axioms

Article

Lie Symmetry and Various Exact Solutions for (3+1)-Dimensional B-Type Kadomtsev–Petviashvili Equation

Ahmed A. Gaber [1], Dalal Alhwikem [2,*] and Abdul-Majid Wazwaz [3]

[1] Department of Mathematics, College of Science Al-Zulfi, Majmaah University, Al Majmaah 15341, Saudi Arabia
[2] Department of Mathematics, College of Science, Qassim University, Burydah 52571, Saudi Arabia
[3] Department of Mathematics, Saint Xavier University, Chicago, IL 60655, USA
* Correspondence: d.alhwikem@qu.edu.sa

Abstract

The (3+1)-dimensional B-type Kadomtsev–Petviashvili (BKP) problem was examined in this paper using the developed Exp-function method (DEFM) and Lie symmetry analysis. The objective of this research is studying the BKP equation to get novel exact solutions. Symmetry analysis has been used to determine similarity variables and vector fields. The governing equation was reduced to five variant ordinary differential equations (ODEs). The DEFM was employed for four of them to obtain several novel exact solutions that contain arbitrary constants. The most appropriate choice of values for these optional constants contributed to the emergence of solutions, such as double waves, multisolitons, kink waves, anti-kink waves, and solitary waves. The obtained exact solutions are presented in a 3D graph. The behavior of the solutions can be utilized to explore the application of the governing equation in fluid dynamics, plasma physics, nonlinear optics, and ocean physics.

Keywords: B-type Kadomtsev–Petviashvili in (3+1) dimensions; Lie-symmetry analysis; differential equations; exact solutions.

MSC: 34A05; 52B15; 58D19; 35E05; 58J70

1. Introduction

Numerous scientific phenomena have led to debates that have arisen in the realms of engineering and physics over the past century. This has led many scientists to focus on studying and understanding these phenomena. During this time, partial differential equations (PDEs) emerged to contribute to the interpretations of these phenomena, including fluid physics, movement in optical fibers, ocean engineering, optics, and other sciences. Therefore, many researchers have developed methods, such as wave and numerical approaches, to solve these equations [1–8]. Symmetry approaches are considered the most important strategies for analyzing PDEs [9–12]. These methods convert PDEs into variant ODEs, which helps to solve them and then obtain various solutions. Lie is considered to have laid the foundation for methods for reducing PDEs. He introduced the Lie-Group method for transforming PDEs into numerous ordinary equations, and subsequently, an extensive literature emerged from Vseyannikov, Olver, Ibrahimov, and others. The Exp-function method [1] is widely recognized as one of the most significant techniques for solving ODEs and has been the focus of extensive research over the past two decades. Ebaid [10] developed this method using an arbitrary function that satisfies the Recati equation to derive multiple solutions. In this research, the DEFM [8] method is used

Check for updates

Academic Editors: Ravi Shankar, Qun Liu and Mingji Zhang

Received: 19 December 2025
Revised: 7 February 2026
Accepted: 17 February 2026
Published: 22 February 2026

 https://doi.org/10.3390/axioms15020156

to find various solutions to ODEs. This method is a modification of Ebaid and has proven effective in the previous literature [7,9].

In this research, the DEFM method is used to find various solutions to ODEs. This method is a modification of the Obeid method and has proven effective in the previous literature.

The primary goals of this study are to examine the symmetries and acquire a variety of exact solutions for the generalized BKP equation in (3+1)dimensions [13,14], which can be expressed as

$$W_{y,t}+3W_{x,x}+3W_{z,z}-3W_x\,W_{x,y}-3W_y\,W_{x,x}-W_{x,x,x,y}=0, \tag{1}$$

where $W=W(x,y,z,t)$. This equation has attracted the attention of many researchers: Li et al. [13] used the Hirota bilinear method for obtaining soliton solutions for governing equations. Ghayad et al. [14] utilized the extended tanh function to obtain many solutions for our equation. Huizhang et al. [15] used Lie symmetry and conservation laws for the governing equation.

2. Symmetry Analysis

In this section, symmetries, vector fields, and similarity reduction for Equation (1) are derived using the symmetry approach [11,12]. The Lie group assumes that infinitesimal transformations with a single parameter are provided as

$$\begin{aligned} t^*&=t+\varepsilon A(t,x,y,z,W)+o(\varepsilon^2), x^*=x+\varepsilon B(t,x,y,z,W)+o(\varepsilon^2),\\ y^*&=y+\varepsilon C(t,x,y,z,W)+o(\varepsilon^2), z^*=z+\varepsilon E(t,x,y,z,W)+o(\varepsilon^2).\\ W^*&=W+\varepsilon\phi(t,x,y,z,W)+o(\varepsilon^2)\end{aligned} \tag{2}$$

The group of transformations (1) stated above has an infinitesimal generator V that may be expressed as

$$V=A\frac{\partial}{\partial t}+B\frac{\partial}{\partial x}+C\frac{\partial}{\partial y}+E\frac{\partial}{\partial z}+\phi\frac{\partial}{\partial W}, \tag{3}$$

The condition of invariance [16–18] is given as

$$\Gamma^{(4)}(\Delta)=0, \tag{4}$$

$$\Delta=W_{y,t}+3W_{x,x}+3W_{z,z}-3W_x\,W_{x,y}-3W_y\,W_{x,x}-W_{x,x,x,y}=0, \tag{5}$$

where the independent variables for the function U are z, x, y and t. The operator V's fifth-order prolongation is represented by $\Gamma^{(4)}$

$$\begin{aligned}\Gamma^{(4)}&=V+\phi_{[x]}\frac{\partial}{\partial W_x}+\phi_{[y]}\frac{\partial}{\partial W_y}+\phi_{[xx]}\frac{\partial}{\partial W_{xx}}+\phi_{[yt]}\frac{\partial}{\partial W_{yt}}+\phi_{[zz]}\frac{\partial}{\partial W_{zz}}\\ &\quad+\phi_{[xy]}\frac{\partial}{\partial W_{xy}}+\phi_{[xx]}\frac{\partial}{\partial W_{xx}}+\phi_{[xxxy]}\frac{\partial}{\partial W_{xxxy}},\end{aligned} \tag{6}$$

where elements $\phi^{[x]}, \phi^{[y]}, \phi^{[xy]}, \phi^{[xx]}$.... may be ascertained using the following phrases:

$$\begin{aligned}\phi^{[x]}&=D_x\phi-W_tD_xA-W_xD_xB-W_yD_xC-W_zD_xE,\\ \phi^{[y]}&=D_y\phi-W_tD_yA-W_xD_yB-W_yD_yC-W_zD_yE\\ \phi^{[yt]}&=D_t\phi_{[x]}-W_{ty}D_tA-W_{xy}D_tB-W_{yy}D_tC-W_{zy}D_tE\\ \phi^{[xx]}&=D_t\phi_{[x]}-W_{tx}D_xA-W_{xx}D_xB-W_{yx}D_xC-W_zD_xE\\ \phi^{[xy]}&=D_y\phi_{[x]}-W_{tx}D_yA-W_{xx}D_yB-W_{yx}D_yC-W_{zx}D_yE\\ \phi^{[zz]}&=D_z\phi_{[x]}-W_{tz}D_zA-W_{xz}D_zB-W_{yz}D_zC-W_{zz}D_zE\\ \phi^{[xxxy]}&=D_y\phi_{[xxx]}-W_{txxx}D_yA-W_{xxxx}D_yB-W_{yxxx}D_yC-W_{zxxx}D_yE\end{aligned} \tag{7}$$

Substituting (6) into invariance condition (4) provides the following formula

$$\begin{aligned}&\phi^{[yt]}+3\phi^{[xx]}\ +3\phi^{[zz]}-3(W_{x,x}\,\phi^{[y]}+\phi^{[xx]}\,W_y\\&+W_{x,y}\,\phi^{[x]}+\phi^{[xy]}\,W_x)-\phi^{[xxxy]}=0\end{aligned}\tag{8}$$

Solving the differential equations resulting from gathering the W coefficients, we get

$$\begin{aligned}&A=c_4t+c_5,\ B=\frac{1}{3}c_4x+\gamma_1(t),\ C=\frac{-1}{6}c_3z+(2c_1-c_4)y+c_6,\\&E=c_1z+c_3t+c_2,\phi=2c_1y-\frac{1}{3}(W+2y)c_4-\frac{\gamma_1'(t)}{3}x+\gamma_2(t)z+\gamma_3(t)\end{aligned}\tag{9}$$

The vector field operator ψ from (9) is readily expressed as

$$\chi=X_1(c_1)+X_2(c_2)+X_3(c_3)+X_4(c_4)+X_5(c_5)+X_6(c_6)+X_7(\gamma_1)+X_8(\gamma_2)+X_9(\gamma_2),\tag{10}$$

where

$$\begin{aligned}&X_1=2y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}+2y\frac{\partial}{\partial W},\ \ X_2=\frac{\partial}{\partial z},X_5=\frac{\partial}{\partial t}\\&X_3=\frac{-1}{6}z\frac{\partial}{\partial y}+t\frac{\partial}{\partial z},\ X_6=\frac{\partial}{\partial y},\ \ X_9=\gamma_3(t)\frac{\partial}{\partial W}\\&X_4=t\frac{\partial}{\partial t}+\frac{1}{3}x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}-\frac{1}{3}(W+2y)\frac{\partial}{\partial W},\\&X_7\ =\gamma_1(t)\frac{\partial}{\partial x}-\gamma_1'(t)\frac{x}{3}\frac{\partial}{\partial W},\ X_8=\gamma_2(t)z\frac{\partial}{\partial W},\end{aligned}\tag{11}$$

This study used the following Lie algebraic non-equivalent possibilities derived from the commutator relations in Table 1.

(I) X_1 +X_4.
(II) X_2+X_7.
(III) $X_2+X_5+X_6$ +X_7.
(IV) $X_5+mX_6+nX_7+X_8$.
(V) $X_2+mX_5+nX_6+X_9$.

Table 1. The commutator table.

	X_1	X_2	X_3	X_4	X_5
X_1	0	$-X_2$	$-X_3+\frac{1}{3\gamma_2}X_8$	0	0
X_2	X_2	0	$\frac{-1}{6}X_2$	0	0
X_3	$X_3-\frac{1}{3}z\frac{\partial}{\partial W}$	$\frac{1}{6}X_2$	0	$-X_3+\frac{1}{3\gamma_2}X_8$	$-X_2$
X_4	0	0	$X_3-\frac{1}{3\gamma_2}X_8$	0	$-X_5$
X_5	0	0	X_2	X_5	0
X_6	$2X_6$	0	0	$-X_6-\frac{2}{3\gamma_3}X_9$	0
X_7	0	0	0	$-t\frac{\partial X_7}{\partial t}+\frac{\gamma_1'(t)x}{9\gamma_3}X_9$	X_5
X_8	0	$\frac{-\gamma_2}{\gamma_3}X_9$	0	$-t\frac{\partial X_8}{\partial t}$	$\frac{-\gamma_2'}{\gamma_2}X_8$
X_9	0	0	0	$-t\frac{\partial X_9}{\partial t}$	$\frac{-\gamma_3'}{\gamma_3}X_9$

Table 1. *Cont.*

	X_6	X_7	X_8	X_9
X_1	$-2X_6$	0	0	0
X_2	0	0	$\frac{\gamma_2}{\gamma_3}X_9$	0
X_3	0	0	0	0
X_4	$X_6+\frac{2}{3\gamma_3}X_9$	$t\frac{\partial X_7}{\partial t}-\frac{\gamma_1'(t)x}{9\gamma_3}X_9$	$t\frac{\partial X_8}{\partial t}$	$t\frac{\partial X_9}{\partial t}$
X_5	0	$\frac{\partial X_7}{\partial t}$	$\frac{\gamma_2'}{\gamma_2}X_8$	$\frac{\gamma_3'}{\gamma_3}X_9$
X_6	0	0	0	0
X_7	0	0	0	0
X_8	0	0	0	0
X_9	0	0	0	0

3. Symmetries and Exact Solution

The characteristic equation can be expressed as follows to get the invariant transformation:

$$\frac{dt}{A(t,x,y,z,W)}=\frac{dx}{B(t,x,y,z,W)}=\frac{dy}{C(t,x,y,z,W)}=\frac{dz}{E(t,x,y,z,W)}=\frac{dw}{\phi(t,x,y,z,W)}. \tag{12}$$

Case I: The invariant variables are obtained by substituting ψ_1 + ψ_4 in Equation (12).

$$X=\frac{x}{\sqrt[3]{t}},\ Y=\frac{y}{t},\ Z\ =\frac{z}{t},\ W=\frac{G(X,Y,Z)}{\sqrt[3]{t}}+y. \tag{13}$$

where $\gamma_2=t$.

The following partial differential equation (PDE) results from substituting Equation (13) into Equation (1).

$$\frac{4}{3}G_Y+\frac{1}{3}XG_{X,Y}+XG_{X,Y}++ZG_{Z,Z}-3G_{Z,Z}+3G_{X,X}G_Y+3G_{X,Y}G_X+G_{X,X,X,Y}=0. \tag{14}$$

The infinitesimal variables of Equation (14) can be expressed as

$$\begin{aligned}&\xi_X=\beta_3,\ \ \xi_Y=2Y\beta_1-\frac{1}{6}Z\beta_2\ \ \xi_Z=\beta_1 Z+\beta_2,\\&\xi_F=\frac{1}{9}\beta_3X+\beta_4Z+\beta_5,\end{aligned} \tag{15}$$

where β_i, i = 1…5 are arbitrary constants; we take $\beta_2=\beta_3=\beta_5=0$, $\beta_1,\beta_4\neq 0$. The characteristic equation is as follows

$$\frac{d\xi_X}{0}=\frac{d\xi_Y}{2Y}=\frac{d\xi_Z}{Z}=\frac{df}{Z}. \tag{16}$$

The similarity variables of Equation (14) are $\eta_1=X,\eta_2=\frac{Z}{\sqrt{Y}}$ and $G(X,Y,Z)=f(\eta_1,\eta_2)+Z$. Equation (14) is simplified to the two-dimensional PDE shown below

$$\begin{aligned}&\frac{5}{12}f_{\eta_2}-\frac{1}{6}\eta_1f_{\eta_1\eta_2}+\frac{1}{4}\eta_2f_{\eta_2\eta_2}+\frac{3}{\eta_2}f_{\eta_2\eta_2}+\frac{3}{2}f_{\eta_2}f_{\eta_1\eta_1}+\frac{1}{2}f_{\eta_2\eta_1\eta_1\eta_1}\\&-af_{\eta_1}f_{\eta_1\eta_2}-bf_{\eta_2}f_{\eta_1\eta_1}+cf_{\eta_1}f_{\eta_1\eta_2}+d\ f_{\eta_2\eta_2}f_{\eta_1}=0.\end{aligned} \tag{17}$$

The similarity variables of Equation (14) can be expressed as

$$\lambda_{\eta_1}=\alpha_1,\ \ \lambda_{\eta_2}=0,\lambda_f=\frac{1}{9}\alpha_1\eta_1+\alpha_2. \tag{18}$$

The characteristic equation can be expressed as follows:

$$\frac{d\eta_1}{1} = \frac{d\eta_2}{0} = \frac{df}{\frac{1}{9}\eta_1}. \tag{19}$$

The symmetries of Equation (17) are $\theta = \eta_2$ and $f(\theta) = w(\theta) - \frac{1}{18}\eta_1^2$. Equation (17) is converted to the ordinary differential equation (ODE) that follows

$$w(\theta) = c_1 \operatorname{arcsin} h(\frac{1}{6}\sqrt{3}\theta) + c_2. \tag{20}$$

Thus, the expression is used in the final solution of Equation (1).

$$W_1(x, y, z, t) = \frac{1}{\sqrt[3]{t}}(c_1 \operatorname{arcsin} h(\frac{1}{6}\sqrt{3}\frac{z}{\sqrt{ty}}) - \frac{1}{18}\frac{x^2}{\sqrt[3]{t^2}} + \frac{z}{t} + c_2) + y \tag{21}$$

Case II:

Substituting $\psi_2 + \psi_7$ in Equation (12) results in invariant variables.

$$X = x - \frac{1}{2}t^2,\ Y = y,\ Z\ = z,\ W = F(X, Y, Z) - \frac{1}{3}xt + \frac{1}{9}t^9. \tag{22}$$

where $\gamma_1 = t$.

We infer the following three-dimensional PDE:

$$3F_{XX} + 3F_{ZZ} - 3F_Y F_{XX} + 3F_X F_{XY} - F_{XXXY} = 0. \tag{23}$$

For obtaining the ODE of Equation (23), take $\theta = k_1X + k_2Y + k_3Z$, which in turn gives

$$(k_1^2 + k_3^2)F'' - 6k_1^2k_2F'F'' - k_1^3k_2F'''' = 0. \tag{24}$$

To find the precise solutions to Equation (24), utilize the DEFM [8] as follows

$$F(\theta) = \sum_{i=1}^{m} \frac{a_0 + a_i\Psi(\theta)}{r_0 + r_i\Psi(\theta)}, \tag{25}$$

where the following Riccati equation can be solved using $\Psi(\theta)$.

$$\Psi'(\theta) = A + B\Psi(\theta) + C\Psi^2(\theta). \tag{26}$$

Utilizing (25) in (24) and balancing between the nonlinear term $F''F'$ and the highly linear F'''' term, the solution may be expressed as

$$F(\theta) = \frac{a_0 + a_1\Psi(\theta)}{r_0 + r_1\Psi(\theta)}, \tag{27}$$

Substitute (27) into (24), and then gather the coefficients of all powers of $\Psi(\theta)$ and solve the algebraic equations for $A_0, A_1, B_0, B_1, r_0, r_1, m_1, m_2, k_1, k_2$ and k_3 by using Maple to obtain the following sets:

Set 1:

$$r_0 = \frac{2A}{B^2}(4k_3r_1CA + a_1B - k_3B^2r_1),\ r_0 = \frac{2AA_1}{B},\ k_1 = \frac{-3(k_3^2 + k_2^2)}{k_3^2(4AC - B^2)}, \tag{28}$$

r_1, m_1, m_2, m_3, k_2 and k_3 are arbitrary constants.

Set 2:

$$a_1 = \frac{-1}{r_0}(2k_3r_1^2A - a_0A - 2k_3Br_0r_1 + 2k_3Cr_0^2),\ k_1 = \frac{3(k_3^2 + k_2^2)}{k_3^2(4AC - B^2)}, \tag{29}$$

$a_0, r_0, m_1, m_2, m_3, k_2$ and k_3 are arbitrary constants.

The following is an expression for the precise travelling wave solutions of (1):
Case 1: A, B and C $\neq 0$.

$$W(t,x,y,z) = \frac{a_0 + a_1[\frac{-B}{2C} + \frac{\sqrt{4AC-B^2}}{2C}\tan(\frac{1}{2}(\sqrt{4AC-B^2}(\theta)))]}{r_0 + r_1[\frac{-B}{2C} + \frac{\sqrt{4AC-B^2}}{2C}\tan(\frac{1}{2}(\sqrt{4AC-B^2}(\theta)))]}. \tag{30}$$

where $a_1 = \frac{-1}{r_0}(2k_3r_1^2A - a_0A - 2k_3Br_0r_1 + 2k_3Cr_0^2)$.
Case 2: A $=$ 0, B and C $\neq 0$.

$$W(t,x,y,z) = \frac{a_0 + a_1\left[\frac{-B\exp(B(\theta)+Bd_0)}{C\exp(\theta)+Bd_0-1}\right]}{r_0 + r_1\left[\frac{-B\exp(B(\theta)+Bd_0)}{C\exp(\theta)+Bd_0-1}\right]}. \tag{31}$$

where $a_1 = \frac{-1}{r_0}(-2k_3Br_0r_1 + 2k_3Cr_0^2)$.
Case 3: A and B $\neq 0$, C $= 0$.

$$W(t,x,y,z) = \frac{a_0 + a_1\left[\frac{-A}{B} + \frac{1}{B}\exp(\theta)\right]}{r_0 + r_1\left[\frac{-A}{B} + \frac{1}{B}\exp(\theta)\right]}. \tag{32}$$

where $A_1 = \frac{-1}{r_0}(2k_3r_1^2A - a_0A - 2k_3Br_0r_1)$.

Case III: Substituting vectors $\psi_2 + \psi_5 + \psi_6 + \psi_7$ into Equation (1) gives the invariant variables as

$$X = x - m_1t,\ Y = y - m_2t,\ Z = z - m_3t,\ \ u = F(X,Y,Z) \tag{33}$$

where $\gamma_1 = 1$.

The governing equation can be reduced using the invariant variables from Equation (23) in Equation (1).

$$-m_1F_{XY} - m_2F_{YY} - m_3F_{YZ} + 3F_{XX} + 3F_{ZZ} - 3F_XF_{XY} - 3F_YF_{XY} + F_{XXXY} = 0. \tag{34}$$

Taking $\theta = k_1X + k_2Y + k_3Z$, Equation (34) can be written in the form

$$-k_2(m_1k_1 + m_2k_2 + m_3k_3 + 3k_1^2 + 3k_3^2)F'' + 6k_1^2k_2F'F'' + k_1^3k_2F'''' = 0 \tag{35}$$

Substitute (27) into (35), and then solve the algebraic equations to get the coefficient of all powers of $\Psi(\theta)$. Obtain the following sets using Maple:
Set 1:

$$\begin{aligned} a_1 &= \frac{-1}{r_0}(2k_3r_1^2A - a_0A - 2k_3Br_0r_1 + 2k_3Cr_0^2), \\ k_2 &= \frac{1}{m_3}(4k_1^3AC + 3k_3^2 - k_1m_1 - k_1^3B^2 + 3k_1^2 - k_3m_3), \end{aligned} \tag{36}$$

$a_0, r_0, m_1, m_2, m_3, k_1$ and k_3 are arbitrary constants.

Set 2:

$$a_0 = 2k_1r_1A,\ k_2 = \frac{1}{m_3}(4k_1^3AC + 3k_3^2 - k_1m_1 - k_1^3B^2 + 3k_1^2 - k_3m_3), \tag{37}$$

$r_0 = 0, m_1, m_2, m_3, k_1$ and k_3 are arbitrary constants.

The following is an expression for the exact travelling wave solutions of (1)

Case 1: A, B and C $\neq 0$.

$$W(t,x,y,z) = \frac{a_0 + a_1[\frac{-B}{2C} + \frac{\sqrt{4AC-B^2}}{2C}\tan(\frac{1}{2}(\sqrt{4AC-B^2}(\theta))) +]}{r_0 + r_1[\frac{-B}{2C} + \frac{\sqrt{4AC-B^2}}{2C}\tan(\frac{1}{2}(\sqrt{4AC-B^2}(\theta))) +]}. \tag{38}$$

where $a_1 = \frac{-1}{r_0}(2k_3r_1^2A - a_0A - 2k_3Br_0r_1 + 2k_3Cr_0^2)$, $k_2 = \frac{1}{m_3}(4k_1^3AC + 3k_3^2 - k_1m_1 - k_1^3B^2 + 3k_1^2 - k_3m_3)$.

Case 2: A $= 0$, B and C $\neq 0$.

$$W(t,x,y,z) = \frac{a_0 + a_1\left[\frac{-B\exp(B(\theta)+Bd_0)}{C\exp(\theta)+Bd_0-1}\right]}{r_0 + r_1\left[\frac{-B\exp(B(\theta)+Bd_0)}{C\exp(\theta)+Bd_0-1}\right]}. \tag{39}$$

where $a_1 = \frac{-1}{r_0}(-2k_3Br_0r_1 + 2k_3Cr_0^2)$, $k_2 = \frac{1}{m_3}(3k_3^2 - k_1m_1 - k_1^3B^2 + 3k_1^2 - k_3m_3)$.

Case 3: A, B and C $= 0$.

$$W(t,x,y,z) = \frac{a_0 + a_1\left[\frac{-A}{B} + \frac{1}{B}\exp(\theta)\right]}{r_0 + r_1\left[\frac{-A}{B} + \frac{1}{B}\exp(\theta)\right]}. \tag{40}$$

where $a_1 = \frac{-1}{r_0}(2k_3r_1^2A - a_0A - 2k_3Br_0r_1)$, $k_2 = \frac{1}{m_3}(3k_3^2 - k_1m_1 - k_1^3B^2 + 3k_1^2 - k_3m_3)$.

Case IV: Substituting vectors $\psi_5 + m\psi_6 + n\psi_7 + \psi_8$ into Equation (1) gives the invariant variables as

$$X = x - mt,\ Y = y - nt,\ Z = z,\ \ u = F(X,Y,Z) + z\int \gamma_2(t)dt \tag{41}$$

where $\gamma_1 = 1$.

The governing equation can therefore be simplified to use the invariant variables from Equation (41) in Equation (1).

$$-mF_{XY} - mF_{YY} + 3F_{XX} + 3F_{ZZ} - 3F_XF_{XY} - 3F_YF_{XY} + F_{XXXY} = 0. \tag{42}$$

Taking $\theta = k_1X + k_2Y + k_3Z$, Equation (42) can be written in the form

$$-k_2(mk_1 + nk_2 + 3k_1^2 + 3k_3^2)F'' + 6k_1^2k_2F'F'' + k_1^3k_2F'''' = 0 \tag{43}$$

Substitute (27) into (43), and then solve the algebraic equations to get the coefficient of all powers of $\Psi(\theta)$. Obtain the following sets using Maple:

Set 1:

$$\begin{aligned} a_1 &= \frac{-1}{r_0}(2k_1r_1^2A - a_0A - 2k_1Br_0r_1 + 2k_1Cr_0^2), \\ k_2 &= \frac{1}{n}(4k_1^3AC + 3k_1^2 - k_1m_1 - k_1^3B^2 + 3k_3^2), \\ &a_0, r_0, m, n, k_1 \text{ and } k_3 \text{ are arbitrary constants.} \end{aligned} \tag{44}$$

Set 2:

$$\begin{aligned} &a_0 = 2k_1r_1A,\ k_2 = \frac{1}{m_3}(4k_1^3AC + 3k_3^2 - k_1m_1 - k_1^3B^2 + 3k_1^2 - k_3m_3), \\ &r_0 = 0, m_1, m_2, m_3, k_1 \text{ and } k_3 \text{ are arbitrary constants.} \end{aligned} \tag{45}$$

Case 1: A $= -1$, B $= 2$, and C $= -2$.

$$W(t,x,y,z) = \frac{a_0(1-\coth(\theta)) + a_1\coth(\theta)}{r_0(1-\coth(\theta)) + r_1\coth(\theta)}. \tag{46}$$

where $a_1 = \frac{-1}{r_0}(2k_1r_1^2 - a_0 - 4k_1r_0r_1 - 4k_1r_0^2)$, $k_2 = \frac{1}{n}(-8k_1^3 + 3k_1^2 - k_1m_1 - 4k_1^3 + 3k_3^2)$.

Case 2: A $= \frac{1}{2}$, B $= 0$, and C $= \frac{1}{2}$.

$$W(t,x,y,z) = \frac{a_0 + a_1[\sec(\theta) + \tan(\theta)]}{r_0 + r_1[\sec(\theta) + \tan(\theta)]}. \tag{47}$$

where $a_1 = \frac{-1}{r_0}(k_1r_1^2 - \frac{1}{2}a_0 + k_1r_0^2)$, $k_2 = \frac{1}{n}(3k_1^2 - k_1m_1 + 3k_3^2)$.

Case 3: A $\neq 0$, B $\neq 0$, and C $= 0$.

$$W(t,x,y,z) = \frac{a_0 + a_1\left[\frac{-A}{B} + \frac{1}{B}\exp(\theta)\right]}{r_0 + r_1\left[\frac{-A}{B} + \frac{1}{B}\exp(\theta)\right]}. \tag{48}$$

where $a_1 = \frac{-1}{r_0}(2k_1r_1^2A - a_0A - 2k_1Br_0r_1)$, $k_2 = \frac{1}{n}(3k_1^2 - k_1m_1 - k_1^3B^2 + 3k_3^2)$.

Case V: Substituting vectors $\psi_2 + m\psi_5 + n\psi_6 + \psi_9$ into Equation (1) gives the invariant variables as

$$X = x,\ Y = y - mt,\ Z = z - nt,\ \ u = F(X,Y,Z) + \int \gamma_3(t)dt \tag{49}$$

Therefore, the governing equation can be simplified to Equation (1) using the invariant variables in Equation (49).

$$-mF_{YY} - mF_{YZ} + 3F_{XX} + 3F_{ZZ} - 3F_XF_{XY} - 3F_YF_{XY} + F_{XXXY} = 0. \tag{50}$$

Taking $\theta = k_1X + k_2Y + k_3Z$, Equation (50) can be written in the form

$$-k_2(mk_2 + nk_3 + 3k_1^2 + 3k_3^2)F'' + 6k_1^2k_2F'F'' + k_1^3k_2F'''' = 0 \tag{51}$$

Substitute (27) into (50), and then gather the coefficient of all powers of $\Psi(\theta)$ to solve the algebraic equations. using Maple to obtain the following sets:

Set 1:

$$\begin{aligned} a_1 &= \frac{-1}{r_0}(2k_1r_1^2A - a_0A - 2k_1Br_0r_1 + 2k_1Cr_0^2), \\ k_2 &= \frac{1}{n}(4k_1^3AC + 3k_1^2 - k_1m - k_1^3B^2 + 3k_3^2), \\ &a_0, r_0, n, k_1 \text{ and } k_3 \text{ are arbitrary constants.} \end{aligned} \tag{52}$$

Set 2:

$$\begin{aligned} a_0 &= \frac{2A}{B}(4k_1r_1CA - k_1B^2r_1 + a_1B), \\ k_2 &= \frac{1}{m_3}(4k_1^3AC + 3k_3^2 - k_1m_1 - k_1^3B^2 + 3k_1^2 - k_3m_3), \\ r_0 &= \frac{2Ar_1}{B}, n, r_1, k_1 \text{ and } k_3 \text{ are arbitrary constants.} \end{aligned} \tag{53}$$

The following is an expression for the exact travelling wave solutions of (1).

Case 1: A $= -1$, B $= 2$, and C $= -2$.

$$W(t,x,y,z) = \frac{a_0(1-\coth(\theta)) + a_1\coth(\theta)}{r_0(1-\coth(\theta)) + r_1\coth(\theta)}. \tag{54}$$

where $a_1 = \frac{-1}{r_0}(-2k_1r_1^2 + a_0A - 4k_1r_0r_1 - 4k_1r_0^2)$, $k_2 = \frac{1}{n}(8k_1^3 + 3k_1^2 - k_1m - 4k_1^3 + 3k_3^2)$.

Case 2: A, B $\neq 0$, and C $= 0$.

$$W(t,x,y,z) = \frac{a_0 + a_1\left[\frac{-A}{B} + \frac{1}{B}\exp(\theta)\right]}{r_0 + r_1\left[\frac{-A}{B} + \frac{1}{B}\exp(\theta)\right]}. \tag{55}$$

where $a_1 = \frac{2A}{B}(-k_1B^2r_1 + a_1B)$, $k_2 = \frac{1}{m_3}(3k_3^2 - k_1m_1 - k_1^3B^2 + 3k_1^2 - k_3m_3)$.

4. Discussion

This section shows a graphical representation of the solutions discovered, triggering a discussion of their physical implications. From the perspective of applications, travelling waves are highly intriguing, regardless of whether their solution formulations are explicit or implicit. Most solutions rely on the expression value $4AC - B^2$ with a suitable test for optional constants.

Figure 1: This graph shows the double-wave behavior of solution (21) with a singularity at t = 0 through the values of the arbitrary constants $c_1 = 1, c_2 = -1$ and y = 1, x = 0.

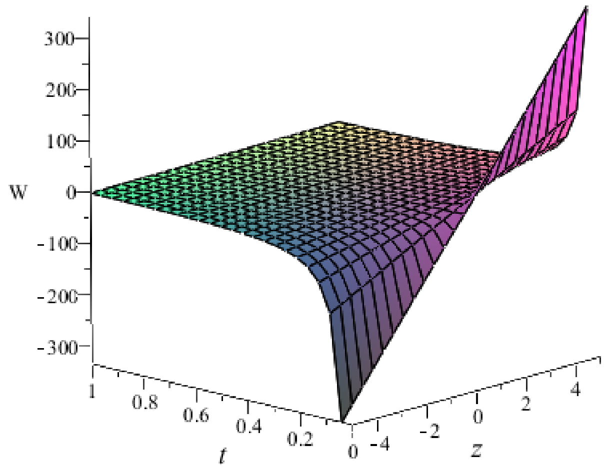

Figure 1. Shows 3D graph of double-wave solution (21).

Figure 2: The multisoliton behavior in this form is solution (30), where $4AC - B^2 > 0$ with the arbitrary constants chosen as $k_2 = -1$, $k_3 = 1$, $a_1 = 2$, $r_1 = 3$ at $y = z = 1$. These solutions are also multisolitons because their vertices do not lie on the same horizontal plane.

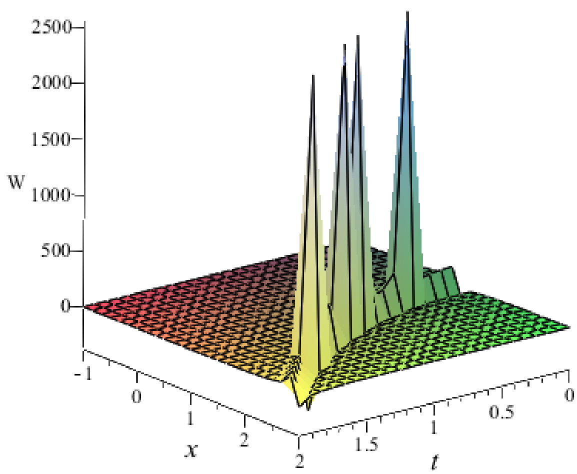

Figure 2. Shows 3D graph of multisoliton solution (30).

Figure 3 describes solution (31) considering the behavior of a kink wave, where $4AC - B^2 < 0$, with an appropriate selection of optional values: $k_2 = 1$, $k_3 = -1$, $r_0 = r_1 = 1$, $a_0 = 1$ at $y = z = 1$.

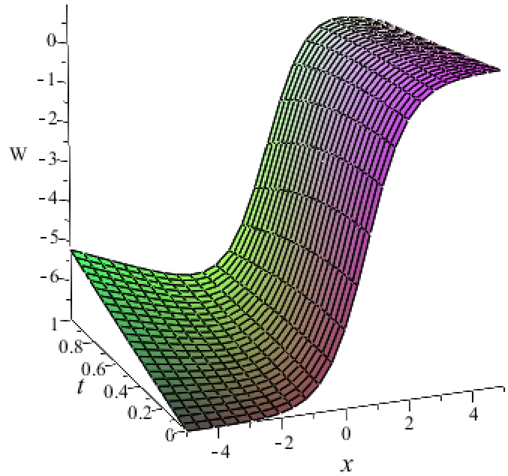

Figure 3. Shows 3D graph of kink-wave solution (32).

Figure 4: Solution (38) is illustrated by the behavior of a soliton wave when the value of $4AC - B^2 > 0$, with perfect choice of arbitrary constants: $k_1 = 1$, $k_3 = -1$, $m_1 = -1$, $m_2 = -1$, $m_3 = 4$, $a_1 = -3$, $r_1 = 4$ at $x = z = 1$.

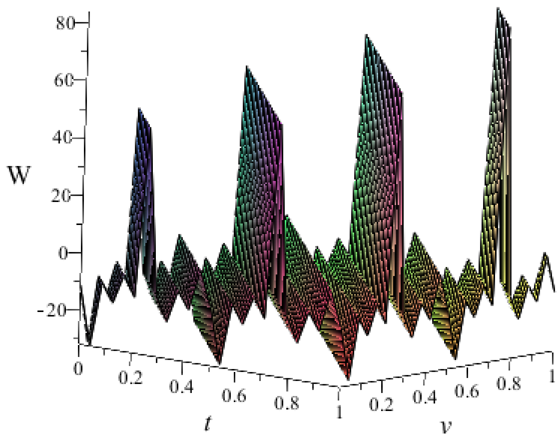

Figure 4. Shows 3D graph of soliton-wave solution (38).

Figure 5 shows the 3D shape of solution (39) considering single-wave behavior when $4AC - B^2 > 0$, with constant values of $k_1 = 1$, $k_3 = 2$, $m_1 = -1$, $m_2 = -1$, $m_3 = 1$, $a_1 = 2$, $r_1 = -3$ at $y = z = 1$.

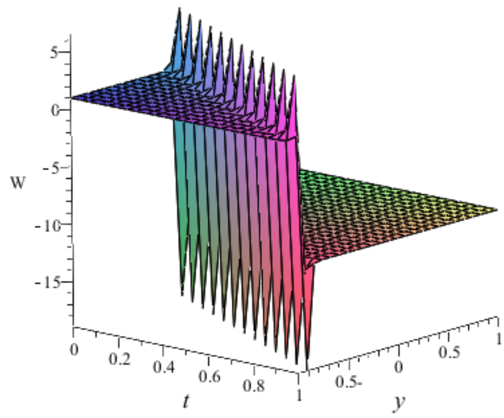

Figure 5. Shows 3D graph of anti-kink wave solution (39).

Figure 6 shows the W-solution in (46) for specific parameter selections when a single wave is produced in a 3D shape. The values of A, B, and C were chosen so that $4AC - B^2 > 0$, with the best choice of the optional values as $k_1 = 1,\ k_3 = 2,\ m = -1,\ n = -1,\ a_1 = 2,\ r_1 = -3$ at $y = z = 1$.

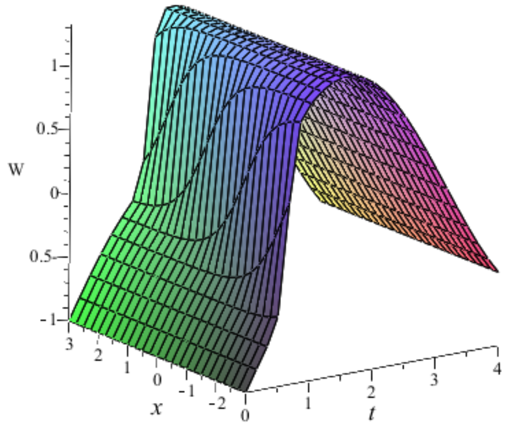

Figure 6. Shows 3D graph of single wave for solution (46).

Figure 7 presents the 3D shape of a solitary wave for solution (48), with a singularity at t = 1.5 and 4.5. The optional values were chosen appropriately to obtain $4AC - B^2 > 0$, with $k_1 = 3,\ k_2 = 1, k_3 = -1, m = -2,\ n = -1,\ a_0 = 1,\ r_1 = -3$ at $y = z = 1$.

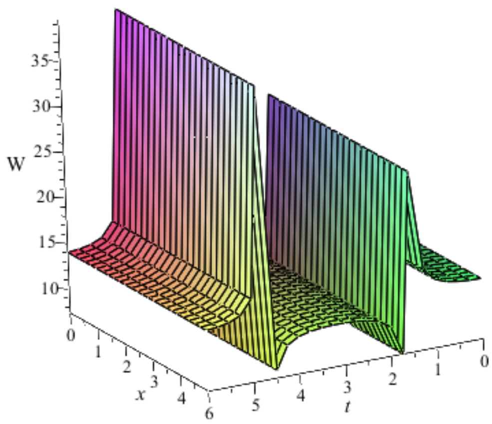

Figure 7. Shows 3D graph of solitary-wave for solution (48).

Figure 8 the exact solution (55) is presented in the 3D graph of a double wave, with $4AC - B^2 > 0, k_1 = 1, k_2 = 2, k_3 = -1, m = 2, n = -1, a_0 = 1, r_1 = -3$ at $y = z = 1$.

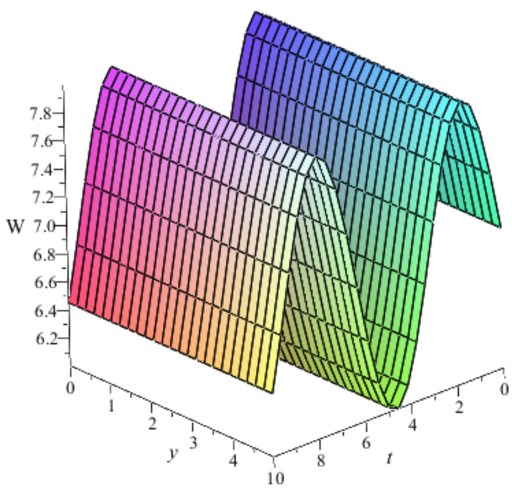

Figure 8. Shows 3D graph of double-wave for solution (55).

5. Conclusions

In this study, the (3+1)-dimensional BKP Eq. was reduced to a number of ODEs via symmetry analysis and similarity transformations. The infinitesimal generators and nine vector generators were obtained. The infinitesimal generators contained arbitrary functions, which made it difficult to use the Lie group to obtain suitable reductions for the governing equation. Linear relationships between vector generators were used to reduce the governing equation to five different PDEs. In case I, symmetry analysis was used twice to obtain an ODE; for the remaining cases, similarity transformations were used to obtain ODEs for each of them. In cases II–V, the DEFM is used to obtain variant exact solutions which have not been obtained in the previous literature. The physical behavior of the solutions was analyzed through 3D shapes. Numerous physical feature solutions are among these, such as double wave, multisoliton, kink wave, anti-kink wave, solitary wave and single wave solutions. In order to illustrate the physical structures of the acquired solutions, the structures of solutions are visually represented in three dimensions.

Comparing these results with those found in [13–15], we deduce that:

* Each of Refs. [13,14] obtained closed solutions using the Hirota bilinear and tanh method, and each represents a special case from what we have obtained.

* Yan et al. [15] deduced vectors X_1, X_2, X_3, X_5, X_6, X_7, X_8, X_9 using the Lie symmetry method, including vectors that we have not obtained, such as vectors X_1, and X_{10}. On the other hand, we obtained the vector X_4. Yan also deduced exact solutions using the tanh method for only one case, which is equivalent to our case II.

Author Contributions: Conceptualization, A.A.G., D.A. and A.-M.W.; methodology, A.A.G., D.A. and A.-M.W.; software, A.A.G., D.A. and A.-M.W.; validation, A.A.G., D.A. and A.-M.W.; formal analysis, A.A.G. and D.A.; investigation, A.A.G. and D.A.; resources, A.A.G. and D.A.; data curation, A.A.G., D.A. and A.-M.W.; writing—original draft preparation, A.A.G.; writing—review and editing, D.A.; visualization, A.A.G., D.A. and A.-M.W.; supervision, A.A.G., D.A. and A.-M.W.; project administration, A.A.G., D.A. and A.-M.W.; funding acquisition, D.A. All authors have read and agreed to the published version of the manuscript.

Funding: The author (Ahmed A. Gaber) extends the appreciation to the Deanship of Postgraduate Studies and Scientific Research at Majmaah University for funding this research work through the project number (R-2026-63). The researcher (Dalal Alhwikem) would like to thank the Deanship of Graduate Studies and Scientific Research at Qassim University for financial support (QU-APC-2026).

Data Availability Statement: No datasets were generated or analyzed during the current study. Data sharing is not applicable.

Acknowledgments: The author (Ahmed A. Gaber) extends the appreciation to the Deanship of Postgraduate Studies and Scientific Research at Majmaah University for funding this research work through the project number (R-2026-63). The researcher (Dalal Alhwikem) would like to thank the Deanship of Graduate Studies and Scientific Research at Qassim University for financial support (QU-APC-2026).

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Gurefe, Y.; Misirli, E. Exp-function method for solving nonlinear evolution equations with higher order nonlinearity. *Comput. Math. Appl.* **2011**, *61*, 2025–2030. [CrossRef]
2. Yao, Z.; Wu, F. Traveling Wave Solutions of the Extended Displacement Shallow-Water Equation. *J. Mar. Sci. Eng.* **2026**, *14*, 263. [CrossRef]
3. Turk, F. Optical Solitons, Optimal Systems and Conserved Quantities of the Schrödinger Equation with Spatio-Temporal and Inter-Modal Dispersions. *Fractal Fract.* **2026**, *10*, 112. [CrossRef]
4. Őzkan, Y.S.; Yasar, E. On the exact solutions of nonlinear evolution equations by the improved tan($\phi/2$)-expansion method. *Pramana J. Phys.* **2020**, *94*, 37. [CrossRef]

5. Miah, M.M.; Ali, H.M.S.; Akbar, M.A.; Wazwaz, A. Some applications of the $(G'/G, 1/G)$-expansion method to find new exact solutions of NLEEs. *The Eur. Phys. J. Plus* **2017**, *132*, 252. [CrossRef]
6. Adnan, M.; Ahmed, N.; Rani, M.; Mohsin, B.B. Generalized kudryashov and extended auxiliary equation methods for novel solitons solutions to (1+1)-dimensional doubly dispersive equation of murnaghan's rod. *Phys. Scr.* **2024**, *99*, 125237. [CrossRef]
7. Bekir, A. Integrability, Similarity Reductions and New Classes of Exact Solutions for (3+1)-D Potential Yu–Toda–Sasa–Fukuyama Equation. *Qual. Theory Dyn. Syst.* **2024**, *23*, 235. [CrossRef]
8. Gaber, A.A.; Shehata, M.H. New approach of MHD boundary layer flow towards a porous stretching sheet via symmetry analysis and the generalized Exp-Function method. *Int. J. Anal. Appl.* **2020**, *5*, 738–747.
9. Moussa, M.H.M.; Gaber, A.A. Symmetry analysis and Solitary wave solutions of nonlinear ion-acoustic waves equation. *Int. J. Anal. Appl.* **2020**, *18*, 448–460. [CrossRef]
10. Ebaid, A. Generalization of He's Exp-Function Method and New Exact Solutions for Burgers Equation. *Z. Naturforschung A* **2009**, *64*, 604–608. [CrossRef]
11. Ovsiannikov, L.V. *Group Analysis of Differential Equations*; Academic Press: New York, NY, USA, 1982.
12. Olver, P.J. *Application of Lie Groups to Differential Equations*; Springer: Berlin/Heidelberg, Germany, 1986.
13. Li, W.; Zhang, Y.; Liu, Y. Exact wave solutions for a (3+1)-dimensional generalized B-type Kadomtsev–Petviashvili equation. *Comput. Math. Appl.* **2019**, *77*, 3087–3101. [CrossRef]
14. Ghayad, M.S.; Ahmed, H.M.; Badra, N.M.; Rezazadeh, H.; Hosseinzadeh, M.A.; Rabie, W.B. New analytical wave structures for generalized B-type Kadomtsev–Petviashvili equation by improved modified extended tanh function method. *Phys. Scr.* **2024**, *99*, 125224. [CrossRef]
15. Yang, H.; Liu, W.; Zhao, Y. Lie Symmetry Analysis, Traveling Wave Solutions, and Conservation Laws to the (3 + 1)-Dimensional Generalized B-Type Kadomtsev-Petviashvili Equation. *Complexity* **2020** , *2020*, 3465860. [CrossRef]
16. Gaber, A.A.; Younis, T.M.; Alharbi, M.F. Symmetries and Novel Exact Solutions for (2+1)-D QZK equation via Lie-Symmetry and Kudryashov-Auxaliry Method. *Eur. J. Pure Appl. Math.* **2025**, *18*, 6030. [CrossRef]
17. Gaber, A.A.; Alsharari, F.; Kumar, S. Some closed-form solutions, conservation laws, and various solitary waves to the (2+1)-D potential B-K equation via Lie symmetry approach. *Int. J. Mod. Phys. B* **2022**, *36*, 2250117. [CrossRef]
18. Kumar, S. Integrability and Dynamic Solutions of the Novel Unified Kairat-II-X Equation via Symmetry Methods. *Qual. Theory Dyn. Syst.* **2026**, *25*, 5. [CrossRef]