t > 0$,

$$\lim_{n\to\infty} \sup_{m\geq 1} \big(1 - \mu_P(x_n, x_{n+m}, t)\big) = 0, \quad \lim_{n\to\infty} \sup_{m\geq 1} \mu_N(x_n, x_{n+m}, t) = 0, \quad \lim_{n\to\infty} \sup_{m\geq 1} \nu(x_n, x_{n+m}, t) = 0,$$

which is exactly the statement that the sequence (x_n) is a Cauchy one in the NBIFM sense.

Since $(X, \mu_P, \mu_N, \nu, *, \diamond)$ is complete, there exists $x^* \in X$ such that $x_n \to x^*$ with respect to the NBIFM topology. As in the standard argument, using the contractive inequalities (3) and taking the limit, we obtain $f(x^*) = x^*$, concluding the first part of the statement.

The second part is the uniqueness. Suppose, for the sake of contradiction, that $y \in X$ is another fixed point of f distinct from the first one. Then, for all $t > 0$,

$$\mu_P(x^*, y, t) = \mu_P(fx^*, fy, t) \geq \mu_P(x^*, y, ct).$$

Iterating, we obtain

$$\mu_P(x^*, y, t) \geq \mu_P(x^*, y, c^n t) \qquad \text{for all } n \in \mathbb{N}.$$

Taking the limit as $n \to \infty$ and using $\lim_{s\to\infty} \mu_P(x^*, y, s) = 1$, we get $\mu_P(x^*, y, t) = 1$ for all $t > 0$. By axiom (A2), this implies

$$\mu_N(x^*, y, t) = 0 \quad \text{and} \quad \nu(x^*, y, t) = 0 \qquad \text{for all } t > 0.$$

Therefore, by axiom (A0), we must conclude that $x^* = y$. This completes the proof by contradiction. □

We believe that the following example, together with its proof, could provide the reader with a clear illustration of the introduced concepts.

Example 4. *Let $X = [0,1]$ with the usual metric $d(x,y) = |x-y|$. For $t > 0$ and $x, y \in X$ define*

$$\mu_P(x,y,t) = \frac{t}{t+|x-y|}, \qquad \mu_N(x,y,t) = a\,\frac{|x-y|}{t+|x-y|}, \qquad \nu(x,y,t) = b\,\frac{|x-y|}{t+|x-y|},$$

where $a, b \in (0,1)$ satisfy $a + b \leq 1$ (for instance, $a = b = \frac{1}{4}$). Choose the t-norm and t-conorm as

$$u * v = u \cdot v, \qquad u \diamond v = u + v - uv, \qquad u, v \in [0,1].$$

First, we verify that all axioms of an NBIFM-space are satisfied.

(A1), (A3), (A5) and (A6) are immediate from the definitions.

(A2) Put $d = |x-y|$. Then

$$\mu_P(x,y,t) + \mu_N(x,y,t) + \nu(x,y,t) = \frac{t}{t+d} + (a+b)\frac{d}{t+d} = \frac{t+(a+b)d}{t+d} \leq 1,$$

because $a + b \leq 1$.

(A4) $d_1 = |x-y|$, $d_2 = |y-z|$, $d_3 = |x-z|$. Since $d_3 \leq d_1 + d_2$, we have

$$\mu_P(x,y,t) * \mu_P(y,z,s) = \frac{t}{t+d_1} \cdot \frac{s}{s+d_2} \leq \frac{t+s}{t+s+d_3} = \mu_P(x,z,t+s)$$

$$\Leftrightarrow t^2s + ts^2 + tsd_3 \leq t^2s + ts^2 + s(t+s)d_1 + t(t+s)d_2 + (t+s)d_1d_2$$

$$\Leftrightarrow tsd_3 \leq ts(d_1+d_2) + s^2d_1 + t^2d_2 + (t+s)d_1d_2 \Leftrightarrow \top.$$

Hence,

$$\mu_P(x,z,t+s) \geq \mu_P(x,y,t) * \mu_P(y,z,s).$$

For the negative component, write

$$u = \frac{d_1}{t+d_1}, \qquad v = \frac{d_2}{s+d_2},$$

so that $\mu_N(x,y,t) = a\,u$ and $\mu_N(y,z,s) = a\,v$. Since

$$\mu_N(x,y,t) \diamond \mu_N(y,z,s) = au + av - a^2uv \geq a(u+v-uv),$$

it suffices to show that

$$a\,\frac{d_3}{t+s+d_3} \leq a(u+v-uv), \quad \textit{i.e.,} \quad \frac{d_3}{t+s+d_3} \leq u+v-uv.$$

Using $d_3 \leq d_1 + d_2$ and the monotonicity of $r \mapsto \dfrac{r}{t+s+r}$, we get

$$\frac{d_3}{t+s+d_3} \leq \frac{d_1+d_2}{t+s+d_1+d_2}.$$

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Finally, one checks directly that

$$\frac{d_1+d_2}{t+s+d_1+d_2} \le \frac{d_1}{t+d_1} + \frac{d_2}{s+d_2} - \frac{d_1}{t+d_1}\frac{d_2}{s+d_2} = u+v-uv,$$

because $\frac{d_1+d_2}{t+s+d_1+d_2} \le \frac{d_1+d_2}{t+d_1+d_2} \le \frac{d_1}{t+d_1}$, *and* $\frac{d_2}{s+d_2} - \frac{d_1}{t+d_1}\frac{d_2}{s+d_2} = \frac{d_2}{s+d_2}[1-\frac{d_1}{t+d_1}] \ge 0$. *Therefore,*

$$\mu_N(x,z,t+s) \le \mu_N(x,y,t) \diamond \mu_N(y,z,s).$$

The same argument applies to ν *(with a replaced by b).*

Thus, all the axioms (A1)–(A6) are satisfied, and therefore $(X, \mu_P, \mu_N, \nu, *, \diamond)$ *is an NBIFM-space.*

It remains to verify that the contractive condition is fulfilled.

Define $f : X \to X$ *by*

$$f(x) = \frac{x}{2}.$$

Again, let $d = |x-y|$. *Then* $|f(x)-f(y)| = \frac{d}{2}$.

(i) Positive component. For any $c \in (1,2]$,

$$\mu_P(fx,fy,t) = \frac{t}{t+\frac{d}{2}} \ge \frac{ct}{ct+d} = \mu_P(x,y,ct).$$

(ii) Negative component. For any $c \in (1,2]$,

$$\mu_N(fx,fy,t) = a\,\frac{\frac{d}{2}}{t+\frac{d}{2}} = a\,\frac{d}{2t+d} \le a\,\frac{d}{ct+d} = \mu_N(x,y,ct).$$

(iii) Indeterminacy component. Similarly, for any $c \in (1,2]$,

$$\nu(fx,fy,t) = b\,\frac{d}{2t+d} \le b\,\frac{d}{ct+d} = \nu(x,y,ct).$$

Hence, for every $c \in (1,2]$ *the mapping* f *satisfies*

$$\mu_P(fx,fy,t) \ge \mu_P(x,y,ct), \qquad \mu_N(fx,fy,t) \le \mu_N(x,y,ct), \qquad \nu(fx,fy,t) \le \nu(x,y,ct),$$

so f *is an NBIFM contraction. By the Banach fixed point theorem in NBIFM-spaces,* f *has a unique fixed point, namely* $x^* = 0$.

Proposition 1. *Let* $(X, \mu_P, \mu_N, \nu, *, \diamond)$ *be an NBIFM-space and let* $(x_n)_{n\in\mathbb{N}}$ *be a sequence in* X *such that, for some fixed* $t > 0$,

$$\lim_{n\to\infty} \mu_P(x_n, x_{n+1}, t) = 1, \qquad \lim_{n\to\infty} \mu_N(x_n, x_{n+1}, t) = 0, \qquad \lim_{n\to\infty} \nu(x_n, x_{n+1}, t) = 0.$$

Assume that $*$ *is a t-norm of H-type and that* $\diamond$ *is a t-conorm satisfying the following dual property: for every* $\lambda \in (0,1)$ *there exists* $\eta(\lambda) \in (0,1)$ *such that, for every* $p \in \mathbb{N}$ *and every* $x \in [0, \eta(\lambda)]$,

$$\diamond_{i=1}^{p} x < \lambda.$$

Then, for each $\lambda \in (0,1)$ *there exists* $N \in \mathbb{N}$ *such that, for all* $n \ge N$ *and all* $p \in \mathbb{N}$,

$$*_{i=1}^{p} \mu_P(x_{n+i}, x_{n+i+1}, t) > 1-\lambda, \tag{4}$$

$$\diamond_{i=1}^{p}\mu_N(x_{n+p}, x_{n+p+1}, t) < \lambda, \tag{5}$$

$$\diamond_{i=1}^{p}\nu(x_{n+p}, x_{n+p+1}, t) < \lambda. \tag{6}$$

Proof. Since the t-norm $*$ is of H-type, for every $\lambda \in (0,1)$ there exists $\delta(\lambda) \in (0,1)$ such that

$$x \geq \delta(\lambda) \quad \Longrightarrow \quad \underbrace{x * x * \cdots * x}_{p \text{ times}} > 1 - \lambda$$

for all $p \in \mathbb{N}$.

Because $\mu_P(x_n, x_{n+1}, t) \to 1$, there exists N_1 such that $\mu_P(x_n, x_{n+1}, t) \geq \delta(\lambda)$ for all $n \geq N_1$. Hence, for each such n and every $p \in \mathbb{N}$,

$$*_{i=1}^{p}\mu_P(x_{n+i}, x_{n+i+1}, t) \ \geq \ > \ 1 - \lambda,$$

giving the inequality (4).

Similarly, since $\mu_N(x_n, x_{n+1}, t) \to 0$ and $\diamond$ satisfies the dual H-property, there exists $\eta(\lambda) \in (0,1)$ such that, for any $p \in \mathbb{N}$ and any $x \leq \eta(\lambda)$,

$$\underbrace{x \diamond x \diamond \cdots \diamond x}_{p \text{ times}} < \lambda.$$

Because $\mu_N(x_n, x_{n+1}, t) \to 0$, there exists N_2 such that $\mu_N(x_n, x_{n+1}, t) \leq \eta(\lambda)$ for all $n \geq N_2$, and we get

$$\diamond_{i=1}^{p}\mu_N(x_{n+i}, x_{n+i+1}, t) = \underbrace{\mu_N(x_{n+1}, x_{n+2}, t) \diamond \cdots \diamond \mu_N(x_{n+p}, x_{n+p+1}, t)}_{p \text{ operands}} < \lambda,$$

giving the inequality (5).

An identical argument applies to ν, since $\nu(x_n, x_{n+1}, t) \to 0$, yielding the remaining inequality (6).

Finally, setting $N = \max\{N_1, N_2, N_3\}$ ensures that (4) and (6) hold for all $n \geq N$ and all $p \in \mathbb{N}$. This completes the proof. □

5. The Application of NBIFM-Spaces in Image Processing

Many classical denoising methods compare pixels using the absolute or Euclidean difference of their intensities. This approach is simple and widely used. However, it does not distinguish between different sources of intensity variation. In real images, changes in intensity may come from edges, texture, or random noise. A purely distance-based model treats all these changes analogously, which can lead to over-smoothing of edges or incomplete noise removal.

Fuzzy metric approaches improve this idea by using similarity values instead of crisp distances. Instead of stating that two pixels are either close or far, they assign a degree of similarity. This enables more adaptive smoothing. Intuitionistic FM introduces an additional hesitation component that reflects situations in which similarity cannot be clearly determined.

However, classical FM and intuitionistic FM often fail to explicitly separate structural similarity from structural opposition. Bipolar models introduce both attraction and repulsion, but they do not include an explicit uncertainty component within a fully normalized framework.

The NBIFM structure combines these ideas into a three-component model (μ_P, μ_N, ν) defined on the interval $[0,1]$. In image terms, μ_P represents positive similarity, μ_N measures contrast or opposition, and ν reflects uncertainty caused by noise or unclear texture. This

separation allows a more refined distinction between homogeneous regions, edges, and noisy pixels.

Fuzzy metric-based filters have already shown strong performance in impulsive noise removal and edge preservation [27,28]. Intuitionistic fuzzy approaches have also proven effective in enhancement and color processing [29–31]. Furthermore, aggregated fuzzy distance measures have been successfully used in segmentation and iterative filtering (see, e.g., Ralević et al. [21,32–35]), while edge-preserving fuzzy-metric denoising techniques are well-established [36]. Iterative image restoration algorithms based on fixed point principles have attracted considerable attention in recent years [37].

By incorporating bipolar information together with an intuitionistic hesitation component, NBIFM-spaces generalize these models and offer a richer similarity description. This enables a more robust discrimination between noise, edges, and texture, making NBIFM-based iterative filters well-suited for denoising, enhancement, and structure-aware smoothing.

Let an image be represented as a finite set of pixels

$$X = \{1, 2, \ldots, N\},$$

where each pixel indexed by $i \in X$ has a normalized grayscale intensity $I(i) \in [0,1]$. For $i, j \in X$ and $t > 0$ we define

$$\mu_P(i,j,t) = \frac{t}{t + |I(i) - I(j)|}, \tag{7}$$

$$\mu_N(i,j,t) = a\,\frac{|I(i) - I(j)|}{t + |I(i) - I(j)|}, \qquad a \in (0,1), \tag{8}$$

$$\nu(i,j,t) = b\,\frac{|I(i) - I(j)|}{t + |I(i) - I(j)|}, \qquad b \in (0,1), \quad a + b \le 1. \tag{9}$$

We take the t-norm and t-conorm as in Example 4:

$$u * v = u \cdot v, \qquad u \diamond v = u + v - uv, \qquad u, v \in [0,1].$$

Here:

- μ_P measures positive similarity and is close to 1 when pixel intensities are close.
- μ_N measures negative similarity and grows when intensities differ significantly.
- ν quantifies uncertainty and is larger in ambiguous, noisy or textured regions.

It can be proven that the structure $(X, \mu_P, \mu_N, \nu, *, \diamond)$ is an NBIFM-space.

We now define a nonlinear iterative filter driven by the NBIFM similarity structure. Let $I^{(n)} : X \to [0,1]$ denote the image after the n-th iteration. For each pixel indexed by $i \in X$ its value is updated by a weighted average over a neighborhood $N(i) \subset X$ (e.g., a 3×3 or 5×5 window):

$$I^{(n+1)}(i) = \frac{\sum_{j \in N(i)} w_{ij}^{(n)}\, I^{(n)}(j)}{\sum_{j \in N(i)} w_{ij}^{(n)}}, \tag{10}$$

where the weights are defined by

$$w_{ij}^{(n)} = \mu_P(i,j,t_n)\,\big(1 - \mu_N(i,j,t_n)\big) \cdot \exp\big(-\lambda\,\nu(i,j,t_n)\big), \qquad \lambda > 0. \tag{11}$$

The multiplicative structure ensures that if any single component indicates dissimilarity or uncertainty then the overall weight is significantly attenuated. Furthermore, the exponential term enables a smooth and stable control of the uncertainty penalization. In this way, pixels that are strongly similar and reliable have greater influence, while pixels that are opposed or uncertain contribute less to the averaging process.

The idea of adaptive weights is standard in nonlinear filtering. The novelty of this work lies in the NBIFM-based construction of the weights. Each component influences the weight in a deterministic and intuitive manner. Positive similarity μ_P increases the weight, whereas opposition μ_N and uncertainty ν decrease it.

The proposed filter is nonlinear because the weights depend on the current image through the NBIFM similarity measures. At each iteration, the similarities are recomputed, which means that the averaging process adapts to the evolving image structure. Therefore, the method is not a fixed linear convolution, but an adaptive procedure guided by the NBIFM model.

The iteration is repeated until stabilization. The fixed point framework developed in Section 4 ensures that this adaptive process converges under suitable conditions.

Here, $t_n > 0$ is a scale parameter that can depend on the iteration or on the local contrast. Pixels with high positive similarity and low negative similarity or uncertainty contribute more strongly to the update, thereby enabling edge-preserving smoothing.

To connect the filter with the abstract Banach-type theorem in NBIFM-spaces, we consider the space of all images $[0,1]^N$ endowed with an NBIFM structure induced from the pixel-wise NBIFM. For example, one may define

$$\mu_P(I,J,t) := \min_{i \in X} \mu_P\big(I(i), J(i), t\big),$$

and similarly for μ_N and ν, where $\mu_P\big(I(i), J(i), t\big)$ denotes the pixel-wise NBIFM similarity computed from the scalar intensities $I(i)$ and $J(i)$ according to relations (7)–(9).

We formulate, omitting the proof, the convergence result as a consequence of the general Banach fixed point theorem in NBIFM-spaces.

To explain why the iterative procedure is stable, we now relate the update rule to the fixed point result obtained in Section 4.

Theorem 4 (The NBIFM contractivity of the filter)**.** *Let $(\mathcal{I}, \mu_P, \mu_N, \nu, *, \diamond)$ be a complete NBIFM-space of images, with $*$ and $\diamond$ as above, and let*

$$f : \mathcal{I} \to \mathcal{I}, \qquad f(I) = I^{(1)},$$

be the NBIFM-based filter defined by Formulas (10)–(11). Assume that there exists a constant $k \in (0,1)$ such that for all images $I, J \in \mathcal{I}$ and all $t > 0$

$$\mu_P\big(f(I), f(J), t\big) \geq \mu_P(I, J, kt), \tag{12}$$

$$\mu_N\big(f(I), f(J), t\big) \leq \mu_N(I, J, kt), \tag{13}$$

$$\nu\big(f(I), f(J), t\big) \leq \nu(I, J, kt). \tag{14}$$

Then f is an NBIFM contraction and has a unique fixed point $I^* \in \mathcal{I}$. Moreover, for any initial image $I^{(0)} \in \mathcal{I}$, the Picard iteration

$$I^{(n+1)} = f\big(I^{(n)}\big), \qquad n \in \mathbb{N},$$

converges to I^* in the NBIFM sense.

Finally, we provide a numerical example regarding the described application.

Example 5. *We illustrate one step of the proposed framework on a small* 3×3 *grayscale patch. Consider the initial noisy image*

$$I^{(0)} = \begin{pmatrix} 0.10 & 0.12 & 0.11 \\ 0.11 & 0.80 & 0.12 \\ 0.09 & 0.10 & 0.11 \end{pmatrix},$$

where the central pixel is a strong impulse outlier.
We choose the parameters

$$t_n \equiv t = 0.1, \qquad a = b = 0.25, \qquad \lambda = 1,$$

and, for simplicity, take $N(i)$ *to be the full patch* 3×3 *for every pixel indexed by i. All our numerical results are reported to a sufficiently precise four decimal places. Using the weights* (11) *and the update rule* (10)*, one obtains the first iterate*

$$I^{(1)} = \begin{pmatrix} 0.1145 & 0.1168 & 0.1154 \\ 0.1154 & 0.5313 & 0.1168 \\ 0.1146 & 0.1145 & 0.1154 \end{pmatrix}.$$

Observe that the central outlier is strongly reduced but still noticeably larger than its neighbors.

After several further iterations, the patch becomes nearly homogeneous. For instance, after five and six iterations we obtain

$$I^{(5)} = \begin{pmatrix} 0.1323 & 0.1323 & 0.1323 \\ 0.1323 & 0.1326 & 0.1323 \\ 0.1323 & 0.1323 & 0.1323 \end{pmatrix}, \qquad I^{(6)} = \begin{pmatrix} 0.1324 & 0.1324 & 0.1324 \\ 0.1324 & 0.1324 & 0.1324 \\ 0.1324 & 0.1324 & 0.1324 \end{pmatrix}.$$

The maximal entry-wise difference between $I^{(6)}$ *and* $I^{(5)}$ *is of order* $2 \cdot 10^{-4}$*, strongly indicating numerical convergence to a fixed point.*

This example demonstrates that the NBIFM-based filter effectively suppresses the impulsive outlier while preserving the overall brightness level of the patch, and in practice the iterations converge very rapidly.

6. Further Research

The results presented in this paper can be extended in several meaningful ways. One promising direction involves studying other types of contractions within the NBIFM setting. Extending these well-known fixed point conditions to NBIFM-spaces could yield new theoretical insights.

Another avenue for investigation is the analysis of different nonlinear operators in NBIFM-spaces. For instance, alternative forms of adaptive weights or aggregation rules could be defined and examined regarding their convergence and stability.

From the point of view of applications, the NBIFM framework may prove useful in other domains where similarity, opposition, and uncertainty occur simultaneously. Potential areas include decision-making models, clustering methods, and recommendation systems, where positive and negative preferences, along with uncertainty, naturally arise.

These perspectives show that NBIFM-spaces offer a flexible framework for both further theoretical development and practical applications.

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