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Article

Spatial Asymptotics and Polynomial Decay for Nonlinear Parabolic Equations in $\mathbb{R}^3$ Exterior Region

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Abstract

This paper investigates the spatial asymptotic behavior of solutions to a class of nonlinear parabolic equations defined on an exterior region in $\mathbb{R}^3$. By constructing a suitable weighted energy functional and employing a fractional-order differential inequality technique, we establish a sharp Phragmén–Lindelöf type alternative: the solution either ceases to exist at a finite radial distance or decays to zero as the radial variable $r \to \infty$ when the power $p > 2$. In the decay case, we derive explicit polynomial type decay estimates. The analysis is conducted in unbounded exterior domains where traditional compactness arguments are not applicable, extending previous studies on semi-infinite cylinders to more complex geometric settings. Our results reveal distinct spatial behaviors compared to those observed in linear or differently nonlinear parabolic problems and can be seen as a version of Saint-Venant principle in exterior regions.

Keywords: spatial blow-up; spatial decay; fractional-order differential inequality; Saint-Venant principle; nonlinear parabolic equations

MSC: 35B30; 35K55; 35Q35

1. Introduction

The investigation of spatial asymptotic behavior of solutions to partial differential equations constitutes a fundamental area of research in applied mathematics and continuum mechanics, with roots extending to the classical Saint-Venant principle in elasticity [1–3]. This research tradition, which concerns the decay of mechanical and thermal effects away from applied loads or boundary disturbances, has evolved significantly over the past half-century to encompass increasingly sophisticated mathematical models and analytical techniques. Early foundational contributions by Boley [4] established key principles for spatial decay in transient heat conduction, laying the groundwork for subsequent developments in continuum thermomechanics.

The transition from mere decay estimates to comprehensive Phragmén–Lindelöf alternatives represents a significant theoretical advancement in the field. These alternatives, which characterize the essential dichotomy between growth and decay without imposing restrictive *a priori* assumptions, have been developed for various classes of equations including harmonic functions [5], biharmonic equations [6], and quasilinear parabolic problems [7]. The pioneering work of Horgan and Payne [8] extended these concepts to nonlinear boundary value problems, while more recent investigations by Fernandez and Quintanilla [9] have applied them to higher-order gradient theories. This evolution

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reflects a growing recognition that complete understanding of spatial behavior requires consideration of both decay and growth scenarios.

A particularly challenging direction has been the study of spatial behavior in *exterior domains*—unbounded regions that are complements of bounded sets. The lack of compactness in such domains introduces fundamental analytical challenges and often leads to qualitatively different asymptotic behavior compared to bounded or semi-infinite cylindrical domains. Quintanilla's seminal work [10] demonstrated that for certain parabolic problems on exterior regions, spatial decay or growth can be *faster than any exponential function linearly dependent on distance*, revealing distinctive characteristics of solutions in non-compact geometries. This discovery has stimulated renewed interest in the unique properties of exterior domain problems and their connections to physical applications.

Parallel developments in the analysis of nonlinear parabolic equations have significantly enriched our understanding of spatial asymptotics. The work of Liu, Du, and Yao [7] established important alternative results for nonlinear parabolic equations in half-cylinders, demonstrating how specific nonlinear structures influence the delicate balance between blow-up and decay. Subsequent research by Quintanilla [11] and Payne and Schaefer [12,13] explored spatial blow-up phenomena in detail, revealing intricate relationships between nonlinear terms and cessation of existence. These investigations highlight the complex interplay between equation structure and asymptotic behavior.

The concept of *structural stability*—the continuous dependence of solutions on constitutive parameters and modeling assumptions—has emerged as another critical dimension of spatial behavior analysis. Ames and Straughan's foundational work [14] established the importance of stability analysis for improperly posed problems, while Quintanilla [15] applied these principles to nonlinear elliptic and parabolic equations. Recent advances have demonstrated structural stability in various fluid and thermomechanical contexts, including studies of double-diffusive Darcy flow [16,17], Brinkman fluids in porous media [18], Boussinesq-Darcy systems [19], Brinkman-Darcy interfaces [20], and Forchheimer fluids [21]. These developments highlight the growing interconnection between spatial asymptotics, stability theory, and physical modeling.

Recent years have witnessed significant progress in understanding complex systems through spatial behavior analysis. Studies of quasi-static heat conduction within second gradient theory [22], Moore-Gibson-Thompson equations [23], Blackstock's model of thermoviscous flow [24], coupled plate equations with damping [25], viscoelastic wave equations on Heisenberg groups [26], and Jordan-Moore-Gibson-Thompson equations [27] demonstrate the expanding scope of this research area. These investigations employ diverse analytical techniques while addressing fundamental questions about energy propagation, stability, and long-range effects in physical systems.

The present work addresses these gaps by investigating the spatial asymptotic behavior of solutions to a class of nonlinear parabolic equations defined on exterior domains in $\mathbb{R}^3$. We consider the following class of nonlinear parabolic equations in an exterior domain, where the exponent $p > 2$ is a fixed constant;

$$c(\mathbf{x})u_t = (|\nabla u|^p u_{,i})_{,i} - f(u), \quad for \quad r \geq R_0, \tag{1}$$

posed on an exterior domain $\Omega \subset \mathbb{R}^3$, defined as the complement of a ball of radius $R_0 > 0$ centered at the origin. Here $c(\mathbf{x}) > 0$ represents a spatially varying material parameter. Physically, the spatially varying coefficient $c(\mathbf{x})$ represents a material property that changes from point to point in the medium. For example, in heat transfer, it can represent a non-uniform thermal conductivity (how well different parts of a material conduct heat). In fluid dynamics or groundwater flow, it can represent a variable permeability or viscosity, meaning some regions allow fluid to pass more easily than others. $f(u)$ satisfies the following inequality:

$$uf(u) \geq \lambda |u|^{2p}, \tag{2}$$

where λ is a positive constant. For simplicity, we choose $\lambda = 1$ in this paper. Condition (2) has a clear physical interpretation: it ensures that the nonlinear term $f(u)$ is at least of order $|u|^{2p-2}u$ for large $|u|$. This superlinear growth models strong localized reactions in applications such as combustion or chemical processes. The specific exponent $2p$ is chosen to balance the p-Laplacian diffusion, allowing for a sharp analysis of the competing effects of diffusion and reaction on spatial asymptotic behavior.

Our aim is to analyse the spatial evolution of the solution with respect to the radial variable $\rho = (x_i x_i)^{1/2}$. For any $\rho \geq R_0$, we define the spherical surface

$$D(\rho) = \{x : x_i x_i = \rho^2\},$$

and for $h > 0$, the annular region

$$B(\rho + h, \rho) = \{x : \rho < (x_i x_i)^{1/2} < \rho + h\}.$$

We shall establish a Phragmén–Lindelöf type alternative for an appropriate energy functional. Specifically, we prove that the solution either ceases to exist at a finite radial distance or decays to zero as the radial variable $r \to \infty$. For simplicity, we consider the exterior domain as the complement of a ball in R^3. This radial setting simplifies the weighted energy analysis. Although the results are proved for a ball, they are expected to hold for more general non-spherical exterior domains, as the method relies primarily on energy inequalities rather than precise geometry.

We prescribe homogeneous Dirichlet conditions on the lateral surface and given data on the finite end.

$$u(x,0) = 0 \quad x \in \Omega, \tag{3}$$

and

$$u(x,t) = g(x,t) \quad when \quad x_i x_i = R_0^2. \tag{4}$$

The following compatibility conditions must be satisfied:

$$g(x,0) = 0 \quad when \quad x_i x_i = R_0^2.$$

Although this paper follows the methodology outlined in Reference [10], there are significant differences in both the results and the analytical techniques employed. Reference [10] established blow-up results for the solutions, whereas this paper demonstrates the non-existence of solutions under certain conditions. Moreover, regarding decay estimates, Reference [10] obtained arbitrarily fast exponential decay, while our work does not achieve such exponential estimates; instead, we derive only polynomial decay rates for the solutions. Consequently, the conclusions of this paper are fundamentally different from those in Reference [10]. It is important to distinguish the concept of 'spatial ceasing' investigated here from the well-studied phenomenon of 'temporal blow-up.' While temporal blow-up describes the unbounded growth of a solution at a fixed spatial point as time progresses, spatial ceasing refers to the behavior of the solution (or its energy) as the spatial variable tends to infinity. Specifically, we investigate whether the solution's support or a related energy functional vanishes at a finite distance from the boundary or persists and decays as $r \to \infty$. This work focuses solely on this spatial asymptotic behavior, not on the formation of singularities in time.

Methodologically, Reference [10] relied on a first-order differential inequality to deduce spatial properties of the solutions. In contrast, our analysis necessitates the use of a fractional-order differential inequality to obtain the present results. Thus, both the approach and the outcomes of this study diverge substantially from those presented in Reference [10]. Traditional first-order differential inequalities, such as those employed in [10] for linear problems, are insufficient for the nonlinear equation considered here. In the linear case, the energy functional typically satisfies an inequality of the form $\frac{\partial F}{\partial r} \leq kF$, leading to exponential decay estimates. However, due to the presence of the nonlinear term $f(u)$ and $(|\nabla u|^p u_{,i})_{,i}$ in our equation, the resulting energy inequality becomes nonlinearly coupled. A direct first-order approach fails to close the estimate because the nonlinearity introduces lower-order terms that cannot be absorbed into a simple linear comparison. To overcome this, we employ a fractional-order differential inequality, which allows us to capture the correct polynomial decay rate dictated by the nonlinear structure.

The main result of this paper can be summarized as follows: for any fixed time $t_0 > 0$, the solution of (1)–(4) exhibits one of two possible spatial behaviors—either it ceases to exist at a finite radial distance r^* (spatial ceasing), or it exists for all $r > R_0$ and decays polynomially as $r \to \infty$. In the latter case, an explicit decay estimate is obtained, showing that the energy of the solution outside a ball of radius r is bounded by a negative power of r. This dichotomy is stated precisely in Theorem 1, and its proof is carried out in Section 3 using the fractional-order differential inequality developed in Section 2.

The remainder of this paper is organized as follows. Section 2 introduces the mathematical framework, defines the weighted energy functionals, and establishes fundamental differential inequalities. Section 3 presents the main alternative results, distinguishing between spatial ceasing and decay scenarios. Section 4 develops the conclusions and suggests directions for future research.

Throughout this paper, we employ standard mathematical notation: partial differentiation is denoted by comma subscripts (e.g., $u_{,i} = \partial u / \partial x_i$), and the summation convention applies to repeated Latin indices ranging from 1 to 3.

2. Basic Inequality

We begin by introducing an auxiliary functional that will play a central role in our subsequent analysis. This functional is designed to capture the energy distribution of the solution in the spatial domain.

Definition 1. *We define the function*

$$F_w(r,t) = \int_0^t \int_{D(r)} \exp(-ws)|\nabla u|^p u_{,i} u \frac{x_i}{r}\, da\, ds. \tag{5}$$

The weight function w in the definition of $F_w(r,t)$ is not fixed; it can be chosen arbitrarily as long as it satisfies the necessary integrability conditions. The decay estimates obtained in this paper depend explicitly on the choice of w, meaning that different weights yield different decay rates. Thus, w serves as a flexible parameter that allows one to tune the decay estimates according to the specific problem or desired precision.

Having defined this functional, we first establish a fundamental identity that describes its change with respect to the radial parameter.

Proposition 1. *For the function $F_w(r,t)$ defined in (5), we have the following equality:*

$$\begin{aligned} F_w(r+h,t) - F_w(r,t) = {} & \frac{w}{2}\int_0^t \int_{B(r+h,r)} \exp(-ws)c(x)u^2\,dx\,ds \\ & + \int_0^t \int_{B(r+h,r)} \exp(-ws)u\,f(u)\,dx\,ds \\ & + \frac{1}{2}\int_{B(r+h,r)} \exp(-wt)c(x)u^2\,dx \\ & + \int_0^t \int_{B(r+h,r)} \exp(-ws)|\nabla u|^{p+2}\,dx\,ds. \end{aligned} \tag{6}$$

Proof. The proof proceeds by a direct, albeit careful, computation of the difference $F_w(r+h,t) - F_w(r,t)$. We observe that this difference can be expressed as an integral over the annular region $B(r+h,r)$. Applying the divergence theorem and utilizing the differential equation satisfied by u, we obtain a series of terms. A crucial step involves integration by parts in time to handle the term containing $u_{,s}$, which ultimately gives rise to the final expression. The detailed calculation is as follows:

$$\begin{aligned} F_w(r+h,t) - F_w(r,t) &= \int_0^t \int_{B(r+h,r)} \exp(-ws)\frac{\partial}{\partial r}\Big(|\nabla u|^p u u_{,i}\frac{x_i}{r}\Big)dx\,ds \\ &= \int_0^t \int_{B(r+h,r)} \exp(-ws)\frac{\partial}{\partial x_i}(|\nabla u|^p u u_{,i})dx\,ds \\ &= \int_0^t \int_{B(r+h,r)} \exp(-ws)\Big[\frac{\partial}{\partial x_i}(|\nabla u|^p u_{,i})u + |\nabla u|^p u_{,i}u_{,i}\Big]dx\,ds \\ &= \int_0^t \int_{B(r+h,r)} \exp(-ws)\Big[c(x)u_{,s}u + uf(u) + |\nabla u|^{p+2}\Big]dx\,ds \\ &= \frac{w}{2}\int_0^t \int_{B(r+h,r)} \exp(-ws)c(x)u^2\,dx\,ds \\ &\quad + \int_0^t \int_{B(r+h,r)} \exp(-ws)u\,f(u)\,dx\,ds \\ &\quad + \frac{1}{2}\int_{B(r+h,r)} \exp(-wt)c(x)u^2\,dx \\ &\quad + \int_0^t \int_{B(r+h,r)} \exp(-ws)|\nabla u|^{p+2}\,dx\,ds. \end{aligned} \tag{7}$$

This completes the proof. □

From the integral identity established in Proposition 1, we can naturally derive a differential form by considering the limit as $h \to 0$. This leads to the following result concerning the radial derivative of F_w.

Proposition 2. *For the function $F_w(r,t)$ defined in (5), we have the following equality:*

$$\begin{aligned} \frac{\partial F_w(r,t)}{\partial r} = {} & \frac{w}{2}\int_0^t \int_{D(r)} \exp(-ws)c(x)u^2\,da\,ds \\ & + \frac{1}{2}\int_{D(r)} \exp(-wt)c(x)u^2\,da \\ & + \int_0^t \int_{D(r)} \exp(-ws)u\,f(u)\,da\,ds \\ & + \int_0^t \int_{D(r)} \exp(-ws)|\nabla u|^{p+2}\,da\,ds. \end{aligned} \tag{8}$$

Proof. Starting from the result of Proposition 1, we divide both sides of Equation (6) by h and examine the limit as $h \to 0$. An application of the Mean Value Theorem for integrals allows us to rewrite the quotient in terms of an intermediate value $\xi \in (r, r+h)$. Subsequently, taking the limit $h \to 0$ yields the derivative expression on the left-hand side and replaces the volume integrals over $B(r+h, r)$ with surface integrals over $D(r)$ on the right-hand side. Explicitly,

$$\begin{aligned}\frac{F_w(r+h,t) - F_w(r,t)}{h} &= \frac{w}{2h} \int_0^t \int_{B(r+h,r)} \exp(-ws) c(x) u^2 \, dx \, ds \\ &+ \frac{1}{2h} \int_{B(r+h,r)} \exp(-wt) c(x) u^2 \, dx \\ &+ \frac{1}{h} \int_0^t \int_{B(r+h,r)} \exp(-ws) u f(u) \, dx \, ds \\ &+ \frac{1}{h} \int_0^t \int_{B(r+h,r)} \exp(-ws) |\nabla u|^{p+2} \, dx \, ds.\end{aligned} \tag{9}$$

By the Mean Value Theorem, there exists $\xi \in (r, r+h)$ such that

$$\begin{aligned}\frac{F_w(r+h,t) - F_w(r,t)}{h} &= \frac{w}{2} \int_0^t \int_{D(\xi)} \exp(-ws) c(x) u^2 \, da \, ds \\ &+ \frac{1}{2} \int_{D(\xi)} \exp(-wt) c(x) u^2 \, da \\ &+ \int_0^t \int_{D(\xi)} \exp(-ws) u f(u) \, da \, ds \\ &+ \int_0^t \int_{D(\xi)} \exp(-ws) |\nabla u|^{p+2} \, da \, ds.\end{aligned} \tag{10}$$

Taking the limit as $h \to 0$, we have $\xi \to r$, and the desired Formula (8) follows immediately. □

Equipped with the derivative formula, we now establish a key differential inequality that relates the functional F_w to its own derivative. This inequality is the cornerstone for the subsequent spatial ceasing and decay analysis.

Proposition 3. *For the function $F_w(r,t)$ defined in (5), there exist positive constants a and b, and a positive function $k_1(t)$, such that the following inequality holds:*

$$|F_w(r,t)| \leq k_1(t) r^a \left(\frac{\partial F_w(r,t)}{\partial r} \right)^b. \tag{11}$$

Moreover, these constants satisfy $0 < a < b < 1$.

Proof. To prove this estimate, we start from the definition of F_w and apply a sequence of standard integral inequalities. First, Hölder's inequality is used to separate the terms involving $|\nabla u|$ and u. Then, a suitable interpolation inequality followed by another application of Hölder's inequality allows us to bound the term involving $|u|^{p+2}$. Finally, we employ the derivative expression from Proposition 2 to relate the resulting bound to $\partial F_w / \partial r$. The explicit computation is as follows:

$$
\begin{aligned}
|F_w(r,t)| &\le \left(\int_0^t\int_{D(r)} \exp(-ws)|\nabla u|^{p+2}\,da\,ds\right)^{\frac{p+1}{p+2}} \\
&\quad\cdot\left(\int_0^t\int_{D(r)} \exp(-ws)|u|^{p+2}\,da\,ds\right)^{\frac{1}{p+2}} \\
&\le \left(\int_0^t\int_{D(r)} \exp(-ws)|\nabla u|^{p+2}\,da\,ds\right)^{\frac{p+1}{p+2}} \\
&\quad\cdot\left(\int_0^t\int_{D(r)} \exp(-ws)|u|^{2p}\,da\,ds\right)^{\frac{1}{2p}} \\
&\quad\cdot\left(\int_0^t\int_{D(r)} \exp(-ws)\,da\,ds\right)^{\frac{p-2}{2p(p+2)}} \\
&\le \left[\frac{1}{w}(1-\exp(-wt))|D(r)|\right]^{\frac{p-2}{2p(p+2)}}\left(\frac{\partial F_w(r,t)}{\partial r}\right)^{\frac{p+1}{p+2}+\frac{1}{2p}} \\
&= \left[\frac{1}{w}(1-\exp(-wt))\right]^{\frac{p-2}{2p(p+2)}}(4\pi r^2)^{\frac{p-2}{2p(p+2)}}\left(\frac{\partial F_w(r,t)}{\partial r}\right)^{\frac{p+1}{p+2}+\frac{1}{2p}} \\
&= k_1(t)r^a\left(\frac{\partial F_w(r,t)}{\partial r}\right)^b,
\end{aligned}
\tag{12}
$$

where

$$
k_1(t) = (4\pi)^{\frac{p-2}{2p(p+2)}}\left[\frac{1}{w}(1-\exp(-wt))\right]^{\frac{p-2}{2p(p+2)}},\quad a = \frac{p-2}{p(p+2)},\quad b = \frac{p+1}{p+2}+\frac{1}{2p}.
$$

Since $p > 2$, we easily obtain $0 < a < 1$. Moreover, $\frac{1}{2p} < \frac{1}{p+2}$. We thus obtain $0 < b = \frac{p+1}{p+2} + \frac{1}{2p} < 1$. Since $a = \frac{p-2}{p(p+2)} < \frac{p-2}{p+2} < \frac{p+1}{p+2} < \frac{p+1}{p+2} + \frac{1}{2p} = b$, we can easily confirm that $0 < a < b < 1$. This completes the proof. □

Remark 1. *The constants a and b appearing in the decay estimates are derived from the fractional-order differential inequality and depend explicitly on the nonlinear exponent p. While these constants are sharp within the framework of our current method, it remains an open question whether they are optimal in the sense of capturing the exact asymptotic behavior. A possible theoretical pathway to sharper estimates would involve constructing matched upper and lower solutions or using a more refined weighted inequality to narrow the gap between the decay rate and the true behavior of the solution.*

3. Alternative Results

Based on the fundamental inequality derived in Proposition 3, we can now investigate the long-term behavior of the solution. The analysis naturally bifurcates into two distinct cases, depending on the sign of the functional F_w. Each case leads to a different qualitative outcome: either finite-time blow-up in the spatial variable or spatial decay at infinity.

The following discussions will be divided into two cases:

Case 1. For any fixed $t_0 > 0$, if there exists a $r_0 > 0$ such that $F_w(r_0, t_0) > 0$. Since $\frac{\partial F_w(r,t)}{\partial r} \ge 0$, we have $F_w(r, t_0) > 0$ for all $r > r_0$.

From inequality (11), we obtain

$$
F_w(r,t_0) \le k_1(t_0) r^a \left(\frac{dF_w(r,t_0)}{dr}\right)^b. \tag{13}
$$

We rewrite (13) in a form suitable for separation of variables:

$$
(F_w(r,t_0))^{\frac{1}{b}} \le (k_1(t_0))^{\frac{1}{b}} r^{\frac{a}{b}} \frac{dF_w(r,t_0)}{dr}. \tag{14}
$$

Separating the variables in Equation (14) yields

$$(F_w(r,t_0))^{-\frac{1}{b}}\,dF_w(r,t_0) \geq (k_1(t_0))^{-\frac{1}{b}} r^{-\frac{a}{b}}\,dr. \tag{15}$$

Integrating (15) from r_0 to r, we obtain

$$\frac{b}{b-1}\left((F_w(r,t_0))^{\frac{b-1}{b}} - (F_w(r_0,t_0))^{\frac{b-1}{b}}\right) \geq \frac{b(k_1(t_0))^{-\frac{1}{b}}}{b-a}\left(r^{\frac{b-a}{b}} - r_0^{\frac{b-a}{b}}\right). \tag{16}$$

Since $\frac{b}{b-1} < 0$ (as $0 < b < 1$), we can rearrange to get

$$(F_w(r,t_0))^{\frac{b-1}{b}} - (F_w(r_0,t_0))^{\frac{b-1}{b}} \leq \frac{(b-1)(k_1(t_0))^{-\frac{1}{b}}}{b-a}\left(r^{\frac{b-a}{b}} - r_0^{\frac{b-a}{b}}\right). \tag{17}$$

From (17), we deduce that

$$(F_w(r,t_0))^{\frac{b-1}{b}} \leq \frac{(b-1)(k_1(t_0))^{-\frac{1}{b}}}{b-a}\left(r^{\frac{b-a}{b}} - r_0^{\frac{b-a}{b}}\right) + (F_w(r_0,t_0))^{\frac{b-1}{b}}. \tag{18}$$

Observe that $\frac{b-1}{b-a} < 0$. Consequently, as $r \to +\infty$, the term $\frac{(b-1)(k_1(t_0))^{-\frac{1}{b}}}{b-a}\left(r^{\frac{b-a}{b}} - r_0^{\frac{b-a}{b}}\right)$ $\to -\infty$. This forces the right-hand side of (18) to become negative for sufficiently large r, implying $F_w(r,t_0) < 0$ (since $\frac{b-1}{b} < 0$). This contradicts the original assumption that $F_w(r,t_0) > 0$ for $r > r_0$. Moreover, there exists a finite radial distance $r^* = \left(\frac{b-a}{1-b}\right)^{\frac{b}{b-a}}(F_w(r_0,t_0))^{\frac{b-1}{b-a}} + r_0$, the solution ceases to exist as $r > r^*$. Therefore, this scenario indicates that the solution cannot persist for all r; it must cease to exist at a finite value of the spatial radius r.

Case 2. For any fixed $t_0 > 0$, $F_w(r,t_0) \leq 0$ for all $r \geq R_0$. We consider $-F_w(r,t_0) \geq 0$. Inequality (11) gives

$$-F_w(r,t_0) \leq k_1(t_0) r^a \left(\frac{dF_w(r,t_0)}{dr}\right)^b. \tag{19}$$

We rewrite (19) as

$$(-F_w(r,t_0))^{\frac{1}{b}} \leq (k_1(t_0))^{\frac{1}{b}} r^{\frac{a}{b}} \frac{dF_w(r,t_0)}{dr}. \tag{20}$$

Separating the variables yields

$$(-F_w(r,t_0))^{-\frac{1}{b}}\,d(-F_w(r,t_0)) \leq -(k_1(t_0))^{-\frac{1}{b}} r^{-\frac{a}{b}}\,dr. \tag{21}$$

Integrating (21) from R_0 to r ($r > R_0$), we obtain

$$\frac{b}{b-1}\left((-F_w(r,t_0))^{\frac{b-1}{b}} - (-F_w(R_0,t_0))^{\frac{b-1}{b}}\right) \leq -\frac{b(k_1(t_0))^{-\frac{1}{b}}}{b-a}\left(r^{\frac{b-a}{b}} - R_0^{\frac{b-a}{b}}\right). \tag{22}$$

Again, using $\frac{b}{b-1} < 0$, this implies

$$(-F_w(r,t_0))^{\frac{b-1}{b}} - (-F_w(R_0,t_0))^{\frac{b-1}{b}} \geq \frac{(1-b)(k_1(t_0))^{-\frac{1}{b}}}{b-a}\left(r^{\frac{b-a}{b}} - R_0^{\frac{b-a}{b}}\right). \tag{23}$$

From (23), we have

$$(-F_w(r,t_0))^{\frac{b-1}{b}} \geq \frac{(1-b)(k_1(t_0))^{-\frac{1}{b}}}{b-a}\left(r^{\frac{b-a}{b}} - R_0^{\frac{b-a}{b}}\right) + (-F_w(R_0,t_0))^{\frac{b-1}{b}}. \tag{24}$$

Since $\frac{b-1}{b} < 0$, raising both sides to the power $\frac{b}{b-1}$ (a negative exponent) reverses the inequality:

$$-F_w(r,t_0) \leq \left[\frac{(1-b)(k_1(t_0))^{-\frac{1}{b}}}{b-a}\left(r^{\frac{b-a}{b}} - R_0^{\frac{b-a}{b}}\right) + (-F_w(R_0,t_0))^{\frac{b-1}{b}}\right]^{-\frac{b}{1-b}}. \tag{25}$$

From (25), we conclude that $-F_w(r,t_0) \to 0$ as $r \to \infty$, because $\frac{b}{1-b} > 0$ and the bracket tends to $+\infty$. This indicates a decay property. To interpret this decay in terms of the solution's energy, we recall the identity from Proposition 1 in the limit as the outer radius tends to infinity. Integrating (6) (or its derivative form) from r to ∞ yields

$$\begin{aligned} -F_w(r,t_0) = {} & \frac{w}{2}\int_0^{t_0}\int_{B(\infty,r)} \exp(-ws)c(x)u^2\,dx\,ds \\ & + \frac{1}{2}\int_{B(\infty,r)} \exp(-wt_0)c(x)u^2\,dx \\ & + \int_0^{t_0}\int_{B(\infty,r)} \exp(-ws)u\,f(u)\,dx\,ds \\ & + \int_0^{t_0}\int_{B(\infty,r)} \exp(-ws)|\nabla u|^{p+2}\,dx\,ds. \end{aligned} \tag{26}$$

In (26), the first and second terms represent the potential energies (from material heterogeneity). The third term represents the nonlinear source energy. The last term represents the dissipation energy (from gradients).

Summarizing all the above discussions, we can now state our main theorem which encapsulates the two possible asymptotic behaviors.

Theorem 1. *Let u be a solution of the initial-boundary value problem determined by (1) and satisfying the relevant hypotheses (2)–(4). Then, for any fixed $t_0 > 0$, one of the following alternatives holds:*

(i) The solution ceases to exist for a finite value of the spatial variable r, or

(ii) The solution exists for all $r \geq R_0$ and satisfies the following spatial decay estimate for all $r \geq R_0 > 0$:

$$\begin{aligned} & \frac{w}{2}\int_0^{t_0}\int_{B(\infty,r)} \exp(-ws)c(x)u^2\,dx\,ds + \frac{1}{2}\int_{B(\infty,r)} \exp(-wt_0)c(x)u^2\,dx \\ & + \int_0^{t_0}\int_{B(\infty,r)} \exp(-ws)u\,f(u)\,dx\,ds + \int_0^{t_0}\int_{B(\infty,r)} \exp(-ws)|\nabla u|^{p+2}\,dx\,ds \\ & \leq \left[\frac{(1-b)(k_1(t_0))^{-\frac{1}{b}}}{b-a}\left(r^{\frac{b-a}{b}} - R_0^{\frac{b-a}{b}}\right) + (-F_w(R_0,t_0))^{\frac{b-1}{b}}\right]^{-\frac{b}{1-b}}. \end{aligned} \tag{27}$$

Remark 2. *The estimate (27) provides an explicit algebraic decay rate in r for the energy of the solution outside a ball of radius r, confirming its spatially localized nature in this case. In contrast to linear problems where the energy decays exponentially, our nonlinear case yields a polynomial decay in (27). This comparison highlights the slowing effect of the nonlinearity. The assumptions in Theorem 1 are essential for the results: $p > 2$ ensures $0 < a < 1$ in Proposition 3, which is key to the fractional-order inequality. $c(x) > 0$ guarantees the coercivity of the energy func-*

tional F_w. $uf(u) \geq |u|^{2p}$ provides the necessary lower bound to link the nonlinear source to the energy estimates.

4. Conclusions

This paper has investigated the spatial asymptotic behavior of solutions to a class of nonlinear parabolic equations defined on exterior domains in $\mathbb{R}^3$. By constructing a suitable weighted energy functional and employing a novel fractional-order differential inequality technique, we established a sharp Phragmén–Lindelöf-type alternative: either the solution's energy ceases to exist at a finite radial distance, or it decays algebraically to zero as the spatial variable tends to infinity. In the decay scenario, explicit polynomial-type estimates were derived, revealing distinct spatial localization properties compared to linear or differently nonlinear problems. Our results extend previous analyses beyond semi-infinite cylindrical geometries to more complex exterior domains, highlighting the impact of both nonlinearity and geometric setting on long-range spatial behavior. We have not discussed the stability of the spatial ceasing boundary. In practice, small perturbations in the boundary data $g(x,t)$ may change the cessation point r^* . Whether this shift is small or large depends on the sensitivity of the energy functional near the threshold, which is an interesting issue for future investigation.

Several promising directions for future research emerge from this work. First, extending the analysis to more general nonlinear structures or coupling mechanisms would be valuable. Second, considering time-dependent exterior domains or incorporating stochastic perturbations could lead to more realistic models. Third, investigating the possibility of sharpening the decay rates or establishing more refined blow-up criteria remains an open challenge. Finally, numerical validation of the theoretical estimates and exploration of potential applications in continuum mechanics would further strengthen the practical relevance of these findings.

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