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Multiplicity Result of Solutions to the Fractional Problems with (p,q)-Growth and Hardy Potentials

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Abstract

This paper focuses on establishing the existence of infinitely many solutions for non-local fractional equations characterized by unbalanced growth and Hardy potentials. We prove that these solutions converge to zero in the L^∞-norm, requiring conditions on the nonlinearity only near the origin and dispensing with assumptions at infinity. As far as we are aware, results for non-local fractional (p,q)-Laplacian problems with singular coefficients such as Hardy potentials have not been extensively studied. To address this gap, we employ the dual fountain theorem together with the modified functional method.

Keywords: fractional (p,q)-Laplacian; variational methods; Hardy potential; modified functional method

MSC: 35B38; 35D30; 35J20; 35J60; 47J30

1. Introduction

Investigation on elliptic problems with the non-local fractional Laplacian and more general integro-differential operators has attracted considerable attention due to their relevance in respect applied and pure mathematical theories used for illustrating certain specific phenomena, such as image processing, Levy processes, the thin obstacle problem, minimal surfaces, multiple scattering, and quasi-geostrophic flows. In addition, comprehensive details and examples on this topic can be found in studies such as [1–5].

The present paper is concerned with the following non-local problems involving fractional (p,q)-Laplace operators and Hardy terms:

$$\begin{cases} \mathcal{L}_p^\alpha \varphi + \mathcal{L}_q^\alpha \varphi = \mu\left(\frac{|\varphi|^{p-2}\varphi}{|x|^{\alpha p}} + \frac{|\varphi|^{q-2}\varphi}{|x|^{\alpha q}}\right) + \theta g(x,\varphi) & \text{in} \quad \Omega, \\ \varphi = 0 & \text{on} \quad \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1}$$

where $0 < \alpha < 1$, $1 < p < q < +\infty$, with $\alpha q < N$; $0 \in \Omega$, μ, θ are positive parameters; Ω is an open and bounded set in $\mathbb{R}^N$ $(N \geq 2)$ with a Lipschitz boundary $\partial\Omega$; and a non-negative function g fulfills the Carathéodory condition, which will be given later. Here, $\mathcal{L}_\mathfrak{m}^\alpha$ $(\mathfrak{m} \in \{p,q\})$ is a non-local pointwise operator defined as follows:

$$\mathcal{L}_\mathfrak{m}^\alpha \varphi(x) = 2\int_{\mathbb{R}^N} |\varphi(x) - \varphi(z)|^{\mathfrak{m}-2}(\varphi(x) - \varphi(z))K_\mathfrak{m}(x,z)dz \quad \text{for all } x \in \mathbb{R}^N,$$

where the function $K_\mathfrak{m} : \mathbb{R}^N \times \mathbb{R}^N \to (0, +\infty)$ fulfills the following hypotheses:

$(\mathcal{L}1)$ $\sigma K_\mathfrak{m} \in L^1(\mathbb{R}^N \times \mathbb{R}^N)$, where $\sigma(x,z) = \min\{|x-z|^\mathfrak{m}, 1\}$;

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($\mathcal{L}2$) There are positive real numbers $\tau_{0,m}$ and $\tau_{1,m}$ with $1 \leq \tau_{0,m}$ such that $\tau_{0,m} \leq K_{\mathfrak{m}}(x,z)|x-z|^{N+\alpha\mathfrak{m}} \leq \tau_{1,m}$ for almost all $(x,z) \in \mathbb{R}^N \times \mathbb{R}^N$ with $x \neq z$;
($\mathcal{L}3$) $K_{\mathfrak{m}}(z,x) = K_{\mathfrak{m}}(x,z)$ for all $(z,x) \in \mathbb{R}^N \times \mathbb{R}^N$.

If $K_{\mathfrak{m}}(x,z) = |x-z|^{-(N+\alpha m)}$, then $\mathcal{L}^{\alpha}_{\mathfrak{m}}$ is the fractional $\mathfrak{m}$-Laplacian operator $(-\Delta)^{\alpha}_{\mathfrak{m}}$ defined as follows:

$$(-\Delta)^{\alpha}_{\mathfrak{m}}\,\varphi(x) = 2\lim_{\sigma \searrow 0}\int_{\mathbb{R}^N \setminus B_{\sigma}(x)} \frac{|\varphi(x)-\varphi(z)|^{\mathfrak{m}-2}\,(\varphi(x)-\varphi(z))}{|x-z|^{N+\alpha\mathfrak{m}}}\,dz, \quad x \in \mathbb{R}^N,$$

where $B_{\sigma}(x) := \{z \in \mathbb{R}^N : |x-z| \leq \sigma\}$.

The coupling of the non-local operators $\mathcal{L}^{\alpha}_{p}$ and $\mathcal{L}^{\alpha}_{q}$ featured in Problem (1) constitutes a fractional double-phase framework characterized by unbalanced growth. This class of problems is of significant interest in mathematical physics, as it provides a theoretical basis for modeling fractional super-diffusion, white-noise limits, and quantum mechanics, as detailed in [6]. Notably, the operator $(-\Delta)^{\alpha}_{p} + (-\Delta)^{\alpha}_{q}$ is designated as the fractional (p,q)-Laplace operator. It serves as the non-local counterpart to the classical (p,q)-Laplacian. Its significance stems from its broad utility in modeling complex phenomena in various sciences, ranging from elasticity theory, plasma physics, biophysics, chemical reaction design, and strongly anisotropic materials; see [7,8] for more details. There has been a surge of interest in the investigation of nonlinear fractional (p,q)-Laplacian equations. This field not only broadens the scope of local (p,q)-Laplacian problems but also introduces many new phenomena and applications arising from non-local integral structures. Specifically, several studies [6,9–11] have investigated the multiplicity of solutions for elliptic equations governed by the fractional (p,q)-Laplacian. Furthermore, the existence and uniqueness of positive solutions for Brézis–Oswald-type problems, featuring unbalanced growth and Hardy potentials, are established in [12].

A distinguishing feature of the present investigation is the inclusion of Hardy potentials. Stationary problems characterized by singular coefficients have recently become a focal point of research, largely because they admit a wide range of interpretations in physics and applied economics; additional contributions and detailed examples can be found in [13–15]. Reflecting this growing interest, the subject has been extensively investigated in recent years; we refer to [16–18] for further details. From an analytical perspective, however, elliptic problems involving singular nonlinearities present significant challenges. These difficulties primarily arise from the non-homogeneity of the operator and the lack of compactness in Palais–Smale sequences. The presence of Hardy potentials poses a significant challenge to verifying the Palais–Smale compactness condition in the target function space. To overcome this obstacle, Ferrara-Bisci [16] employed a refined version of Ricceri's variational principle [19] to demonstrate the existence of at least one nontrivial weak solution for nonlinear elliptic equations containing the p-Laplacian and a Hardy potential. Building upon this approach, recent studies [18,20,21] have successfully derived multiplicity results for p-Laplacian problems by leveraging various critical point theorems of Ricceri's type [19,22].

In contrast to the strategies employed in [16,18,20,21], Fiscella [23] applied the classical mountain pass theorem to establish the existence of a nontrivial solution for a Schrödinger–Kirchhoff-type fractional p-Laplacian equation with a Hardy potential. The problem under consideration is given by:

$$\Big(c + d[\varphi]^{p(\theta-1)}_{\alpha,p}\Big)(-\Delta)^{\alpha}_{p}\,\varphi(x) + V(x)|\varphi|^{p-2}\varphi = \mu\frac{|\varphi|^{p-2}\varphi}{|x|^{\alpha p}} + \lambda g(x,\varphi) \quad \text{in} \quad \mathbb{R}^N,$$

where $c > 0$, $d \geq 0$, and μ, $\lambda \in \mathbb{R}$ with $\lambda > 0$; $V : \mathbb{R}^N \to (0, \infty)$ is a potential function; and g is continuous and satisfies the Ambrosetti–Rabinowitz condition [24]. Furthermore, a related existence result for a double-phase problem involving a Hardy potential is presented in [25]. It should be noted that, in the article, the cut-off function method was used to show the Palais–Smale compactness condition, which is the main challenge in considering elliptic equations with a Hardy potential. In a recent study inspired by these developments, the authors of [26] demonstrated several multiplicity results and obtained the L^∞-bound of nontrivial weak solutions to the Kirchhoff–Schrödinger-type double-phase equations involving Hardy terms:

$$\begin{aligned} &-K\left(\int_{\mathbb{R}^N} H_{p,q}(x, |\nabla\varphi|)\, dx\right) \operatorname{div}(|\nabla\varphi|^{p-2}\nabla\varphi + a(x)|\nabla\varphi|^{q-2}\nabla\varphi) \\ &\quad + V(x)(|\varphi|^{p-2}w + a(x)|\varphi|^{q-2}\varphi) = \mu\left(\frac{|\varphi|^{p-2}\varphi}{|x|^p} + a(x)\frac{|\varphi|^{q-2}\varphi}{|x|^q}\right) + g(x, \varphi) \quad \text{in } \mathbb{R}^N, \end{aligned}$$

where the function $H_{p,q} : \mathbb{R}^N \times [0, \infty) \to [0, \infty)$ is defined as

$$H_{p,q}(x, \eta) := \frac{1}{p}\eta^p + \frac{a(x)}{q}\eta^q,$$

$\mu \in \mathbb{R}$, $0 \leq a \in L^\infty(\mathbb{R}^N)$ and

$$N \geq 2,\ 1 < p < q < N \ \text{ and } \ \frac{q}{p} < 1 + \frac{1}{N}.$$

Furthermore, the Carathéodory function $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is assumed not to satisfy the Ambrosetti–Rabinowitz condition. Within this framework, ref. [27] presents a multiplicity result for solutions of Schrödinger–Hardy-type equations involving a fractional p-Laplacian operator.

The primary objective of this paper is to establish the existence of a sequence of infinitely many small-energy solutions that converge to 0 in L^∞-norm. A key feature of our result is that it requires conditions on the nonlinear term g only locally near the origin, without any assumptions at infinity. This work builds upon related studies [28–30], encompassing cases both with and without Hardy potentials (see [27,31]). While it is well known that global oddness and specific asymptotic behaviors at infinity are typically essential for multiplicity results (as seen in [26,32,33]), we derive our conclusion by imposing conditions on $g(\cdot, \eta)$ solely near zero. Specifically, we assume that $g(\cdot, \eta)$ is odd in η for small η, with no restrictions required at infinity. To date, the L^∞-boundedness of weak solutions converging to zero for fractional (p, q)-Laplacian problems involving Hardy potentials remains largely unexplored, with the notable exception of the studies in [28,31]. Distinguishing our work from these Kirchhoff–Hardy-type investigations, we establish multiplicity results specifically for the parameter range $\mu \in (-\infty, \mu^*)$, where μ^* is a positive constant. In this case, our result extends the study carried out in [28,31]. Departing from the global variational formulation utilized in [29,30] (originally introduced in [34]), we adopt a different strategy. Specifically, following the framework of [28,31], we combine the modified functional method with the dual fountain theorem to derive our main result. It is also worth noting that the conditions on g—which will be specified—are imposed near zero, and no asymptotic assumptions are required at infinity.

The remainder of this paper is organized as follows. We begin in Section 2 by recalling essential facts regarding fractional Sobolev spaces. Section 3 discusses the variational setting and auxiliary results for Problem (1). We conclude in Section 4 by demonstrating the existence of a sequence of solutions that converges to 0 in L^∞-norm.

2. Preliminaries

Let $0 < \alpha < 1 < r < +\infty$ be real numbers with $\alpha r < N$, and define the fractional Sobolev space $W^{\alpha,r}(\mathbb{R}^N)$ as

$$W^{\alpha,r}(\mathbb{R}^N) := \left\{ \varphi \in L^r(\mathbb{R}^N) : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\varphi(x) - \varphi(z)|^r}{|x - z|^{N+\alpha r}} \, dz \, dx < +\infty \right\},$$

which is endowed with the norm

$$\|\varphi\|_{W^{\alpha,r}(\mathbb{R}^N)} := \left(\|\varphi\|^r_{L^r(\mathbb{R}^N)} + |\varphi|^r_{W^{\alpha,r}(\mathbb{R}^N)} \right)^{\frac{1}{r}},$$

where

$$\|\varphi\|^r_{L^r(\mathbb{R}^N)} := \int_{\mathbb{R}^N} |\varphi(x)|^r \, dx \quad \text{and} \quad |\varphi|^r_{W^{\alpha,r}(\mathbb{R}^N)} := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\varphi(x) - \varphi(z)|^r}{|x - z|^{N+\alpha r}} \, dz \, dx.$$

Throughout this paper, let $\Omega \subset \mathbb{R}^N$ be a bounded open set with a Lipschitz boundary. We define the fractional Sobolev space $W^{\alpha,r}_0(\Omega)$ as the collection of functions in $W^{\alpha,r}(\mathbb{R}^N)$ that vanish almost everywhere outside Ω:

$$W^{\alpha,r}_0(\Omega) := \left\{ \varphi \in W^{\alpha,r}(\mathbb{R}^N) : \varphi(x) = 0 \text{ a.e. in } \mathbb{R}^N \backslash \Omega \right\}.$$

This space is endowed with the norm

$$\|\varphi\|_{W^{\alpha,r}_0(\Omega)} := \left(\|\varphi\|^r_{L^r(\Omega)} + [\varphi]^r_{\alpha,r} \right)^{\frac{1}{r}},$$

where the Gagliardo seminorm is given by

$$[\varphi]^r_{\alpha,r} := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\varphi(x) - \varphi(z)|^r}{|x - z|^{N+\alpha r}} \, dz \, dx.$$

It is well known that $W^{\alpha,r}_0(\Omega)$ is a reflexive and separable Banach space, and that $C^\infty_0(\Omega)$ is dense in $W^{\alpha,r}_0(\Omega)$ (refer to [35,36]).

Lemma 1 ([36]). *Let $0 < \alpha < 1 < r < +\infty$. Then, it holds the continuous embeddings:*

$$\begin{aligned} &W^{\alpha,r}_0(\Omega) \hookrightarrow L^\kappa(\Omega) \qquad \text{for all } \kappa \in [1, r^*_\alpha], \qquad \text{if } \alpha r < N; \\ &W^{\alpha,r}_0(\Omega) \hookrightarrow L^\kappa(\Omega) \qquad \text{for every } \kappa \in [1, \infty), \qquad \text{if } \alpha r = N. \end{aligned}$$

Notably, $W^{\alpha,r}_0(\Omega)$ is compactly embedded into the Lebesgue space $L^\kappa(\Omega)$ for any κ in the range $[1, r^*_\alpha)$. The critical exponent r^*_α is given by

$$r^*_\alpha := \begin{cases} \frac{Nr}{N - \alpha r} & \text{if } \alpha r < N, \\ +\infty & \text{if } \alpha r \geq N. \end{cases}$$

We introduce the fractional Sobolev space $W^{\alpha,r}_K(\mathbb{R}^N)$ associated with a kernel function $K_r : \mathbb{R}^N \times \mathbb{R}^N \setminus \{(0,0)\} \to (0, +\infty)$, which satisfies hypotheses ($\mathcal{L}1$)–($\mathcal{L}3$). This space is defined as:

$$W^{\alpha,r}_{K_r}(\mathbb{R}^N) := \left\{ \psi \in L^r(\mathbb{R}^N) : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\psi(x) - \psi(z)|^r K_r(x,z) \, dx \, dz < +\infty \right\}.$$

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Under condition ($\mathcal{L}1$), it follows that the mapping

$$(x,z) \mapsto K_r^{\frac{1}{r}}(x,z)(\psi(x)-\psi(z)) \in L^r(\mathbb{R}^N)$$

belongs to $L^r(\mathbb{R}^{2N})$ for any $\psi \in C_0^\infty(\mathbb{R}^N)$. We further define the closed linear subspace $W_{K_r}^{\alpha,r}(\mathbb{R}^N)$ by

$$W_{0,K_r}^{\alpha,r}(\Omega) := \left\{\psi \in W_{K_r}^{\alpha,r}(\mathbb{R}^N) : \psi(x) = 0 \text{ a.e. in } \mathbb{R}^N \backslash \Omega\right\},$$

endowed with the norm

$$\|\psi\|_{W_{0,K_r}^{\alpha,r}(\Omega)} := \left(\|\psi\|_{L^r(\Omega)}^r + [\psi]_{\alpha,r,K_r}^r\right)^{\frac{1}{r}},$$

where the Gagliardo seminorm is given by

$$[\psi]_{\alpha,r,K_r}^r := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\psi(x)-\psi(z)|^r K_r(x,z)\,dx\,dz.$$

Lemma 2 ([37]). *If $\psi \in W_{0,K_r}^{\alpha,r}(\Omega)$, then $\psi \in W_0^{\alpha,r}(\Omega)$. In addition,*

$$\|\psi\|_{W_0^{\alpha,r}(\Omega)} \le \max\{1, \tau_{0,r}^{-\frac{1}{r}}\}\|\psi\|_{W_{0,K_r}^{\alpha,r}(\Omega)},$$

where $\tau_{0,r}$ is given in ($\mathcal{L}2$).

As a direct consequence of Lemmas 1 and 2, we obtain the following result:

Lemma 3 ([37]). *For any $\psi \in W_0^{\alpha,r}(\Omega)$ and $1 \le \kappa \le p_\alpha^*$, there is a positive real number $\mathcal{D}_r = C(\alpha,r,N)$ such that*

$$\begin{aligned}\|\psi\|_{L^\kappa(\Omega)}^\kappa &\le \mathcal{D}_r \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\psi(x)-\psi(z)|^r}{|x-z|^{N+\alpha\mathfrak{m}}}dx\,dz \\ &\le \frac{\mathcal{D}_r}{\tau_{0,r}} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\psi(x)-\psi(z)|^r K_r(x,z)\,dx\,dz,\end{aligned}$$

where $\tau_{0,r}$ is given in ($\mathcal{L}2$). In addition, the continuous embedding $W_{0,K_r}^{\alpha,r}(\Omega) \hookrightarrow L^\kappa(\Omega)$ holds for any $\kappa \in [1, r_\alpha^*]$, while the embedding

$$W_{0,K_r}^{\alpha,r}(\Omega) \hookrightarrow\hookrightarrow L^\kappa(\Omega)$$

is compact for $\kappa \in (1, r_\alpha^*)$.

Throughout the sequel, assume that the exponents satisfy $0 < \alpha < 1 < p < q < p_\alpha^*$ with the restriction $\alpha q < N$. For $\mathfrak{m} \in \{p,q\}$, let $K_\mathfrak{m} : \mathbb{R}^N \times \mathbb{R}^N \setminus \{(0,0)\} \to (0,\infty)$ be a kernel function satisfying hypotheses ($\mathcal{L}1$)–($\mathcal{L}3$). To investigate Problem (1), we introduce the Banach space

$$\mathbb{X} := W_{0,K_p}^{\alpha,p}(\Omega) \cap W_{0,K_q}^{\alpha,q}(\Omega),$$

endowed with the norm

$$\|\varphi\|_{\mathbb{X}} := \|\varphi\|_{W_{0,K_p}^{\alpha,p}(\Omega)} + \|\varphi\|_{W_{0,K_q}^{\alpha,q}(\Omega)}.$$

By combining Lemmas 1 and 3, we arrive at the following conclusion:

Lemma 4. *If $\varphi \in \mathbb{X}$, then $\varphi \in W_0^{\alpha,p}(\Omega) \cap W_0^{\alpha,q}(\Omega)$. Furthermore, there is a real number $\mathcal{D}_1 = C_1(\alpha, p, q, N) > 0$ such that the estimate*

$$\|\varphi\|^{\kappa}_{L^{\kappa}(\Omega)} \leq \mathfrak{C}_0 \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\varphi(x) - \varphi(z)|^p K_p(x,z) + |\varphi(x) - \varphi(z)|^q K_q(x,z)\, dz\, dx \right)$$

is valid for any $\kappa \in [1, p^*_\alpha]$ and $\varphi \in \mathbb{X}$. As a result, the continuous embedding $\mathbb{X} \hookrightarrow L^{\kappa}(\Omega)$ holds for any $\kappa \in [1, p^*_\alpha]$. In addition, the embedding $\mathbb{X} \hookrightarrow L^{\kappa}(\Omega)$ is compact whenever $\kappa \in (1, p^*_\alpha)$.

A crucial tool for our analysis is the fractional Hardy inequality, as presented in [38].

Lemma 5. *For any $\varphi \in W^{\alpha,m}_{0,K_{\mathfrak{m}}}(\Omega)$, where $\alpha\mathfrak{m} < N$, and for all $\varphi \in W^{\alpha,\mathfrak{m}}_{0,K_{\mathfrak{m}}}(\Omega) \backslash \{0\}$, where $\alpha\mathfrak{m} > N$, we have*

$$\begin{aligned} \|\varphi\|^{\mathfrak{m}}_{\mathcal{H}_{\mathfrak{m}}} := \int_\Omega \frac{|\varphi(x)|^{\mathfrak{m}}}{|x|^{\alpha\mathfrak{m}}} dx &\leq d_{\mathcal{H}_{\mathfrak{m}}} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\varphi(x) - \varphi(z)|^{\mathfrak{m}}}{|x - z|^{N + \alpha\mathfrak{m}}} dz\, dx, \\ &\leq \frac{d_{\mathcal{H}_{\mathfrak{m}}}}{\tau_{0,m}} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\varphi(x) - \varphi(z)|^{\mathfrak{m}} K_{\mathfrak{m}}(x,z) dz\, dx, \end{aligned}$$

where $d_{\mathcal{H}_{\mathfrak{m}}}$ is an optimal positive constant. In particular, if $\mathfrak{m} \in \{p, q\}$, then for any $\varphi \in \mathbb{X}$, where $\alpha q < N$, and for all $\varphi \in \mathbb{X} \backslash \{0\}$, where $\alpha q > N$, we know that

$$\int_\Omega \frac{|\varphi(x)|^{\mathfrak{m}}}{|x|^{\alpha\mathfrak{m}}} dx \leq \max\left\{ d_{\mathcal{H}_p} \tau^{-1}_{0,p}, d_{\mathcal{H}_q} \tau^{-1}_{0,q} \right\} \left([\varphi]^p_{\alpha,p,K_p} + [\varphi]^q_{\alpha,q,K_q} \right). \tag{2}$$

3. Variational Setting and Auxiliary Results

In this section, we establish the variational formulation for Problem (1) and present several necessary lemmas.

Definition 1. *We say that $\varphi \in \mathbb{X}$ is a weak solution of (1) if it satisfies the following:*

$$\begin{aligned} &\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\varphi(x) - \varphi(z)|^{p-2} (\varphi(x) - \varphi(z))(\psi(x) - \psi(z)) K_p(x,z)\, dz\, dx \\ &+ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\varphi(x) - \varphi(z)|^{q-2} (\varphi(x) - \varphi(z))(\psi(x) - \psi(z)) K_q(x,z)\, dz\, dx \\ &= \mu \left(\int_\Omega \frac{|\varphi(x)|^{p-2}\varphi(x)}{|x|^{\alpha p}} \psi(x)\, dx + \int_\Omega \frac{|\varphi(x)|^{q-2}\varphi(x)}{|x|^{\alpha q}} \psi(x)\, dx \right) + \theta \int_\Omega g(x, \varphi)\psi(x)\, dx \end{aligned} \tag{3}$$

for any $\psi \in \mathbb{X}$.

We observe that relation (3) is the Equation (1) written in the distributional sense. Hence, relation (3) represents the weak formulation of (1). Note that, to write such a weak formulation, we need to assume condition ($\mathcal{L}3$).

Consider the functional $\Phi_\mu : \mathbb{X} \to \mathbb{R}$ given by the following:

$$\Phi_\mu(\varphi) := \Phi(\varphi) - \mu \Phi_{\mathcal{H}_{p,q}}(\varphi),$$

where Φ and $\Phi_{\mathcal{H}_{p,q}}$ are defined as:

$$\begin{aligned} \Phi(\varphi) :=& \frac{1}{p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\varphi(x) - \varphi(z)|^p K_p(x,z)\, dx\, dz \\ &+ \frac{1}{q} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\varphi(x) - \varphi(z)|^q K_q(x,z)\, dx\, dz \end{aligned}$$

and

$$\Phi_{\mathcal{H}_{p,q}}(\varphi) := \frac{1}{p}\int_\Omega \frac{|\varphi(x)|^p}{|x|^{\alpha p}}\,dx + \frac{1}{q}\int_\Omega \frac{|\varphi(x)|^q}{|x|^{\alpha q}}\,dx.$$

It is straightforward to verify that Φ_μ is well-defined on $\mathbb{X}$. Following a similar strategy to the proof of Lemma 2 in [39], we derive the result below.

Lemma 6. *The functional $\Phi_\mu : \mathbb{X} \to \mathbb{R}$ is of class $C^1(\mathbb{X}, \mathbb{R})$, and its Fréchet derivative is*

$$\begin{aligned}\left\langle \Phi'_\mu(\varphi), v\right\rangle = &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\varphi(x)-\varphi(z)|^{p-2}(\varphi(x)-\varphi(z))(v(x)-v(z))K_p(x,z)dx\,dz\\ &+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\varphi(x)-\varphi(z)|^{q-2}(\varphi(x)-\varphi(z))(v(x)-v(z))K_q(x,z)dx\,dz\\ &-\mu\left(\int_\Omega \frac{|\varphi(x)|^{p-2}\varphi(x)}{|x|^{\alpha p}}v(x)\,dx + \int_\Omega \frac{|\varphi(x)|^{q-2}\varphi(x)}{|x|^{\alpha q}}v(x)\,dx\right)\end{aligned} \tag{4}$$

for any $\varphi, v \in \mathbb{X}$. Here, let us denote $\langle\cdot,\cdot\rangle$ with the pairing of $\mathbb{X}$ and its dual $\mathbb{X}^*$.

Now, suppose that the following conditions for g hold:

(G1) $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the following subcritical growth condition: there exist a non-negative function $\rho_1 \in L^\infty(\Omega)$ and a positive constant ρ_2 such that

$$|g(x,\eta)| \le \rho_1(x) + \rho_2|\eta|^{\ell-1}$$

for all $\eta \in \mathbb{R}$ and a.e. $x \in \Omega$. Here, the exponent ℓ satisfies $q < \ell < p^*_\alpha$;

(G2) For some small $\eta_1 > 0$, the function $g(x,\eta)$ is odd with respect to η whenever $|\eta| < \eta_1$. Furthermore, it satisfies the strict inequality $pG(x,\eta) > g(x,\eta)\eta$ for $0 < |\eta| < \eta_1$ and almost all $x \in \Omega$, where $G(x,\eta) = \int_0^\eta g(x,t)\,dt$;

(G3) $\lim_{|\eta|\to 0} \frac{g(x,\eta)}{|\eta|^{p-2}\eta} = +\infty$ uniformly for almost all $x \in \Omega$.

Under (G2) and (G3), for given $\mathcal{D}_1 \ge 1$, there exists $\eta_2 \in (0,\eta_1)$ such that

$$G(x,\eta) \ge \mathcal{D}_1|\eta|^p \quad \text{for almost everywhere } x \in \Omega \text{ and all } |\eta| < \eta_2. \tag{5}$$

Fix $\eta_3 \in (0, \eta_2/2)$ and let $\phi \in C^1(\mathbb{R},\mathbb{R})$ be such that ϕ is even, $\phi(\eta) = 1$ for $|\eta| \le \eta_3$, $\phi(\eta) = 0$ for $|\eta| \ge 2\eta_3$, $|\phi'(\eta)| \le 2/\eta_3$, and $\phi'(\eta)\eta \le 0$. Then, let us define the modified function $\widetilde{g} : \Omega \times \mathbb{R} \to \mathbb{R}$ as

$$\widetilde{g}(x,\eta) := \frac{\partial}{\partial\eta}\widetilde{G}(x,\eta),$$

where

$$\widetilde{G}(x,\eta) := \phi(\eta)G(x,\eta) + (1-\phi(\eta))\xi|\eta|^p \tag{6}$$

for some fixed $\xi \in \left(0, \min\left\{\frac{1}{p}, \frac{\widetilde{\mathcal{D}}}{2q}\right\}\right)$, with

$$\widetilde{\mathcal{D}} = \min\left\{1, \frac{\tau_{0,p}}{\mathcal{D}_p}, \frac{\tau_{0,q}}{\mathcal{D}_q}\right\}, \tag{7}$$

where $\tau_{0,m}$ and $\mathcal{D}_m$ ($\mathfrak{m} \in \{p,q\}$) are given in ($\mathcal{L}2$) and Lemma 3, respectively. Clearly, $\widetilde{G}$ is even in η,

$$\widetilde{g}(x,\eta) = \phi'(\eta)G(z,\eta) + \phi(\eta)g(x,\eta) - \phi'(\eta)\xi|\eta|^p + (1-\phi(\eta))\xi p|\eta|^{p-2}\eta \tag{8}$$

and

$$p\widetilde{G}(z,\eta) - \widetilde{g}(x,\eta)\eta = \phi(\eta)[pG(x,\eta) - g(x,\eta)\eta] - \phi'(\eta)\eta[G(z,\eta) - \xi|\eta|^p].$$

Then, the definition of ϕ and (5) yield

$$p\widetilde{G}(x,\eta) - \widetilde{g}(x,\eta)\eta \geq 0 \text{ for all } \eta \in \mathbb{R} \text{ and for almost all } x \in \Omega \tag{9}$$

and

$$p\widetilde{G}(x,\eta) - \widetilde{g}(x,\eta)\eta = 0 \quad \text{iff} \quad \eta \equiv 0 \quad \text{or} \quad |\eta| \geq 2\eta_3. \tag{10}$$

Let us introduce the modified energy functional $\mathcal{E}_{\mu,\theta} : \mathbb{X} \to \mathbb{R}$ as follows:

$$\mathcal{E}_{\mu,\theta}(\varphi) := \Phi_\mu(\varphi) - \theta\Psi(\varphi),$$

where the functional Ψ is given by

$$\Psi(\varphi) = \int_\Omega \widetilde{G}(x,\varphi)\,dx.$$

It is readily verified that $\mathcal{E}_{\mu,\theta}$ belongs to $C^1(\mathbb{X},\mathbb{R})$ and is an even functional. Its Fréchet derivative is expressed as

$$\left\langle \mathcal{E}'_{\mu,\theta}(\varphi), v \right\rangle = \left\langle \Phi'_\mu(\varphi), v \right\rangle - \theta \int_\Omega \widetilde{g}(x,\varphi)v\,dx$$

for any $v \in \mathbb{X}$.

In the sequel, we present several auxiliary results essential for establishing our main assertion.

Lemma 7. *Assume that* (G1)–(G3) *hold. Then, there exist a positive constant* μ^* *and an interval* Λ *such that* $\mathcal{E}_{\mu,\theta}$ *is coercive for every* $\theta \in \Lambda$ *and any* $\mu \in (-\infty, \mu^*)$.

Proof. Firstly, let us put

$$\mu^* = \frac{p\widetilde{\mathcal{D}} \min\left\{ d_{\mathcal{H}_p}^{-1}\tau_{0,p}, d_{\mathcal{H}_q}^{-1}\tau_{0,q} \right\}}{2q}, \tag{11}$$

where $\tau_{0,m}$, $\widetilde{\mathcal{D}}$ and $d_{\mathcal{H}_\mathfrak{m}}$ ($\mathfrak{m} \in \{p,q\}$) are given in ($\mathcal{L}2$), (7) and Lemma 5, respectively. Let $\varphi \in \mathbb{X}$ with $\|\varphi\|_\mathbb{X} \geq 1$. By (G1), (6) and the definition of ϕ, there exists a positive constant C_1 such that

$$|\widetilde{G}(x,\eta)| \leq C_1 + \xi|\eta|^p \tag{12}$$

for all $\eta \in \mathbb{R}$ and for almost all $x \in \Omega$. Let us take $\mu \in (0,\mu^*)$. Since $\mu < \mu^*$, taking Lemmas 3 and 5, (G1), (6), and the definition of ϕ into account, we get

$$
\begin{aligned}
\mathcal{E}_{\mu,\theta}(\varphi) &\geq \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\varphi(x)-\varphi(z)|^p K_p(x,z)\,dx\,dz \\
&\quad + \frac{1}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\varphi(x)-\varphi(z)|^q K_q(x,z)\,dx\,dz \\
&\quad - \frac{\mu}{p}\left(\int_\Omega \frac{|\varphi|^p}{|x|^{\alpha p}}\,dx + \int_\Omega \frac{|\varphi|^q}{|x|^{\alpha q}}\,dx\right) - \theta\int_\Omega \left|\widetilde{G}(x,\varphi)\right| dx \\
&\geq \frac{1}{q}\left([\varphi]^p_{\alpha,p,K_p} + [\varphi]^q_{\alpha,q,K_q}\right) - \frac{\mu}{p\min\left\{d^{-1}_{\mathcal{H}_p}\tau_{0,p}, d^{-1}_{\mathcal{H}_q}\tau_{0,q}\right\}}\left([\varphi]^p_{\alpha,p,K_p} + [\varphi]^q_{\alpha,q,K_q}\right) \\
&\quad - \theta\int_\Omega \left|\widetilde{G}(x,\varphi)\right| dx \\
&\geq \frac{\widetilde{\mathcal{D}}}{2q}\left([\varphi]^p_{\alpha,p,K_p} + [\varphi]^q_{\alpha,q,K_q} + \|\varphi\|^p_{L^p(\Omega)} + \|\varphi\|^q_{L^q(\Omega)}\right) \\
&\quad - \frac{\mu}{p\min\left\{d^{-1}_{\mathcal{H}_p}\tau_{0,p}, d^{-1}_{\mathcal{H}_q}\tau_{0,q}\right\}}\left([\varphi]^p_{\alpha,p,K_p} + [\varphi]^q_{\alpha,q,K_q}\right) - \theta\int_\Omega \left|\widetilde{G}(x,\varphi)\right| dx \\
&\geq \left(\frac{\widetilde{\mathcal{D}}}{2q} - \frac{\mu}{p\min\left\{d^{-1}_{\mathcal{H}_p}\tau_{0,p}, d^{-1}_{\mathcal{H}_q}\tau_{0,q}\right\}}\right)\|\varphi\|^p_{\mathbb{X}} - \theta\int_\Omega (C_1 + \xi|\varphi|^p)\,dx \\
&\geq \left(\frac{\widetilde{\mathcal{D}}}{2q} - \frac{\mu}{p\min\left\{d^{-1}_{\mathcal{H}_p}\tau_{0,p}, d^{-1}_{\mathcal{H}_q}\tau_{0,q}\right\}}\right)\|\varphi\|^p_{\mathbb{X}} - \theta C_1|\Omega| - \theta\xi\|\varphi\|^p_{\mathbb{X}} \\
&\geq \left[\frac{p\widetilde{\mathcal{D}}\min\left\{d^{-1}_{\mathcal{H}_p}\tau_{0,p}, d^{-1}_{\mathcal{H}_q}\tau_{0,q}\right\} - 2q\mu}{2pq\min\left\{d^{-1}_{\mathcal{H}_p}\tau_{0,p}, d^{-1}_{\mathcal{H}_q}\tau_{0,q}\right\}} - \theta\xi\right]\|\varphi\|^p_{\mathbb{X}} - \theta C_1|\Omega|. \qquad (13)
\end{aligned}
$$

Additionally, if $\mu \in (-\infty, 0]$, then it follows (13), in an analogous way, such that

$$
\begin{aligned}
\mathcal{E}_{\mu,\theta}(\varphi) &\geq \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\varphi(x)-\varphi(z)|^p K_p(x,z)\,dx\,dz \\
&\quad + \frac{1}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\varphi(x)-\varphi(z)|^q K_q(x,z)\,dx\,dz \\
&\quad - \theta\int_\Omega \left|\widetilde{G}(x,\varphi)\right| dx \\
&\geq \left[\frac{\widetilde{\mathcal{D}}}{2q} - \theta\xi\right]\|\varphi\|^p_{\mathbb{X}} - \theta C_1|\Omega|. \qquad (14)
\end{aligned}
$$

Set

$$
\Lambda_1 = \left(0, \frac{p\widetilde{\mathcal{D}}\min\left\{d^{-1}_{\mathcal{H}_p}\tau_{0,p}, d^{-1}_{\mathcal{H}_q}\tau_{0,q}\right\} - 2q\mu}{2pq\min\left\{d^{-1}_{\mathcal{H}_p}\tau_{0,p}, d^{-1}_{\mathcal{H}_q}\tau_{0,q}\right\}\xi}\right)
$$

and

$$
\Lambda_2 = \left(0, \frac{\widetilde{\mathcal{D}}}{2q\xi}\right).
$$

Therefore, we derive through (13) and (14) that $\mathcal{E}_{\mu,\theta}$ is coercive in $\mathbb{X}$, that is, $\mathcal{E}_{\mu,\theta}(\varphi) \to \infty$ as $\|\varphi\|_{\mathbb{X}} \to \infty$ for any $\theta \in \Lambda$ and for any $\mu \in (-\infty, \mu^*)$, where Λ is either Λ_1 or Λ_2. □

Lemma 8. *Assume that conditions* (G1)–(G3) *are satisfied. The derivative functional* Ψ' *is sequentially weakly-strongly continuous from* $\mathbb{X}$ *to* $\mathbb{X}^*$.

Proof. Let $\{\varphi_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathbb{X}$ such that $\varphi_k \rightharpoonup \varphi$ in $\mathbb{X}$ as $k \to \infty$. By the boundedness of $\{\varphi_k\}_{k\in\mathbb{N}}$ and the compact embedding in Lemma 4, there exists a subsequence $\left\{\varphi_{k_j}\right\}_{j\in\mathbb{N}}$ such that

$$\varphi_{k_j}(x) \to \varphi(x) \text{ a.e. in } \Omega \text{ and } \varphi_{k_j} \to \varphi \text{ in } L^m(\Omega) \text{ as } j \to \infty, \tag{15}$$

where $1 \le m < p_s^*$. By Theorem 4.9 in [40], we can find a further subsequence (still denoted by $\{\varphi_{k_j}\}_{j\in\mathbb{N}}$) and and a dominating function $w \in L^p(\Omega)$ such that $\varphi_{k_j}(x) \to \varphi(x)$ as $j \to \infty$ for almost all $x \in \Omega$ and $|\varphi_{k_j}(x)| \le w(x)$ for all $j \in \mathbb{N}$ and for almost all $x \in \Omega$. For any $u \in \mathbb{X}$, we have

$$\left|\langle \Psi'(\varphi_{k_j}) - \Psi'(\varphi), u\rangle\right| = \left| \int_\Omega \left(\widetilde{g}(x, \varphi_{k_j}) - \widetilde{g}(x, \varphi)\right) u \, dx\right|.$$

By the definition of ϕ and the assumption (G1), we deduce from (8) that

$$|\widetilde{g}(x, \eta)| \le C_2\left(1 + |\eta|^{p-1}\right) \tag{16}$$

for some positive constant C_2. Owing to (16), we obtain

$$\begin{aligned}\left|\int_\Omega \left(\widetilde{g}(x, \varphi_{k_j}) - \widetilde{g}(x, \varphi)\right) u \, dx\right| &\le \int_\Omega \left|\left(\widetilde{g}(x, \varphi_{k_j}) - \widetilde{g}(x, \varphi)\right) u\right| dx \\ &\le C_2 \int_\Omega \left(2 + \left|\varphi_{k_j}\right|^{p-1} + |\varphi|^{p-1}\right) |u| \, dx \\ &\le C_2 \int_\Omega \left(2 + |w|^{p-1} + |\varphi|^{p-1}\right) |u| \, dx \end{aligned} \tag{17}$$

for some positive constant C_2. Invoking (15)–(17) and the convergence principle, one has

$$\left|\left(\widetilde{g}(x, \varphi_{k_j}) - \widetilde{g}(x, \varphi)\right) u\right| \le h(x)$$

for almost all $x \in \Omega$ and for some $h \in L^1(\Omega)$, and also $\left|\left(\widetilde{g}(x, \varphi_{k_j}) - \widetilde{g}(x, \varphi)\right) u\right| \to 0$ as $j \to \infty$ for almost all $x \in \Omega$. Together with the Lebesgue Dominated convergence theorem, this yields

$$\begin{aligned}\|\Psi'(\varphi_{k_j}) - \Psi'(\varphi)\|_{\mathbb{X}^*} &= \sup_{\|u\|_{\mathbb{X}} \le 1} \left|\left\langle \Psi'(\varphi_{k_j}) - \Psi'(\varphi), u\right\rangle\right| \\ &= \sup_{\|u\|_{\mathbb{X}} \le 1} \left| \int_\Omega \left(\widetilde{g}(x, \varphi_{k_j}) - \widetilde{g}(x, \varphi)\right) u \, dx\right| \to 0\end{aligned}$$

as $j \to \infty$. Therefore, we derive that $\Psi'(\varphi_{k_j}) \to \Psi'(\varphi)$ in $\mathbb{X}^*$ as $j \to \infty$, as previously claimed. □

Now, we show that the energy functional $\mathcal{E}_{\mu,\theta}$ satisfies the Cerami condition at level $c \in \mathbb{R}$ (hereafter denoted as the $(C)_c$-condition), i.e., for any $c \in \mathbb{R}$, any sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathbb{X}$ such that

$$\mathcal{E}_{\mu,\theta}(\varphi_n) \to c \quad \text{and} \quad \|\mathcal{E}'_{\mu,\theta}(\varphi_n)\|_{\mathbb{X}^*}(1 + \|\varphi_n\|_{\mathbb{X}}) \to 0 \quad \text{as} \quad n \to \infty \tag{18}$$

has a convergent subsequence. The fundamental strategy for proving the following results relies on arguments analogous to those presented in [23]; for further reference, see [27].

Lemma 9. *Assume that* (G1)–(G3) *hold. Then, for any* $\mu \in (0, \mu^*)$, *the functional* $\mathcal{E}_{\mu,\theta}$ *ensures the* $(C)_c$*-condition for every* $\theta \in \Lambda$, *where* μ^* *and* Λ *are given in Lemma 7.*

Proof. For any $c \in \mathbb{R}$, let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a $(C)_c$-sequence in $\mathbb{X}$ satisfying (18). The coercivity of $\mathcal{E}_{\mu,\theta}$ implies that $\{\varphi_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{X}$, which, by the reflexivity of the space, ensures the existence of a weakly convergent subsequence. Without loss of generality, we may assume that $\varphi_n \rightharpoonup \varphi$ in $\mathbb{X}$ as $n \to \infty$. By invoking Lemmas 4 and 5, and passing to a further subsequence (still denoted by $\{\varphi_n\}_{n\in\mathbb{N}}$), we obtain the following convergence properties:

$$\begin{aligned} &\varphi_n \rightharpoonup \varphi \text{ in } \mathbb{X}, \quad \varphi_n \rightharpoonup \varphi \text{ in } L^p(\Omega, |x|^{-\alpha p}), \quad \varphi_n \rightharpoonup \varphi \text{ in } L^q(\Omega, |x|^{-\alpha q}), \\ &\varphi_n \to \varphi \text{ a.e. in } \mathbb{R}^N, \quad \varphi_n \to \varphi \text{ in } L^\nu(\Omega), \quad \|\varphi_n - \varphi\|_{H_p} + \|\varphi_n - \varphi\|_{H_q} \to \lambda \end{aligned} \tag{19}$$

for any $\nu \in [p, p_\alpha^*)$. Then, for $\mathfrak{m} \in \{p, q\}$, the sequence

$$\left\{ |\varphi_n(x) - \varphi_n(z)|^{\mathfrak{m}-2}(\varphi_n(x) - \varphi_n(z)) K_{\mathfrak{m}}(x,z)^{\frac{1}{\mathfrak{m}'}} \right\}_{n\in\mathbb{N}}$$

is bounded in $L^{\mathfrak{m}'}(\mathbb{R}^N \times \mathbb{R}^N)$, and

$$\begin{aligned} \Upsilon_n(x,z) &:= |\varphi_n(x) - \varphi_n(z)|^{\mathfrak{m}-2}(\varphi_n(x) - \varphi_n(z)) K_{\mathfrak{m}}(x,z)^{\frac{1}{\mathfrak{m}'}} \\ &\to \Upsilon(x,z) := |\varphi(x) - \varphi(z)|^{\mathfrak{m}-2}(\varphi(x) - \varphi(z)) K_{\mathfrak{m}}(x,z)^{\frac{1}{\mathfrak{m}'}} \end{aligned}$$

almost everywhere in $\mathbb{R}^N \times \mathbb{R}^N$ as $n \to \infty$. Thus, passing to a further subsequence if necessary, we infer that $\Upsilon_n \rightharpoonup \Upsilon$ in $L^{\mathfrak{m}'}(\mathbb{R}^N \times \mathbb{R}^N)$ as $n \to \infty$. Hence, since $(x,z) \mapsto |\varrho(x) - \varrho(z)| \cdot K_{\mathfrak{m}}(x,z)^{\frac{1}{\mathfrak{m}}} \in L^{\mathfrak{m}}(\mathbb{R}^N \times \mathbb{R}^N)$, we assert that for any $\varrho \in \mathbb{X}$,

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\varphi_n(x) - \varphi_n(z)|^{\mathfrak{m}-2}(\varphi_n(x) - \varphi_n(z))(\varrho(x) - \varrho(z)) K_{\mathfrak{m}}(x,z)\, dxdz \\ &\to \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\varphi(x) - \varphi(z)|^{\mathfrak{m}-2}(\varphi(x) - \varphi(z))(\varrho(x) - \varrho(z)) K_{\mathfrak{m}}(x,z)\, dxdz \end{aligned} \tag{20}$$

as $n \to \infty$.

Moreover, the sequence

$$\left\{ \frac{|\varphi_n(x)|^{\mathfrak{m}-2}\varphi_n(x)}{|x|^{\frac{\alpha\mathfrak{m}}{\mathfrak{m}'}}} \right\}_{n\in\mathbb{N}}$$

is bounded in $L^{\mathfrak{m}'}(\Omega)$ as well as

$$\frac{|\varphi_n(x)|^{\mathfrak{m}-2}\varphi_n(x)}{|x|^{\frac{\alpha\mathfrak{m}}{\mathfrak{m}'}}} \to \frac{|\varphi(x)|^{\mathfrak{m}-2}\varphi(x)}{|x|^{\frac{\alpha\mathfrak{m}}{\mathfrak{m}'}}}$$

almost everywhere in Ω as $n \to \infty$. By (19), we have

$$|\varphi_n|^{\mathfrak{m}-2}\varphi_n \rightharpoonup |\varphi|^{\mathfrak{m}-2}\varphi \text{ in } L^{\mathfrak{m}'}(\mathbb{R}^N, |x|^{-\alpha\mathfrak{m}}),$$

so that

$$\lim_{n\to\infty} \int_\Omega \frac{|\varphi_n|^{\mathfrak{m}-2}\varphi_n}{|x|^{\alpha\mathfrak{m}}} \varrho\, dx = \int_\Omega \frac{|\varphi|^{\mathfrak{m}-2}\varphi}{|x|^{\alpha\mathfrak{m}}} \varrho\, dx \tag{21}$$

for any $\varrho \in \mathbb{X}$.

Consequently, we arrive by (20) and (21) that

$$\begin{aligned} &\lim_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\varphi_n(x) - \varphi_n(z)|^{\mathfrak{m}-2}(\varphi_n(x) - \varphi_n(z))(\varphi(x) - \varphi(z)) K_{\mathfrak{m}}(x,z)\, dx\, dz \\ &\quad = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\varphi(x) - \varphi(z)|^{\mathfrak{m}} K_{\mathfrak{m}}(x,z)\, dx\, dz = [\varphi]^{\mathfrak{m}}_{\alpha,m,K_{\mathfrak{m}}} \end{aligned} \tag{22}$$

and

$$\lim_{n\to\infty}\int_\Omega \frac{|\varphi_n|^{\mathfrak{m}-2}\varphi_n}{|x|^{\alpha\mathfrak{m}}}\varphi\,dx = \int_\Omega \frac{|\varphi|^{\mathfrak{m}}}{|x|^{\alpha\mathfrak{m}}}dx \tag{23}$$

for $\mathfrak{m}\in\{p,q\}$.

By also taking Lemma 4, (19), (16), and the Hölder inequality into account, we know that

$$\begin{aligned}\left|\int_\Omega \widetilde{g}(x,\varphi_n)(\varphi_n-\varphi)dx\right| &\le C_1\int_\Omega\Big(1+|\varphi_n|^{p-1}\Big)|\varphi_n-\varphi|dx\\ &\le C_3\Big(1+\|\varphi_n\|^{p-1}_{L^p(\Omega)}\Big)\|\varphi_n-\varphi\|_{L^p(\Omega)}\to 0\end{aligned} \tag{24}$$

as $n\to\infty$ for a positive constant C_3. Furthermore, using (19), (20), and the Brézis and Lieb lemma in (Theorem 1, [41]), we obtain

$$\begin{aligned}[\varphi_n]^p_{\alpha,p,K_p}+[\varphi_n]^q_{\alpha,q,K_q}-[\varphi_n-\varphi]^p_{\alpha,p,K_p}-[\varphi_n-\varphi]^q_{\alpha,q,K_q} &= [\varphi]^p_{\alpha,p,K_p}+[\varphi]^q_{\alpha,q,K_q}+o(1),\\ \|\varphi_n\|^p_{H_p}+\|\varphi_n\|^q_{H_q}-\|\varphi_n-\varphi\|^p_{H_p}-\|\varphi_n-\varphi\|^q_{H_q} &= \|\varphi\|^p_{H_p}+\|\varphi\|^q_{H_q}+o(1)\end{aligned} \tag{25}$$

as $n\to\infty$. Thus, by Lemma 3, (18) and (22)–(25), we obtain

$$\begin{aligned}o(1) &= \Big\langle \mathcal{E}'_{\mu,\lambda}(\varphi_n),\varphi_n-\varphi\Big\rangle\\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|\varphi_n(x)-\varphi_n(z)|^pK_p(x,z)\,dxdz\\ &\quad+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|\varphi_n(x)-\varphi_n(z)|^qK_q(x,z)\,dxdz\\ &\quad-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|\varphi_n(x)-\varphi_n(z)|^{p-2}(\varphi_n(x)-\varphi_n(z))(\varphi(x)-\varphi(z))K_p(x,z)\,dxdz\\ &\quad-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|\varphi_n(x)-\varphi_n(z)|^{q-2}(\varphi_n(x)-\varphi_n(z))(\varphi(x)-\varphi(z))K_q(x,z)\,dxdz\\ &\quad-\mu\left(\int_\Omega\frac{|\varphi_n|^{p-2}\varphi_n}{|x|^{\alpha p}}(\varphi_n-\varphi)dx+\int_\Omega\frac{|\varphi_n|^{q-2}\varphi_n}{|x|^{\alpha q}}(\varphi_n-\varphi)dx\right)\\ &\quad-\theta\int_\Omega\widetilde{g}(x,\varphi_n)(\varphi_n-\varphi)\,dx+o(1)\\ &\ge [\varphi_n]^p_{\alpha,p,K_p}+[\varphi_n]^q_{\alpha,q,K_q}-[\varphi]^p_{\alpha,p,K_p}-[\varphi]^q_{\alpha,q,K_q}\\ &\quad-\mu\left(\int_\Omega\frac{|\varphi_n|^p}{|x|^{\alpha p}}dx+\int_\Omega\frac{|\varphi_n|^q}{|x|^{\alpha q}}dx-\int_\Omega\frac{|\varphi|^p}{|x|^{\alpha p}}dx-\int_\Omega\frac{|\varphi|^q}{|x|^{\alpha q}}dx\right)+o(1)\\ &\ge \Big([\varphi_n-\varphi]^p_{\alpha,p,K_p}+[\varphi_n-\varphi]^q_{\alpha,q,K_q}\Big)\\ &\quad-\mu\Big(\Big(\|\varphi_n\|^p_{H_p}-\|\varphi\|^p_{H_p}\Big)+\Big(\|\varphi_n\|^q_{H_q}-\|\varphi\|^q_{H_q}\Big)\Big)+o(1)\\ &\ge \frac{\widetilde{\mathcal{D}}}{2}\Big([\varphi_n-\varphi]^p_{\alpha,p,K_p}+[\varphi_n-\varphi]^q_{\alpha,q,K_q}+\|\varphi_n-\varphi\|^p_{L^p(\Omega)}+\|\varphi_n-\varphi\|^q_{L^q(\Omega)}\Big)\\ &\quad-\mu\Big(\|\varphi_n-\varphi\|^p_{H_p}+\|\varphi_n-\varphi\|^q_{H_q}\Big)+o(1)\end{aligned} \tag{26}$$

as $n\to\infty$, where $\widetilde{\mathcal{D}}$ is given in (7). Hence, from (19), we arrive

$$\begin{aligned}&\widetilde{\mathcal{D}}\Big([\varphi_n-\varphi]^p_{\alpha,p,K_p}+[\varphi_n-\varphi]^q_{\alpha,q,K_q}+\|\varphi_n-\varphi\|^p_{L^p(\Omega)}+\|\varphi_n-\varphi\|^q_{L^q(\Omega)}\Big)\\ &\quad\le 2\mu\Big(\|\varphi_n-\varphi\|^p_{H_p}+\|\varphi_n-\varphi\|^q_{H_q}\Big)+o(1)\\ &\quad= 2\mu\lambda+o(1)\end{aligned} \tag{27}$$

as $n \to \infty$. Now, suppose, for contradiction, that $\lambda > 0$. Then, from Lemma 5, (27), and the fact that $2\mu < 2\mu^* < \widetilde{\mathcal{D}} \min\left\{ d_{\mathcal{H}_p}^{-1} \tau_{0,p}, d_{\mathcal{H}_q}^{-1} \tau_{0,q} \right\}$, we have

$$\begin{aligned}
\widetilde{\mathcal{D}} \lim_{n\to\infty} \Big(& [\varphi_n - \varphi]_{\alpha,p,K_p}^p + [\varphi_n - \varphi]_{\alpha,q,K_q}^q + \|\varphi_n - \varphi\|_{L^p(\Omega)}^p + \|\varphi_n - \varphi\|_{L^q(\Omega)}^q \Big) \\
& \leq 2\mu \lim_{n\to\infty} \left(\|\varphi_n - \varphi\|_{H_p}^p + \|\varphi_n - \varphi\|_{H_q}^q \right) \\
& < \widetilde{\mathcal{D}} \min\left\{ d_{\mathcal{H}_p}^{-1} \tau_{0,p}, d_{\mathcal{H}_q}^{-1} \tau_{0,q} \right\} \lim_{n\to\infty} \left(\|\varphi_n - \varphi\|_{H_p}^p + \|\varphi_n - \varphi\|_{H_q}^q \right) \\
& \leq \widetilde{\mathcal{D}} \lim_{n\to\infty} \left(d_{\mathcal{H}_p}^{-1} \tau_{0,p} \|\varphi_n - \varphi\|_{H_p}^p + d_{\mathcal{H}_q}^{-1} \tau_{0,q} \|\varphi_n - \varphi\|_{H_q}^q \right) \\
& \leq \widetilde{\mathcal{D}} \lim_{n\to\infty} \Big([\varphi_n - \varphi]_{\alpha,p,K_p}^p + [\varphi_n - \varphi]_{\alpha,q,K_q}^q + \|\varphi_n - \varphi\|_{L^p(\Omega)}^p + \|\varphi_n - \varphi\|_{L^q(\Omega)}^q \Big),
\end{aligned}$$

which is absurd. Therefore, $\lambda = 0$; so, by (27), we have $\varphi_n \to \varphi$ in $\mathbb{X}$. This completes the proof. □

4. Main Result

Our goal in this section is to show that Problem (1) possesses a sequence of infinitely many weak solutions with L^∞-convergence to zero. The dual fountain theorem serves as our main tool, the application of which relies on a specific decomposition of the space. Let $\mathbb{E}$ be a separable, reflexive Banach space. According to [42,43], there exists a biorthogonal system $\{e_n, h_n^*\}$ such that $\mathbb{E}$ and $\mathbb{E}^*$ are the closed spans of $\{e_n\}$ and $\{h_n^*\}$, respectively. Denoting $\mathbb{E}_n = \operatorname{span}\{e_n\}$, we decompose $\mathbb{E}$ into the following subspaces for our variational framework:

$$\mathbb{Y}_n = \bigoplus_{k=1}^{n} \mathbb{E}_k \quad \text{and} \quad \mathbb{H}_n = \overline{\bigoplus_{k=n}^{\infty}} \mathbb{E}_k.$$

Lemma 10 (Dual Fountain Theorem [33])**.** *Assume that $(\mathbb{E}, \|\cdot\|)$ is a Banach space, and $F \in C^1(\mathbb{E}, \mathbb{R})$ is an even functional. If there is $n_0 > 0$ so that, for each $n \geq n_0$, there exist $\delta_n > \gamma_n > 0$ such that the following hold:*

($\mathcal{C}1$) $\inf\{F(\omega) : \omega \in \mathbb{H}_n, \|\omega\| = \delta_n\} \geq 0$;
($\mathcal{C}2$) $\phi_n := \max\{F(\omega) : \omega \in \mathbb{Y}_n, \|\omega\| = \gamma_n\} < 0$;
($\mathcal{C}3$) $\varrho_n := \inf\{F(\omega) : \omega \in \mathbb{H}_n, \|\omega\| \leq \delta_n\} \to 0$ *as* $n \to \infty$;
($\mathcal{C}4$) *F fulfills the $(C)_c^*$-condition for every $c \in [\varrho_{n_0}, 0)$,*

then F admits a sequence of negative critical values $\varrho_n < 0$ satisfying $\varrho_n \to 0$ as $n \to \infty$.

Definition 2. *Suppose that $(\mathbb{E}, \|\cdot\|)$ is a real reflexive and separable Banach space, $F \in C^1(\mathbb{E}, \mathbb{R})$, $c \in \mathbb{R}$. Then, F fulfills the $(C)_c^*$-condition (with respect to $\mathbb{Y}_n$) if any sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathbb{E}$ for which $\varphi_n \in \mathbb{Y}_n$, for any $n \in \mathbb{N}$,*

$$F(\varphi_n) \to c \quad \text{and} \quad \|(F|_{\mathbb{Y}_n})'(\varphi_n)\|_{\mathbb{E}^*}(1 + \|\varphi_n\|_{\mathbb{E}}) \to 0 \text{ as } n \to \infty,$$

has a subsequence converging to a critical point of F.

Lemma 11. *Let us denote*

$$\kappa_{v,n} = \sup_{\|v\|=1, v\in\mathbb{H}_n} \|v\|_{L^v(\Omega)}$$

and

$$\kappa_n = \max\{\kappa_{q,n}, \kappa_{p,n}\}. \tag{28}$$

Then, $\kappa_n \to 0$ as $n \to \infty$ (see [33]).

From Lemmas 7–9 and Proposition 1, we obtain the following consequence.

Theorem 1. *Assume that conditions* (G1)–(G3) *are satisfied. Then, for any* $\mu \in (-\infty, \mu^*)$, *there exists a sequence of nontrivial solutions* $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathbb{X}$ *satisfying* $\mathcal{E}_{\mu,\theta}(\varphi_n) \to 0$ *as* $n \to \infty$ *for every* $\theta \in \Lambda$. *Here,* μ^* *and* Λ *are as specified in Lemma 7.*

Proof. If all conditions ($\mathcal{C}1$)–($\mathcal{C}4$) of Lemma 10 are satisfied, then for each $\theta \in \Lambda$ and any $\mu \in (-\infty, \mu^*)$, the functional $\mathcal{E}_{\mu,\theta}$ admits a sequence of negative critical values c_n such that $c_n \to 0$ as $n \to \infty$. To this end, we shall verify that the requirements of Lemma 10 are indeed fulfilled.

($\mathcal{C}1$): Let $\kappa_n < 1$ for a large enough n, where κ_n was given in Lemma 11. First, we consider the case where $\mu \in (0, \mu^*)$. By virtue of (12) and applying reasoning similar to that used in (13), for any $\varphi \in \mathbb{X}$ with $\|\varphi\|_{\mathbb{X}} \geq 1$, we obtain

$$
\begin{aligned}
\mathcal{E}_{\mu,\theta}(\varphi) &\geq \left(\frac{\widetilde{\mathcal{D}}}{2q} - \frac{\mu}{p\min\left\{d_{\mathcal{H}_p}^{-1}\tau_{0,p}, d_{\mathcal{H}_q}^{-1}\tau_{0,q}\right\}}\right)\|\varphi\|_{\mathbb{X}}^p - \theta\int_\Omega \left|\widetilde{G}(x,\varphi)\right| dx \\
&\geq \left(\frac{\widetilde{\mathcal{D}}}{2q} - \frac{\mu}{p\min\left\{d_{\mathcal{H}_p}^{-1}\tau_{0,p}, d_{\mathcal{H}_q}^{-1}\tau_{0,q}\right\}}\right)\|\varphi\|_{\mathbb{X}}^p - \theta\int_\Omega (|G(x,\varphi)| + \xi|\varphi|^p)\, dx \\
&\geq \left(\frac{\widetilde{\mathcal{D}}}{2q} - \frac{\mu}{p\min\left\{d_{\mathcal{H}_p}^{-1}\tau_{0,p}, d_{\mathcal{H}_q}^{-1}\tau_{0,q}\right\}}\right)\|\varphi\|_{\mathbb{X}}^p \\
&\quad - \theta\int_\Omega \rho_1(x)|\varphi| + \frac{\rho_2}{\ell}|\varphi|^\ell\, dx - \theta\xi\int_\Omega |\varphi|^p\, dx \\
&\geq \left(\frac{\widetilde{\mathcal{D}}}{2q} - \frac{\mu}{p\min\left\{d_{\mathcal{H}_p}^{-1}\tau_{0,p}, d_{\mathcal{H}_q}^{-1}\tau_{0,q}\right\}}\right)\|\varphi\|_{\mathbb{X}}^p - \theta\|\rho_1\|_{L^\infty(\Omega)}\|\varphi\|_{L^1(\Omega)} \\
&\quad - \frac{\theta\rho_2}{\ell}\|\varphi\|_{L^\ell(\Omega)}^\ell - \theta\xi\kappa_n^p\|\varphi\|_{\mathbb{X}}^p \\
&\geq \left(\frac{\widetilde{\mathcal{D}}}{2q} - \frac{\mu}{p\min\left\{d_{\mathcal{H}_p}^{-1}\tau_{0,p}, d_{\mathcal{H}_q}^{-1}\tau_{0,q}\right\}}\right)\|\varphi\|_{\mathbb{X}}^p - \theta\|\rho_1\|_{L^\infty(\Omega)}\kappa_n\|\varphi\|_{\mathbb{X}} \\
&\quad - \frac{\theta\rho_2}{\ell}\kappa_n^\ell\|\varphi\|_{\mathbb{X}}^\ell - \theta\xi\kappa_n^p\|\varphi\|_{\mathbb{X}}^p \\
&\geq \left(\frac{\widetilde{\mathcal{D}}}{2q} - \frac{\mu}{p\min\left\{d_{\mathcal{H}_p}^{-1}\tau_{0,p}, d_{\mathcal{H}_q}^{-1}\tau_{0,q}\right\}}\right)\|\varphi\|_{\mathbb{X}}^p - \theta\|\rho_1\|_{L^\infty(\Omega)}\kappa_n\|\varphi\|_{\mathbb{X}} \\
&\quad - \frac{\theta\rho_2}{\ell}\kappa_n^p\|\varphi\|_{\mathbb{X}}^\ell - \theta\xi\kappa_n^p\|\varphi\|_{\mathbb{X}}^p \\
&\geq \left(\frac{\widetilde{\mathcal{D}}}{2q} - \frac{\mu}{p\min\left\{d_{\mathcal{H}_p}^{-1}\tau_{0,p}, d_{\mathcal{H}_q}^{-1}\tau_{0,q}\right\}}\right)\|\varphi\|_{\mathbb{X}}^p - \theta\|\rho_1\|_{L^\infty(\Omega)}\kappa_n\|\varphi\|_{\mathbb{X}} \\
&\quad - \theta\left(\frac{\rho_2}{\ell} + \xi\right)\kappa_n^p\|\varphi\|_{\mathbb{X}}^\ell \\
&\geq \left[\frac{p\widetilde{\mathcal{D}}\min\left\{d_{\mathcal{H}_p}^{-1}\tau_{0,p}, d_{\mathcal{H}_q}^{-1}\tau_{0,q}\right\} - 2q\mu}{2pq\min\left\{d_{\mathcal{H}_p}^{-1}\tau_{0,p}, d_{\mathcal{H}_q}^{-1}\tau_{0,q}\right\}} - \theta\left(\frac{\rho_2}{\ell} + \xi\right)\kappa_n^p\|\varphi\|_{\mathbb{X}}^{\ell-p}\right]\|\varphi\|_{\mathbb{X}}^p \\
&\quad - \theta\|\rho_1\|_{L^\infty(\Omega)}\kappa_n\|\varphi\|_{\mathbb{X}}
\end{aligned}
\tag{29}
$$

for sufficiently large n. Setting $\delta_{1,n}$ as in

$$\delta_{1,n} = \left[\theta\Big(\frac{\rho_2}{\ell} + \xi\Big)\frac{4pq\min\Big\{d_{\mathcal{H}_p}^{-1}\tau_{0,p}, d_{\mathcal{H}_q}^{-1}\tau_{0,q}\Big\}\kappa_n^p}{p\widetilde{\mathcal{D}}\min\Big\{d_{\mathcal{H}_p}^{-1}\tau_{0,p}, d_{\mathcal{H}_q}^{-1}\tau_{0,q}\Big\} - 2q\mu}\right]^{\frac{1}{p-2\ell}}, \tag{30}$$

we consider $\varphi \in \mathbb{H}_n$ with $\|\varphi\|_{\mathbb{X}} = \delta_{1,n}$. Note that for a sufficiently large n, the condition $\delta_{1,n} > 1$ is satisfied. Since $\mu \in (0, \mu^*)$ and $\delta_{1,n} \to \infty$ as $n \to \infty$, we can choose $n_0 \in \mathbb{N}$ such that

$$\begin{aligned}
\mathcal{E}_{\mu,\theta}(\varphi) &\geq \left[\frac{p\widetilde{\mathcal{D}}\min\Big\{d_{\mathcal{H}_p}^{-1}\tau_{0,p}, d_{\mathcal{H}_q}^{-1}\tau_{0,q}\Big\} - 2q\mu}{2pq\min\Big\{d_{\mathcal{H}_p}^{-1}\tau_{0,p}, d_{\mathcal{H}_q}^{-1}\tau_{0,q}\Big\}} - \theta\Big(\frac{\rho_2}{\ell} + \xi\Big)\kappa_n^p\|\varphi\|_{\mathbb{X}}^{\ell-p}\right]\|\varphi\|_{\mathbb{X}}^p \\
&\quad - \theta\|\rho_1\|_{L^\infty(\Omega)}\kappa_n\|\varphi\|_{\mathbb{X}} \\
&\geq \frac{p\widetilde{\mathcal{D}}\min\Big\{d_{\mathcal{H}_p}^{-1}\tau_{0,p}, d_{\mathcal{H}_q}^{-1}\tau_{0,q}\Big\} - 2q\mu}{2pq\min\Big\{d_{\mathcal{H}_p}^{-1}\tau_{0,p}, d_{\mathcal{H}_q}^{-1}\tau_{0,q}\Big\}}\delta_{1,n}^p \\
&\quad - \theta\|\rho_1\|_{L^\infty(\Omega)}\left[\theta\Big(\frac{\rho_2}{\ell} + \xi\Big)\frac{4pq\min\Big\{d_{\mathcal{H}_p}^{-1}\tau_{0,p}, d_{\mathcal{H}_q}^{-1}\tau_{0,q}\Big\}}{p\widetilde{\mathcal{D}}\min\Big\{d_{\mathcal{H}_p}^{-1}\tau_{0,p}, d_{\mathcal{H}_q}^{-1}\tau_{0,q}\Big\} - 2q\mu}\right]^{\frac{1}{p-2\ell}}\kappa_n^{\frac{2(p-\ell)}{p-2\ell}} \\
&\geq 0
\end{aligned}$$

for all $n \in \mathbb{N}$ with $n \geq n_0$.

Moreover, if $\mu \in (-\infty, 0]$ and $\|\varphi\|_{\mathbb{X}} \geq 1$, then it follows from the definition of κ_n, (G1) and arguments analogous to those in (29) that

$$\begin{aligned}
\mathcal{E}_{\mu,\theta}(\varphi) &\geq \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|\varphi(x) - \varphi(z)|^p K_p(x,z)\,dx\,dz \\
&\quad + \frac{1}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|\varphi(x) - \varphi(z)|^q K_q(x,z)\,dx\,dz \\
&\quad - \theta\int_\Omega\Big|\widetilde{G}(x,\varphi)\Big|\,dx \\
&\geq \left[\frac{\widetilde{\mathcal{D}}}{2q} - \theta\Big(\frac{\rho_2}{\ell} + \xi\Big)\kappa_n^p\|\varphi\|_{\mathbb{X}}^{\ell-p}\right]\|\varphi\|_{\mathbb{X}}^p - \theta\|\rho_1\|_{L^\infty(\Omega)}\kappa_n\|\varphi\|_{\mathbb{X}}
\end{aligned} \tag{31}$$

for a sufficiently large n. We set $\delta_{2,n}$ as follows:

$$\delta_{2,n} = \left(\frac{4q\theta}{\widetilde{\mathcal{D}}}\Big(\frac{\rho_2}{\ell} + \xi\Big)\kappa_n^p\right)^{\frac{1}{p-2\ell}}. \tag{32}$$

It is evident that $\delta_{2,n} \to \infty$ as $n \to \infty$. For sufficiently large n, let $\varphi \in \mathbb{H}_n$ satisfy $\|\varphi\|_{\mathbb{X}} = \delta_{2,n} > 1$. In view of (31), there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\begin{aligned}
\mathcal{E}_{\mu,\theta}(\varphi) &\geq \left[\frac{\widetilde{\mathcal{D}}}{2q} - \theta\Big(\frac{\rho_2}{\ell} + \xi\Big)\kappa_n^p\|\varphi\|_{\mathbb{X}}^{\ell-p}\right]\|\varphi\|_{\mathbb{X}}^p - \theta\|\rho_1\|_{L^\infty(\Omega)}\kappa_n\|\varphi\|_{\mathbb{X}} \\
&\geq \frac{\widetilde{\mathcal{D}}}{4q}\delta_{2,n}^p - \theta\|\rho_1\|_{L^\infty(\Omega)}\left[\frac{4q\theta}{\widetilde{\mathcal{D}}}\Big(\frac{\rho_2}{\ell} + \xi\Big)\right]^{\frac{1}{p-2\ell}}\kappa_n^{\frac{2(p-\ell)}{p-2\ell}} \\
&\geq 0
\end{aligned}$$

for all $n \in \mathbb{N}$ with $n \geq n_0$.

Let δ_n denote either $\delta_{1,n}$ or $\delta_{2,n}$ as defined in (30) and (32), respectively. Consequently, for any $\mu \in (-\infty, \mu^*)$, we conclude that

$$\inf\{\mathcal{E}_{\mu,\theta}(\varphi) : \varphi \in \mathbb{H}_n, \|\varphi\|_{\mathbb{X}} = \delta_n\} \geq 0.$$

($\mathcal{C}2$): Note that $\|\cdot\|_{L^\infty(\Omega)}, \|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_{\mathbb{X}}$ are equivalent on $\mathbb{Y}_n$. Then, there are constants $\widetilde{\nu}_{1,n} > 0$ and $\widetilde{\nu}_{2,n} > 0$ such that

$$\widetilde{\nu}_{1,n}\|\varphi\|_{L^\infty(\Omega)} \leq \|\varphi\|_{\mathbb{X}} \leq \widetilde{\nu}_{2,n}\|\varphi\|_{L^p(\Omega)} \tag{33}$$

for any $\varphi \in \mathbb{Y}_n$. From (G2) and (G3), for any $\mathcal{D}_3 > 0$, there exists $\eta_4 \in (0, \eta_3/2)$ such that

$$G(x,\eta) \geq \frac{\mathcal{D}_3\widetilde{\nu}_{2,n}^p}{p}|\eta|^p \tag{34}$$

for almost all $x \in \Omega$ and all $|\eta| \leq \eta_4$. Choose $\gamma_n := \min\{\frac{1}{2}, \eta_4\widetilde{\nu}_{1,n}\}$ for all $n \in \mathbb{N}$. Then, we can determine that $\|\varphi\|_{L^\infty(\Omega)} \leq \eta_3$ for $\varphi \in \mathbb{Y}_n$ with $\|\varphi\|_{\mathbb{X}} = \gamma_n$, and thus, $\widetilde{G}(x,\varphi) = G(x,\varphi)$.

First, we consider the case where $\mu \in (0, \mu^*)$. Then, for any $\varphi \in \mathbb{Y}_n$ with $\|\varphi\|_{\mathbb{X}} = \gamma_n$, we derive by (33) and (34) that

$$\begin{aligned}
\mathcal{E}_{\mu,\theta}(\varphi) &\leq \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\varphi(x)-\varphi(z)|^p K_p(x,z)\,dx\,dz \\
&\quad + \frac{1}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\varphi(x)-\varphi(z)|^q K_q(x,z)\,dx\,dz \\
&\quad - \frac{\mu}{q}\left(\int_\Omega \frac{|\varphi|^p}{|x|^{\alpha p}}\,dx + \int_\Omega \frac{|\varphi|^q}{|x|^{\alpha q}}\,dx\right) - \theta\int_\Omega \left|\widetilde{G}(x,\varphi)\right|dx \\
&\leq \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\varphi(x)-\varphi(z)|^p K_p(x,z)\,dx\,dz \\
&\quad + \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\varphi(x)-\varphi(z)|^q K_q(x,z)\,dx\,dz - \theta\int_\Omega \frac{\mathcal{D}_3\widetilde{\nu}_{2,n}^p}{p}|\varphi|^p\,dx \\
&\leq \frac{1}{p}\|\varphi\|_{\mathbb{X}}^p - \frac{\theta\mathcal{D}_3\widetilde{\nu}_{2,n}^p}{p}\|\varphi\|_{L^p(\Omega)}^p \\
&\leq \frac{1}{p}\|\varphi\|_{\mathbb{X}}^p - \frac{\theta\mathcal{D}_3}{p}\|\varphi\|_{\mathbb{X}}^p \\
&\leq (1-\theta\mathcal{D}_3)\gamma_n^p.
\end{aligned} \tag{35}$$

Next, if $\mu \in (-\infty, 0]$, it follows from Lemma 5 that

$$\begin{aligned}
\mathcal{E}_{\mu,\theta}(\varphi) &\leq \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\varphi(x)-\varphi(z)|^p K_p(x,z)\,dx\,dz \\
&\quad + \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\varphi(x)-\varphi(z)|^q K_q(x,z)\,dx\,dz \\
&\quad - \frac{\mu}{q}\left(\int_\Omega \frac{|\varphi|^p}{|x|^{\alpha p}}\,dx + \int_\Omega \frac{|\varphi|^q}{|x|^{\alpha q}}\,dx\right) - \theta\int_\Omega \left|\widetilde{G}(x,\varphi)\right|dx \\
&\leq \frac{1}{p}\left(1 - \frac{p\mu}{q\min\left\{d_{\mathcal{H}_p}^{-1}\tau_{0,p}, d_{\mathcal{H}_q}^{-1}\tau_{0,q}\right\}}\right)\|\varphi\|_{\mathbb{X}}^p - \theta\mathcal{D}_3\widetilde{\nu}_{2,n}^p\|\varphi\|_{L^p(\Omega)}^p \\
&\leq \left(\frac{q\min\left\{d_{\mathcal{H}_p}^{-1}\tau_{0,p}, d_{\mathcal{H}_q}^{-1}\tau_{0,q}\right\} - p\mu}{pq\min\left\{d_{\mathcal{H}_p}^{-1}\tau_{0,p}, d_{\mathcal{H}_q}^{-1}\tau_{0,q}\right\}} - \theta\mathcal{D}_3\right)\gamma_n^p.
\end{aligned} \tag{36}$$

By selecting a sufficiently large constant $\mathcal{D}_3$ such that either $1 < \theta\mathcal{D}_3$ or

$$\frac{q\min\left\{d_{\mathcal{H}_p}^{-1}\tau_{0,p}, d_{\mathcal{H}_q}^{-1}\tau_{0,q}\right\} - p\mu}{pq\min\left\{d_{\mathcal{H}_p}^{-1}\tau_{0,p}, d_{\mathcal{H}_q}^{-1}\tau_{0,q}\right\}} < \theta\mathcal{D}_3$$

holds, it follows from (35) and (36) that

$$\phi_n = \max\{\mathcal{E}_{\mu,\theta}(\varphi) : \varphi \in \mathbb{Y}_n, \|\varphi\|_{\mathbb{X}} = \gamma_n\} < 0$$

for any $\mu \in (-\infty, \mu^*)$. We can further adjust n_0 to a larger value to guarantee that $\delta_n > \gamma_n > 0$ holds for every $n \geq n_0$.

($\mathcal{C}3$): Let δ_n be either $\delta_{1,n}$ or $\delta_{2,n}$, which is given in (30) and (32), respectively. Because $\mathbb{Y}_n \cap \mathbb{H}_n \neq \emptyset$ and $0 < \gamma_n < \delta_n$, we have $\varrho_n \leq \phi_n < 0$ for all $n \geq n_0$. Let $\varphi \in \mathbb{H}_n$ with $\|\varphi\|_{\mathbb{X}} = 1$ and $0 < \lambda < \delta_n$. With an argument analogous to that in (29), we have for any $\mu \in (0, \mu^*)$

$$\begin{aligned}
\mathcal{E}_{\mu,\theta}(\lambda\varphi) &\geq \frac{1}{q}\Bigg(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\lambda\varphi(x) - \lambda\varphi(z)|^p K_p(x,z)\, dx\, dz \\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\lambda\varphi(x) - \lambda\varphi(z)|^q K_q(x,z)\, dx\, dz\Bigg) \\
&\quad - \frac{\mu}{p}\left(\int_\Omega \frac{|\lambda\varphi|^p}{|x|^{\alpha p}}\, dx + \int_\Omega \frac{|\lambda\varphi|^q}{|x|^{\alpha q}}\, dx\right) - \theta\int_\Omega \left|\widetilde{G}(x,\lambda\varphi)\right| dx \\
&\geq \left(\frac{\widetilde{\mathcal{D}}}{2q} - \frac{\mu}{p\min\left\{d_{\mathcal{H}_p}^{-1}\tau_{0,p}, d_{\mathcal{H}_q}^{-1}\tau_{0,q}\right\}}\right)\|\lambda\varphi\|_{\mathbb{X}}^p - \theta\int_\Omega \left|\widetilde{G}(x,\lambda\varphi)\right| dx \\
&\geq -\theta\int_\Omega (|G(x,\lambda\varphi)| + \xi|\lambda\varphi|^p)\, dx \\
&\geq -\theta\int_\Omega \rho_1(x)|\lambda\varphi| + \frac{\rho_2}{\ell}|\lambda\varphi|^\ell\, dx - \theta\xi\int_\Omega |\lambda\varphi|^p dx \\
&\geq -\theta\|\rho_1\|_{L^\infty(\Omega)}\delta_{1,n}\|\varphi\|_{L^1(\Omega)} - \frac{\theta\rho_2}{\ell}\delta_{1,n}^\ell\|\varphi\|_{L^\ell(\Omega)}^\ell - \theta\xi\delta_{1,n}^p\|\varphi\|_{L^p(\Omega)}^p \\
&\geq -\theta\|\rho_1\|_{L^\infty(\Omega)}\delta_{1,n}\kappa_n - \frac{\theta\rho_2}{\ell}\delta_{1,n}^\ell\kappa_n^\ell - \theta\xi\delta_{1,n}^\ell\kappa_n^p \\
&\geq -\theta\|\rho_1\|_{L^\infty(\Omega)}\delta_{1,n}\kappa_n - \theta\left(\frac{\rho_2}{\ell} + \xi\right)\delta_{1,n}^\ell\kappa_n^p,
\end{aligned} \tag{37}$$

where κ_n comes from (28). Consequently, based on the above estimate and the definition of $\delta_{1,n}$, we deduce

$$\begin{aligned}
0 > \varrho_n &\geq -\theta\|\rho_1\|_{L^\infty(\Omega)}\delta_{1,n}\kappa_n - \theta\left(\frac{\rho_2}{\ell} + \xi\right)\delta_{1,n}^\ell\kappa_n^p, \\
&\geq -\theta\|\rho_1\|_{L^\infty(\Omega)}\left[\theta\left(\frac{\rho_2}{\ell} + \xi\right)\frac{4pq\min\left\{d_{\mathcal{H}_p}^{-1}\tau_{0,p}, d_{\mathcal{H}_q}^{-1}\tau_{0,q}\right\}}{p\widetilde{\mathcal{D}}\min\left\{d_{\mathcal{H}_p}^{-1}\tau_{0,p}, d_{\mathcal{H}_q}^{-1}\tau_{0,q}\right\} - 2q\mu}\right]^{\frac{1}{p-2\ell}}\kappa_n^{\frac{2(p-\ell)}{p-2\ell}} \\
&\quad - \theta\left(\frac{\rho_2}{\ell} + \xi\right)\left[\theta\left(\frac{\rho_2}{\ell} + \xi\right)\frac{4pq\min\left\{d_{\mathcal{H}_p}^{-1}\tau_{0,p}, d_{\mathcal{H}_q}^{-1}\tau_{0,q}\right\}}{p\widetilde{\mathcal{D}}\min\left\{d_{\mathcal{H}_p}^{-1}\tau_{0,p}, d_{\mathcal{H}_q}^{-1}\tau_{0,q}\right\} - 2q\mu}\right]^{\frac{\ell}{p-2\ell}}\kappa_n^{\frac{p(p-\ell)}{p-2\ell}}.
\end{aligned} \tag{38}$$

Moreover, let $\mu \in (-\infty, 0]$. Then, by proceeding in a manner analogous to that in (37), we obtain that

$$\begin{aligned}\mathcal{E}_{\mu,\theta}(\lambda\varphi) &\geq \frac{1}{q}\bigg(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\lambda\varphi(x) - \lambda\varphi(z)|^p K_p(x,z)\, dx\, dz \\ &\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\lambda\varphi(x) - \lambda\varphi(z)|^q K_q(x,z)\, dx\, dz\bigg) \\ &\quad - \theta\int_\Omega \left|\widetilde{G}(x,\lambda\varphi)\right| dx \\ &\geq \frac{\widetilde{\mathcal{D}}}{2q}\|\lambda\varphi\|_{\mathbb{X}}^p - \theta\int_\Omega \left|\widetilde{G}(x,\lambda\varphi)\right| dx \\ &\geq -\theta\|\rho_1\|_{L^\infty(\Omega)}\delta_{2,n}\kappa_n - \theta\Big(\frac{\rho_2}{\ell} + \xi\Big)\delta_{2,n}^\ell \kappa_n^p\end{aligned}$$

for a large enough n. Combining this with the definition of $\delta_{2,n}$, we obtain the following estimate for ϱ_n:

$$\begin{aligned}0 > \varrho_n &\geq -\theta\|\rho_1\|_{L^\infty(\Omega)}\delta_{2,n}\kappa_n - \theta\Big(\frac{\rho_2}{\ell} + \xi\Big)\delta_{2,n}^\ell \kappa_n^p \\ &\geq -\theta\|\rho_1\|_{L^\infty(\Omega)}\left[\frac{4q\theta}{\widetilde{\mathcal{D}}}\Big(\frac{\rho_2}{\ell} + \xi\Big)\kappa_n^p\right]^{\frac{1}{p-2\ell}} \kappa_n^{\frac{2(p-\ell)}{p-2\ell}} \\ &\quad - \theta\Big(\frac{\rho_2}{\ell} + \xi\Big)\left[\frac{4q\theta}{\widetilde{\mathcal{D}}}\Big(\frac{\rho_2}{\ell} + \xi\Big)\kappa_n^p\right]^{\frac{\ell}{p-2\ell}} \kappa_n^{\frac{p(p-\ell)}{p-2\ell}}. \qquad (39)\end{aligned}$$

Taking into account that $\kappa_n \to 0$ and $p < q$, we conclude from (38) and (39) that $\varrho_n \to 0$ as $n \to \infty$ for all $\mu \in (0, \mu^*)$. Specifically,

$$\varrho_n = \{\mathcal{E}_{\mu,\theta}(\varphi) : \varphi \in \mathbb{H}_n, \|\varphi\|_{\mathbb{X}} \leq \delta_n\} \to 0.$$

($\mathcal{C}4$): For $c \in \mathbb{R}$, let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{X}$ such that $\varphi_n \in \mathbb{Y}_n$ for each $n \in \mathbb{N}$, satisfying

$$\mathcal{E}_{\mu,\theta}(\varphi_n) \to c \quad \text{and} \quad \|(\mathcal{E}_{\mu,\theta}|_{\mathbb{Y}_n})'(\varphi_n)\|_{\mathbb{X}^*}(1 + \|\varphi_n\|_{\mathbb{X}}) \to 0 \text{ as } n \to \infty.$$

For any $\mu \in (0, \mu^*)$ and $\theta \in \Lambda$, Lemma 7 ensures that $\mathcal{E}_{\mu,\theta}$ is coercive, which implies the boundedness of $\{\varphi_n\}_{n\in\mathbb{N}}$ in $\mathbb{X}$. Consequently, by passing to a subsequence if necessary (still denoted by $\{\varphi_n\}_{n\in\mathbb{N}}$), we obtain a limit function $\varphi \in \mathbb{X}$ such that the convergence properties in (19) are satisfied.

To complete this proof, we will show that $\varphi_n \to \varphi$ in $\mathbb{X}$ as $n \to \infty$, and also that φ is a critical point of $\mathcal{E}_{\mu,\theta}$. As $\mathbb{X} = \overline{\bigcup_{n\in\mathbb{N}} \mathbb{Y}_n}$, for $n \in \mathbb{N}$, we can choose $u_n \in \mathbb{Y}_n$ such that $u_n \to \varphi$ as $n \to \infty$. Hence, we have

$$\langle \mathcal{E}'_{\mu,\theta}(\varphi_n), \varphi_n - \varphi\rangle = \langle (\mathcal{E}_{\mu,\theta}|_{\mathbb{Y}_n})'(\varphi_n), \varphi_n - u_n\rangle + \langle (\mathcal{E}_{\mu,\theta}|_{\mathbb{Y}_n})'(w_n), u_n - \varphi\rangle.$$

Since $(\mathcal{E}_{\mu,\theta}|_{\mathbb{Y}_n})'(\varphi_n) \to 0$, $u_n \to \varphi$ and $\varphi_n - u_n \to 0$ in $\mathbb{Y}_n$ as $n \to \infty$, we have

$$\langle \mathcal{E}'_{\mu,\theta}(\varphi_n), \varphi_n - \varphi\rangle \to 0 \text{ as } n \to \infty.$$

This, together with (18) and (22)–(25), yields relation (26). Since $\mathcal{E}'_{\mu,\theta}$ is a mapping of type (S_+), we can state that $\varphi_n \to \varphi$ as $n \to \infty$. In addition, we have $\mathcal{E}'_{\mu,\theta}(\varphi_n) \to \mathcal{E}'_{\mu,\theta}(\varphi)$ as $n \to \infty$. Let us verify that φ is a critical point of $\mathcal{E}_{\mu,\theta}$. To this end, let $n_0 \in \mathbb{N}$ be fixed and choose an arbitrary $u \in \mathbb{Y}_{n_0}$. For any $n \geq n_0$, we observe that

$$\begin{aligned}\langle \mathcal{E}'_{\mu,\theta}(\varphi), u\rangle &= \langle \mathcal{E}'_{\mu,\theta}(\varphi) - \mathcal{E}'_{\mu,\theta}(\varphi_n), u\rangle + \langle \mathcal{E}'_{\mu,\theta}(\varphi_n), u\rangle \\ &= \langle \mathcal{E}'_{\mu,\theta}(\varphi) - \mathcal{E}'_{\mu,\theta}(\varphi_n), u\rangle + \langle (\mathcal{E}_{\mu,\theta}|_{\mathbb{Y}_n})'(\varphi_n), u\rangle.\end{aligned}$$

Taking the limit as $n \to \infty$ on the right-hand side, it follows that

$$\langle \mathcal{E}'_{\mu,\theta}(\varphi), u\rangle = 0 \text{ for all } u \in \mathbb{Y}_{n_0}.$$

By the density of $\bigcup_{n\in\mathbb{N}} \mathbb{Y}_n$ in $\mathbb{X}$ and the arbitrariness of n_0, it follows that $\mathcal{E}'_{\mu,\theta}(\varphi) = 0$, as previously claimed. Hence, we know $\varphi_n \to \varphi$ in $\mathbb{X}$ as $n \to \infty$, and φ is also a critical point of $\mathcal{E}_{\mu,\theta}$. As a consequence, we derive that the functional $\mathcal{E}_{\mu,\theta}$ assures the $(C)^*_c$-condition for every $\theta \in \Lambda$ and for any $\mu \in (0, \mu^*)$. Condition $(\mathcal{C}4)$ is proved. The proof is complete. □

The following regularity-type result is established by adapting the methodologies found in [27,31].

Proposition 1. *Suppose that condition* (G1) *is satisfied. If φ is a weak solution to Problem* (1)*, then there exist positive constants κ and $\mathcal{C}$, both independent of φ, such that*

$$\|\varphi\|_{L^\infty(\Omega)} \le \mathcal{C}\|\varphi\|^\kappa_{L^\ell(\Omega)}$$

for any $\mu \in (-\infty, \mu^*)$, where μ^* is the constant defined in Lemma 7.

Lemma 12. *If* (G1)–(G3) *holds, then we have*

$$\mathcal{E}_{\mu,\theta}(\varphi) = \left\langle \mathcal{E}'_{\mu,\theta}(\varphi), \varphi \right\rangle = 0 \text{ if and only if } \varphi = 0 \tag{40}$$

for any $\mu \in (-\infty, 0]$.

Proof. Let $\mathcal{E}_{\mu,\theta}(\varphi) = \left\langle \mathcal{E}'_{\mu,\theta}(\varphi), \varphi \right\rangle = 0$. Then, we see that

$$\begin{aligned}0 &= -p\mathcal{E}_{\mu,\theta}(\varphi) \\ &\ge -\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\varphi(x) - \varphi(z)|^p K_p(x,z)\, dx\, dz \\ &\quad -\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\varphi(x) - \varphi(z)|^q K_q(x,z)\, dx\, dz \\ &\quad +\mu\left(\int_\Omega \frac{|\varphi|^p}{|x|^{\alpha p}}\, dx + \int_\Omega \frac{|\varphi|^q}{|x|^{\alpha q}}\, dx\right) + \theta p \int_\Omega \widetilde{G}(x,\varphi)\varphi\, dx\end{aligned} \tag{41}$$

and

$$\begin{aligned}\left\langle \mathcal{E}'_{\mu,\theta}(\varphi), \varphi \right\rangle &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\varphi(x) - \varphi(z)|^p K_p(x,z)\, dx\, dz \\ &\quad +\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\varphi(x) - \varphi(z)|^q K_q(x,z)\, dx\, dz \\ &\quad -\mu\left(\int_\Omega \frac{|\varphi|^p}{|x|^{\alpha p}}\, dx + \int_\Omega \frac{|\varphi|^q}{|x|^{\alpha q}}\, dx\right) - \theta\int_\Omega \widetilde{g}(x,\varphi)\varphi\, dx = 0.\end{aligned} \tag{42}$$

It follows from relations (41) and (42) that

$$\int_\Omega \left(p\widetilde{G}(x,\varphi) - \widetilde{g}(x,\varphi)\varphi\right) dx \le 0.$$

Consequently, relation (9) implies that $\varphi = 0$. The converse is clear from the definition of $\mathcal{E}_{\mu,\theta}$. □

Remark 1. *Regrettably, it cannot be guaranteed that $\varphi = 0$ even under the conditions $\mathcal{E}_{\mu,\theta}(\varphi) = 0$ and $\langle \mathcal{E}'_{\mu,\theta}(\varphi), \varphi \rangle = 0$ for $\mu \in (0, \mu^*)$, where μ^* is as defined in Lemma 7. Consequently, 0 is not necessarily the only critical point of the functional $\mathcal{E}_{\mu,\theta}$ for any $\mu \in (0, \mu^*)$.*

Equipped with the results from Lemmas 9 and 12, Theorem 1, and Proposition 1, we are now able to complete the proof of our main theorem.

Theorem 2. *Assume that conditions* (G1)–(G3) *are satisfied. Then, for any $\mu \in (-\infty, 0]$, Problem* (1) *admits a sequence of nontrivial solutions $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathbb{X}$ such that $\mathcal{E}_{\mu,\theta}(\varphi_n) \to 0$ and $\|\varphi_n\|_{L^\infty(\Omega)} \to 0$ as $n \to \infty$ for every $\theta \in \Lambda$, where the set Λ is specified in Lemma 7.*

Proof. By virtue of Theorem 1, for each $\theta \in \Lambda$ and any $\mu \in (-\infty, 0]$, the functional $\mathcal{E}_{\mu,\theta}$ admits a sequence of negative critical values c_n converging to 0 as $n \to \infty$. This, in conjunction with Lemma 9, implies that for any sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq \mathbb{X}$ satisfying $\mathcal{E}_{\mu,\theta}(\varphi_n) = c_n$ and $\|\mathcal{E}'_{\mu,\theta}(\varphi_n)\|_{\mathbb{X}^*} = 0$, it follows that $\{\varphi_n\}_{n\in\mathbb{N}}$ is a $(C)_0$-sequence of $\mathcal{E}_{\mu,\theta}$ and possesses a convergent subsequence. Thus, up to a subsequence, still denoted by $\{\varphi_n\}_{n\in\mathbb{N}}$, one has $\varphi_n \to \varphi$ in $\mathbb{X}$ as $n \to \infty$. In accordance with Lemma 12, we infer that 0 is the unique critical point with zero energy, the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ has to converge to $\varphi = 0$ in $\mathbb{X}$; so, $\|\varphi_n\|_{L^r(\Omega)} \to 0$ as $n \to \infty$ for any r with $r \in [p, p^*_\alpha]$. By means of Proposition 1, any weak solution ψ of (1) belongs to the space $L^\infty(\Omega)$ and there exist positive constants κ and $\mathcal{C}$ independent of ψ such that

$$\|\psi\|_{L^\infty(\Omega)} \leq \mathcal{C}\|\psi\|^\kappa_{L^\ell(\Omega)}.$$

Based on this observation, it follows that $\|\varphi_n\|_{L^\infty(\Omega)} \to 0$ as $n \to \infty$. Consequently, we conclude that $\|\varphi_n\|_{L^\infty(\Omega)} \leq \eta_3$ for sufficiently large n. This establishes that $\{\varphi_n\}_{n\in\mathbb{N}}$ is indeed a sequence of weak solutions to Problem (1) for all n large enough, as desired. □

5. Conclusions and Future Work

This paper is dedicated to deriving the existence of a sequence of infinitely many small-energy solutions that converge to zero in the L^∞-norm. A key feature of our approach is that the conditions on the nonlinearity g are imposed solely on its local behavior, with no requirements at infinity. While traditional multiplicity results often rely on global symmetry and asymptotic conditions at infinity (see, e.g., [26,32,33]), we derive our conclusions under purely local assumptions; specifically, $g(\cdot, \eta)$ is assumed to be odd in η only for small values of η. This localization of hypotheses constitutes a primary novelty of our work. As far as we know, such results for nonlinear elliptic problems involving singular coefficients like Hardy potentials have not been comprehensively investigated.

Furthermore, a promising research direction in powerful relation is the investigation of the fractional $(p(\cdot), q(\cdot))$-Laplacian problem with Hardy-type potentials, expressed as follows:

$$\mathcal{L}_{p(\cdot,\cdot)}\varphi + \mathcal{L}_{q(\cdot,\cdot)}\varphi = \mu\left(\frac{|\varphi|^{p(x)-2}\varphi}{|x|^{\alpha p(x)}} + \frac{|\varphi|^{q(x)-2}\varphi}{|x|^{\alpha q(x)}}\right) + \theta g(x, \varphi) \quad \text{in} \quad \Omega, \tag{43}$$

where operator $\mathcal{L}$ is defined by

$$\mathcal{L}_{\mathfrak{m}(\cdot,\cdot)}\varphi(x) = 2\lim_{\sigma\searrow 0}\int_{\mathbb{R}^N\setminus B_\sigma(x)} \frac{|\varphi(x) - \varphi(z)|^{\mathfrak{m}(x,z)-2}(\varphi(x) - \varphi(z))}{|x - z|^{N+\alpha\mathfrak{m}(x,z)}}\, dx, \quad x \in \mathbb{R}^N,$$

where $\alpha \in (0,1)$ and $B_\sigma(x) := \{z \in \mathbb{R}^N : |x-z| \leq \sigma\}$. To our knowledge, the existence of solutions to fractional $(p(\cdot), q(\cdot))$-Laplacian problems with Hardy potentials remains unexplored due to the absence of fractional Hardy inequalities in variable exponent spaces. Nevertheless, the Hardy–Leray and associated inequalities recently established in [44] for these spaces offer a promising direction. Leveraging the results from [44] is expected to provide the necessary framework for addressing the existence of solutions to problem (43).

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