axioms

MDPI

Article

Boundedness and Applications of Fractional Integral Operators in Nonlocal Problems with Fractional Laplacians

Saba Mehmood [1,*], Dušan J. Simjanović [2] and Branislav M. Randjelović [3,*]

1 Department of Mathematics, School of Science, University of Management and Technology, Lahore 54700, Pakistan
2 Faculty of Information Technology, Belgrade Metropolitan University, 11000 Belgrade, Serbia; dusan.simjanovic@metropolitan.ac.rs
3 Faculty of Electronic Engineering, University of Niš,18000 Niš, Serbia
* Correspondence: saba.mehmood@umt.edu.pk (S.M.); bane@elfak.ni.ac.rs (B.M.R.)

Abstract

In this paper, we investigate the properties of the boundedness of fractional integral operators K_α defined on general measure metric spaces. We study their action in Lebesgue spaces $L^p(Y)$, Morrey spaces $L^p_\varphi(Y)$, and extend our analysis to fractional Sobolev spaces $W^{\alpha,p}(Y)$. Using classical dyadic decomposition and the Hardy–Littlewood maximal operator, we establish sharp bounds for K_α in terms of kernel parameters and the geometric structure of the space. A significant contribution of this work is the proof that K_α is bounded from $W^{\alpha,p}(Y)$ to $L^q(Y)$, where thus linking our operator-theoretic framework with the theory of nonlocal and fractional partial differential equations. These results provide valuable tools for studying regularity, a priori estimates, and solution mappings in nonlocal problems involving the fractional Laplacian and related operators on irregular or non-Euclidean domains.

Keywords: fractional integral; Lebesgue spaces; Morrey spaces; Sobolev spaces; fractional Laplacian

MSC: 42B25; 42B35; 46E30; 47B34

1. Introduction

The fractional integral operator, commonly known as the Riesz potential, is a singular integral operator defined by convolution with a Riesz kernel. These operators have been widely used to analyze the structure of metric measure spaces. Previous research has investigated the boundedness properties of fractional integral operators in various function spaces [1,2].

Ho studied the boundedness of fractional integral operators on grand Morrey spaces and grand Hardy-Morrey spaces in the Euclidean setting [3]. Similarly, Sugano and Tanaka established boundedness results for fractional integral operators on generalized Morrey spaces [4]. The weighted inequalities for fractional maximal functions and integral operators on weighted Morrey spaces are given by Pan and Sun [5].

Eridani et al. established the boundedness of fractional integral operators, specifically Riesz potentials, on Morrey spaces over quasi-metric measure spaces [6]. They derived Sobolev, trace, and power-weight inequalities [7]. Meanwhile, Burenkov and his co-authors provided necessary and sufficient conditions for the boundedness of fractional maximal operators M_α in local Morrey spaces [8]. Burenkov and Goldman also characterized the

Check for updates

Academic Editor: Roman Dmytryshyn

Received: 4 January 2026
Revised: 6 March 2026
Accepted: 9 March 2026
Published: 16 March 2026

boundedness of the Hardy–Littlewood maximal operator M from Lebesgue spaces L^p to Morrey spaces L^p_ϕ [9].

Previous work has shown the boundedness of Bessel–Riesz operators, which are a special type, on Lebesgue and Morrey spaces over measure metric spaces. This was done by Mehmood and co-authors using dyadic decomposition and maximal operators [10–12]. Idris and colleagues demonstrated similar kernel-controlled boundedness for Bessel–Riesz operators on Morrey spaces through weighted Young's inequalities [13,14]. Earlier, Chiarenza established the boundedness for fractional integrals connected to the Hardy–Littlewood maximal function in Morrey spaces [15]. Additional results on weighted and generalized Morrey spaces can be found in the work of Gürbüz [16–18].

In this paper, we present a fresh approach to proving boundedness for general fractional integral operators, called Riesz potentials, on Lebesgue and Morrey spaces over quasi-metric measure spaces. What makes our work different is clear and simple. Earlier studies, such as that by Mehmood et al., focused only on the special case of Bessel–Riesz operators in measurement metric spaces using dyadic methods [10]. Eridani et al. handled fractional integrals on quasi-metric Morrey spaces but did not directly control the operator norm by the Riesz kernel norm [6]. Ho and Sugano–Tanaka worked on grand or generalized Morrey spaces but stayed in the Euclidean setting [3,4,19].

In contrast, we prove that for general Riesz potentials, not just Bessel–Riesz, the operator norm is directly controlled by the norm of the Riesz kernel itself. We achieve this using Hardy–Littlewood maximal operators rather than relying on Young's inequality, which has limitations in general spaces [20]. This direct kernel control works in broader quasi-metric measure spaces and provides sharper estimates than before.

The recent study by Mehmood et al. focused specifically on the Bessel–Riesz operators [10,21,22]. Here, we extend the analysis to all Riesz potentials with precise control over kernel parameters. Our mapping results from fractional Sobolev spaces $W^{\alpha,p}(Y)$ to Lebesgue spaces $L^q(Y)$ build on previous work [4,6,10,23] and support solutions to nonlocal fractional PDEs on non-Euclidean domains [24,25]. These bounds help us obtain regularity estimates and a priori controls.

2. Preliminaries

Definition 1 ([11])**.** *Σ, which is a collection of subsets of a non-empty set Y, is said to be a sigma algebra if*

1. *If $E \in \Sigma$ then $E^c \in \Sigma$.*
2. *Suppose that a countable collection of subsets $E_1, E_2, E_3 \ldots$ is in Σ, then their union $U_{i=1}^{\infty} E_i \in \Sigma$.*
3. *The intersection of a countable collection of subsets $\{E_i\}_{i=1}^{\infty}$ is an element of the sigma-algebra Σ, that is, $\bigcap_{i=1}^{\infty} E_i \in \Sigma$.*

Definition 2. *A function $\nu : \Sigma \to R^+$ is a measure on Σ such that*

1. *$\nu(\phi) = 0$, where ϕ is the empty set .*
2. *If $E_1, E_2, E_3 \ldots$ are disjoint sets in Σ, then $\nu(\bigcup_{i=1}^{\infty} E_i) = \sum_1^{\infty} \nu(E_i)$.*

Definition 3 ([10])**.** *Let $Y := (Y, \sigma, \nu)$ be a topological space equipped with a complete measure ν, such that the space of compactly supported continuous functions is dense in $L^1(Y, \nu)$. Additionally, there exists a function $\sigma : Y \times Y \to [0, \infty)$, referred to as a quasi-metric function, which satisfies the following conditions:*

1. *$\sigma(y, z) > 0\ \forall y \neq z \in Y$ and $\sigma(y, y) = 0$ for all $y \in Y$;*
2. *There exists a constant $c_0 \geq 1$ such that, for any $y, z \in Y$, $\sigma(y, z) \leq c_0 \sigma(z, y)$.*

3. *For any $x, y, z \in Y$, there exists a constant $c_1 \geq 1$, such that $\sigma(x,z) \leq c_1(\sigma(x,y) + \sigma(y,z))$).*

The balls $G(b,r) = \{y \in Y : \sigma(b,y) < r\}$ are assumed to be ν measurable and for $b \in Y$, $r > 0, 0 < \nu(G(b,r)) < \infty$. Suppose V is a neighborhood of $y \in Y$, then there exists $0 < r$, such that the ball $G(y,r) \subset V$. For any $b \in Y, 0 < r_1 < r_2 < \infty$, we assume that $\nu(Y) = \infty, \nu(b) = 0$. A quasi-metric measure space will be defined as the triple (Y, σ, ν).

Definition 4 ([13])**.** *We shall have the properties $\nu(G) \sim r^\beta$ for each ball $G = G(b,r)$ in measurement metric spaces (Y, σ, ν). In the case $0 < \alpha < \beta$, we define*

$$K_\alpha(y,z) = \sigma(y,z)^{\alpha-\beta}; \quad y, z \in Y.$$

Here $\sigma(y,z)^{\alpha-1}$ (used later in the operator) follows the standard Riesz potential convention, while K_α accounts for the space's doubling dimension β. Suppose that $a \in Y$ and $r > 0$, then

$$\int_Y |K_\alpha| d\nu(y) = \sum_{m\in\mathbb{Z}} \int_{2^m r \leq \sigma(a,y) < 2^{m+1} r} |K_\alpha| d\nu(y) \sim \sum_{m\in\mathbb{Z}} (2^m r)^\alpha.$$

As long as $0 < \alpha$, $K_\alpha \in L^1(\nu)$.

When $z = y$, $\sigma(y,y) = 0$ makes $\sigma(y,z)^{\alpha-1}$ singular since $\alpha - 1 < 0$. Here $\epsilon > 0$ is a small radius that excludes the singularity. We introduce the fractional integral operator in the principal value sense.

$$(K^\alpha h)(y) := \lim_{\epsilon\to 0^+} \int_{\{z:\sigma(y,z)>\epsilon\}} \sigma(y,z)^{\alpha-1} h(z)\, d\nu(z); \tag{1}$$

$$y \in Y,\ 0 < \alpha < 1. \tag{2}$$

This excludes the singularity at $z = y$. The integral converges because near $z = y$, $\nu(G(y,r)) \sim r^\beta$ and

$$\int_0^r t^{\alpha-1} t^\beta \frac{dt}{t} \sim \int_0^r t^{\alpha+\beta-2}\, dt < \infty \tag{3}$$

since $\alpha + \beta - 1 > 0$. The dyadic decomposition automatically starts from $\sigma(y,z) \geq \epsilon > 0$.

Definition 5. *Suppose that $1 \leq p < \infty$. The Lebesgue space is denoted by $L^p(\nu)$ and is defined as a collection of measurable functions h such that the following is written:*

$$\|h : L^p_\nu\| = \left(\int_Y |h(y)|^p d\nu(y)\right)^{1/p}. \tag{4}$$

We introduce the following operator (the fractional integral operator):

$$(K_\alpha h(y)) := \int_Y \sigma(y,z)^{\alpha-1} h(z) d\nu(z);\ y \in Y;\ 0 < \alpha < 1. \tag{5}$$

Definition 6 ([26])**.** *Let $1 \leq q < p \leq \infty$. Suppose that there is a function ϕ defined from $\mathbb{R}^+$ to $\mathbb{R}^+$, i.e., $\phi : \mathbb{R}^+ \to \mathbb{R}^+$. We use the notation $L^p_\phi(\nu) = L^p_\phi(Y, \nu)$ to define Morrey spaces as the collection of all functions $f \in L^p_{loc}(Y)$ such that*

$$\|f\|_{L^p_\phi(\nu)} := \sup_G \frac{1}{\phi(G)} \left(\frac{1}{\nu(G)} \int_G |f(y)|^p\, d\nu(y)\right)^{1/p} < \infty, \tag{6}$$

where $\phi(G) = \phi(\nu(G))$.

https://doi.org/10.3390/axioms15030220

3. Main Results

Theorem 1. *Let $1 < p < \infty$ and $0 < \alpha < 1$. Then the fractional integral operator K_α is bounded from $L^p(\nu)$ to $L^p(\nu)$:*

$$\|K_\alpha h\|_{L^p(\nu)} \leq C\|K_\alpha\|_{L^1(\nu)}\|h\|_{L^p(\nu)}$$

for all $h \in L^p(\nu)$, where $C > 0$ is a universal constant.

Proof. The Hardy–Littlewood maximal operator is defined as

$$Mh(y) = \sup_{r>0} \frac{1}{\nu(B(y,r))} \int_{B(y,r)} |h(z)|\, d\nu(z), \quad B(y,r) = \{z \in Y : \sigma(y,z) < r\}.$$

Assume $h \geq 0$. For each $r > 0$, $y \in Y$,

$$K_\alpha h(y) = \int_Y \sigma(y,z)^{\alpha-1} h(z)\, d\nu(z) = I_1(y,r) + I_2(y,r),$$

where $I_1(y,r) = \int_{\sigma(y,z)<r}$ and $I_2(y,r) = \int_{\sigma(y,z)\geq r}$.

For $I_1(y,r)$, dyadic decomposition gives:

$$\begin{aligned} I_1(y,r) &= \sum_{m=-\infty}^{-1} \int_{2^m r \leq \sigma(y,z) < 2^{m+1} r} \sigma(y,z)^{\alpha-1} h(z)\, d\nu(z) \\ &\lesssim \sum_{m=-\infty}^{-1} (2^m r)^{\alpha-1} \int_{B(y,2^{m+1}r)} h(z)\, d\nu(z) \\ &\leq \|K_\alpha\|_{L^1(\nu)} Mh(y), \end{aligned}$$

where the doubling condition $\nu(B(y,2^{m+1}r)) \sim (2^m r)^\beta$ justifies the estimates.

For $I_2(y,r)$,

$$\begin{aligned} I_2(y,r) &\lesssim \sum_{m=0}^{\infty} (2^m r)^{\alpha-1} \int_{B(y,2^{m+1}r)} h(z)\, d\nu(z) \\ &\leq \|h\|_{L^p(\nu)} \sum_{m=0}^{\infty} (2^m r)^{\alpha-1} \nu(B(y,2^{m+1}r))^{1-1/p} \\ &\lesssim r^{-1/p} \|K_\alpha\|_{L^1(\nu)} \|h\|_{L^p(\nu)}. \end{aligned}$$

Choose $r > 0$ such that $Mh(y) = r^{-1/p}\|h\|_{L^p(\nu)}$. Then

$$K_\alpha h(y) \leq C\|K_\alpha\|_{L^1(\nu)} Mh(y).$$

Since $M : L^p(\nu) \to L^p(\nu)$ is bounded, integrate to obtain

$$\|K_\alpha h\|_{L^p(\nu)} \leq C\|K_\alpha\|_{L^1(\nu)}\|Mh\|_{L^p(\nu)} \leq C\|K_\alpha\|_{L^1(\nu)}\|h\|_{L^p(\nu)}.$$

For general h, write $h = h^+ - h^-$ and use linearity with triangle inequality. □

Note: The proof assumes $h \geq 0$ simplification of estimates involving the positive kernel. For $h < 0$, split $h = h^+ - h^-$ (both ≥ 0), apply linearity: $K^\alpha h = K^\alpha h^+ - K^\alpha h^-$, and use the bound on each part plus the triangle inequality to get the full result.

Theorem 2. *Suppose that $0 < \alpha < 1$, $1 < p < \infty$, and ϕ is a decreasing function. Then there exists a positive constant C, such that*

$$\|K_\alpha h : L^p_\phi(\nu)\| \le C\|K_\alpha : L^1(\nu)\| \cdot \|h : L^p_\phi(\mu)\|. \tag{7}$$

Proof. Let $\tilde{G} = G(b, 2r)$, $G = G(b, r)$. Write $h = h_1 + h_2$ with $h_1 = h\chi_{\tilde{G}}$, $h_2 = h\chi_{\tilde{G}^c}$.

Case 1: h_1. Since ϕ is decreasing,

$$\|h_1\|_{L^p(\nu)} \le \phi(\tilde{G})\nu(\tilde{G})^{1/p}\|h\|_{L^p_\phi(\nu)} \le \phi(G)(2\nu(G))^{1/p}\|h\|_{L^p_\phi(\nu)},$$

by doubling property $\nu(\tilde{G}) \le 2\nu(G)$. Boundedness of $K_\alpha : L^p(\nu) \to L^p(\nu)$ gives

$$\int_G |K_\alpha h_1|^p d\nu \le C\|K_\alpha\|^p_{L^1(\nu)}\|h_1\|^p_{L^p(\nu)} \le C\phi(G)^p\nu(G)\|h\|^p_{L^p_\phi(\nu)}.$$

Case 2: h_2. For $y \in G(b, r)$,

$$|K_\alpha h_2(y)| \le \sum_{m=1}^{\infty} \int_{2^m r \le \sigma(y,z) < 2^{m+1} r} \sigma(y,z)^{\alpha-1}|h(z)|d\nu(z).$$

On each annulus, $\sigma(y,z)^{\alpha-1} \le (2^m r)^{\alpha-1}$, so

$$|K_\alpha h_2(y)| \lesssim \sum_{m=1}^{\infty} (2^m r)^{\alpha-1} \int_{G(y, 2^{m+1} r)} |h| d\nu.$$

By L^p_ϕ definition, $\int_{G(y,2^{m+1}r)} |h|d\nu \le C\phi(2^{m+1}r)\|h\|_{L^p_\phi(\nu)}$. Thus,

$$|K_\alpha h_2(y)| \lesssim \|h\|_{L^p_\phi(\nu)} \sum_{m=1}^{\infty} (2^m r)^{\alpha-1}\phi(2^{m+1}r).$$

Since ϕ decreasing, $\phi(2^{m+1}r) \le \phi(r)$, and

$$\sum_{m=1}^{\infty} (2^m r)^{\alpha-1}\phi(2^{m+1}r) \le \phi(r) \sum_{m=1}^{\infty} 2^{m(\alpha-1)} = \phi(r) \cdot \frac{2^{\alpha-1}}{1 - 2^{\alpha-1}} < \infty,$$

geometric series converges as $2^{\alpha-1} < 1$. So $|K_\alpha h_2(y)| \le C\phi(r)\|h\|_{L^p_\phi(\nu)}$.

Both terms bounded by $C\phi(r)\|h\|_{L^p_\phi(\nu)}$. Taking sup over balls gives the result. □

Theorem 3. *Suppose that $1 < p < \infty$, $0 < \alpha < 1$, and ϕ is a decreasing function. Assume $K_\alpha : L^p(\nu) \to L^p(\nu)$ bounded via kernel*

$$|K_\alpha h(y)| \lesssim \int \sigma(y,z)^{\alpha-1}|h(z)|d\nu(z) \le C\|h\|_{L^p},$$

with Hölder weight $\sigma(y,z)^{1+p(\alpha-1)}$ on annuli. Then there exists $C > 0$ such that

$$\|K_\alpha h : L^p_\phi(\nu)\| \le C\|K_\alpha : L^p(\nu)\| \cdot \|h : L^p_\phi(\nu)\|. \tag{8}$$

Proof. For $G := G(b, r)$, $\tilde{G} := G(b, 2r)$, let $h = h_1 + h_2 := h\chi_{\tilde{G}} + h\chi_{\tilde{G}^c}$, $h \in L^p_\phi(\nu)$.

Since $\|h_1 : L^p(\nu)\| \leq \phi(\tilde{G})\nu(\tilde{G})^{1/p}\|h : L^p_\phi(\nu)\| < \infty$, we have $h_1 \in L^p(\nu)$. Then

$$\left(\int_G |K_\alpha h_1(z)|^p \, d\nu(z)\right)^{1/p} = \|K_\alpha h_1 : L^p(\nu)\| \leq C\|K_\alpha : L^p(\nu)\| \cdot \|h_1 : L^p(\nu)\|$$
$$\leq C\phi(G)\nu(G)^{1/p}\|K_\alpha : L^p(\nu)\| \cdot \|h : L^p_\phi(\nu)\|.$$

For h_2, $y \in G(b,r)$: Hölder gives

$$\begin{aligned}|K_\alpha h_2(y)| &\leq \sum_{m=1}^{\infty} \int_{2^m r \leq \sigma(y,z) < 2^{m+1} r} \sigma(y,z)^{\alpha-1}|h(z)|d\nu(z) \\ &\leq \sum_{m=1}^{\infty} (2^m r)^{1+p(\alpha-1)} \int_{G(y,2^{m+1}r)} |h(z)|d\nu(z) \\ &\leq C\|h : L^p_\phi(\nu)\| \sum_{m=1}^{\infty} (2^m r)^{1+p(\alpha-1)}\phi(2^{m+1}r) \\ &\leq C\phi(r)\|h : L^p_\phi(\nu)\| \cdot \|K_\alpha : L^p(\nu)\|,\end{aligned}$$

since ϕ decreasing and geometric series $\sum 2^{m(1+p(\alpha-1))} < \infty$ as $1 + p(\alpha - 1) < 0$.

This proves the theorem. □

Lemma 1. *If $\nu(G(b,r)) \sim r^\beta$, then*

$$\int_Y K_\alpha(y,z)d\nu(z) < \infty \tag{9}$$

where $K_\alpha(y,z) = \sigma(y,z)^{\alpha-\beta}$.

Proof.
$$\int_Y K_\alpha(y,z)d\nu(z) = \int_{\sigma(y,z)<r} K_\alpha(y,z)d\nu(z) + \int_{\sigma(y,z)\geq r} K_\alpha(y,z)d\nu(z)$$

By simplifying,

$$\begin{aligned}\int_{\sigma(y,z)<r}(K_\alpha(y,z)d\nu(z)) &= \sum_{m=-\infty}^{-1} \int_{2^m r \leq \sigma(y,z) < 2^{m+1} r} (K_\alpha(y,z)d\nu(z)) \\ &\sim \sum_{m=-\infty}^{-1} (2^m r)^{\alpha-\beta} \int_{\sigma(y,z)<2^{m+1}r} (K_\alpha(y,z)d\nu(z)) \\ &\leq C_1 \sum_{m=-\infty}^{-1} (2^m r)^{\alpha-\beta}\nu(\beta_{2^{m+1}r}(y)) \\ &= C_1 \sum_{m=-\infty}^{-1} (2^m r)^{\alpha} \leq C_1^\alpha < \infty \\ \int_{\sigma(y,z)\geq r} K_\alpha(y,z)d\nu(z) &= \sum_{m=0}^{\infty} \int_{2^m r \leq \sigma(y,z) < 2^{m+1} r} (K_\alpha(y,z)d\nu(z)) \\ &\sim \sum_{m=0}^{\infty} (2^m r)^{\alpha-\beta} \leq C_2 r^{\alpha-\beta} < \infty.\end{aligned}$$

So for each $y \in Y$ and $r > 0$,

$$\int_Y K_\alpha(y,z)d\nu(z) \leq C\left(r^\alpha + r^{\alpha-\beta}\right),$$
$$\int_Y K_\alpha(y,z)d\nu(z) < \infty.$$

The following theorem may be used to demonstrate that K_α is a member of some Lebesgue spaces. □

Theorem 4. *Given $0 < \alpha < \beta$, then for every $0 < r$, we have*

$$||K_\alpha : L^p||^p \sim \sum_{m\in\mathbb{Z}} (2^m r)^{(\alpha-\beta)p+\beta}, \; 1 \le \beta/(\beta-\alpha) = p. \tag{10}$$

Proof. For each $r > 0$ we have

$$\begin{aligned}\int_Y |K_\alpha(y,z)|^p d\nu(z) &= \int_{0\le\sigma(y,z)} |K_\alpha(y,z)|^p d\nu(y)\\ &= \sum_{m\in\mathbb{Z}} \int_{2^m r\le\sigma<2^{m+1}r} \sigma(y,z)^{(\alpha-\beta)p} d\nu(z)\\ &\sim \sum_{m\in\mathbb{Z}} (2^m r)^{(\alpha-\beta)p} \int_{2^m r\le\sigma<2^{m+1}r} d\nu(z)\\ &\sim \sum_{m\in\mathbb{Z}} (2^m R)^{(\alpha-\beta)p+\beta}.\end{aligned}$$

By Lemma (1), for $1 \le p < \infty$ we specify $K_\alpha \in L^p(\mu)$, if and only if

$$||K_\alpha : L^p|| = \left(\int_{R^n} |K_\alpha(y,z)|^p d\nu(z)\right)^{1/p} < \infty. \tag{11}$$

As may be seen from the definition above,

$$||K_\alpha : L^p|| < \infty \iff 1 \le \beta/(\beta-\alpha) = p. \tag{12}$$

Using the fact that $K_\alpha \in L^p(Y)$, the boundedness of fractional operators $K_\alpha h(y)$ can be established on Lebesgue spaces $L^p(Y)$ using the well-known inequality of Young. □

Theorem 5. *Suppose that $p, q, r \in [1,\infty]$ such that*

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1.$$

For a function $h_1 \in L^p(\mathbb{R}^n)$ and a function $h_2 \in L^q(\mathbb{R}^n)$, then $h_1 * h_2$ exists and is a member of $L^r(\mathbb{R}^n)$. Furthermore,

$$||h_1 * h_2||_r \le ||h_1||_p ||h_2||_q. \tag{13}$$

Boundedness of Fractional Integral Operator on Morrey Space:
In this part, we will explore the boundedness of the fractional integral operator on Morrey spaces within measure metric spaces. As noted earlier, $K_\alpha \in L^p(Y)$, where $1 \le \frac{\beta}{\beta-\alpha} = p$. Considering the inclusion property of Morrey spaces, it follows that $K_\alpha \in L^p_\phi(\nu)$. Consequently, the following theorem can be established.

Theorem 6. *Let $\beta > \alpha > 0, 1 < p_1 < q_1 < p$, $p \ge s \ge 1$, with interpolation parameters*

$$\frac{1}{p_2} = \frac{1}{p_1} - \frac{q_1}{p_1 p'}, \quad \frac{1}{q_2} = \frac{1}{q_1} - \frac{1}{p'},$$

where $\frac{\beta}{\beta-\alpha} = p'$ *and* $M_q : L^{p_1,q_1} \to L^{p_1,q_1}$ *bounded. Then*

$$\|K_\alpha h\|_{L^{p_2,q_2}(\nu)} \le C\|K_\alpha\|_{L^p_\phi(\nu)}\|h\|_{L^{p_1,q_1}(\nu)} \tag{14}$$

for all $h \in L^{p_1,q_1}(\nu)$.

Proof. Decompose for each $y \in Y$, $r > 0$:

$$|K_\alpha h(y)| = I_1(y) + I_2(y),$$
$$I_1(y) = \int_{\sigma(y,z)<r} \sigma(y,z)^{\alpha-\beta} h(z)d\nu(z),$$
$$I_2(y) = \int_{\sigma(y,z)\ge r} \sigma(y,z)^{\alpha-\beta} h(z)d\nu(z).$$

Step 1: I_1 estimate (near field). Dyadic decomposition:

$$|I_1(y)| \le \sum_{m=-\infty}^{-1} (2^m r)^{\alpha-\beta} \int_{2^m r\le\sigma(y,z)<2^{m+1}r} |h(z)|d\nu(z).$$

By Hölder with exponents s, s':

$$\int |h|d\nu \le \|h\|^{q_1/p_1}_{L^{p_1}}\|1\|_{L^{s'}} \le CM_q h(y)(2^m r)^{\beta/s},$$

so

$$|I_1(y)| \le CM_q h(y) \sum_{m=-\infty}^{-1} (2^m r)^{\alpha-\beta+\beta/s}(2^m r)^{\beta/s'} r^{-\beta/s}.$$

the series converges: $\alpha - \beta + \beta/s + \beta/s' = \alpha - \beta < 0$. Thus

$$|I_1(y)| \le C\|K_\alpha\|_{L^p_\phi} M_q h(y) r^{\beta/p'}.$$

Step 2: I_2 estimate (far field).

$$|I_2(y)| \le \sum_{m=0}^{\infty} (2^m r)^{\alpha-\beta} \int_{2^m r\le\sigma(y,z)<2^{m+1}r} |h(z)|d\nu(z).$$

Lorentz L^{p_1,q_1} gives the

$$\int |h|d\nu \le C\|h\|_{L^{p_1,q_1}}(2^m r)^{\beta(1/p_1-1/q_1)},$$

so

$$|I_2(y)| \le C\|h\|_{L^{p_1,q_1}} \sum_{m=0}^{\infty} (2^m r)^{\alpha-\beta+\beta(1/p_1-1/q_1)}.$$

Exponent: $\alpha - \beta + \beta/p_1 - \beta/q_1 = \beta(1/p' - 1/q_1) < 0$, series converges. Thus

$$|I_2(y)| \le C\|K_\alpha\|_{L^p_\phi}\|h\|_{L^{p_1,q_1}} r^{\beta(1/p'-1/q_1)}.$$

Step 3: Lorentz norm. Choose $r^{\beta/q_1} = \|h\|_{L^{p_1,q_1}}/M_q h(y)$:

$$|K_\alpha h(y)| \le C\|K_\alpha\|_{L^p_\phi}\|h\|^{q_1/p'}_{L^{p_1,q_1}} M_q h(y)^{1-q_1/p'}.$$

Lorentz interpolation parameters give $q_1/p' = 1 - p_1/p_2$. For $B(y,r)$:

$$\left(\int_{B(y,r)} |K_\alpha h|^{p_2} d\nu\right)^{1/p_2} \le C\|K_\alpha\|_{L^p_\phi}\|h\|^{1-p_1/p_2}_{L^{p_1,q_1}}\left(\int_{B(y,r)} |M_q h|^{p_1} d\nu\right)^{1/p_2}.$$

Normalize by $r^{\beta/p_2-\beta/q_2}$ and taking the supremum:

$$\|K_\alpha h\|_{L^{p_2,q_2}} \le C\|K_\alpha\|_{L^p_\phi}\|h\|_{L^{p_1,q_1}}.$$

By Chiarenza-Frasca, $M_q : L^{p_1,q_1} \to L^{p_1,q_1}$ bounded. □

Corollary 1. *Suppose that we have $p_1 = q_1, p_2 = q_2$ and $s = p, 1 \le \beta/\beta - \alpha = p$. Suppose for a positive constant $C_1, \nu(G(b,r)) \le C_1 r^\beta$, $h \in L^{p_1}(\nu)$ and $K_\alpha \in L^s(\nu)$, then there exists a positive constant C_2, such that*

$$||K_\alpha h(y) : L^{q_1}(\nu)|| \le C_2||K_\alpha : L^s(\nu)||.||h : L^{p_1}(\nu)||s. \tag{15}$$

According to the corollary above, $K_\alpha h(y)$ is a bounded operator from $L^{p_1}(\nu)$ to $L^{q_1}(\nu)$. Furthermore, the norm of $K_\alpha h(y)$ is controlled by the norm of the Riesz kernel, which ensures the boundedness of $K_\alpha h(y)$.

4. Implications and Analytical Foundations for Fractional Integral Operators

Theorem 7. *The fractional integral operator K_α provides the explicit embedding $W^{\alpha,p}(Y) \hookrightarrow L^q(Y)$ where*

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\beta}.$$

For the fractional Poisson equation $(-\Delta)^\alpha u = f$, the weak solution is $u = (-\Delta)^{-\alpha} f$. Since the inverse fractional Laplacian equals the Riesz potential:

$$(-\Delta)^{-\alpha} f(y) = K_\alpha f(y) = \int_Y \frac{f(z)}{\sigma(y,z)^{\beta-\alpha}} d\nu(z).$$

Proof. Consider the fractional integral operator

$$u(y) = K_\alpha f(y) = c_{\alpha,\beta}\int_Y \frac{f(z)}{\sigma(y,z)^{\beta-\alpha}}\, d\nu(z).$$

Decompose at radiu:$r > 0$:

$$I_1(y) = \int_{\{\sigma(y,z)<r\}} \frac{f(z)}{\sigma(y,z)^{\beta-\alpha}}\, d\nu(z) \lesssim r^\alpha Mf(y),$$
$$I_2(y) = \int_{\{\sigma(y,z)\ge r\}} \frac{f(z)}{\sigma(y,z)^{\beta-\alpha}}\, d\nu(z) \lesssim r^\alpha Mf(y).$$

Optimal choice $r = (Mf(y))^{-1/\beta}$ yields

$$|u(y)| \lesssim (Mf(y))^{1-\alpha/\beta}.$$

With parameter relation $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\beta}$ (i.e., $q(1-\alpha/\beta) = p$) and $M : L^p \to L^p$ bounded, we obtain the embedding

$$\|u\|_{L^q(Y)} \le C\|f\|_{W^{\alpha,p}(Y)}.$$

This establishes the explicit Sobolev embedding $W^{\alpha,p}(Y) \hookrightarrow L^q(Y)$ for the fractional Poisson equation $(-\Delta)^\alpha u = f$. $\square$

Theorem 8. *The inverse fractional Laplacian satisfies*

$$(-\Delta)^{-\alpha} f(y) = K_\alpha f(y) = \int_Y \frac{f(z)}{\sigma(y,z)^{\beta-\alpha}} d\nu(z). \tag{16}$$

Theorem 3 proves: $\|K_\alpha f\|_{L^q} \le C\|f\|_{W^{\alpha,p}}$, giving precise a priori bounds for weak solutions of $(-\Delta)^\alpha u = f$.

Proof. For $(-\Delta)^\alpha u = f$, apply the following inverse operator:

$$u = (-\Delta)^{-\alpha} f = K_\alpha f. \tag{17}$$

Split integral at radius $r > 0$:

$$K_\alpha f(y) = \int_{\sigma(y,z)<r} + \int_{\sigma(y,z)\ge r} = I_1(y) + I_2(y). \tag{18}$$

Local term estimate:

$$I_1(y) \lesssim r^\alpha M f(y) \quad \text{(dyadic annuli)}. \tag{19}$$

Nonlocal term estimate:

$$I_2(y) \lesssim r^\alpha M f(y) \quad \text{(volume growth } \nu(B) \sim r^\beta). \tag{20}$$

Optimal radius choice:

$$r = (Mf(y))^{-1/\beta} \quad \Longrightarrow \quad |K_\alpha f(y)| \lesssim (Mf(y))^{1-\alpha/\beta}. \tag{21}$$

Parameter scaling $q(1-\alpha/\beta) = p$ and $M : L^p \to L^p$ bounded gives:

$$\|K_\alpha f\|_{L^q} \le C\|f\|_{W^{\alpha,p}}. \tag{22}$$

Thus $u = K_\alpha f$ satisfies explicit L^q regularity bound. $\square$

The Riesz potential solves the fractional Poisson equation:

$$(-\Delta)^\alpha u = f \quad \Longleftrightarrow \quad u = K_\alpha f, \quad K_\alpha f(y) = c_{\alpha,\beta} \int_Y \frac{f(z)}{\sigma(y,z)^{\beta-\alpha}}\, d\nu(z). \tag{23}$$

Dyadic decomposition yields the pointwise estimate:

$$|K_\alpha f(y)| \lesssim (Mf(y))^{1-\alpha/\beta}. \tag{24}$$

The relation $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\beta}$ with $M : L^p \to L^p$ bounded gives:

$$K_\alpha : L^p(Y) \to L^q(Y), \quad \mathcal{L}^\Phi(Y) \to \mathcal{L}^\Psi(Y), \quad L^{p,\tau}(Y) \to L^{q,\tau}(Y), \quad \mathcal{M}^\lambda_p(Y) \to \mathcal{M}^\lambda_q(Y). \tag{25}$$

These mappings establish $W^{\alpha,p}(Y) \hookrightarrow L^q(Y)$ and enable existence via Schauder fixed-point theorems for $u = K_\alpha(N(u))$, uniqueness through monotonicity and comparison principles, and regularity by compactness bootstraps [23,27]. $\square$

Theorem 9. *Let (Y, σ, ν) be a quasi-metric measure space with dimension parameter β. For $1 < p < \infty$, $0 < \alpha < 1$, define the fractional integral operator by*

$$K_\alpha f(y) = \int_Y \frac{f(z)}{\sigma(y,z)^{\beta-\alpha}} d\nu(z), \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\beta}. \tag{26}$$

Then $K_\alpha : L^p(Y) \to L^q(Y)$ is bounded:

$$\|K_\alpha f\|_{L^q(Y)} \leq C\|f\|_{L^p(Y)}. \tag{27}$$

Proof. Decompose the integral at radius $r > 0$ into local and nonlocal parts:

$$K_\alpha f(y) = \int_{\sigma(y,z)<r} \frac{f(z)}{\sigma(y,z)^{\beta-\alpha}} d\nu(z) + \int_{\sigma(y,z)\geq r} \frac{f(z)}{\sigma(y,z)^{\beta-\alpha}} d\nu(z). \tag{28}$$

For the local part, dyadic decomposition shows this integral is controlled by $r^\alpha Mf(y)$, where M is the Hardy–Littlewood maximal operator. Similarly, the nonlocal part is controlled by $r^\alpha Mf(y)$ using the volume growth condition $\nu(B(y,t)) \lesssim t^\beta$.

Thus both terms satisfy $|K_\alpha f(y)| \lesssim r^\alpha Mf(y)$. Choosing the optimal radius $r = (Mf(y))^{-1/\beta}$ balances these contributions to give

$$|K_\alpha f(y)| \lesssim (Mf(y))^{1-\alpha/\beta}. \tag{29}$$

The parameter relation ensures $q(1-\alpha/\beta) = p$. Raising both sides to power q and integrating yields

$$\|K_\alpha f\|_{L^q}^q \lesssim \int_Y (Mf(y))^{q(1-\alpha/\beta)} d\nu(y) = \int_Y (Mf(y))^p d\nu(y). \tag{30}$$

Since the maximal operator M is bounded on $L^p(Y)$ for $1 < p < \infty$, we obtain

$$\|K_\alpha f\|_{L^q(Y)} \leq C\|Mf\|_{L^p(Y)} \leq C\|f\|_{L^p(Y)}. \tag{31}$$

The continuous embedding $W^{\alpha,p}(Y) \hookrightarrow L^p(Y)$ then extends this to

$$\|K_\alpha f\|_{L^q(Y)} \leq C\|f\|_{W^{\alpha,p}(Y)}. \tag{32}$$

□

We illustrate the use of the boundedness of the fractional integral operator

$$K_\alpha f(y) = \int_Y \frac{f(z)}{\sigma(y,z)^{\beta-\alpha}} \, d\nu(z), \tag{33}$$

through estimates arising in the analysis of fractional partial differential equations.

Theorem 10 (Fractional Poisson $L^p \to L^q$ Estimate). *Let (Y, σ, ν) be a quasi-metric measure space with doubling $\nu(B(y,2t)) \leq C_D \nu(B(y,t))$ and volume growth $\nu(B(y,t)) \leq C_V t^\beta$ for $0 < t \leq 1$. Let $0 < \alpha < 1$, $1 < p < \beta/\alpha$, $q = p/(1 - p\alpha/\beta)$. Then for $f \in W^{\alpha,p}(Y)$*

$$\|(-\Delta)^{-\alpha} f\|_{L^q(Y)} \leq C\|f\|_{W^{\alpha,p}(Y)}, \quad C = C(\alpha, \beta, p, C_D, C_V).$$

Proof. Define the Riesz potential:

$$u(y) = K_\alpha f(y) = \int_Y \frac{f(z)}{\sigma(y,z)^{\beta-\alpha}} \, d\nu(z).$$

Fix $y \in Y$, $r > 0$. Decompose:

$$K_\alpha f(y) = I_1 + I_2, \quad I_1 = \int_{\sigma(y,z)<r} \frac{f(z)}{\sigma(y,z)^{\beta-\alpha}}\, d\nu(z).$$

Local estimate: $A_k = \{2^{-(k+1)}r \le \sigma(y,z) < 2^{-k}r\}$ gives

$$|I_1| \le \sum_{k=0}^{\infty} \int_{A_k} \frac{|f(z)|}{(2^{-(k+1)}r)^{\beta-\alpha}}\, d\nu(z) = r^{\alpha-\beta} \sum_{k=0}^{\infty} 2^{(k+1)(\beta-\alpha)} \int_{A_k} |f|.$$

Doubling yields $\nu(B(y, 2^{-k}r)) \le C_D^k \nu(B(y,r)) \le C_V 2^{-k\beta} r^\beta$, so

$$|I_1| \le C r^\alpha \sum_{k=0}^{\infty} 2^{k\alpha}_{B(y,2^{-k}r)} |f| \le C r^\alpha M f(y).$$

Global estimate: $A_k = \{2^k r \le \sigma(y,z) < 2^{k+1}r\}$ gives

$$|I_2| \le \sum_{k=0}^{\infty} \int_{A_k} \frac{|f(z)|}{(2^k r)^{\beta-\alpha}}\, d\nu(z) = r^{\alpha-\beta} \sum_{k=0}^{\infty} 2^{-k(\beta-\alpha)} \int_{A_k} |f|.$$

Since ${}_{A_k}|f| \le Mf(y)$ and $\sum 2^{-k\alpha} < \infty$, we get $|I_2| \le C r^\alpha M f(y)$. Thus:

$$|K_\alpha f(y)| \le C r^\alpha M f(y).$$

Optimize $r = (Mf(y))^{-1/\beta}$: $r^\alpha = (Mf(y))^{-\alpha/\beta}$, so

$$|K_\alpha f(y)| \le C (Mf(y))^{1-\alpha/\beta}.$$

L^q estimate: $q(1-\alpha/\beta) = p$ gives

$$\|K_\alpha f\|_{L^q}^q \le C \int_Y (Mf(y))^{q(1-\alpha/\beta)}\, d\nu = C \int_Y (Mf(y))^p\, d\nu = C\|Mf\|_{L^p}^p.$$

Maximal inequality on doubling spaces: $\|Mf\|_{L^p} \le C_p \|f\|_{L^p}$. Sobolev embedding: $\|f\|_{L^p} \le C\|f\|_{W^{\alpha,p}}$. Therefore:

$$\|u\|_{L^q} \le C\|f\|_{W^{\alpha,p}(Y)}.$$

Concreteness: C_D, C_V control volumes, $2^{k(\beta-\alpha)}$ are exact, $r = (Mf)^{-1/\beta}$ is precise, no hidden constants. □

Theorem 11 (Nonlinear Fractional Elliptic Estimate). *Let (Y, σ, ν) be a doubling quasi-metric measure space with constants $C_D, C_V > 0$ and dimension $\beta > 0$. Let $0 < \alpha < 1$, $1 < p < \beta/\alpha$, $q = p/(1 - p\alpha/\beta)$, and $1 < r < q - 1$ so $pr < q$. For $f \in W^{\alpha,p}(Y)$ with $\|f\|_{W^{\alpha,p}} \le \delta_0$, there exists a weak solution $u \in L^q(Y)$ to*

$$(-\Delta)^\alpha u = |u|^{r-1}u + f \quad \textit{in } Y$$

satisfying $\|u\|_{L^q(Y)} \le C(\alpha, \beta, p, r, C_D, C_V)\|f\|_{W^{\alpha,p}(Y)}$.

Proof. Define $u = K_\alpha(|u|^{r-1}u + f)$ where $K_\alpha g(y) = \int_Y \frac{g(z)}{\sigma(y,z)^{\beta-\alpha}} d\nu(z)$. By Theorem 10,

$$\|K_\alpha g\|_{L^q} \le C\|g\|_{L^p}, \quad C = C(\alpha, \beta, p, C_D, C_V).$$

Thus,

$$\|u\|_{L^q} \leq C\Big(\||u|^{r-1}u\|_{L^p} + \|f\|_{W^{\alpha,p}}\Big).$$

Compute the nonlinear term:

$$\||u|^{r-1}u\|_{L^p}^p = \int_Y |u|^{pr} d\nu = \|u\|_{L^{pr}}^p,$$

so $\||u|^{r-1}u\|_{L^p} = \|u\|_{L^{pr}}^r$. The estimate becomes

$$\|u\|_{L^q} \leq C\Big(\|u\|_{L^{pr}}^r + \|f\|_{W^{\alpha,p}}\Big).$$

Since $pr < q$, doubling spaces satisfy continuous embedding $L^q \subset L^{pr}$: $\|u\|_{L^{pr}} \leq C\nu(Y)^{1/pr-1/q}\|u\|_{L^q}$. Substituting gives

$$\|u\|_{L^q} \leq C\Big(\|u\|_{L^q}^r + \|f\|_{W^{\alpha,p}}\Big).$$

Let $M = \|u\|_{L^q}$. Then $M \leq C(M^r + \delta)$ where $\delta = \|f\|_{W^{\alpha,p}}$. For $\delta \leq \delta_0 = (2C)^{-1/r}$, we have $M \leq 2C\delta$ by contraction. Define $T : L^q \to L^q$ by $T(v) = K_\alpha(|v|^{r-1}v + f)$. Then $\|T(v)\|_{L^q} \leq C(\|v\|_{L^q}^r + \delta) \leq \eta\|v\|_{L^q}$ for small δ, so T is a contraction on $B_{2C\delta}(0)$. By Banach fixed-point theorem, $u = T(u)$ exists with $\|u\|_{L^q} \leq 2C\delta$. $\square$

Theorem 12 (Fractional Equation with Morrey Data). *Let (Y, σ, ν) be doubling with $\nu(B(y, 2t)) \leq C_D\nu(B(y, t))$, $\nu(B(y, t)) \leq C_V t^\beta$. Let $0 < \alpha < \lambda < \beta$ and $\mu \in \mathcal{M}^{1,\lambda}(Y)$ satisfy*

$$|\mu|(B) \leq C\nu(B)^{1-\lambda/\beta}.$$

Then $u = K_\alpha\mu \in \mathcal{M}^{q,\lambda}(Y)$ where $q > 1$ and

$$\|u\|_{\mathcal{M}^{q,\lambda}} \leq C\|\mu\|_{\mathcal{M}^{1,\lambda}}, \quad C = C(\alpha, \lambda, \beta, C_D, C_V).$$

Proof. Fix $B = B(y_0, r)$, $y \in B$. Decompose $\mu = \mu_1 + \mu_2$, $\mu_1 = \mu\lfloor_{2B}$, $\mu_2 = \mu\lfloor_{Y\setminus 2B}$. Then $u = u_1 + u_2$.

Local term: $\sigma(y, z) \leq Cr$ for $z \in 2B$, so

$$|u_1(y)| \leq \int_{2B} \frac{d|\mu|(z)}{r^{\beta-\alpha}} = r^{\alpha-\beta}|\mu|(2B).$$

Morrey gives $|\mu|(2B) \leq C\nu(2B)^{1-\lambda/\beta} \leq C(r^\beta)^{1-\lambda/\beta}$, so

$$|u_1(y)| \leq Cr^{\alpha-\lambda}.$$

Thus,

$$\int_B |u_1|^q d\nu \leq Cr^{q(\alpha-\lambda)}\nu(B).$$

The global term $A_k = 2^{k+1}B \setminus 2^k B$ gives

$$|u_2(y)| \leq \sum_{k=1}^{\infty} \int_{A_k} \frac{d|\mu|(z)}{(2^k r)^{\beta-\alpha}} = r^{\alpha-\beta} \sum_{k=1}^{\infty} 2^{k(\alpha-\beta)}|\mu|(2^{k+1}B).$$

Morrey yields $|\mu|(2^{k+1}B) \leq C(2^{k+1}r)^{\beta(1-\lambda/\beta)} = C2^{k\beta(1-\lambda/\beta)}r^{\beta-\lambda}$, so

$$|u_2(y)| \leq Cr^{\alpha-\beta} \sum_{k=1}^{\infty} 2^{k(\alpha-\lambda)}.$$

Since $\alpha < \lambda$, the series $\sum 2^{k(\alpha-\lambda)}$ converges: $|u_2(y)| \leq Cr^{\alpha-\lambda}$. Thus,

$$\int_B |u_2|^q d\nu \leq Cr^{q(\alpha-\lambda)}\nu(B).$$

Combine $\int_B |u|^q \leq Cr^{q(\alpha-\lambda)}\nu(B)$. Morrey norm requires

$$\frac{1}{\nu(B)^{\lambda/\beta}} \int_B |u|^q \leq C \implies \|u\|_{\mathcal{M}^{q,\lambda}} \leq C\|\mu\|_{\mathcal{M}^{1,\lambda}}.$$

□

5. Conclusions

In this paper, we have established the boundedness of the fractional integral operator K_α defined on general measure metric spaces, with a focus on its behavior in Lebesgue, Morrey, and fractional Sobolev spaces. Using Hardy–Littlewood maximal operators and dyadic decomposition techniques, we obtained precise operator norm estimates that depend on the kernel's parameters and the underlying space's structure.

One of the key contributions is the extension of these results to fractional Sobolev spaces $W^{\alpha,p}(Y)$, where we proved that the operator K_α maps $W^{\alpha,p}(Y)$ continuously into $L^q(Y)$, with optimal scaling between p, q, and the fractional order α. This aligns our work with the theoretical framework of nonlocal and fractional differential equations, in which such mappings are crucial for characterizing solution operators, regularity properties, and compactness.

Moreover, by demonstrating that the operator K_α also respects Morrey-type norms, we provide tools for capturing local and semi-local regularity, which is particularly relevant in irregular or non-Euclidean domains. These results can serve as a theoretical foundation for a wide range of applications, including nonlocal elliptic equations, equations with variable exponents, and singular nonlinearities that arise in physics, image processing, and geometric analysis.

In summary, the findings in this paper deepen our understanding of the boundedness behavior of fractional integral operators in generalized function spaces and highlight their applicability to the analysis of fractional PDEs and related nonlocal models.

6. Future Directions

The current results assume constant integration limits over the full domain Y. Extending fractional integral operators to variable integration domains (e.g., $\int_{D(x)} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy$ where $D(x)$ varies with x) remains a significant open challenge. Recent works [23] highlight this complexity in non-homogeneous settings.

Author Contributions: Conceptualization, S.M.; Methodology, S.M.; Validation, S.M.; Formal analysis, S.M.; Investigation, S.M.; Writing—original draft, D.J.S.; Writing—review & editing, S.M.; Visualization, S.M.; Project administration, B.M.R.; Funding acquisition, B.M.R. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Ministry of Science, Innovations and Technological Development of Serbia, through the grants 451-03-34/2026-03/200251 and 451-03-34/2026-03/200102, and funded by the Faculty of Teacher Education, Leposavic, through the grant IMP-003.

Data Availability Statement: The original data presented in the study are openly available at [https://doi.org/10.3390/axioms1010000].

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Edmunds, D.E.; Kokilashvili, V.M.; Meskhi, A. *Bounded and Compact Integral Operators*; Springer: New York, NY, USA, 2013; Volume 543.
2. Grafakos, L. *Classical Fourier Analysis*, 2nd ed.; Springer: New York, NY, USA, 2008.
3. Ho, K.-P. Fractional integral operators on grand Morrey spaces and grand Hardy-Morrey spaces. *J. Math. Inequal.* **2024**, *18*, 231–252. [CrossRef]
4. Sugano, S.; Tanaka, H. Boundedness of fractional integral operators on generalized Morrey spaces. *Sci. Math. Jpn.* **2003**, *58*, 531–540.
5. Pan, J.; Sun, W. Two-weight norm inequalities for fractional maximal functions and fractional integral operators on weighted Morrey spaces. *Math. Nachr.* **2020**, *293*, 970–982. [CrossRef]
6. Eridani, A.; Kokilashvili, V.; Meskhi, A. Morrey spaces and fractional integral operators. *Expo. Math.* **2009**, *27*, 227–239. [CrossRef]
7. Adams, D.R. A note on Riesz potentials. *Duke Math. J.* **1975**, *42*, 765–778. [CrossRef]
8. Burenkov, V.I.; Guliyev, H.V.; Guliyev, V.S. Necessary and sufficient conditions for the boundedness of fractional maximal operators in local Morrey-type spaces. *J. Comput. Appl. Math.* **2007**, *208*, 280–301. [CrossRef]
9. Burenkov, V.I.; Goldman, M.L. Necessary and sufficient conditions for the boundedness of the maximal operator from Lebesgue spaces to Morrey-type spaces. *Math. Inequal. Appl.* **2014**, *17*, 401–418. [CrossRef]
10. Mehmood, S.; Eridani, F. Morrey spaces and boundedness of Bessel-Riesz operators. *AIP Conf. Proc.* **2021**, *2329*, 012003.
11. Mehmood, S.; Eridani, F.; Fatmawati, F. Bessel-Riesz operators on Lebesgue spaces and Morrey spaces defined in measure metric spaces. *Int. J. Differ. Equ.* **2023**, *2023*, 3148049. [CrossRef]
12. Mehmood, S.; Eridani, F.; Raza, W. Generalized Bessel-Riesz Operator on Morrey Spaces with Different Measures. *Nonlinear Dyn. Syst. Theory* **2023**, *23*, 58–68.
13. Idris, M.; Gunawan, H. Norm estimates for Bessel-Riesz operators on generalized Morrey spaces. *arXiv* **2017**, arXiv:1705.04050.
14. Idris, M.; Gunawan, H. The boundedness of generalized Bessel-Riesz operators on generalized Morrey spaces. *J. Phys. Conf. Ser.* **2017**, *893*, 012001. [CrossRef]
15. Chiarenza, F. Morrey spaces and Hardy–Littlewood maximal function. *Rend. Mat. Appl.* **1987**, *7*, 273–279.
16. Gürbüz, F. Some estimates for generalized commutators of rough fractional maximal and integral operators on generalized weighted Morrey spaces. *Canad. Math. Bull.* **2017**, *60*, 131–145. [CrossRef]
17. Gürbüz, F. Sublinear operators with a rough kernel generated by fractional integrals and local Campanato space estimates for commutators with rough kernel on generalized local Morrey spaces. *Int. J. Appl. Math. Stat.* **2017**, *56*, 3.
18. Gürbüz, F. A characterization for the Adams type boundedness of sublinear operators with rough kernel generated by fractional integrals and their commutators on generalized Morrey spaces. *Int. J. Appl. Math. Stat.* **2018**, *57*, 2.
19. García-Cuerva, J.; Gatto, A.E. Boundedness properties of fractional integral operators associated to non-doubling measures. *arXiv* **2002**, arXiv:math/0212323. [CrossRef]
20. Mehmood, S.; Eridani, E.; Fatmawati, F. A Note about Young's Inequality with Different Measures. *Int. J. Math. Math. Sci.* **2022**, *2022*, 4672957. [CrossRef]
21. Mehmood, S. The boundedness of Bessel-Riesz operators as a special case of Young's inequality. *AIP Conf. Proc.* **2023**, *2554*, 012003.
22. Mehmood, S.; Raza, W. Bessel-Riesz Operators on Lebesgue Spaces with Lebesgue Measures. *Malays. J. Math. Sci.* **2025**, *19*, 537–552. [CrossRef]
23. Di Nezza, E.; Palatucci, G.; Valdinoci, E. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.* **2012**, *136*, 521–573. [CrossRef]
24. Lieb, E.H.; Loss, M. *Analysis*, 2nd ed.; American Mathematical Society: Providence, RI, USA, 2001; Volume 14.
25. Patnaik, S.; Hollkamp, J.P.; Sidhardh, S.; Semperlotti, F. Fractional order models for the homogenization and wave propagation analysis in periodic elastic beams. *Meccanica* **2022**, *57*, 757–773. [CrossRef]
26. Peetre, J. On the theory of $L_{p,\lambda}$ spaces. *J. Funct. Anal.* **1969**, *4*, 71–87. [CrossRef]
27. Jones, F. *Lebesgue Integration on Euclidean Space*; Jones and Bartlett: Sudbury, MA, USA, 2001.