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A Unified Rotation-Minimizing Darboux Framework for Curves and Relativistic Ruled Surfaces in Minkowski Three-Space

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Abstract

We propose a comprehensive rotation-minimizing (RM) Darboux framework for the study of curve theory and relativistic ruled surfaces in Minkowski three-space $\mathbb{E}_1^3$. The construction merges the adaptability of the classical Darboux frame to surface geometry with the reduced rotational behavior characteristic of RM frames, yielding a natural geometric description of curves in a Lorentzian environment. For unit speed non-null curves, the governing equations of the RM Darboux frame are derived, and precise connections between the RM curvature functions and the classical Frenet and Darboux invariants are obtained, thereby elucidating the geometric significance of RM curvatures in Lorentzian geometry. Within this setting, multiple classes of ruled surfaces are generated using RM Darboux frame vector fields. Necessary and sufficient conditions for developability, minimality, and flatness are formulated exclusively in terms of RM curvature quantities. The role of the causal character of the generating curve is analyzed in detail, revealing distinct geometric behaviors for space-like and time-like cases. These findings indicate that the RM Darboux framework constitutes a flexible and effective approach for modeling curve-induced surface geometries in Minkowski space, with potential relevance to relativistic kinematics, world sheet constructions, and geometric problems arising in mathematical physics.

Keywords: Minkowski three space; Lorentzian geometry; rotation minimizing frame; RM Darboux frame; ruled surfaces; causal character; curve surface geometry

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1. Introduction

The geometry of curves and surfaces in Lorentzian manifolds plays a central role in differential geometry and relativistic physics. Among such spaces, Minkowski three-space $\mathbb{E}_1^3$ provides the simplest yet most fundamental model for spacetime geometry, serving as a natural arena for the investigation of relativistic trajectories, worldlines, and ruled world sheets. Classical curve theory in $\mathbb{E}_1^3$ has been extensively developed through Lorentzian extensions of the Frenet–Serret formalism, uncovering a wide range of geometric phenomena governed by curvature and torsion invariants [1–3]. Nevertheless, the Frenet frame requires non-vanishing curvature and exhibits intrinsic rotational behavior, which may limit its effectiveness in both theoretical investigations and applied contexts.

For curves constrained to surfaces, the Darboux frame offers a natural alternative by incorporating surface geometry via geodesic curvature, normal curvature, and geodesic

torsion [1,4]. In Lorentzian geometry, Darboux frames have been successfully employed in the analysis of space-like and time-like curves on surfaces, as well as in the study of the associated ruled surfaces [5]. Despite these advantages, the Darboux frame still inherits rotational features from Frenet theory, which can obscure geometric interpretation, particularly in settings where causal structure is essential.

Rotation-minimizing (RM) frames, originally introduced by Bishop in the Euclidean context, provide an alternative framing that suppresses unnecessary rotation of the normal plane along a curve [6]. RM frames have found applications in geometric modeling, computer graphics, elastic rod theory, and numerical curve evolution [7,8]. Their extension to Lorentzian and pseudo-Riemannian manifolds has revealed smoother behavior and improved analytical stability when compared with Frenet frames [3,9], making them especially suitable for the study of relativistic curves and surfaces.

Inspired by these developments, ruled surfaces generated by RM frame vector fields have been investigated in both Euclidean and Lorentzian settings [10,11]. Such surfaces naturally arise in swept surface modeling, spacetime world sheet descriptions, and flux surfaces in relativistic field theories [12–16]. In Minkowski space, the causal nature of the generating curve significantly affects both intrinsic and extrinsic surface geometry, producing behaviors absent in the Euclidean case [17,18]. However, a unified RM-based formulation adapted to curves lying on surfaces, analogous to the classical Darboux theory, has not yet been fully developed.

More recently, RM Darboux frames have been introduced as a synthesis of Darboux surface adaptation and RM minimal rotation [11,19,20]. These frames yield simplified curvature expressions and provide clearer insight into curve surface interactions in Lorentzian manifolds. They have also demonstrated potential applications in relativistic kinematics, magnetic curve theory, and geometric evolution problems in pseudo-Riemannian spaces [21]. These considerations have motivated the formulation of a unified and systematic RM Darboux framework in Minkowski three-space.

The objective of this work is to establish such a unified rotation-minimizing Darboux framework for curves and relativistic ruled surfaces in $\mathbb{E}_1^3$. We derive the evolution equations of the Lorentzian RM Darboux frame for unit speed non-null curves and obtain explicit relations linking RM curvature functions with classical Frenet and Darboux invariants. Using this framework, several families of ruled surfaces are constructed via RM frame vectors, and complete characterizations of developability, minimality, and flatness are formulated purely in terms of RM curvature data. The results emphasize the decisive role of causal character in shaping surface geometry and demonstrate that the RM Darboux approach provides a powerful and geometrically meaningful tool for modeling curve-driven geometries in Lorentzian space.

The novelty of the present work lies in the development of a fully unified rotation minimizing (RM) Darboux framework for curves and ruled surfaces in Minkowski three-space. Unlike existing RM-based approaches, the proposed formulation systematically integrates surface adaptation with minimal normal plane rotation in a Lorentzian setting. Within this framework, we derive explicit relations between RM curvature functions and the classical Frenet and Darboux invariants, and we establish the necessary and sufficient conditions for developability, minimality, and flatness expressed purely in terms of RM curvature quantities. Furthermore, a complete geometric classification of RM ruled surfaces is obtained via the behavior of their parametric curve families, providing a unified and transparent description that extends and generalizes the previously known results in Minkowski space. In particular, the unified RM Darboux formulation and the resulting classification criteria are genuinely new, while several existing RM- and Darboux-based ruled surface constructions appear as special cases of the present theory. The

incremental gain over the prior works lies in the systematic characterization of geometric properties using RM curvature functions alone, independent of specific causal types or ad hoc assumptions on the generating curves.

The structure of the paper is as follows. Section 2 presents the necessary geometric background in Minkowski three-space $\mathbb{E}_1^3$ and introduces a unified treatment of the Frenet, Darboux, and Lorentzian RM Darboux frames for unit speed non-null curves, together with their evolution equations and invariant relations. In Section 3, a general family of RM ruled surfaces is constructed using RM Darboux frame vector fields, and explicit formulas for the fundamental forms, unit normal vector, and curvature invariants are obtained. Developability, minimality, and flatness conditions are characterized in terms of RM curvature functions and director coefficients. Section 4 is devoted to the geometric classification of RM ruled surfaces through the analysis of their parametric curves, providing criteria for geodesic, asymptotic, and principal curve families. Section 5 presents several explicit space-like and time-like examples that illustrate the theoretical results and highlight the impact of causal character on surface geometry. Finally, Section 6 summarizes the main conclusions and outlines possible directions for future research.

2. Geometric Preliminaries and RM Darboux Frames

This section introduces the fundamental concepts from the theory of curves and surfaces in Minkowski three-space $\mathbb{E}_1^3$ that will be required in the sequel. We briefly review Lorentzian, Frenet, and Darboux frames and then present the rotation-minimizing (RM) Darboux frame, which plays a central role in our construction. Standard surface invariants and curvature formulas are also recalled for completeness.

Let $\mathbb{E}_1^3 = (\mathbb{R}^3, \langle \cdot, \cdot \rangle)$ denote Minkowski three-space equipped with the Lorentzian inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = -u_1 v_1 + u_2 v_2 + u_3 v_3, \qquad \mathbf{u}, \mathbf{v} \in \mathbb{R}^3,$$

which induces a pseudo-Euclidean geometry of signature $(-, +, +)$.

A vector $\mathbf{u} \in \mathbb{E}_1^3 \setminus \{0\}$ is said to be time-like, space-like, or light-like accordingly as

$$\langle \mathbf{u}, \mathbf{u} \rangle < 0, \qquad \langle \mathbf{u}, \mathbf{u} \rangle > 0, \qquad \langle \mathbf{u}, \mathbf{u} \rangle = 0,$$

respectively [1,2].

Consider a smooth non-null curve $\delta : I \to \mathbb{E}_1^3$ parametrized by arc length s, satisfying

$$\langle \delta'(s), \delta'(s) \rangle = \varepsilon_{\mathrm{T}}, \qquad \varepsilon_{\mathrm{T}} \in \{-1, +1\}.$$

The unit tangent vector of δ is denoted by $\mathrm{T}(s) = \delta'(s)$. If the curvature $\kappa(s)$ of δ does not vanish, the principal normal vector N and the binormal vector B are defined by

$$\mathrm{N} = \frac{\mathrm{T}'}{\sqrt{|\langle \mathrm{T}', \mathrm{T}' \rangle|}}, \qquad \mathrm{B} = \mathrm{T} \times \mathrm{N},$$

with causal signatures

$$\langle \mathrm{N}, \mathrm{N} \rangle = \varepsilon_{\mathrm{N}}, \qquad \langle \mathrm{B}, \mathrm{B} \rangle = \varepsilon_{\mathrm{B}}, \qquad \varepsilon_{\mathrm{T}} \varepsilon_{\mathrm{N}} \varepsilon_{\mathrm{B}} = -1.$$

The Lorentzian Frenet–Serret equations are given by [1,4]

$$\begin{aligned} \mathrm{T}' &= \varepsilon_{\mathrm{N}} \kappa \, \mathrm{N}, \\ \mathrm{N}' &= -\varepsilon_{\mathrm{T}} \kappa \, \mathrm{T} + \varepsilon_{\mathrm{B}} \tau \, \mathrm{B}, \\ \mathrm{B}' &= -\varepsilon_{\mathrm{N}} \tau \, \mathrm{N}, \end{aligned} \tag{1}$$

where $\kappa(s)$ and $\tau(s)$ denote the curvature and torsion of δ, respectively.

Let $M \subset \mathbb{E}_1^3$ be an oriented regular surface with a unit normal vector field n. For a curve δ lying on M, the associated Darboux frame $\{\mathrm{T}, g, n\}$ is defined by

$$g = n \times \mathrm{T}.$$

The evolution of this frame is governed by the Lorentzian Darboux equations

$$\begin{aligned} \mathrm{T}' &= k_g\, g + k_n\, n, \\ g' &= -\varepsilon_{\mathrm{T}} k_g\, \mathrm{T} + \tau_g\, n, \\ n' &= -\varepsilon_{\mathrm{T}} k_n\, \mathrm{T} - \tau_g\, g, \end{aligned} \tag{2}$$

where k_g, k_n, and τ_g denote the geodesic curvature, normal curvature, and geodesic torsion, respectively [1,4].

A different choice of moving frame is obtained by minimizing the rotation of the normal plane along the curve. The Lorentzian rotation-minimizing (RM) Darboux frame $\{\mathrm{T}, \mathrm{U}, \mathrm{V}\}$ satisfies

$$\begin{aligned} \mathrm{T}' &= \kappa_1\, \mathrm{U} + \kappa_2\, \mathrm{V}, \\ \mathrm{U}' &= -\varepsilon_{\mathrm{T}} \kappa_1\, \mathrm{T}, \\ \mathrm{V}' &= -\varepsilon_{\mathrm{T}} \kappa_2\, \mathrm{T}, \end{aligned} \tag{3}$$

where $\kappa_1(s)$ and $\kappa_2(s)$ are the RM curvature functions. In contrast to the Frenet and Darboux frames, this framing avoids superfluous rotation of the normal vectors and remains well defined in degenerate situations [3,6,9,11,19].

Let $\Phi(u, v)$ be a regular parametrization of a surface $M \subset \mathbb{E}_1^3$. The first fundamental form of M is expressed as

$$I = E\, du^2 + 2F\, du\, dv + G\, dv^2,$$

with coefficients

$$E = \langle \Phi_u, \Phi_u \rangle, \qquad F = \langle \Phi_u, \Phi_v \rangle, \qquad G = \langle \Phi_v, \Phi_v \rangle.$$

The second fundamental form is given by

$$II = e\, du^2 + 2f\, du\, dv + g\, dv^2,$$

where

$$e = \langle \Phi_{uu}, n \rangle, \quad f = \langle \Phi_{uv}, n \rangle, \quad g = \langle \Phi_{vv}, n \rangle,$$

and n denotes the unit normal vector field of M. The Gaussian curvature K and mean curvature H are defined by

$$K = \varepsilon_n \frac{LN - M^2}{EG - F^2}, \qquad H = \varepsilon_n \frac{EN - 2FM + GL}{2(EG - F^2)},$$

whenever $EG - F^2 \neq 0$. Here, the factor $\varepsilon_n = \langle n, n \rangle$ accounts for the causal character of the unit normal vector and ensures that the curvature expressions are valid for both space-like and time-like surfaces in Minkowski space. These quantities describe the intrinsic and extrinsic geometry of surfaces in Minkowski space [2,4].

Definition 1. *Let $\delta : I \to \mathbb{E}_1^3$ be a smooth non-null curve and let $\mathbf{d} : I \to \mathbb{E}_1^3$ be a smooth non-zero vector field along δ. A surface $M \subset \mathbb{E}_1^3$ is said to be a* ruled surface *if it admits a parametrization*

$$\Phi(s, v) = \delta(s) + v\,\mathbf{d}(s), \qquad s \in I,\ v \in \mathbb{R},$$

where δ is called the base curve and $\mathbf{d}$ is referred to as the director vector field [22–25].

Remark 1. *The Frenet frame considered here is defined for non-null curves whose principal normal vector is also non-null. Space-like curves admitting a null principal normal, commonly referred to as pseudo-null curves, are not included in this setting. Since the standard Frenet construction degenerates in that case, such curves fall outside the scope of the present analysis.*

Throughout this paper, we work with curves that are non-null and parametrized by arc length, unless explicitly stated otherwise. The corresponding RM Darboux frame is assumed to be orthonormal with respect to the Minkowski metric. All geometric quantities are considered only on parameter domains where the relevant curvature functions are well defined and no denominators vanish. Accordingly, the formulas for the first and second fundamental forms, as well as for the Gaussian and mean curvatures of the associated ruled surfaces, are valid only away from singularities of the parametrization. It is also noted that the causal type of the ruled surface may depend on the parameters (s, v), and all geometric expressions are interpreted locally according to this causal behavior.

The symbols ε_T, ε_U, and ε_V indicate the causal characters of the RM Darboux frame vectors T, U, and V, respectively, with $\varepsilon_T, \varepsilon_U, \varepsilon_V \in \{-1, +1\}$ satisfying $\varepsilon_T\varepsilon_U\varepsilon_V = -1$. Similarly, ε_d and ε_n denote the causal characters of the director vector field $\mathbf{d}$ and the unit normal vector field n of the ruled surface. The introduction of these sign parameters enables a uniform formulation of the results for both space-like and time-like configurations within the Lorentzian framework of $\mathbb{E}_1^3$.

Remark 2. *The scope of the present work is confined to non-null curves and ruling directions in Minkowski three-space. Configurations involving light-like elements are excluded, since the RM Darboux frame cannot be constructed in a regular manner in such cases. Furthermore, all statements and classifications are to be understood on regular regions of the parameter domain, excluding points where the ruled surface parametrization becomes degenerate.*

3. Construction of General RM Ruled Surfaces with Lorentzian Normalization

Let $\delta : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a unit speed non null curve endowed with the Lorentzian RM Darboux frame $\{\mathrm{T}, \mathrm{U}, \mathrm{V}\}$, where

$$\langle \mathrm{T}, \mathrm{T} \rangle = \varepsilon_{\mathrm{T}}, \quad \langle \mathrm{U}, \mathrm{U} \rangle = \varepsilon_{\mathrm{U}}, \quad \langle \mathrm{V}, \mathrm{V} \rangle = \varepsilon_{\mathrm{V}}, \qquad \varepsilon_{\mathrm{T}}\varepsilon_{\mathrm{U}}\varepsilon_{\mathrm{V}} = -1.$$

Let $u_i : I \to \mathbb{R}$, $i = 1, 2, 3$, be smooth functions satisfying the Lorentzian normalization condition

$$\varepsilon_{\mathrm{T}} u_1^2(s) + \varepsilon_{\mathrm{U}} u_2^2(s) + \varepsilon_{\mathrm{V}} u_3^2(s) = \varepsilon_{\mathbf{d}}, \qquad \varepsilon_{\mathbf{d}} \in \{-1, +1\}.$$

A surface $M \subset \mathbb{E}_1^3$ is called a general RM ruled surface if it admits a parametrization

$$\Phi(s, v) = \delta(s) + v\,\mathbf{d}(s),$$

where the director vector field $\mathbf{d}$ is given by

$$\mathbf{d}(s) = u_1(s)\,\mathrm{T}(s) + u_2(s)\,\mathrm{U}(s) + u_3(s)\,\mathrm{V}(s).$$

The ruled surface is classified as spacelike or time-like according to the causal character of the induced metric on its tangent plane, or equivalently by the sign of $\langle n, n\rangle$, where n denotes the unit normal vector field of the surface. The causal character of the director vector field $\mathbf{d}$ influences the surface geometry but is not sufficient on its own to define the causal type of the surface.

Using the RM Darboux equations, the derivative of $\mathbf{d}$ is computed as

$$\mathbf{d}' = \big(u_1' + u_2\kappa_1 + u_3\kappa_2\big)\mathrm{T} + \big(u_2' - \varepsilon_{\mathrm{T}}u_1\kappa_1\big)\mathrm{U} + \big(u_3' - \varepsilon_{\mathrm{T}}u_1\kappa_2\big)\mathrm{V}. \tag{4}$$

For $\Phi(s, v) = \delta(s) + v\mathbf{d}(s)$, we have

$$\Phi_s = \mathrm{T} + v\mathbf{d}', \qquad \Phi_v = \mathbf{d}.$$

Then the coefficients of the first fundamental form

$$I = E\,ds^2 + 2F\,ds\,dv + G\,dv^2$$

are given by

$$\begin{aligned} E &= \langle \Phi_s, \Phi_s\rangle = \varepsilon_{\mathrm{T}} + 2v\,\varepsilon_{\mathrm{T}}\alpha + v^2\Big(\varepsilon_{\mathrm{T}}\alpha^2 + \varepsilon_{\mathrm{U}}\beta^2 + \varepsilon_{\mathrm{V}}\gamma^2\Big),\\ F &= \langle \Phi_s, \Phi_v\rangle = \varepsilon_{\mathbf{d}} + v(\varepsilon_{\mathrm{T}}u_1\alpha + \varepsilon_{\mathrm{U}}u_2\beta + \varepsilon_{\mathrm{V}}u_3\gamma),\\ G &= \langle \Phi_v, \Phi_v\rangle = \varepsilon_{\mathbf{d}}, \end{aligned} \tag{5}$$

where

$$\begin{aligned} \alpha &= u_1' + u_2\kappa_1 + u_3\kappa_2,\\ \beta &= u_2' - \varepsilon_{\mathrm{T}}u_1\kappa_1,\\ \gamma &= u_3' - \varepsilon_{\mathrm{T}}u_1\kappa_2. \end{aligned} \tag{6}$$

Lemma 1. *Let $\Phi(s, v) = \delta(s) + v\,\mathbf{d}(s)$ be a general RM ruled surface. The (unnormalized) normal vector field of Φ is defined by*

$$\widetilde{n}(s, v) = \Phi_s \times \Phi_v,$$

where $\times$ denotes the Lorentzian cross product in $\mathbb{E}_1^3$. The corresponding unit normal vector field is

$$n(s, v) = \frac{1}{\sqrt{|\Delta|}}\{v(\beta u_3 - \gamma u_2)\mathrm{T} + [v\gamma u_1 - (v\alpha + 1)u_3]\mathrm{U} + [(v\alpha + 1)u_2 - \beta u_1]\mathrm{V}\},\ \langle n, n\rangle = \varepsilon_n,$$

where

$$\Delta = \varepsilon_{\mathrm{U}}u_3^2 + \varepsilon_{\mathrm{V}}u_2^2 + v^2\Big[\varepsilon_{\mathrm{T}}(\beta u_3 - \gamma u_2)^2 + \varepsilon_{\mathrm{U}}(\gamma u_1 - \alpha u_3)^2 + \varepsilon_{\mathrm{V}}(\alpha u_2 - \beta u_1)^2\Big], \tag{7}$$

and α, β and γ are given by (6).

Proof. Let

$$\Phi(s, v) = \delta(s) + v\,\mathbf{d}(s), \qquad \mathbf{d}(s) = u_1\mathrm{T} + u_2\mathrm{U} + u_3\mathrm{V}$$

be a general RM ruled surface along the unit speed curve δ endowed with the Lorentzian RM Darboux frame $\{\mathrm{T}, \mathrm{U}, \mathrm{V}\}$. Differentiating Φ with respect to s and v, we obtain

$$\Phi_s = \mathrm{T} + v\,\mathbf{d}', \qquad \Phi_v = \mathbf{d}.$$

Using the RM Darboux equations, the derivative of the director field is written as

$$\mathbf{d}' = \alpha\,\mathrm{T} + \beta\,\mathrm{U} + \gamma\,\mathrm{V},$$

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where

$$\alpha = u_1' + u_2\kappa_1 + u_3\kappa_2, \qquad \beta = u_2' - \varepsilon_{\mathrm{T}} u_1\kappa_1, \qquad \gamma = u_3' - \varepsilon_{\mathrm{T}} u_1\kappa_2.$$

By definition, the unnormalized normal vector field of Φ is

$$\widetilde{n} = \Phi_s \times \Phi_v = (\mathrm{T} + v\mathbf{d}') \times \mathbf{d} = \mathrm{T} \times \mathbf{d} + v\,\mathbf{d}' \times \mathbf{d}.$$

Substituting $\mathbf{d}$ and $\mathbf{d}'$ and using the Lorentzian cross-product relations of the RM frame, we obtain

$$\mathrm{T} \times \mathbf{d} = u_2\mathrm{V} - u_3\mathrm{U},$$

and

$$\mathbf{d}' \times \mathbf{d} = (\beta u_3 - \gamma u_2)\mathrm{T} + (\gamma u_1 - \alpha u_3)\mathrm{U} + (\alpha u_2 - \beta u_1)\mathrm{V}.$$

Hence,

$$\widetilde{n} = v(\beta u_3 - \gamma u_2)\mathrm{T} + \big[v\gamma u_1 - (v\alpha + 1)u_3\big]\mathrm{U} + \big[(v\alpha + 1)u_2 - \beta u_1\big]\mathrm{V}.$$

Using the orthonormality of the RM Darboux frame, we compute

$$\begin{aligned}\Delta &= \langle \widetilde{n}, \widetilde{n} \rangle \\ &= \varepsilon_{\mathrm{U}} u_3^2 + \varepsilon_{\mathrm{V}} u_2^2 \\ &\quad + v^2\Big[\varepsilon_{\mathrm{T}}(\beta u_3 - \gamma u_2)^2 + \varepsilon_{\mathrm{U}}(\gamma u_1 - \alpha u_3)^2 + \varepsilon_{\mathrm{V}}(\alpha u_2 - \beta u_1)^2\Big],\end{aligned}$$

which coincides with (7). Provided that $\Delta \neq 0$, the unit normal vector field is defined by

$$n(s, v) = \frac{\widetilde{n}(s, v)}{\sqrt{|\Delta|}}.$$

By construction, it satisfies

$$\langle n, n \rangle = \varepsilon_n,$$

which completes the proof. □

Remark 3. *The explicit expression of the normal vector $n(s, v)$ shows that its direction is governed by both the intrinsic geometry of the base curve through the RM curvatures κ_1 and κ_2 and the choice of the director field coefficients u_i. In particular, the causal character of the normal vector may vary along the surface depending on the sign of Δ, which reflects the interaction between the ruling direction and the RM evolution of the normal plane. Moreover, when $v = 0$, the normal vector reduces to*

$$n(s, 0) = -u_3\,\mathrm{U} + u_2\,\mathrm{V},$$

showing that the normal direction along the striction curve depends only on the normal components of the ruling and is independent of the RM curvature functions. Geometrically, the condition $\Delta \neq 0$ ensures the non-degeneracy of the tangent plane along the surface. Variations in the sign of Δ may lead to changes in the causal character of the unit normal vector, reflecting the interaction between the ruling direction and the evolution of the RM normal plane along the base curve. Accordingly, all curvature expressions and classification results are understood to hold only on parameter domains where $\Delta \neq 0$, while points with $\Delta = 0$ correspond to singular or degenerate configurations and are excluded from the analysis.

Lemma 2. *Let $\Phi(s,v) = \delta(s) + v\,\mathbf{d}(s)$ be a general RM ruled surface with unit normal vector n. Then the coefficients of the second fundamental form are given by*

$$\begin{aligned} L &= \frac{\varepsilon_{\mathrm{U}}\kappa_1(\gamma u_1 - \alpha u_3) + \varepsilon_{\mathrm{V}}\kappa_2(\alpha u_2 - \beta u_1) + v\langle \mathbf{d}'', \widetilde{n}\rangle}{\sqrt{|\Delta|}}, \\ M &= \frac{\varepsilon_{\mathrm{T}}\alpha(\beta u_3 - \gamma u_2) + \varepsilon_{\mathrm{U}}\beta(\gamma u_1 - \alpha u_3) + \varepsilon_{\mathrm{V}}\gamma(\alpha u_2 - \beta u_1)}{\sqrt{|\Delta|}}, \\ N &= 0, \end{aligned} \tag{8}$$

where Δ given by (7).

Proof. Let $\Phi(s,v) = \delta(s) + v\,\mathbf{d}(s)$ be a general RM ruled surface along a unit speed non-null curve δ endowed with the Lorentzian RM Darboux frame $\{\mathrm{T}, \mathrm{U}, \mathrm{V}\}$. The first partial derivatives of Φ are

$$\Phi_s = \mathrm{T} + v\,\mathbf{d}', \qquad \Phi_v = \mathbf{d}.$$

Differentiating once more with respect to s and v yields

$$\Phi_{ss} = \mathrm{T}' + v\,\mathbf{d}'', \qquad \Phi_{sv} = \mathbf{d}', \qquad \Phi_{vv} = 0.$$

Using the RM Darboux equations, the derivative of the tangent vector is

$$\mathrm{T}' = \kappa_1 \mathrm{U} + \kappa_2 \mathrm{V},$$

and the derivative of the director field is written as

$$\mathbf{d}' = \alpha\,\mathrm{T} + \beta\,\mathrm{U} + \gamma\,\mathrm{V},$$

where

$$\alpha = u_1' + u_2\kappa_1 + u_3\kappa_2, \qquad \beta = u_2' - \varepsilon_{\mathrm{T}} u_1\kappa_1, \qquad \gamma = u_3' - \varepsilon_{\mathrm{T}} u_1\kappa_2.$$

Let n denote the unit normal vector field of the surface. By definition, the coefficients of the second fundamental form are given by

$$L = \langle \Phi_{ss}, n\rangle, \qquad M = \langle \Phi_{sv}, n\rangle, \qquad N = \langle \Phi_{vv}, n\rangle.$$

Since $\Phi_{vv} = 0$, it follows immediately that

$$N = \langle 0, n\rangle = 0.$$

Using $\Phi_{sv} = \mathbf{d}'$ and the expression

$$n = \frac{\widetilde{n}}{\sqrt{|\Delta|}},$$

we obtain

$$M = \langle \mathbf{d}', n\rangle = \frac{\langle \mathbf{d}', \widetilde{n}\rangle}{\sqrt{|\Delta|}}.$$

Substituting the explicit expressions of $\mathbf{d}'$ and $\widetilde{n}$ and using the orthonormality of the RM frame, we compute

$$\langle \mathbf{d}', \widetilde{n}\rangle = \varepsilon_{\mathrm{T}}\alpha(\beta u_3 - \gamma u_2) + \varepsilon_{\mathrm{U}}\beta(\gamma u_1 - \alpha u_3) + \varepsilon_{\mathrm{V}}\gamma(\alpha u_2 - \beta u_1),$$

which yields the stated expression for M. From $\Phi_{ss} = \mathrm{T}' + v\,\mathbf{d}''$ and $\mathrm{T}' = \kappa_1\mathrm{U} + \kappa_2\mathrm{V}$, we have

$$L = \langle \mathrm{T}' + v\mathbf{d}'', n\rangle = \frac{\langle \kappa_1\mathrm{U} + \kappa_2\mathrm{V}, \widetilde{n}\rangle + v\langle \mathbf{d}'', \widetilde{n}\rangle}{\sqrt{|\Delta|}}.$$

A direct computation gives

$$\langle \kappa_1\mathrm{U} + \kappa_2\mathrm{V}, \widetilde{n}\rangle = \varepsilon_{\mathrm{U}}\kappa_1(\gamma u_1 - \alpha u_3) + \varepsilon_{\mathrm{V}}\kappa_2(\alpha u_2 - \beta u_1).$$

Substituting this into the expression of L yields the desired formula. Combining the above computations completes the proof. □

Theorem 1. *Let $\Phi(s,v) = \delta(s) + v\,\mathbf{d}(s)$ be a regular general RM ruled surface in $\mathbb{E}_1^3$. Then the Gaussian and the mean curvature of Φ is given by*

$$K = -\frac{\Psi^2}{|\Delta|\left[\varepsilon_{\mathbf{d}}\varepsilon_{\mathrm{T}} + 2v\,\varepsilon_{\mathbf{d}}\varepsilon_{\mathrm{T}}\alpha + v^2\varepsilon_{\mathbf{d}}\left(\varepsilon_{\mathrm{T}}\alpha^2 + \varepsilon_{\mathrm{U}}\beta^2 + \varepsilon_{\mathrm{V}}\gamma^2\right) - \varepsilon_{\mathrm{T}}^2u_1^2\right]}, \tag{9}$$

$$H = \frac{\varepsilon_{\mathbf{d}}\left[\varepsilon_{\mathrm{U}}\kappa_1(\gamma u_1 - \alpha u_3) + \varepsilon_{\mathrm{V}}\kappa_2(\alpha u_2 - \beta u_1) + v\Theta\right] - 2\varepsilon_{\mathrm{T}}u_1\,\Psi}{2\sqrt{|\Delta|}\left[\varepsilon_{\mathbf{d}}\varepsilon_{\mathrm{T}} + 2v\,\varepsilon_{\mathbf{d}}\varepsilon_{\mathrm{T}}\alpha + v^2\varepsilon_{\mathbf{d}}\left(\varepsilon_{\mathrm{T}}\alpha^2 + \varepsilon_{\mathrm{U}}\beta^2 + \varepsilon_{\mathrm{V}}\gamma^2\right) - \varepsilon_{\mathrm{T}}^2u_1^2\right]}, \tag{10}$$

where Δ given by (7) and

$$\begin{aligned}\Theta &= \varepsilon_{\mathrm{T}}\alpha'(\beta u_3 - \gamma u_2) + \varepsilon_{\mathrm{U}}(\beta + \alpha\kappa_1)(\gamma u_1 - \alpha u_3) + \varepsilon_{\mathrm{V}}(\gamma + \alpha\kappa_2)(\alpha u_2 - \beta u_1),\\ \Psi &= \varepsilon_{\mathrm{T}}\alpha(\beta u_3 - \gamma u_2) + \varepsilon_{\mathrm{U}}\beta(\gamma u_1 - \alpha u_3) + \varepsilon_{\mathrm{V}}\gamma(\alpha u_2 - \beta u_1).\end{aligned} \tag{11}$$

Theorem 2. *A general RM ruled surface $\Phi(s,v) = \delta(s) + v\mathbf{d}(s)$ is developable if and only if*

$$\det\left(\mathrm{T}, \mathbf{d}, \mathbf{d}'\right) = 0.$$

Equivalently, this condition can be expressed as

$$\varepsilon_{\mathrm{U}}u_2\kappa_2 - \varepsilon_{\mathrm{V}}u_3\kappa_1 + \varepsilon_{\mathrm{T}}u_1\left(\varepsilon_{\mathrm{U}}u_2'\kappa_2 - \varepsilon_{\mathrm{V}}u_3'\kappa_1\right) = 0.$$

Proof. A ruled surface is developable if and only if the tangent plane remains constant along each ruling, which is equivalent to $\det(\delta', \mathbf{d}, \mathbf{d}') = 0$. Substituting $\delta' = \mathrm{T}$ and using (4), the stated condition follows by direct computation, taking into account the causal signs of the RM frame. □

Corollary 1. *If $u_1 = 0$ and u_2, u_3 are constants, then the general RM ruled surface is developable if and only if*

$$\varepsilon_{\mathrm{U}}u_2\kappa_2 - \varepsilon_{\mathrm{V}}u_3\kappa_1 = 0.$$

Proof. For constant coefficients, all derivative terms vanish, and the developability condition reduces immediately to the stated relation. □

Theorem 3. *Let $\Phi(s,v) = \delta(s) + v\,\mathbf{d}(s)$ be a regular general RM ruled surface in $\mathbb{E}_1^3$. Then the Gaussian curvature of Φ along the striction curve $(v = 0)$ is given by*

$$K\big|_{v=0} = -\frac{\Psi^2}{|\Delta_0|\left(\varepsilon_{\mathbf{d}}\varepsilon_{\mathrm{T}} - \varepsilon_{\mathrm{T}}^2u_1^2\right)}, \tag{12}$$

where

$$\Delta_0 = \varepsilon_U u_3^2 + \varepsilon_V u_2^2 \quad \textit{and} \quad \Psi = \varepsilon_T \alpha(\beta u_3 - \gamma u_2) + \varepsilon_U \beta(\gamma u_1 - \alpha u_3) + \varepsilon_V \gamma(\alpha u_2 - \beta u_1).$$

Theorem 4. *Let $\Phi(s,v) = \delta(s) + v\,\mathbf{d}(s)$ be a regular general RM ruled surface in $\mathbb{E}_1^3$. Then Φ is developable along the striction curve $(v = 0)$ if and only if*

$$\Psi = 0.$$

Proof. Along the striction curve, the Gaussian curvature is given by (12). Since Φ is regular, the denominator of (12) is non-zero. Therefore,

$$K\big|_{v=0} = 0 \quad \Longleftrightarrow \quad \Psi^2 = 0 \quad \Longleftrightarrow \quad \Psi = 0.$$

This proves the claim. □

Theorem 5. *Let $\Phi(s,v) = \delta(s) + v\,\mathbf{d}(s)$ be a regular general RM ruled surface in $\mathbb{E}_1^3$. Then Φ is minimal if and only if*

$$\varepsilon_{\mathbf{d}}\Big[\varepsilon_U \kappa_1(\gamma u_1 - \alpha u_3) + \varepsilon_V \kappa_2(\alpha u_2 - \beta u_1) + v\Theta\Big] - 2\varepsilon_T u_1\,\Psi = 0, \tag{13}$$

for all (s,v), where Θ and Ψ are defined in (11).

Corollary 2. *A regular general RM ruled surface is minimal if and only if*

$$\begin{cases} \Theta = 0, \\ \varepsilon_{\mathbf{d}}\big[\varepsilon_U \kappa_1(\gamma u_1 - \alpha u_3) + \varepsilon_V \kappa_2(\alpha u_2 - \beta u_1)\big] - 2\varepsilon_T u_1\,\Psi = 0. \end{cases} \tag{14}$$

Corollary 3. *Along the striction curve $(v = 0)$, the ruled surface Φ is minimal if and only if*

$$\varepsilon_{\mathbf{d}}\big[\varepsilon_U \kappa_1(\gamma u_1 - \alpha u_3) + \varepsilon_V \kappa_2(\alpha u_2 - \beta u_1)\big] - 2\varepsilon_T u_1\,\Psi = 0.$$

Corollary 4. *If the director field $\mathbf{d}$ has constant components u_i then a general RM ruled surface is minimal if and only if*

$$\Theta = \varepsilon_U \alpha \kappa_1(\gamma u_1 - \alpha u_3) + \varepsilon_V \alpha \kappa_2(\alpha u_2 - \beta u_1) = 0$$

and

$$\varepsilon_{\mathbf{d}}\big[\varepsilon_U \kappa_1(\gamma u_1 - \alpha u_3) + \varepsilon_V \kappa_2(\alpha u_2 - \beta u_1)\big] - 2\varepsilon_T u_1\,\Psi = 0.$$

Corollary 5. *If the ruling direction is orthogonal to the tangent vector $(u_1 = 0)$ then Φ is minimal if and only if*

$$\Theta = 0 \quad \textit{and} \quad \varepsilon_U \kappa_1(\gamma u_1 - \alpha u_3) + \varepsilon_V \kappa_2(\alpha u_2 - \beta u_1) = 0.$$

Remark 4. *The minimality conditions show that RM ruled surfaces balance the curvature of the base curve and the variation of the ruling direction. In particular, minimality forces a strong compatibility between the RM curvatures (κ_1, κ_2) and the director field coefficients (u_1, u_2, u_3), generalizing classical results for Frenet based ruled surfaces to the Lorentzian RM framework.*

Theorem 6. *Let $\Phi(s,v) = \delta(s) + v\,\mathbf{d}(s)$ be a regular general RM ruled surface in $\mathbb{E}_1^3$. The mean curvature of Φ along the striction curve $(v = 0)$ is given by*

$$H\big|_{v=0} = \frac{\varepsilon_{\mathbf{d}}\left[\varepsilon_{\mathrm{U}}\kappa_1(\gamma u_1 - \alpha u_3) + \varepsilon_{\mathrm{V}}\kappa_2(\alpha u_2 - \beta u_1)\right] - 2\varepsilon_{\mathrm{T}}u_1\,\Psi}{2\sqrt{|\Delta_0|}(\varepsilon_{\mathbf{d}}\varepsilon_{\mathrm{T}} - \varepsilon_{\mathrm{T}}^2 u_1^2)}, \tag{15}$$

where

$$\Delta_0 = \varepsilon_{\mathrm{U}}u_3^2 + \varepsilon_{\mathrm{V}}u_2^2$$

and

$$\Psi = \varepsilon_{\mathrm{T}}\alpha(\beta u_3 - \gamma u_2) + \varepsilon_{\mathrm{U}}\beta(\gamma u_1 - \alpha u_3) + \varepsilon_{\mathrm{V}}\gamma(\alpha u_2 - \beta u_1).$$

Moreover, Φ is minimal along the striction curve if and only if

$$\varepsilon_{\mathbf{d}}\left[\varepsilon_{\mathrm{U}}\kappa_1(\gamma u_1 - \alpha u_3) + \varepsilon_{\mathrm{V}}\kappa_2(\alpha u_2 - \beta u_1)\right] - 2\varepsilon_{\mathrm{T}}u_1\,\Psi = 0. \tag{16}$$

Proof. From the general expression of the mean curvature

$$H = \frac{GL - 2FM}{2(EG - F^2)},$$

substituting the fundamental coefficients evaluated along the striction curve $(v = 0)$ yields the explicit Formula (15). Since Φ is regular, the denominator in (15) is non-zero. Therefore, $H\big|_{v=0} = 0$ if and only if the numerator of (15) vanishes, which leads directly to condition (16). This completes the proof. □

Corollary 6. *If the ruling direction is orthogonal to the tangent vector $(u_1 = 0)$ then Φ is minimal along the striction curve if and only if*

$$\varepsilon_{\mathrm{U}}\kappa_1(\gamma u_1 - \alpha u_3) + \varepsilon_{\mathrm{V}}\kappa_2(\alpha u_2 - \beta u_1) = 0.$$

Corollary 7. *If the director coefficients u_i are constant then minimality along the striction curve is equivalent to*

$$\varepsilon_{\mathbf{d}}\left[\varepsilon_{\mathrm{U}}\kappa_1(\gamma u_1 - \alpha u_3) + \varepsilon_{\mathrm{V}}\kappa_2(\alpha u_2 - \beta u_1)\right] = 2\varepsilon_{\mathrm{T}}u_1\,\Psi.$$

Remark 5. *Along the striction curve, the contribution of the second derivative $\mathbf{d}''$ disappears from the mean curvature. Thus, minimality at $v = 0$ is governed purely by the interaction between the RM curvature functions (κ_1, κ_2) and the orientation of the ruling direction encoded in (u_1, u_2, u_3).*

4. Classification of RM Ruled Surfaces by Curve Properties

This section is devoted to the geometric analysis of rotation-minimizing (RM) ruled surfaces in Minkowski three-space $\mathbb{E}_1^3$ from both intrinsic and extrinsic points of view. The classification is carried out by examining the geometric nature of the parametric foliations induced by the surface coordinates. In particular, we determine precise conditions under which the coordinate curves associated with the base parameter and the ruling parameter constitute geodesics, asymptotic curves, or principal curvature lines. The characterization is obtained through the fundamental forms expressed with respect to the RM Darboux frame, allowing all criteria to be written explicitly in terms of the RM curvature functions and the director field components. This approach generalizes classical classifications based on Frenet frames and demonstrates the effectiveness of RM techniques in Lorentzian geometry.

Let

$$\Phi(s,v) = \delta(s) + v\,\mathbf{d}(s)$$

be a regular general RM ruled surface in $\mathbb{E}_1^3$, where the director field is expanded relative to the RM Darboux frame as

$$\mathbf{d}(s) = u_1(s)\mathrm{T}(s) + u_2(s)\mathrm{U}(s) + u_3(s)\mathrm{V}(s), \qquad \varepsilon_{\mathbf{d}} = \varepsilon_{\mathrm{T}}u_1^2 + \varepsilon_{\mathrm{U}}u_2^2 + \varepsilon_{\mathrm{V}}u_3^2.$$

Theorem 7. *The parametric curves corresponding to the s-parameter are geodesics on Φ if and only if the condition*

$$\varepsilon_{\mathrm{T}}\alpha u_1 + \varepsilon_{\mathrm{U}}\beta u_2 + \varepsilon_{\mathrm{V}}\gamma u_3 = 0$$

is satisfied.

Proof. A curve on a surface is geodesic precisely when its acceleration has no component in the tangent direction orthogonal to the curve. For the s-curves, the second derivative is given by $\Phi_{ss} = \delta'' + v\mathbf{d}''$. Requiring the tangential projection orthogonal to Φ_s to vanish leads to the scalar condition

$$\langle \Phi_{ss}, \Phi_v \rangle = 0.$$

Substituting the RM frame decomposition of $\mathbf{d}'$ and simplifying yields the stated relation. □

Theorem 8. *The ruling curves (v-curves) of Φ are geodesics if and only if the director field is constant along the base curve; that is,*

$$\mathbf{d}'(s) = \mathbf{0}, \quad \textit{or equivalently} \quad \alpha = \beta = \gamma = 0.$$

Proof. A surface curve is geodesic when its acceleration vector is normal to the surface. Along each ruling, one has $\Phi_{vv} = 0$ and $\Phi_{sv} = \mathbf{d}'(s)$. Hence, the absence of tangential acceleration occurs exactly when $\mathbf{d}'(s)$ vanishes identically, which is equivalent to $\alpha = \beta = \gamma = 0$. □

Corollary 8. *Whenever the ruling curves of Φ are geodesics, the surface reduces to a cylindrical ruled surface with a constant ruling direction.*

Proof. From Theorem 8, $\mathbf{d}'(s) = 0$, so the director field does not vary along δ. Consequently, the surface is cylindrical. □

Theorem 9. *The s-curves of Φ define asymptotic directions if and only if*

$$L = 0 \quad \textit{for all } (s, v).$$

Equivalently, the following conditions hold:

$$\varepsilon_{\mathrm{U}}\kappa_1(\gamma u_1 - \alpha u_3) + \varepsilon_{\mathrm{V}}\kappa_2(\alpha u_2 - \beta u_1) = 0, \qquad \langle \mathbf{d}'', \widetilde{n} \rangle = 0.$$

Proof. The normal curvature along the s-curves is expressed as

$$k_n^{(s)} = \frac{L}{E}.$$

Thus the asymptotic condition $k_n^{(s)} = 0$ is equivalent to $L = 0$. Since L depends affinely on the ruling parameter v, its vanishing for all (s, v) forces both the constant and linear coefficients to be zero, yielding the stated conditions. □

Theorem 10. *For every regular general RM ruled surface, the ruling curves are asymptotic lines.*

Proof. The normal curvature in the direction of the rulings is

$$k_n^{(v)} = \frac{N}{G}.$$

For ruled surfaces, the second fundamental coefficient N vanishes identically. Hence, $k_n^{(v)} = 0$, and all v-curves are asymptotic. □

Theorem 11. *The coordinate curves corresponding to both parameters form principal curvature lines if and only if*

$$M = 0, \quad \textit{equivalently} \quad \Psi = 0.$$

Proof. For a parametrized surface, the condition that the coordinate curves coincide with principal directions is equivalent to the vanishing of the mixed coefficient of the second fundamental form. Thus $M = 0$, which is equivalent to $\Psi = 0$. □

Corollary 9. *If both families of parametric curves on a general RM ruled surface are principal curvature lines then the surface is developable.*

Remark 6. *From a physical viewpoint, ruled surfaces in Minkowski space may be interpreted as world sheets traced by the transport of a spatial or temporal direction along a worldline. Within this interpretation, the rulings represent instantaneous propagation directions, while the s-curves describe the evolution of the generating trajectory. The preceding result shows that when both foliations align with principal directions, the induced world sheet has zero Gaussian curvature, corresponding to a flat, non-accelerating relativistic surface.*

Remark 7. *The classification (Table 1) reveals that RM ruled surfaces retain the intrinsic flatness characteristic of ruled geometries, while the RM Darboux frame enables a concise and transparent description of geodesic, asymptotic, and principal directions solely in terms of RM curvature data and director coefficients. This constitutes a natural Lorentzian extension of classical Frenet-based results.*

Table 1. Geometric classification of general RM ruled surfaces according to the behavior of their parametric curve families. All characterizing conditions are expressed solely in terms of RM curvature functions and director field coefficients.

Curve Family	Geometric Property	Characterizing Condition
s-curves	Asymptotic	$L = 0$
s-curves	Geodesic	$\varepsilon_{\mathrm{T}}\alpha u_1 + \varepsilon_{\mathrm{U}}\beta u_2 + \varepsilon_{\mathrm{V}}\gamma u_3 = 0$
s-curves	Principal	$\Psi = 0$
v-curves	Asymptotic	Identically ($N = 0$)
v-curves	Geodesic	$\alpha = \beta = \gamma = 0$
v-curves	Principal	$\Psi = 0$

5. Explicit Examples and Visualizations

In this section, we illustrate the theoretical developments by constructing several explicit examples of rotation-minimizing (RM) ruled surfaces in Minkowski three-space $\mathbb{E}_1^3$. Both space-like and time-like base curves are considered. For each case, the corresponding RM Darboux frame $\{\mathrm{T}, \mathrm{U}, \mathrm{V}\}$ is determined explicitly, together with the associated RM curvature functions, and the resulting ruled surfaces are visualized.

Example 1. *This example illustrates RM ruled surfaces generated by spacelike base curves and highlights the role of constant RM curvature functions in simplifying the surface geometry.*

Let δ be a space-like curve lying on a space-like surface M, defined by

$$\delta(s) = \big(\sinh s, \cosh s, a\ \sinh 2s\big), \qquad a > 0.$$

The Darboux frame of δ takes the form

$$\begin{aligned}
\mathrm{T}(s) &= \frac{1}{\psi(s)}\big(\cosh s, \sinh s, 2a\cosh 2s\big),\\
n(s) &= \frac{1}{\psi(s)}\big(2a\cosh s\cosh 2s, 2a\sinh s\cosh 2s, 1\big),\\
g(s) &= \frac{1}{\psi(s)}\big(\sinh s, \cosh s, 2a\cosh 2s\big),
\end{aligned}$$

where $\psi = \sqrt{-1 + 4a^2\cosh^2 2s}$. A direct computation yields

$$\kappa_g = \frac{1}{\psi}, \qquad \kappa_n = \frac{4a\sinh 2s}{\psi}, \qquad \tau_g = 0.$$

Hence, the angle function $\theta(s)$ reduces to a constant θ_0, and one may choose $\mathrm{U}(s) = g(s)$ and $\mathrm{V}(s) = n(s)$. The resulting RM ruled surfaces generated by U *and* V *are given by*

$$\Phi_g(s,v) = \begin{pmatrix} \sinh s + \dfrac{v\sinh s}{\psi} \\ \cosh s + \dfrac{v\cosh s}{\psi} \\ a\sinh 2s - \dfrac{2a\,v\cosh 2s}{\psi} \end{pmatrix},$$

$$\Phi_n(s,v) = \begin{pmatrix} \sinh s + \dfrac{2a\,v\cosh s\cosh 2s}{\psi} \\ \cosh s + \dfrac{2a\,v\sinh s\cosh 2s}{\psi} \\ a\sinh 2s + \dfrac{v}{\psi} \end{pmatrix},$$

which are depicted in Figures 1–3.

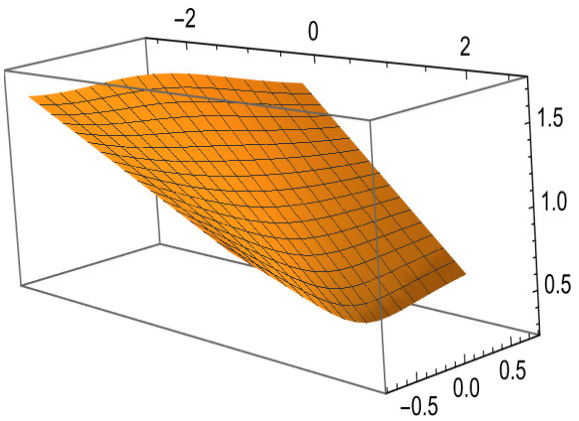


Figure 1. The RM ruled surface $\Phi_g(s,v)$.

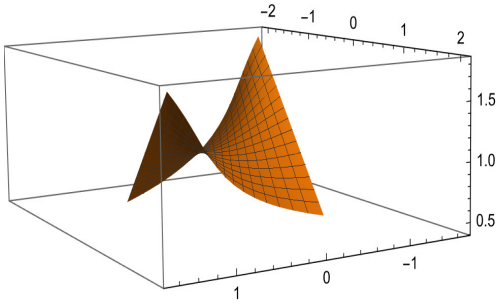


Figure 2. The RM ruled surface $\Phi_n(s, v)$.

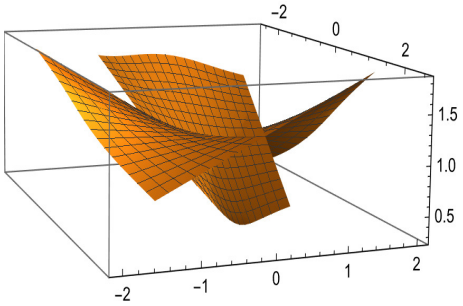


Figure 3. Combination of RM ruled surface $\Phi_g(s, v)$ together with $\Phi_n(s, v)$.

Example 2. *This example considers a space-like base curve contained in a time-like plane and illustrates how the causal character of the ambient surface influences the associated RM Darboux frame and the geometry of the resulting RM ruled surface. In particular, it highlights the interaction between a space-like generating curve and a time-like supporting plane within the unified RM Darboux framework.*

Consider the space-like curve

$$\mu(s) = \left(a, as^2, bs^3\right), \qquad a, b > 0,$$

contained in the time-like plane $M: \ x = a$. *The Darboux frame of* μ *is given by*

$$\begin{aligned} \mathrm{T}(s) &= \frac{1}{\rho(s)}\left(0, 2as, 3bs^2\right), \\ n(s) &= (1, 0, 0), \\ g(s) &= \frac{1}{\rho(s)}\left(0, -3bs^2, 2as\right), \end{aligned}$$

where $\rho = \sqrt{4a^2s^2 + 9b^2s^4}$. *The corresponding Darboux invariants satisfy*

$$\kappa_g = \frac{6abs^2}{4a^2s^2 + 9b^2s^4}, \qquad \kappa_n = 0, \qquad \tau_g = 0.$$

Thus $\theta(s) = \theta_0$ *is constant, and the RM Darboux frame coincides with*

$$\begin{aligned} \mathrm{T}(s) &= \frac{1}{\rho(s)}\left(0, 2as, 3bs^2\right), \\ \mathrm{U}(s) &= g(s), \\ \mathrm{V}(s) &= n(s), \\ \kappa_1(s) &= \kappa_g(s), \quad \kappa_2(s) = 0. \end{aligned}$$

Choosing the director field $\mathbf{d}(s) = \cosh s\,\mathrm{U}(s) + \sinh s\,\mathrm{V}(s)$, *the RM ruled surface is expressed as*

$$\Gamma(s,v) = \begin{pmatrix} a + v\sinh s \\ as^2 - \dfrac{3vbs^2\cosh s}{\rho} \\ bs^3 + \dfrac{2avs\cosh s}{\rho} \end{pmatrix}, \quad \rho = \sqrt{4a^2s^2 + 9b^2s^4},$$

as illustrated in Figures 4–6.

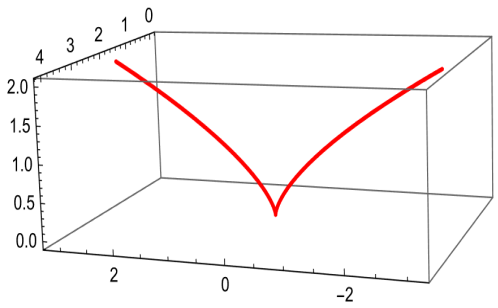


Figure 4. The base curve $\mu(s)$.

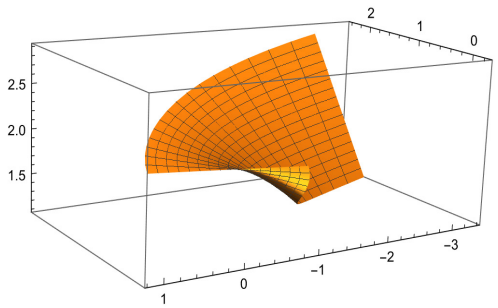


Figure 5. The RM ruled surface $\Gamma(s,v)$ with $a = 1$, $b = 2$, $s \in [-1,-0.5]$ and $v \in [-1,1]$.

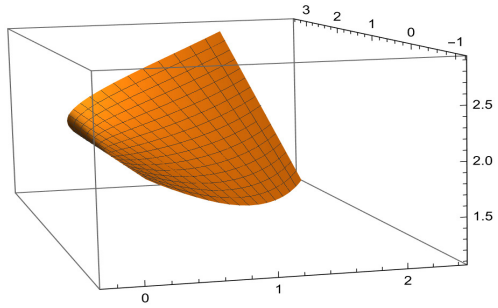


Figure 6. The RM ruled surface $\Gamma(s,v)$ with $a = 1$, $b = 2$, $s \in [0.5,-1]$ and $v \in [-1,1]$.

Example 3. *This example illustrates the general theory for RM ruled surfaces with variable director coefficients and provides explicit conditions for developability and minimality within the unified RM Darboux framework.*

Let ω *be the space-like curve*

$$\omega(s) = \big(\cosh s, \sinh s, as\big), \qquad a \neq 0,$$

lying on a spacelike surface M. Its Darboux frame is given by

$$\begin{aligned}
\mathrm{T}(s) &= \frac{1}{\lambda}\big(\sinh s, \cosh s, a\big),\\
n(s) &= (\cosh s, \sinh s, 0),\\
g(s) &= \frac{1}{\lambda}\big(a\sinh s, a\cosh s, -1\big),
\end{aligned}$$

where $\lambda = \sqrt{1+a^2}$. *The associated invariants satisfy*

$$\kappa_g = 0, \qquad \kappa_n = \tau_g = \frac{1}{\lambda},$$

and, hence, $\theta(s) = \int_0^s \tau_g\, ds = \frac{s}{\lambda}$. *The RM Darboux frame vectors and curvature functions are then obtained as*

$$\begin{aligned}
\mathrm{U}(s) &= \Big(\frac{a}{\lambda}\sinh s\cosh\theta - \cosh s\sinh\theta,\ \frac{a}{\lambda}\cosh s\cosh\theta - \sinh s\sinh\theta,\ -\cosh\theta\Big),\\
\mathrm{V}(s) &= \Big(-\frac{a}{\lambda}\sinh s\sinh\theta + \cosh s\cosh\theta,\ -\frac{a}{\lambda}\cosh s\sinh\theta + \sinh s\cosh\theta,\ \sinh\theta\Big),\\
\kappa_1(s) &= -\frac{1}{\lambda}\sinh\theta, \qquad \kappa_2(s) = \frac{1}{\lambda}\cosh\theta.
\end{aligned}$$

The RM ruled surface generated by ω *is defined by*

$$\Omega(s,v) = \omega(s) + v\,\mathbf{d}(s),$$

where

$$\mathbf{d}(s) = u_1(s)\mathrm{T}(s) + u_2(s)\mathrm{U}(s) + u_3(s)\mathrm{V}(s).$$

Its parametrization is (see Figures 7–9)

$$\Omega(s,v) = \begin{pmatrix}
\cosh s + \frac{v\,u_1\sinh s}{\lambda} + v\,u_1[\frac{a}{\lambda}\sinh s\cosh\theta - \cosh s\sinh\theta]\\
+v\,u_3[-\frac{a}{\lambda}\sinh s\sinh\theta + \cosh s\cosh\theta]\\
\sinh s + \frac{v\,u_1\cosh s}{\lambda} + v\,u_2[\frac{a}{\lambda}\cosh s\cosh\theta - \sinh s\sinh\theta]\\
+v\,u_3[-\frac{a}{\lambda}\cosh s\sinh\theta + \sinh s\cosh\theta]\\
a\,s - \frac{a\,v\,u_1}{\lambda} - v\,u_2\cosh\theta + v\,u_3\sinh\theta
\end{pmatrix}. \tag{17}$$

The Gaussian curvature and mean curvatures can be computed as (see Figuers 10 and 11)

$$\begin{aligned}
K(s,v) &= -\frac{\Big[u_2u_3' - u_3u_2' + \frac{u_1}{\lambda}\big(u_2\cosh\theta + u_3\sinh\theta\big)\Big]^2}{\Sigma^2},\\
H(s,v) &= \frac{1}{2}\frac{u_2u_3' - u_3u_2' + \frac{u_1}{\lambda}\big(u_2\cosh\theta + u_3\sinh\theta\big)}{\sqrt{\Sigma}},
\end{aligned} \tag{18}$$

where

$$\begin{aligned}
\Sigma = &\Big(u_1^2 - u_2^2 + u_3^2\Big)\\
&+ 2v\Big[\frac{u_1}{\lambda}(-u_2\sinh\theta + u_3\cosh\theta) - \big(u_2u_2' + u_3u_3'\big)\Big]\\
&+ v^2\Big[\Big(u_2' - \frac{u_1}{\lambda}\sinh\theta\Big)^2 - \Big(u_3' + \frac{u_1}{\lambda}\cosh\theta\Big)^2\Big],
\end{aligned} \tag{19}$$

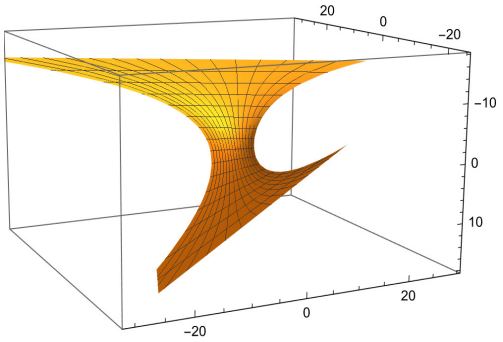


Figure 7. The RM ruled surface $\Omega(s, v)$ associated with the base curve $\omega(s)$.

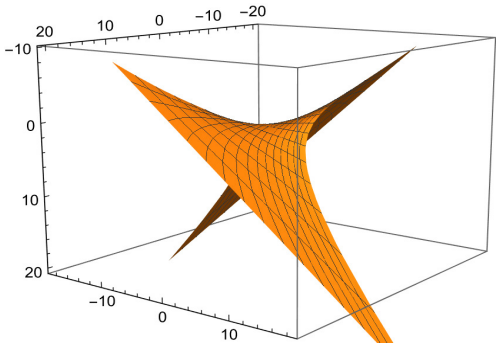


Figure 8. The RM ruled surface $\Omega(s, v)$ with $u_1 = u_3 = 0$, $u_2 = \frac{1}{\sqrt{2}}$, $v \in [-3, 3]$.

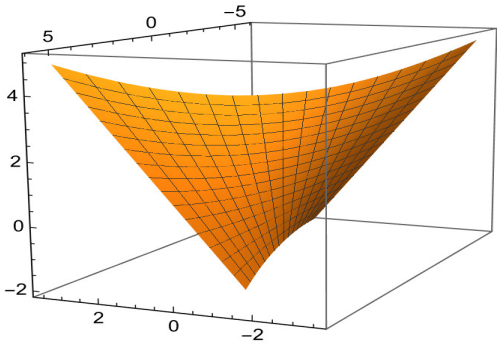


Figure 9. The RM ruled surface $\Omega(s, v)$ with $u_1 = u_2 = 0$, $u_3 = \frac{1}{2}$, $v \in [-3, 3]$.

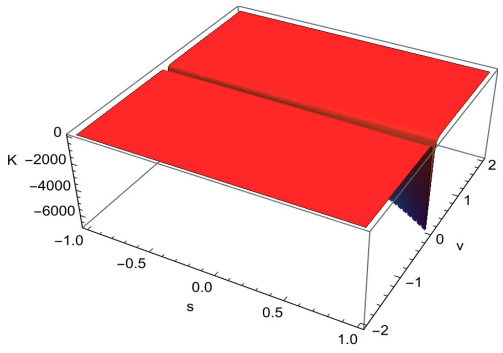


Figure 10. The Gaussian curvature of RM ruled surface $\Omega(s, v)$.

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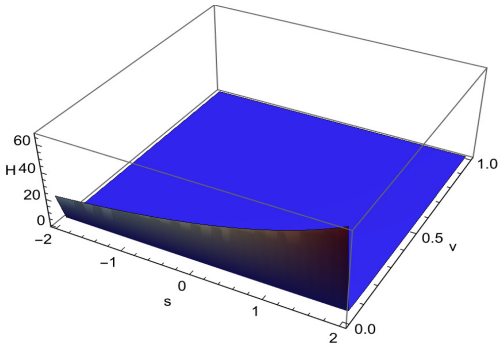


Figure 11. The mean curvature of RM ruled surface $\Omega(s, v)$.

As a comment on the results obtained in the above example, the RM ruled surface Ω satisfies the following properties:

(a) Ω is developable if and only if

$$u_2 u_3' - u_3 u_2' + \frac{u_1}{\lambda}(u_2 \cosh\theta + u_3 \sinh\theta) = 0.$$

(b) Ω is minimal if and only if the same quantity vanishes.
(c) If $u_1 \equiv 0$, then the director field lies in span$\{U, V\}$ and the surface is non-tangent along ω.
(d) If $u_2 \equiv 0$ or $u_3 \equiv 0$, then Ω reduces to a U-type or V-type RM ruled surface, respectively, and is developable.
(e) If $u_2'(s) = u_3'(s) = 0$, then Ω is a cylindrical surface and hence developable.

6. Conclusions

This work has introduced a comprehensive geometric framework based on the rotation minimizing (RM) Darboux construction for the study of curves and ruled surfaces in Minkowski three-space $\mathbb{E}_1^3$. By combining the surface adapted features of the classical Darboux frame with the minimal twisting property inherent to RM frames the resulting formulation offers a coherent and conceptually clear approach for analyzing curve-induced surface geometries within a Lorentzian context.

For unit speed non-null curves, the Lorentzian RM Darboux frame was constructed and its evolution equations were established explicitly. The corresponding RM curvature functions were related in a precise manner to the Frenet and Darboux invariants, thereby clarifying their geometric interpretation in Minkowski space. These relations emphasize the stability and interpretability of RM frames, especially in degenerate situations such as vanishing curvature or cases where classical frames exhibit unnecessary rotational behavior.

Using the RM Darboux frame as a generating mechanism, a broad family of ruled surfaces was defined through general director vector fields expressed in terms of RM frame vectors. For this class of RM ruled surfaces, closed form expressions for the first and second fundamental forms, the surface normal, and the Gaussian and mean curvatures were derived. Developability, minimality, and flatness were characterized by necessary and sufficient conditions written entirely in terms of RM curvature functions and director coefficients. These criteria extend well known results for ruled surfaces and demonstrate how the causal nature of both the base curve and the ruling direction governs the intrinsic and extrinsic geometry of the surface.

A further contribution of this work is the geometric classification of RM ruled surfaces via the behavior of their parametric curves. Explicit conditions were obtained under which the u- and v-curves defined geodesics, asymptotic directions, or principal curvature lines. This analysis provides a unified Lorentzian generalization of classical Frenet-based classifi-

cations and shows that RM ruled surfaces naturally reflect intrinsic flatness while allowing a transparent description of principal and asymptotic foliations through RM invariants.

Several illustrative space-like and time-like examples were presented to validate the theoretical development and to highlight the practical advantages of the RM Darboux approach. These examples confirm that the proposed framework leads to simplified curvature representations and facilitates a clearer geometric understanding of relativistic ruled surfaces.

In summary, the RM Darboux formulation developed here constitutes a flexible and effective tool for the geometric modeling and analysis of ruled surfaces generated by curves in Minkowski space. Future investigations may address RM-based surface evolutions, inextensible and variational flows, extensions to higher-dimensional Lorentzian manifolds, and potential applications to relativistic kinematics, world sheet geometry, and related structures arising in mathematical physics.

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References

1. O'Neill, B. *Semi–Riemannian Geometry with Applications to Relativity*; Academic Press: New York, NY, USA, 1983.
2. López, R. Differential geometry of curves and surfaces in Lorentz–Minkowski space. *Int. Electron. J. Geom.* **2014**, *7*, 44–107. [CrossRef]
3. Etayo, F. Rotation minimizing vector fields and frames in Riemannian manifolds. In *Geometry, Algebra and Applications: From Mechanics to Cryptography*; Springer Proceedings in Mathematics & Statistics ; Springer: Cham, Switzerland, 2016; Volume 161.
4. do Carmo, M.P. *Differential Geometry of Curves and Surfaces*, 2nd ed.; Dover Publications: Mineola, NY, USA, 2016.
5. Ali, A.T.; López, R. Slant helices in Minkowski space E_1^3. *J. Korean Math. Soc.* **2011**, *48*, 159–167. [CrossRef]
6. Bishop, L.R. There is more than one way to frame a curve. *Am. Math. Mon.* **1975**, *82*, 246–251. [CrossRef]
7. Hanson, A.J.; Ma, H. Parallel transport approach to curve framing. *Indiana Univ. Math. J.* **1995**, *44*, 651–669.
8. Farouki, R.T.; Giannelli, C.; Sampoli, M.L.; Sestini, A. Rotation–minimizing osculating frames. *Comput. Aided Geom. Des.* **2014**, *31*, 27–42. [CrossRef]
9. da Silva, L.C.B. Rotation minimizing frames and spherical curves in simply isotropic and pseudo-isotropic 3-spaces. *Tamkang J. Math.* **2020**, *51*, 31–52. [CrossRef]
10. Li, Y.; Chen, Z.; Nazra, S.H.; Abdel-Baky, R.A. Singularities for timelike developable surfaces in Minkowski 3-space. *Symmetry* **2023**, *15*, 277. [CrossRef]
11. Solouma, E.; Al-Dayel, I.; Khan, M.A.; Abdelkawy, M. Investigation of special type-Π Smarandache ruled surfaces due to the rotation minimizing Darboux frame in E^3. *Symmetry* **2023**, *15*, 2207. [CrossRef]
12. Capovilla, R.; Guven, J. Geometry of relativistic strings. *Class. Quantum Grav.* **2020**, *37*, 075003.
13. Guven, J. Conformal geometry of curves and surfaces. *Phys. Rev. D* **2019**, *99*, 025013.. [CrossRef]
14. Solouma, E.; Al-Dayel, I.; Abdelkawy, M.A. Ruled surfaces and their geometric invariants via the orthogonal modified frame in Minkowski 3-space. *Mathematics* **2025**, *13*, 2557. [CrossRef]
15. Solouma, E.; Saber, S.; Marin, M.; Baskonus, H.M. Geometric invariants and evolution of RM–Hasimoto surfaces in Minkowski 3-space $\mathbb{E}_1^3$. *Mathematics* **2025**, *13*, 3420. [CrossRef]

16. Messaoudi, M.; Solouma, E.; Alshehri, M.N.; Aljohani, A.F.; Marin, M. Lorentzian structure and curvature analysis of osculating type–2 ruled surfaces via the type–2 Bishop frame. *Mathematics* **2025**, *13*, 3464. [CrossRef]
17. Alghanemi, A.; Matar, G.; Saloom, A. The differential geometry of a space curve via a constant vector in $\mathbb{R}^3$. *Axioms* **2025**, *14*, 190. [CrossRef]
18. Monterde, J. Relativistic elastic curves in Minkowski space. *J. Geom. Phys.* **2022**, *173*, 104446.
19. Solouma, E.; Al-Dayel, I.; Khan, M.A.; Lazer, Y.A.A. Characterization of imbricate–ruled surfaces via the rotation minimizing Darboux frame in Minkowski 3-space E_1^3. *AIMS Math.* **2024**, *9*, 13028–13042. [CrossRef]
20. Solouma, E.; Saber, S.; Baskonus, H.M. Exploring harmonic evolute geometries derived from tubular surfaces in Minkowski 3-space using the RM Darboux frame. *Mathematics* **2025**, *13*, 2329. [CrossRef]
21. Chen, B.-Y.; Aloui, F.; Khan, M.A.; Choudhary, M.A. Magnetic curves in differential geometry: A comprehensive survey. *Mathematics* **2025**, *13*, 3849. [CrossRef]
22. Struik, D.J. *Lectures on Classical Differential Geometry*; Addison-Wesley: Boston, MA, USA, 1961.
23. Aldossary, M.T.; Abdel-Baky, R.A. Sweeping surface due to the rotation minimizing Darboux frame in Euclidean 3-space E^3. *AIMS Math.* **2023**, *8*, 447–462. [CrossRef]
24. Li, Y.; Abdel-Aziz, H.S.; Serry, H.M.; El-Adawy, F.M.; Saad, M.K. Geometric visualization of evolved ruled surfaces via an alternative frame in Lorentz–Minkowski 3-space. *AIMS Math.* **2024**, *9*, 25619–25635. [CrossRef]
25. Turgut, A.; Hacisalihoglu, H.H. Spacelike ruled surfaces in Minkowski 3-space. *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.* **1997**, *46*, 83–91.

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