axioms

MDPI

Article

General Convergence Rates by the Delayed Sums Method

Cheng Hu [1], Shangshang Yang [1] and Tonghui Wang [2,*]

[1] School of Mathematics and Statistics, Shandong Normal University, Jinan 250014, China
[2] Department of Mathematical Sciences, New Mexico State University, Las Cruces, NM 88003, USA
* Correspondence: twang@nmsu.edu

Abstract

In this study, we propose a delayed sums method to investigate the convergence rates of partial sums. This approach enables general and systematic treatment of the convergence behavior of partial sums, encompassing and extending classical results such as the law of large numbers, the law of logarithm, and the law of the iterated logarithm, as well as convergence with respect to the general norming factors. By establishing almost sure convergence of appropriately defined delayed sums, the proposed method yields explicit convergence rates across a wide range of probabilistic settings. As a result, many convergence problems that were previously treated in isolation can be analyzed within a single coherent theoretical structure.

Keywords: delayed sums; convergence rates; law of large numbers; law of logarithm; law of the iterated logarithm; general norming factors

MSC: 60F15

1. Introduction

Limit theorems form the foundational framework of probability theory and mathematical statistics, as they reveal the long-term asymptotic behavior of random sequences. As a natural extension of the limit theorems for almost sure convergence, convergence rates address a deeper question: how rapidly a sequence approaches its limiting value. Over the decades, numerous classical contributions have advanced this field. Baum and Katz (1965) [1] established convergence rates for the law of large numbers; Davis (1968a) [2] derived the convergence rates for the law of logarithm; Davis (1968b) [3] investigated the convergence rates associated with the law of the iterated logarithm; and Norvaiša and Račkauskas (1984) [4] further studied the convergence rates in the topological space.

As a natural extension of partial sums, delayed sums (also known as moving averages) play a significant role in probability and statistics, particularly in change-point analysis and signal detection (see Lai (2004) [5] for details). Let $\{X_n,\ n = 1, 2, \cdots\}$ be a sequence of random variables with partial sums $S_t = \sum_{i=1}^{\lfloor t \rfloor} X_i$ for $t \geq 1$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. For non-negative real numbers (u and v), the delayed sums are defined as

$$S_{u,v} = S_{\lfloor u+v \rfloor} - S_{\lfloor u \rfloor} = \sum_{i=\lfloor u \rfloor+1}^{\lfloor u+v \rfloor} X_i,$$

This formulation effectively captures moving segments of a sequence, allowing for the examination of its local asymptotic behavior. Consequently, the delayed sums provides

Check for updates

Academic Editors: Rui Santos, Miguel Felgueiras and João Paulo Martins

Received: 10 November 2025
Revised: 18 January 2026
Accepted: 22 January 2026
Published: 26 January 2026

Copyright: © 2026 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license.

a powerful analytical tool for studying non-stationary stochastic processes and for deriving precise convergence properties beyond global averages.

Numerous limit theorems for almost sure convergence of delayed sums have been established. Let $\{X_n,\ n = 1, 2, \cdots\}$ be a sequence of independent and identically distributed (i.i.d.) random variables with $E[X_1] = 0$. Chow (1973) [6] established the strong laws of large numbers of delayed sums, showing that for any $0 < p < 2$,

$$E|X_1|^p < \infty \iff P\left(\lim_{n\to\infty} \frac{\max_{1\le j\le n} |S_{n,j}|}{n^{1/p}} = 0\right) = 1, \tag{1}$$

and for any $p \ge 1, 0 < \alpha < \min(2/p, 1)$,

$$E|X_1|^p < \infty \iff P\left(\lim_{n\to\infty} \frac{\max_{1\le j\le n^\alpha} |S_{n,j}|}{n^{1/p}} = 0\right) = 1. \tag{2}$$

With further assumptions on $\{X_n,\ n = 1, 2, \cdots\}$ of finite variance $E[X_1^2] = \sigma^2$, Lai (1974) [7] derived the law of logarithm of delayed sums, proving that for any $0 < \beta < 1$,

$$E\Big(|X_1|^{2/\beta}(\log|X_1|)^{-1/\beta}\Big) < \infty \iff P\left(\limsup_{n\to\infty} \frac{|S_{n,n^\beta}|}{\sqrt{2(1-\beta)n^\beta \log n}} = \sigma\right) = 1. \tag{3}$$

For a general norming factor whose order lies between $\sqrt{n \log\log n}$ and $\sqrt{n \log n}$, Gut et al. (2010) [8] further proved the extension of the law of the iterated logarithm of delayed sums, that is

$$E\Big(f^{-1}(X_1^2)\Big) < \infty \iff P\left(\limsup_{n\to\infty} \frac{|S_{n,n/L(n)}|}{\sqrt{2h(n)n/L(n)}} = \sigma\right) = 1, \tag{4}$$

where $L(x)$ is a slowly varying function (see, for example, Gut (2007) [9]; the definition of a slowly varying function is given in Section 2), with $L(x)$ differentiable, $L(x) \uparrow \infty$, and $xL'(x)/L(x)$ decreasing as $x \to \infty$. And $h(n) = \log L(n) + \log\log n$, $f(n) = \min\{h(n)n/L(n), n\}$, with $f(\cdot)$ an increasing interpolating function (that is, $f(x) = f_{\lfloor x \rfloor}$ for $x > 0$) and $f^{-1}(\cdot)$ is the corresponding (suitably defined) inverse function.

Motivated by the convergence rates established by Lai (1974) [7], we propose a new delayed sums method for analyzing convergence rates. By employing the theory of slowly varying functions and regularly varying functions, this approach unifies the treatment of various convergence rate problems by reducing them to the corresponding almost sure convergence of delayed sums.

Using the delayed sums method proposed in this work, we leverage existing almost sure convergence results for delayed sums to recover known convergence rates and to derive new ones. More importantly, by applying this method, we establish convergence rates concerning the general norming factor, whose order lies between $\sqrt{n \log\log n}$ and $\sqrt{n \log n}$. This yields a novel correspondence between moment conditions and convergence rates under such general norming factors.

The remainder of this paper is organized as follows. Section 2 introduces some basic concepts and lemmas. Section 3 presents the core theorem establishing the relationship between the convergence rates of partial sums and the almost sure convergence of delayed sums. Several specific convergence rate results are then derived as corollaries, some of which are consistent with the existing results. Building on the key result in Section 3, Section 4 provides a new theorem of convergence rates for the general norming factor and illustrates it with some examples. Section 5 presents the discussion and future di-

rections. Throughout, we use the following conventions: $\log x = \ln(x \vee e)$ for $x \geq 0$; $f(x) \sim g(x)$ as $x \to \infty$ means $\lim_{x\to\infty} f(x)/g(x) = 1$; $f(x) = o(g(x))$ as $x \to \infty$ means $\lim_{x\to\infty} f(x)/g(x) = 0$; and $f^{-1}(x)$ denotes the inverse function of $f(x)$.

2. Basic Setting

To establish our main results, the definitions of slowly varying functions and regularly varying functions are given as follows:

Definition 1. *(i) The function $l(x) > 0$ defined on $(0, \infty)$ is said to be a slowly varying function at infinity if*

$$\lim_{x\to\infty} \frac{l(tx)}{l(x)} = 1, \quad \text{for any } t > 0.$$

(ii) The function $f(x) > 0$ defined on $(0, \infty)$ is said to be a regularly varying function with index ρ at infinity ($\rho \in \mathbb{R}$), (we write $f \in R_\rho$) if

$$\lim_{x\to\infty} \frac{f(tx)}{f(x)} = t^\rho, \quad \text{for any } t > 0.$$

Note that $\log x$, $\log\log x$ and $\exp\{(\log x)^\beta\}$ for some $0 < \beta < 1$ are examples of slowly varying functions. The functions x^β and $x^\beta \log x$ for some $\beta > 0$ are examples of regularly varying functions.

It is well known that slowly varying functions and regularly varying functions have many properties. In the following, we list several properties, which are relative to our results, from Bingham et al. (1989) [10], where (i) corresponds to Theorem 1.2.1, (ii) corresponds to Theorem 1.3.1, and (iii) corresponds to Theorem 1.4.1.

Proposition 1. *(i) If $l(x)$ is a slowly varying function, then for any compact subset $K \subset (0, \infty)$,*

$$\lim_{x\to\infty} \frac{l(tx)}{l(x)} = 1, \quad \text{uniformly for all } t \in K.$$

(ii) The function $l(x)$ is a slowly varying function if and only if

$$l(x) = c(x) \exp\left\{ \int_a^x \frac{b(u)}{u} du \right\}, \quad x \geq a,$$

for some $a > 0$, where $c(x) > 0$, $\lim_{x\to\infty} c(x) = c > 0$, and $\lim_{x\to\infty} b(x) = 0$.
(iii) $f \in R_\rho$ ($\rho \in \mathbb{R}$) if and only if

$$f(x) = x^\rho l(x),$$

where $l(x) > 0$ is a slowly varying function.

We indicate the following further properties which will be used in our proofs.

Lemma 1. *(i) If $f(x)$ is differentiable and $f' \in R_\rho$ ($\rho \in \mathbb{R}$), then $f \in R_{\rho+1}$.*
(ii) If $f \in R_\rho$ ($\rho \in \mathbb{R}$), then for any $a, b \in \mathbb{R}$ with $a < b$,

$$\lim_{x\to\infty} \frac{f(x+r)}{f(x)} = 1, \quad \text{uniformly for all } r \in [a, b].$$

Proof. (i) This result can be obtained immediately by noting that

$$\lim_{x\to\infty} \frac{f(tx)}{f(x)} = \lim_{x\to\infty} \frac{tf'(tx)}{f'(x)} = t^{\rho+1}, \qquad \text{for any } t > 0.$$

(ii) Since $f \in R_\rho$, according to part (iii) of Proposition 1, for any $r \in \mathbb{R}$, there exists a slowly varying function $l(x)$ such that $f(x) = x^\rho l(x)$. Then we have

$$\begin{aligned}\lim_{x\to\infty} \frac{f(x+r)}{f(x)} &= \lim_{x\to\infty} \frac{(x+r)^\rho l(x+r)}{x^\rho l(x)} \\ &= \lim_{x\to\infty} (1+\frac{r}{x})^\rho \cdot \frac{l(x(1+r/x))}{l(x)} \\ &= 1, \quad \text{uniformly for all } r \in [a,b].\end{aligned}$$

The last equality follows from part (i) of Proposition 1. □

As a well-known result, The Borel–Cantelli lemma plays a crucial role in proving our main result, and we state it as follows without proof.

Lemma 2 (Borel–Cantelli). *Let $\{A_n\}_{n=1}^\infty$ be a sequence of events on a probability space $(\Omega, \mathcal{F}, P)$.*
(i) If $\sum_{n=1}^\infty P(A_n) < \infty$, then

$$P(A_n \text{ i.o.}) = 0,$$

where A_n i.o. $:= \limsup_{n\to\infty} A_n = \bigcap_{n=1}^\infty \bigcup_{k=n}^\infty A_k$.
(ii) If $\sum_{n=1}^\infty P(A_n) = \infty$ and the events $\{A_n\}$ are mutually independent, then

$$P(A_n \text{ i.o.}) = 1.$$

The next lemma is the uniform Berry–Esseen inequality under a relaxed third-moment assumption, which is a direct corollary of Theorem 2.1 in Chen and Shao (2001) [11].

Lemma 3. *Let $\{X_n,\ n = 1, 2, \cdots\}$ be a sequence of i.i.d. random variables with $E[X_1] = 0$ and $E[X_1^2] = \sigma^2 < \infty$. Assume that $E\big(X_1^2 g(X_1)\big) < \infty$ for some non-negative, even function $g(x)$ defined on $\mathbb{R}$, with $g(x)$ and $x/g(x)$ both non-decreasing on $(0,\infty)$. Then there exists a constant $A > 0$ such that*

$$\sup_{x\in\mathbb{R}} \left| P\left(\frac{S_n}{\sigma\sqrt{n}} > x\right) - (1-\Phi(x)) \right| \le \frac{AE\big(X_1^2 g(X_1)\big)}{\sigma^2 g(\sigma\sqrt{n})}.$$

3. Convergence Rates by the Delayed Sums Method

We now state our main theorem, which shows that convergence rates of partial sums can be deduced directly from the corresponding almost sure convergence of delayed sums.

Theorem 1. *Let $\{X_n,\ n = 1, 2, \cdots\}$ be a sequence of i.i.d. random variables with $E[X_1] = 0$. Let $a(x), d(x), G(x)$ be functions all defined on $(0,\infty)$, with $a(x), d(x), G(x) \uparrow \infty$, $G(x)$ differentiable, $G'(x) \in R_\rho$, and $d(x) \in R_{\rho'}$ $(\rho, \rho' \in \mathbb{R})$. Assume that*

$$a^{-1}\Big(G^{-1}(n)\Big) + G^{-1}(n) \le a^{-1}\Big(G^{-1}(n+1)\Big), \quad n = 1, 2, \cdots \tag{5}$$

(i) If further

$$\lim_{x\to\infty} \frac{G^{-1}(x+1)}{G^{-1}(x)} = K, \quad \text{for some } K \ge 1, \tag{6}$$

then for any $\varepsilon > 0$,

$$P\left(\limsup_{n\to\infty} \frac{\max_{1\le j\le a(n)} |S_{n,j}|}{\sqrt{a(n)d(a(n))}} > \varepsilon\right) = 0, \tag{7}$$

implies

$$\sum_{n=1}^{\infty} G'(n) P\left(\max_{1\le j\le n} |S_j| > K^{\rho'+1}\varepsilon \sqrt{n\, d(n)} \right) < \infty. \tag{8}$$

(ii) If further $E[X_1^2] = \sigma^2 < \infty$ *and* $E\left(X_1^2 g(X_1)\right) < \infty$ *for some non-negative, even function* $g(x)$ *defined on* $\mathbb{R}$*, with* $g(x)$ *and* $x/g(x)$ *both non-decreasing on* $(0,\infty)$*, then for any* $\varepsilon > 0$*,*

$$P\left(\limsup_{n\to\infty} \frac{|S_{n,a(n)}|}{\sqrt{a(n)d(a(n))}} > \varepsilon \right) = 0, \tag{9}$$

together with

$$\lim_{x\to\infty} \frac{\sqrt{d(x)}\exp\{\frac{\varepsilon^2}{2\sigma^2} d(x)\}}{g(\sigma\sqrt{x})} = 0, \tag{10}$$

implies

$$\sum_{n=1}^{\infty} G'(n) P\left(|S_n| > \varepsilon \sqrt{n\, d(n)} \right) < \infty. \tag{11}$$

Proof. (i) For any fixed $\varepsilon > 0$, denoting $\varepsilon' = K^{\rho'+1}\varepsilon$, note that

$$\begin{aligned}
&\int_1^{\infty} G'(t)\, P\left(\max_{1\le j\le t} |S_j| > \varepsilon' \sqrt{t\, d(t)} \right) dt \\
&= \sum_{n=1}^{\infty} \int_n^{n+1} G'(t)\, P\left(\max_{1\le j\le n} |S_j| > \varepsilon' \sqrt{t\, d(t)} \right) dt \\
&\ge \sum_{n=1}^{\infty} \int_n^{n+1} G'(t)\, P\left(\max_{1\le j\le n} |S_j| > \varepsilon' \sqrt{(n+1)\, d(n+1)} \right) dt.
\end{aligned}$$

Since $G'(x)$ and $d(x)$ are both regularly varying functions, according to part (ii) of Lemma 1, we have

$$\lim_{n\to\infty} \frac{G'(n+r)}{G'(n)} = 1, \quad \text{uniformly for all } r \in [0,1], \quad \textit{and} \quad \lim_{n\to\infty} \frac{d(n+1)}{d(n)} = 1.$$

It follows that

$$\int_1^{\infty} G'(t)\, P\left(\max_{1\le j\le t} |S_j| > \varepsilon' \sqrt{t\, d(t)} \right) dt < \infty, \tag{12}$$

implies (8). So we only need to prove (12).

Applying the change in the variable $u = G(t)$ to (12), notice that

$$\begin{aligned}
\int_1^{\infty} G'(t)\, P\left(\max_{1\le j\le t} |S_j| > \varepsilon' \sqrt{t\, d(t)} \right) dt &= \int_1^{\infty} P\left(\max_{1\le j\le t} |S_j| > \varepsilon' \sqrt{t\, d(t)} \right) dG(t) \\
&= \int_{G(1)}^{\infty} P\left(\max_{1\le j\le G^{-1}(u)} |S_j| > \varepsilon' \sqrt{G^{-1}(u)\, d(G^{-1}(u))} \right) du \\
&\le \sum_{m=\lfloor G(1)\rfloor}^{\infty} \int_m^{m+1} P\left(\max_{1\le j\le G^{-1}(u)} |S_j| > \varepsilon' \sqrt{G^{-1}(u)\, d(G^{-1}(u))} \right) du \\
&\le \sum_{m=\lfloor G(1)\rfloor}^{\infty} P\left(\max_{1\le j\le G^{-1}(m+1)} |S_j| > \varepsilon' \sqrt{G^{-1}(m)\, d(G^{-1}(m))} \right).
\end{aligned}$$

Since $\{X_n,\ n = 1, 2, \cdots\}$ are identically distributed, we only need to show

$$\sum_{m=\lfloor G(1)\rfloor+1}^{\infty} P\left(\max_{1\le j\le G^{-1}(m)} |S_{a^{-1}(G^{-1}(m)),j}| > \varepsilon' \sqrt{G^{-1}(m-1)\, d(G^{-1}(m-1))} \right) < \infty. \tag{13}$$

Since (5) implies

$$\lfloor a^{-1}\Big(G^{-1}(m)\Big) + G^{-1}(m)\rfloor < \lfloor a^{-1}\Big(G^{-1}(m+1)\Big)\rfloor + 1, \quad m = 1, 2, \cdots$$

it ensures $S_{a^{-1}(G^{-1}(m)),G^{-1}(m)}$ and $S_{a^{-1}(G^{-1}(m+1)),G^{-1}(m+1)}$ have no common terms for $m = 1, 2, \cdots$, which implies that the sequence $\{S_{a^{-1}(G^{-1}(m)),G^{-1}(m)}, m = 1, 2, \cdots\}$ is independent. According to the Borel–Cantelli lemma, we know that (13) is equivalent to

$$P\left(\limsup_{m\to\infty} \frac{\max_{1\le j\le G^{-1}(m)} |S_{a^{-1}(G^{-1}(m)),j}|}{\sqrt{G^{-1}(m-1)\, d(G^{-1}(m-1))}} > \varepsilon'\right) = 0. \tag{14}$$

On the other hand, taking $n = a^{-1}(G^{-1}(m))$ in (7), we obtain

$$P\left(\limsup_{m\to\infty} \frac{\max_{1\le j\le G^{-1}(m)} |S_{a^{-1}(G^{-1}(m)),j}|}{\sqrt{G^{-1}(m) d(G^{-1}(m))}} > \varepsilon\right) = 0. \tag{15}$$

Then according to part (iii) of Proposition 1, there exists a slowly varying function $l(x)$ such that $d(x) = x^{\rho'} l(x)$. Hence, according to (6),

$$\begin{aligned}
&\lim_{m\to\infty} \frac{G^{-1}(m) d(G^{-1}(m))}{G^{-1}(m-1) d(G^{-1}(m-1))} \\
&= \lim_{m\to\infty} \Big(\frac{G^{-1}(m)}{G^{-1}(m-1)}\Big)^{\rho'+1} \cdot \frac{l(\eta(G^{-1}(m-1))G^{-1}(m-1))}{l(G^{-1}(m-1))} \\
&= K^{\rho'+1},
\end{aligned} \tag{16}$$

where $\lim_{m\to\infty} \eta(G^{-1}(m-1)) = K$, and the last inequality follows from part (i) of Proposition 1. Substituting (16) into (15), we can get (14), and the desired conclusion follows.

(ii) Based on arguments analogous to the proof of part (i), (11) can be obtained if we prove

$$\int_1^\infty P\Big(|S_t| > \varepsilon\sqrt{t\, d(t)}\Big) dG(t) < \infty. \tag{17}$$

Applying the change in the variable $u = G(t)$ to (17), we have

$$\begin{aligned}
&\int_1^\infty P\Big(|S_t| > \varepsilon\sqrt{t\, d(t)}\Big) dG(t) \\
&= \int_{G(1)}^\infty P\Big(|S_{G^{-1}(u)}| > \varepsilon\sqrt{G^{-1}(u)\, d(G^{-1}(u))}\Big) du \\
&\le \sum_{m=\lfloor G(1)\rfloor}^\infty \int_m^{m+1} P\Big(|S_{G^{-1}(u)}| > \varepsilon\sqrt{G^{-1}(u)\, d(G^{-1}(u))}\Big) du.
\end{aligned} \tag{18}$$

According to Lemma 3, we have

$$\left|P\Big(\frac{S_n}{\sigma\sqrt{n}} > \frac{\varepsilon}{\sigma}\sqrt{d(n)}\Big) - \Big(1 - \Phi\Big(\frac{\varepsilon}{\sigma}\sqrt{d(n)}\Big)\Big)\right| \le \frac{AE\big(X_1^2 g(X_1)\big)}{\sigma^2 g(\sigma\sqrt{n})},$$

for some $A > 0$. Hence,

$$\begin{aligned}
&\left|P(|S_n| > \varepsilon\sqrt{nd(n)}) - 2(1-\Phi(\frac{\varepsilon}{\sigma}\sqrt{d(n)}))\right| \\
&= \left|P(\frac{|S_n|}{\sigma\sqrt{n}} > \frac{\varepsilon}{\sigma}\sqrt{d(n)}) - 2(1-\Phi(\frac{\varepsilon}{\sigma}\sqrt{d(n)}))\right| \\
&\le \left|P(\frac{S_n}{\sigma\sqrt{n}} > \frac{\varepsilon}{\sigma}\sqrt{d(n)}) - (1-\Phi(\frac{\varepsilon}{\sigma}\sqrt{d(n)}))\right| \\
&+ \left|P(-\frac{S_n}{\sigma\sqrt{n}} > \frac{\varepsilon}{\sigma}\sqrt{d(n)}) - (1-\Phi(\frac{\varepsilon}{\sigma}\sqrt{d(n)}))\right| \\
&\le \frac{2AE\left(X_1^2 g(X_1)\right)}{\sigma^2 g(\sigma\sqrt{n})}.
\end{aligned} \tag{19}$$

Based on the elementary inequality of normal distribution,

$$\frac{1}{\sqrt{2\pi}}\cdot\frac{x}{1+x^2}e^{-\frac{x^2}{2}} \le 1-\Phi(x) \le \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{x}e^{-\frac{x^2}{2}}, \quad \text{for any } x>0,$$

we have

$$\frac{1}{2\sqrt{2\pi}}\cdot\frac{\sigma}{\varepsilon\sqrt{d(n)}}e^{-\frac{\varepsilon^2}{2\sigma^2}d(n)} \le 1-\Phi\left(\frac{\varepsilon}{\sigma}\sqrt{d(n)}\right) \le \frac{1}{\sqrt{2\pi}}\cdot\frac{\sigma}{\varepsilon\sqrt{d(n)}}e^{-\frac{\varepsilon^2}{2\sigma^2}d(n)},$$

for sufficiently large n. According to (10) and $E\left(X_1^2 g(X_1)\right) < \infty$, there holds

$$\frac{2AE\left(X_1^2 g(X_1)\right)}{\sigma^2 g(\sigma\sqrt{n})} = o\left(1-\Phi\left(\frac{\varepsilon}{\sigma}\sqrt{d(n)}\right)\right), \quad \text{as } n\to\infty.$$

Then according to (19),

$$P\left(|S_n| > \varepsilon\sqrt{nd(n)}\right) \sim 2\left(1-\Phi(\frac{\varepsilon}{\sigma}\sqrt{d(n)})\right), \quad \text{as } n\to\infty. \tag{20}$$

Based on the monotonicity of $1-\Phi\left(\frac{\varepsilon}{\sigma}\sqrt{d(n)}\right)$ and the regular variation of $d(x)$, we have, for any $r\in[0,1]$,

$$\begin{aligned}
&P\left(|S_{G^{-1}(m+r)}| > \varepsilon\sqrt{G^{-1}(m+r)\,d(G^{-1}(m+r))}\right) \\
&\sim P\left(|S_{G^{-1}(m+r)}| > \varepsilon\sqrt{\lfloor G^{-1}(m+r)\rfloor\, d(\lfloor G^{-1}(m+r)\rfloor)}\right) \\
&\sim 2\left(1-\Phi\left(\frac{\varepsilon}{\sigma}\sqrt{d(\lfloor G^{-1}(m+r)\rfloor)}\right)\right) \\
&\le 2\left(1-\Phi\left(\frac{\varepsilon}{\sigma}\sqrt{d(\lfloor G^{-1}(m)\rfloor)}\right)\right) \\
&\sim P\left(|S_{G^{-1}(m)}| > \varepsilon\sqrt{\lfloor G^{-1}(m)\rfloor\, d(\lfloor G^{-1}(m)\rfloor)}\right) \\
&\sim P\left(|S_{G^{-1}(m)}| > \varepsilon\sqrt{G^{-1}(m)\,d(G^{-1}(m))}\right),
\end{aligned} \tag{21}$$

as $m\to\infty$. Combining (21) with (18), based on the identical distribution of $\{X_n,\, n=1,2,\cdots\}$, we only need to prove

$$\sum_{m=\lfloor G(1)\rfloor}^{\infty} P\left(|S_{a^{-1}(G^{-1}(m)),G^{-1}(m)}| > \varepsilon\sqrt{G^{-1}(m)\,d(G^{-1}(m))}\right) < \infty. \tag{22}$$

Also, based on the independence of $\{S_{a^{-1}(G^{-1}(m)),G^{-1}(m)}, m = 1,2,\cdots\}$ and the Borel–Cantelli lemma, we know that (22) is equivalent to

$$P\left(\limsup_{m\to\infty}\frac{|S_{a^{-1}(G^{-1}(m)),G^{-1}(m)}|}{\sqrt{G^{-1}(m)\,d(G^{-1}(m))}} > \varepsilon\right) = 0. \tag{23}$$

On the other hand, taking $n = a^{-1}(G^{-1}(m))$ in (9), we can get (23), and the desired conclusion follows. □

Remark 1. *Theorem 1 provides two approaches for obtaining convergence rates under different settings. In the first approach, if one can establish the almost sure convergence of the maximum delayed sums* $\max_{1\le j\le a(n)}|S_{n,j}|$*, then the convergence rate of the maximum partial sums* $\max_{1\le j\le n}|S_j|$ *can be derived under condition (6), which does not require additional moment conditions. In the second approach, if the almost sure convergence of the delayed sums* $S_{n,a(n)}$ *is proven, then the corresponding convergence rate of the partial sums* S_n *can be obtained under condition (10), which requires a moment of order greater than two.*

Remark 2. *Based on the proof of part (i) of Theorem 1, if we consider the case of maximum delayed sums* $\max_{1\le j\le a(n)} S_{n,j}$ *instead of* $\max_{1\le j\le a(n)}|S_{n,j}|$ *and the maximum partial sums* $\max_{1\le j\le n} S_j$ *instead of* $\max_{1\le j\le n}|S_j|$*, under the same assumptions, we can also determine that for any* $\varepsilon > 0$*,*

$$P\left(\limsup_{n\to\infty}\frac{\max_{1\le j\le a(n)} S_{n,j}}{\sqrt{a(n)d(a(n))}} > \varepsilon\right) = 0, \tag{24}$$

implies

$$\sum_{n=1}^{\infty} G'(n)P\left(\max_{1\le j\le n} S_j > K^{\rho'+1}\varepsilon\sqrt{n\,d(n)}\right) < \infty. \tag{25}$$

Remark 3. *Assumption (5) is the key condition in Theorem 1. It ensures the independence of the delayed sums constructed in the proof, so that the Borel–Cantelli lemma remains applicable. This is a technical requirement but is not difficult to fulfill. Two typical cases are as presented below:*

Case 1: Suppose $a^{-1}(x) = Cx$ *for some* $C > 0$*. Choosing* $G(x) = \log a^{-1}(x)$*, we have* $G^{-1}(x) = a(e^x)$*. Then (5) follows from the elementary inequality:*

$$e^n + e^n \le e^{n+1}, \quad n = 1,2,\cdots$$

Case 2: Suppose $a^{-1}(x)/x \to \infty$ *monotonically. Taking* $G(x) = a^{-1}(x)/x$*, we have* $a^{-1}(x) = xG(x)$*. Consequently, (5) is satisfied because*

$$nG^{-1}(n) + G^{-1}(n) \le (n+1)G^{-1}(n+1), \quad n = 1,2,\cdots$$

These two cases also illustrate how $G(x)$ *can be chosen once the form of* $a^{-1}(x)$ *is known.*

Remark 4. *The main purpose of Assumption (6) is to prevent the function* $G(x)$ *from varying too slowly, which would lead to overly rapid growth in* $G^{-1}(x)$*. For instance, if* $G(x) = \log x$*, then (6) holds since* $\lim_{x\to\infty}(e^{x+1}/e^x) = e$*. However, if* $G(x) = \log\log x$*, then (6) fails as* $\lim_{x\to\infty}(e^{e^{x+1}}/e^{e^x}) = \infty$*.*

Remark 5. *Assumption (10) is a technical requirement intended to apply the uniform Berry–Esseen inequality to control the asymptotic rate of* $P\left(|S_n| > \varepsilon\sqrt{nd(n)}\right)$. *This condition can be satisfied by choosing an appropriate function* $d(x)$ *based on* $g(x)$. *For example, taking*

$$d(x) = \frac{2\sigma^2}{\varepsilon^2}\left(\log g(\sigma\sqrt{x}) - \beta \log\log g(\sigma\sqrt{x})\right),$$

for some $\beta > \frac{1}{2}$, *it can be verified that (10) holds. Then we only need to prove that* $d(x)$ *meets the other conditions of Theorem 1.*

Remark 6. *The core of applying Theorem 1 to obtain the convergence rates lies in the determination of the functions* $a(x)$, $d(x)$ *and* $G(x)$. *One can first determine the function* $d(x)$ *through the norming factors of the convergence rates for our target, then identify the function* $a(x)$ *by proving the corresponding almost sure convergence of delayed sums, and finally select an appropriate* $G(x)$ *such that (5) holds, thereby obtaining the convergence rates* $G'(n)$.

Based on Theorem 1 and the existing results on almost sure convergence of delayed sums listed in Section 1, we choose appropriate functions $G(x)$ by identifying $a(x)$ and $d(x)$ in (1)–(3). This yields several convergence rate results as corollaries, some of which coincide with known results.

Corollary 1. *Let* $\{X_n, n = 1, 2, \cdots\}$ *be a sequence of i.i.d. random variables with* $E[X_1] = 0$.
(i) If $E|X_1|^p < \infty$ *for some* $0 < p < 2$, *then for any* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\le j\le n} |S_j| > \varepsilon n^{1/p}\right) < \infty. \tag{26}$$

(ii) If $E|X_1|^p < \infty$ *for some* $\beta > 1/2$, $p > 1/\beta$, *then for any* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\beta p-2} P\left(\max_{1\le j\le n} |S_j| > \varepsilon n^{\beta}\right) < \infty. \tag{27}$$

Proof. (i) The moment condition in (1) is satisfied, so (1) indicates that for any $\varepsilon > 0$,

$$P\left(\limsup_{n\to\infty} \frac{\max_{1\le j\le n} |S_{n,j}|}{n^{1/p}} > \varepsilon\right) = 0. \tag{28}$$

Assuming that in Theorem 1,

$$a(x) = x, \quad d(x) = x^{2/p-1}, \quad G(x) = \log x,$$

all defined on $(0, \infty)$, we have

$$a^{-1}(x) = x, \quad G^{-1}(x) = e^x, \quad G'(x) = x^{-1}.$$

It is easy to check that $G'(x) \in R_{-1}$, $d(x) \in R_{2/p-1}$, and

$$\lim_{x\to\infty} \frac{G^{-1}(x+1)}{G^{-1}(x)} = e.$$

In addition, (5) holds according to Remark 3.

So all the conditions of part (i) of Theorem 1 are satisfied, then (28) implies

$$\sum_{n=1}^{\infty} n^{-1} P\Big(\max_{1\le j\le n} |S_j| > e^{2/p}\varepsilon n^{1/p}\Big) < \infty,$$

which implies (26) based on the arbitrariness of $\varepsilon > 0$.
(ii) The moment condition in (2) is satisfied, so (2) with $\alpha = 1/(\beta p) < \min(2/p, 1)$ indicates that for any $\varepsilon > 0$,

$$P\left(\limsup_{n\to\infty} \frac{\max_{1\le j\le n^{1/(\beta p)}} |S_{n,j}|}{n^{1/p}} > \varepsilon\right) = 0. \tag{29}$$

Assuming that in Theorem 1,

$$a(x) = x^{1/(\beta p)}, \quad d(x) = x^{2\beta-1}, \quad G(x) = x^{\beta p-1},$$

all defined on $(0,\infty)$, we have

$$a^{-1}(x) = x^{\beta p}, \quad G^{-1}(x) = x^{1/(\beta p-1)}, \quad G'(x) = (\beta p-1)x^{\beta p-2}.$$

It is easy to check that $G'(x) \in R_{\beta p-2}$, $d(x) \in R_{2\beta-1}$, and

$$\lim_{x\to\infty} \frac{G^{-1}(x+1)}{G^{-1}(x)} = 1.$$

In addition, (5) holds according to Remark 3.

So all the conditions of part (i) of Theorem 1 are satisfied, then (29) implies (27). □

Remark 7. *The moment condition and convergence rate in part (i) of Corollary 1 are the same as those in Theorem 1 of Baum and Katz (1965) [1].*

Corollary 2. *Let* $\{X_n, n = 1, 2, \cdots\}$ *be i.i.d. random variables with* $E[X_1] = 0$ *and* $E[X_1^2] = \sigma^2 < \infty$. *If*

$$E\Big(|X_1|^{2/\beta}(\log|X_1|)^{-1/\beta}\Big) < \infty \quad \textit{for some } \frac{2}{3} \le \beta < 1,$$

then for any $\varepsilon > \sigma\sqrt{2(1-\beta)/\beta}$,

$$\sum_{n=1}^{\infty} n^{1/\beta-2} P\Big(|S_n| > \varepsilon\sqrt{n\log n}\Big) < \infty. \tag{30}$$

Proof. The moment condition in (3) is satisfied, so (3) indicates that for any $\varepsilon > 0$,

$$P\left(\limsup_{n\to\infty} \frac{|S_{n,n^\beta}|}{\frac{\sigma}{\varepsilon}\sqrt{2(1-\beta)n^\beta \log n}} > \varepsilon\right) = 0. \tag{31}$$

Assuming that in Theorem 1,

$$a(x) = x^{\beta}, \quad G(x) = x^{1/\beta-1}, \quad d(x) = \frac{2\sigma^2}{\varepsilon^2}\big(\log g(\sigma\sqrt{x}) - \log\log g(\sigma\sqrt{x})\big),$$

all defined on $(0,\infty)$ where $g(x) = |x|^{2/\beta-2}(\log|x|)^{-1/\beta}$ defined on $\mathbb{R}$, we have

$$a^{-1}(x) = x^{1/\beta}, \quad G^{-1}(x) = x^{\beta/(1-\beta)}, \quad G'(x) = (1/\beta-1)x^{1/\beta-2},$$

and

$$d(x) \sim \frac{2(1-\beta)\sigma^2}{\beta\varepsilon^2} \cdot \log x, \quad a(x)d(a(x)) \sim \frac{2(1-\beta)\sigma^2}{\varepsilon^2} \cdot x^{\beta} \log x, \quad \text{as } x \to \infty. \tag{32}$$

It is easy to check that $G'(x) \in R_{1/\beta-2}$, $d(x)$ is a slowly varying function, (5) holds according to Remark 3, and (10) holds from the construction of $d(x)$. All the conditions of part (ii) of Theorem 1 are satisfied, then combining (31) with (32), we have

$$\sum_{n=1}^{\infty} n^{1/\beta-2} P\left(|S_n| > \sigma \sqrt{\frac{2(1-\beta)}{\beta} \cdot n \log n} \right) < \infty,$$

which implies that (30) holds for any $\varepsilon > \sigma\sqrt{2(1-\beta)/\beta}$. □

4. The Convergence Rates for the General Norming Factors

We first present the following lemma, which gives an important asymptotic property of slowly varying functions and an estimate for the asymptotic rate of $G'(x)$ when an explicit expression for $G(x)$ is not available in the setting of the general norming factors.

Lemma 4. *Assume that the function $L(x)$, defined and differentiable on $(0, \infty)$, satisfies the following: $L(x) \uparrow \infty$, $x/L(x) \uparrow \infty$ as $x \to \infty$,*

$$\lim_{x\to\infty} \frac{L(x)L'(x)}{L'(x/L(x))} = 1, \tag{33}$$

and

$$\lim_{x\to\infty} xL'(x)/L(x) = 0. \tag{34}$$

(i) If $L(x)$ is a slowly varying function, then we have

$$\lim_{x\to\infty} \frac{L(x)}{L(xL(x))} = \lim_{x\to\infty} \frac{L(x/L(x))}{L(x)} = 1. \tag{35}$$

(ii) Let $a(x) = x/L(x)$ and $G(x) = a^{-1}(x)/x$. Then we have

$$\lim_{x\to\infty} \frac{G'(x)}{L'(x)} = 1. \tag{36}$$

Proof. (i) The conclusion holds by observing the following two equations:

$$\begin{aligned} \lim_{x\to\infty} \frac{L(x/L(x))}{L(x)} &= \lim_{x\to\infty} \frac{L'(x/L(x)) \cdot \frac{L(x)-xL'(x)}{L^2(x)}}{L'(x)} \\ &= \lim_{x\to\infty} \frac{L'(x/L(x))}{L(x)L'(x)} \cdot (1 - \frac{xL'(x)}{L(x)}) = 1, \end{aligned}$$

where the first equality follows from *L'Hôpital's* rule.

$$\begin{aligned} \lim_{x\to\infty} \frac{L(x)}{L(xL(x))} &= \lim_{y\to\infty} \frac{L(y/L(y))}{L(yL(y/L(y))/L(y))} \\ &= \lim_{y\to\infty} \frac{L(y/L(y))}{L(y)} = 1, \end{aligned}$$

where the second-to-last equality follows from $\lim_{y\to\infty} L(y/L(y))/L(y) = 1$ and part (i) of Proposition 1.

(ii) Denote $u = a^{-1}(x)$, then $x = a(u) = u/L(u)$. It follows that

$$\left(a^{-1}(x)\right)' = \frac{1}{(a(u))'} = \frac{L^2(u)}{L(u) - uL'(u)} = \frac{L(u)}{1 - uL'(u)/L(u)}.$$

Hence,

$$G'(x) = \frac{\left(a^{-1}(x)\right)' x - a^{-1}(x)}{x^2} = \frac{uL'(u)}{x(1 - uL'(u)/L(u))}.$$

According to (33) we have

$$uL'(u) \sim xL'(x) \quad \text{as } x \to \infty.$$

Combining this with (34), we can obtain (36). □

Next, according to Theorem 1, we obtain a new theorem of convergence rates for the general norming factor whose order lies between $\sqrt{n \log\log n}$ and $\sqrt{n \log n}$.

Theorem 2. *Let $\{X_n, n = 1, 2, \cdots\}$ be i.i.d. random variables with $E[X_1] = 0$ and $E[X_1^2] = \sigma^2 < \infty$. Assume that the function $L(x)$, defined and differentiable on $(0, \infty)$, satisfies the following: $L'(x) \in R_{-1}$, $L(x) \uparrow \infty$, $x/L(x) \uparrow \infty$, $xL'(x)/L(x)$ decreases as $x \to \infty$, (33) and (34) hold, and $L(x)/(\log x)^\alpha \to \infty$ as $x \to \infty$ for some $\alpha > 0$. Then we have for any $\varepsilon > \sqrt{2C}\sigma$,*

$$E\left[\frac{X_1^2 L(X_1^2)}{\log L(X_1^2)}\right] < \infty \quad \textit{implies} \quad \sum_{n=1}^{\infty} L'(n) P(|S_n| > \varepsilon \sqrt{n \log L(n)}) < \infty, \tag{37}$$

where

$$C = \begin{cases} 1, & \textit{if } \lim_{x\to\infty} \log L(x)/\log\log x = \infty, \\ 1 + 1/\beta, & \textit{if } \lim_{x\to\infty} \log L(x)/\log\log x = \beta. \end{cases} \tag{38}$$

Proof. Define

$$h(x) = \log L(x) + \log\log x, \quad f(x) = \min\{h(x)x/L(x), x\},$$

both on $(0, \infty)$. It is obvious that the moment condition $E[f^{-1}(X_1^2)] < \infty$, where $f(x) = \min\{h(x)x/L(x), x\}$ is equivalent to the moment condition in (4), where $f(x)$ is defined by floor interpolating on the sequence $f(n) = \min\{h(n)n/L(n), n\}$. Note that $L'(x) \in R_{-1}$ implies that $L(x)$ is a slowly varying function based on part (i) of Lemma 1.

According to $L(x)/(\log x)^\alpha \to \infty$, as $x \to \infty$ for some $\alpha > 0$, we have

$$h(x) \sim C \log L(x), \quad f(x) \sim Cx \log L(x)/L(x), \quad \textit{as} \quad x \to \infty,$$

where C satisfies (38). Based on part (i) of Lemma 4, (35) holds. Noting that

$$1 = \lim_{x\to\infty} \frac{L(x)}{L(xL(x))} \le \lim_{x\to\infty} \frac{L(x)}{L(xL(x)/\log L(x))} \le 1,$$

and

$$1 = \lim_{x\to\infty} \frac{L(x/L(x))}{L(x)} \le \lim_{x\to\infty} \frac{L(x \log L(x)/L(x))}{L(x)} \le 1,$$

it can be easily verified that

$$f^{-1}(x) \sim \frac{1}{C} xL(x)/\log L(x), \quad as\ x \to \infty.$$

So the moment condition $E\left[\frac{X_1^2 L(X_1^2)}{\log L(X_1^2)}\right] < \infty$ is equivalent to $E[f^{-1}(X_1^2)] < \infty$. Then (4) indicates that for any $\varepsilon > 0$,

$$P\left(\limsup_{n\to\infty} \frac{|S_{n,n/L(n)}|}{\frac{\sigma}{\varepsilon}\sqrt{\frac{2}{C}h(n)n/L(n)}} > \sqrt{C}\varepsilon\right) = 0. \tag{39}$$

Assuming that in Theorem 1,

$$a(x) = x/L(x), \quad G(x) = a^{-1}(x)/x,$$

both defined on $(0, \infty)$, based on part (ii) of Lemma 4, we have

$$G'(x) \sim L'(x), \quad \text{and then } G(x) \sim L(x), \quad as\ x \to \infty.$$

It is obvious that $G'(x) \in R_{-1}$, and (5) holds according to Remark 3.

Assuming that in Theorem 1,

$$g(x) = Cf^{-1}(x^2)/x^2, \quad d(x) = \frac{2\sigma^2}{\varepsilon^2}\left(\log g(\sigma\sqrt{x}) - \log\log g(\sigma\sqrt{x})\right),$$

defined on $\mathbb{R}$ and $(0, \infty)$, respectively, (10) holds from the construction of $d(x)$, and

$$g(x) \sim L(x^2)/\log L(x^2), \quad d(x) \sim \frac{2\sigma^2}{\varepsilon^2} \cdot \log L(x) \sim \frac{2\sigma^2}{C\varepsilon^2} \cdot h(x), \quad as\ x \to \infty.$$

Moreover, according to (35) we have

$$d(a(x)) \sim \frac{2\sigma^2}{\varepsilon^2} \cdot \log L(x/L(x)) \sim \frac{2\sigma^2}{\varepsilon^2} \cdot \log L(x) \sim d(x), \quad as\ x \to \infty.$$

Hence according to (39), we get (9) under the definitions of $a(x)$ and $d(x)$ here. All the conditions of part (ii) of Theorem 1 are satisfied. Then we have

$$\sum_{n=1}^{\infty} L'(n) P\left(|S_n| > \sigma\sqrt{2Cn\log L(n)}\right) < \infty,$$

which implies that (37) holds for any $\varepsilon > \sqrt{2C}\sigma$. □

Remark 8. *In Theorem 2, the order of* $\log L(n)$ *lies between* $\log\log n$ *and* $\log n$. *The minimum order of the norming factor* $\sqrt{n\log L(n)}$ *in Theorem 2 and the norming factor* $\sqrt{n}\varphi(n)$ *in Theorem 3 of Davis (1968b) [3], where* φ *is a non-negative increasing function satisfying* $\int_1^\infty t^{-1}\varphi(t)e^{-\varphi^2(t)/2}\,dt < \infty$, *are both* $\sqrt{n\log\log n}$.

Next, we provide two specific examples to illustrate Theorem 2.

Example 1. *Set* $L(x) = (\log x)^\alpha (\log\log x)^\beta$, $\alpha > 0$ *and* $\beta \in \mathbb{R}$, *defined on* $(0, \infty)$ *in Theorem 2. Then* $L(x) \uparrow \infty$, $x/L(x) \uparrow \infty$, *as* $x \to \infty$,

$$\lim_{x\to\infty} \log L(x)/\log\log x = \alpha,$$

$$L'(x) \sim \alpha(\log x)^{\alpha-1}(\log\log x)^{\beta}/x, \quad as \quad x \to \infty,$$

which is regularly varying with index -1, *and*

$$xL'(x)/L(x) \sim \alpha/\log x, \quad as \quad x \to \infty,$$

which is decreasing. It can be easily verified that (33) and (34) hold.

Therefore, according to Theorem 2, if for some $\alpha > 0$ *and* $\beta \in \mathbb{R}$,

$$E[X_1^2(\log|X_1|)^{\alpha}(\log\log|X_1|)^{\beta-1}] < \infty,$$

then for any $\varepsilon > \sqrt{2(1+1/\alpha)}\sigma$,

$$\sum_{n=1}^{\infty} \frac{(\log n)^{\alpha-1}(\log\log n)^{\beta}}{n} P\Big(|S_n| > \varepsilon\sqrt{n\log\log n}\Big) < \infty.$$

Remark 9. *Taking* $\alpha = 1$ *and* $\beta = 2$ *in Example 1, we conclude that if*

$$E[X_1^2 \log|X_1| \log\log|X_1|] < \infty,$$

then for any $\varepsilon > 2\sigma$,

$$\sum_{n=1}^{\infty} \frac{(\log\log n)^2}{n} P\Big(|S_n| > \varepsilon\sqrt{n\log\log n}\Big) < \infty.$$

The moment condition and convergence rate here coincide with those in Theorem 3 of Davis (1968b) [3] for the case where the order of the norming factor is $\sqrt{n\log\log n}$.

Example 2. *Set* $L(x) = (\log x)^{\alpha}\exp\{(\log x)^{\beta}\}$, $\alpha \in \mathbb{R}$ *and* $0 < \beta < 1/2$, *defined on* $(0,\infty)$ *in Theorem 2. Then* $L(x) \uparrow \infty$, $x/L(x) \uparrow \infty$, *as* $x \to \infty$,

$$\lim_{x\to\infty} \log L(x)/\log\log x = \infty,$$

$$L'(x) = [\alpha(\log x)^{\alpha-1} + \beta(\log x)^{\alpha+\beta-1}]\exp\{(\log x)^{\beta}\}/x,$$

which is a regularly varying function with index -1, *and*

$$xL'(x)/L(x) = \alpha(\log x)^{-1} + \beta(\log x)^{\beta-1},$$

which is decreasing. It can be easily verified that (33) and (34) hold.

Therefore, according to Theorem 2, if for some $\alpha \in \mathbb{R}$ *and* $0 < \beta < 1/2$,

$$E\Big(X_1^2(\log|X_1|)^{\alpha-\beta}\exp\{(2\log|X_1|)^{\beta}\}\Big) < \infty,$$

then for any $\varepsilon > \sqrt{2}\sigma$,

$$\sum_{n=1}^{\infty} \frac{(\log n)^{\alpha+\beta-1}\exp\{(\log n)^{\beta}\}}{n} P\Big(|S_n| > \varepsilon\sqrt{n(\log n)^{\beta}}\Big) < \infty.$$

Remark 10. *Taking* $\alpha = 0$ *and* $\beta = 1/3$ *in Example 2, we conclude that if*

$$E\left(X_1^2 \frac{\exp\{(2\log|X_1|)^{1/3}\}}{(\log|X_1|)^{1/3}}\right) < \infty,$$

then for any $\varepsilon > \sqrt{2}\sigma$,

$$\sum_{n=1}^{\infty} \frac{\exp\{(\log n)^{1/3}\}}{n(\log n)^{2/3}} P\left(|S_n| > \varepsilon\sqrt{n(\log n)^{1/3}}\right) < \infty.$$

5. Discussion and Future Directions

Theorem 1 provides a general mechanism for deriving convergence rates of partial sums by reducing the problem to almost sure convergence of appropriately constructed delayed sums. This perspective shifts the focus of convergence rate analysis from direct probability bounds of partial sums to structural properties of delayed sums.

One key advantage of this approach is its flexibility with respect to norming factors. Classical convergence rate results typically rely on carefully tailored arguments for each specific asymptotic regime. In contrast, Theorem 1 allows a wide class of norming factors to be treated within a single framework, provided suitable $a(x)$, $d(x)$, and $G(x)$ can be identified.

Moreover, the delayed sums framework separates analytical difficulties into two distinct components: establishing almost sure convergence of delayed sums, and verifying the regular variation and the other conditions required by Theorem 1. Once the former is achieved—often using existing limit theorems—the corresponding convergence rates of partial sums follow in a systematic manner. This separation clarifies why many classical convergence rate results exhibit similar structures despite being derived by different techniques.

An important limitation of the present work is the assumption of independence. While delayed sums have been extensively studied under independence, extending Theorem 1 to dependent sequences poses significant challenges. In particular, the independence of delayed sums plays a crucial role in the application of the Borel–Cantelli lemma. In the dependent case, the absence of the Borel–Cantelli lemma makes it unclear whether the delayed sums method remains valid or what its specific form would be.

Another open problem in this paper concerns the key condition (5) in Theorem 1. This is a technical requirement, and it remains unknown whether this condition is necessary. In addition, we only emphasize the existence of a function $G(x)$ that satisfies this condition: as long as a function $G(x)$ meeting this condition is found, the convergence rate can be derived. However, we do not discuss its uniqueness—that is, if there exists another function $\overline{G}$ that also satisfies this condition, whether $G(x)$ and $\overline{G}$ are equivalent in some sense, and whether the convergence rates $G'(x)$ and $\overline{G}'(x)$ are of the same order.

The other remaining problem of this paper is that for the general convergence rates presented in Theorem 2, the order of the corresponding norming factor lies between $\sqrt{n \log\log n}$ and $\sqrt{n \log n}$. This is because the existing literature only provides conclusions of general almost sure convergence of delayed sums with norming factors of this order. If we can prove the generalized limit theorem of delayed sums where the norming factor has an order between $\sqrt{n \log n}$ and $\sqrt{n^{\beta}}$, $\beta > 1$, then we can extend Theorem 2 to this case.

Author Contributions: Methodology, C.H. and T.W.; validation, C.H., S.Y. and T.W.; formal analysis, T.W.; writing—original draft preparation, S.Y.; writing—review and editing, C.H. and T.W. All authors have read and agreed to the published version of the manuscript.

Funding: This work was partially supported by the Outstanding Youth Innovation Team Program of Shandong Province (Grant No. 2024KJG010), the National Natural Science Foundation of China (Grant Nos. 12371148, 42007141) and the Natural Science Foundation of Shandong Province (Grant No. ZR2019BA038).

Data Availability Statement: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Acknowledgments: The authors would like to thank the editor and five referees for their valuable comments, which improved the manuscript.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Baum, L.; Katz, M. Convergence rates in the law of large numbers. *Trans. Am. Math. Soc.* **1965**, *120*, 108–123. [CrossRef]
2. Davis, J. Convergence rates for probabilities of moderate deviations. *Ann. Math. Stat.* **1968**, *39*, 2016–2028. [CrossRef]
3. Davis, J. Convergence rates for the law of the iterated logarithm. *Ann. Math. Stat.* **1968**, *39*, 1479–1485. [CrossRef]
4. Norvaiša, R.; Račkauskas, A. Law of large numbers with respect to a quasinorm. I. *Lith. Math. J.* **1984**, *24*, 149–159. [CrossRef]
5. Lai, T. Limit theorems for moving averages. In *Probability, Finance and Insurance*; World Scientific: Singapore, 2004; pp. 1–14.
6. Chow, Y. Delayed sums and Borel summability of independent, identically distributed random variables. *Bull. Inst. Math. Acad. Sin.* **1973**, *1*, 207–220.
7. Lai, T. Limit theorems of delayed sums. *Ann. Probab.* **1974**, *2*, 432–440. [CrossRef]
8. Gut, A.; Jonsson, F.; Stadtmüller, U. Between the LIL and the LSL. *Bernoulli* **2010**, *16*, 1–22. [CrossRef]
9. Gut, A. *Probability: A Graduate Course*, 2nd ed.; Springer: New York, NY, USA, 2007.
10. Bingham, N.; Goldie, C.; Teugels, J. *Regular Variation*; Cambridge University Press: Cambridge, UK, 1989; Volume 27.
11. Chen, L.; Shao, Q. A non-uniform Berry–Esseen bound via Stein's method. *Probab. Theory Relat. Fields* **2001**, *120*, 236–254. [CrossRef]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.