

Article

Data-Driven Safe Controller Design for LPV Systems with Constant Input Matrix Under Polyhedral and Ellipsoidal Constraints

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Abstract

This paper proposes a data-driven control design framework for linear parameter-varying (LPV) systems with polyhedral and ellipsoidal constraints, where the input matrix of the system is constant. In contrast to traditional model-based approaches, the proposed method ensures closed-loop stability and safety invariance solely through measured data, eliminating the need for explicit system identification. For polyhedral constraints, a sufficient condition is given to ensure the λ -contractivity of perturbed LPV systems. Under ellipsoidal constraints, input limitations are incorporated through a linearization approach, which allows the λ -contractivity property of LPV systems to be verified using linear matrix inequalities (LMIs). Numerical examples demonstrate the effectiveness of the proposed method.

Keywords: data-driven control; safe control; linear parameter-varying systems; contractive sets

MSC: 34A05; 37A55; 49N05; 81P50

1. Introduction

In modern control systems, ensuring stable operation under safety constraints is paramount, particularly in safety-critical domains such as robotics, autonomous vehicles, aerospace systems, and industrial process control. These systems must meet stringent safety specifications in complex work environments, including collision avoidance, operational envelope preservation, and input amplitude constraints [1–4].

In the field of formal synthesis of safety controllers, safe control design methods typically depend on reachability analysis [1–3], control barrier functions (CBFs) [4,5], and model predictive control techniques [6,7]. These control design methods require a clear system model as a prerequisite for designing the corresponding controller to ensure stability and safety. However, in practical applications, accurate modeling poses significant challenges [8]. Systems may contain unmodeled dynamics, parametric uncertainties, or external disturbances, which can significantly degrade the performance of model-based approaches [8,9]. Inaccurate models usually result in the loss of invariance properties and violation of constraints [10]. To address this issue, recent studies have introduced



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data-driven approaches (e.g., Refs. [10–13]), which derive control laws directly from input–output data without relying on prior knowledge of the system model.

In modern control theory, data-driven control strategies have emerged as a significant research direction. These approaches construct control laws directly from experimental system data, eliminating the conventional model identification phase. Early works such as [14] considered the problem of designing robust state feedback controllers for discrete-time linear time-invariant systems, while [15] presented three data-driven approaches based on the notion of control as interconnection. Subsequent research extended these foundations to more complex scenarios: Ref. [16] proposed a data-driven LMI control method to design a controller with bounded trajectories, optimal performance, and noise robustness for discrete time-varying systems with unknown dynamics. Ref. [17] developed hierarchical LPV control, while [18] advanced data-driven control by developing a data-based state feedback design approach for linear discrete-time systems with uncertain delays, and [19] considered the design of a data-based feedback controller for switched discrete-time linear systems. The parallel developments in nonlinear systems [20,21] have substantially expanded the applicability of this methodology, enabling it to address a broad range of complex dynamic systems encountered in real-world scenarios. Uncertain dynamics are prevalent in nonlinear control systems, and neural networks serve as a conventional method for addressing uncertainties in control systems. This study investigates the application of deep neural network-based output-feedback intermittent control to non-strict-feedback nonlinear systems and successfully resolves the finite-time stabilization problem for such systems under uncertain dynamic conditions [22].

Ensuring that data-driven controllers meet safety constraints while effectively handling external disturbances and input constraints without relying on system models remains a critical and unresolved challenge in the field of control theory. For linear time-invariant (LTI) systems, refs. [23,24] proposed an innovative data-driven framework that directly utilizes input–output data to design a gain-scheduled controller. This controller strictly guarantees that the system state remains within a polyhedral safe set while satisfying given convex polyhedral constraints. However, the main results are presented under the nominal assumption of noise-free input and state data, with only preliminary results provided for scenarios involving noisy data. To more flexibly characterize and address the nonlinear dynamics widely encountered in practical engineering applications, researchers often adopt the linear parameter-varying (LPV) system as a modeling framework [25–28]. By incorporating time-varying scheduling parameters to encapsulate nonlinear characteristics, LPV systems combine the flexibility of model representation with the convenience of controller design based on linear theory, thereby serving as an effective bridge between nonlinear system analysis and synthesis. In this direction, ref. [29] represents a significant extension. This work proposes a solution to the data-based safe scheduling control problem. For nonlinear systems represented in LPV form, the method can identify a safe controller relying solely on measured data. The research extends data-driven safe control methods from simple polyhedral safe sets to more general scenarios involving closed convex polyhedral and ellipsoidal sets. Related work on data-driven control methodologies and the treatment of polyhedral and ellipsoidal constraints can be found in [30–32]. This advancement not only enhances the method’s ability to describe safety constraints, enabling it to accommodate more complex real-world safety specifications, but also creatively addresses the controller design challenge when collected data samples fail to meet the classical “persistent excitation” condition. This breakthrough holds substantial practical significance, as satisfying strict persistent excitation conditions in real-world data collection is often costly or difficult to achieve. By introducing relaxation techniques or new data sufficiency conditions, ref. [29] ensures that a controller meeting safety requirements can still be synthesized with limited

or even “non-ideal” data. This significantly improves the practicality of data-driven safe control methods and opens new avenues for addressing the safe control of complex systems with unknown models, external disturbances, and limited data quality. Reference [33] proposes a high-confidence data-driven safe tracking control method for stochastic linear discrete-time systems, which achieves high-probability safe and stable tracking under limited data conditions by constructing a data-driven controller based on probabilistic λ -contractivity and a dynamic reference governor.

This paper investigates the problem of designing a corresponding safety controller based on limited input-state data to ensure that the controlled LPV system satisfies the specified safety constraints, namely, polyhedral constraints and ellipsoidal set constraints. Building upon the foundational work presented in [23,29], the main contributions of this study are as follows:

- (1) For LPV systems with exogenous disturbances, we investigate the design of safety-guaranteed gain controllers under polyhedral safe set constraints;
- (2) For disturbance-free LPV systems with bounded control inputs, we address the constrained controller design problem, ensuring the invariance of ellipsoidal safe sets.

The structure of this paper is organized as follows: Section 2 presents the preliminary knowledge of this paper. Section 3 introduces the two research problems (i.e., Problem 1 and Problem 2). Data-driven solutions to Problem 1 and Problem 2 are provided in Sections 3 and 4, respectively. Section 5 demonstrates numerical examples, and Section 6 concludes the paper.

Notations: The following symbols are used in this paper. Let $\mathbb{R}^n$ be the n -dimensional Euclidean space and $\mathbb{R}^{n \times m}$ represent the set of $n \times m$ real matrices. The all-ones matrix with appropriate dimension is denoted by $\mathbf{1}$. The m -dimensional identity matrix is I_m , with e_i representing its i -th column. The transpose of the matrix A is written as $A^\top$. For a symmetric matrix A , $A \succeq 0$ ($\succ 0$) indicates that the matrix A is a semi-positive definite matrix (positive definite matrix). $A \otimes B$ denotes the Kronecker product between A and B . For matrices C and D with equal column dimensions, $C \odot D$ represents their Khatri–Rao product. Given a set $\mathcal{M}$ and $\lambda \geq 0$, define $\lambda\mathcal{M} := \{\lambda x \mid x \in \mathcal{M}\}$. For a polyhedron $\mathcal{S}$, $\text{vert}(\mathcal{S})$ represents the vertices of the polyhedron $\mathcal{S}$.

2. Preliminaries

As this article is a brief paper, we will not elaborate on existing results and methods in this section. Instead, we will merely list the definitions and lemmas utilized in the proof of the main theorems presented in this paper. For a more comprehensive introduction to the conclusions and methods, please refer to the relevant literature [34].

Definition 1 (C-set [34]). A C-set is a convex and compact subset of $\mathbb{R}^n$ including the origin as an interior point.

Definition 2 (Minkowski function [34]). Given a C-set $\mathcal{S}$, its Minkowski function is as follows:

$$\Psi_{\mathcal{S}}(x) = \inf\{\lambda \geq 0 : x \in \lambda\mathcal{S}\}. \quad (1)$$

Definition 3 (Contractive set [34]). The C-set $\mathcal{S}$ is contractive for the system

$$x(t+1) = f(x(t), u(t), w(t)),$$

where $w \in \mathcal{W}$, $u \in \mathcal{U}$, $\mathcal{W}$ is the disturbance set, and $\mathcal{U}$ is the admissible control input set, if and only if there exists a control function $u(x) \in \mathcal{U}$ such that, for every $x \in \mathcal{S}$, the following condition holds:

$$\Psi_{\mathcal{S}}(f(x, u, w)) \leq \lambda \Psi_{\mathcal{S}}(x), \quad \forall w \in \mathcal{W},$$

where $\Psi_{\mathcal{S}}(x)$ is the Minkowski function of the set $\mathcal{S}$, for some $0 \leq \lambda < 1$. In this case, the set $\mathcal{S}$ is said to be λ -contractive.

Definition 4 (Polyhedral C-set [35]). A polyhedral C-set is a set of the form

$$\mathcal{S}(H, h) = \{x \in \mathbb{R}^n : Hx \leq h\} = \{x \in \mathbb{R}^n : H_i x \leq h_i, i = 1, 2, \dots, l\}, \quad (2)$$

where H_i denotes the i -th row of the $l \times n$ matrix H and $h_i > 0$ denotes the i -th component of the $l \times 1$ vector h .

Remark 1. A polyhedral set includes the origin if and only if $h \geq 0$ (i.e., $h_i \geq 0, \forall i = 1, 2, \dots, l$) and includes the origin as an interior point if and only if $h > 0$ (i.e., $h_i > 0, \forall i = 1, 2, \dots, l$).

Definition 5 (Ellipsoidal set [34]). Given a vector x_0 , named the center, and a positive definite matrix P , an ellipsoid is a set of the form

$$\mathcal{E}(x_0, P, 1) = \{x \in \mathbb{R}^n : \sqrt{(x - x_0)^T P (x - x_0)} \leq 1\}. \quad (3)$$

If $x_0 = 0$, the above-mentioned ellipsoidal set in (3) becomes the ellipsoidal C-set $\mathcal{E}(P, 1) = \{x \in \mathbb{R}^n : \sqrt{x^T P x} \leq 1\}$, which represents the set of ellipsoids with their center at the origin.

Lemma 1 (Schur complement [34]). Given the symmetric matrix $S = \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix}$, where

$S_1 \in \mathbb{R}^{n \times n}$, the following three conditions are equivalent:

- (i) $S < 0$;
- (ii) $S_1 < 0, S_4 - S_3 S_1^{-1} S_2 < 0$;
- (iii) $S_4 < 0, S_1 - S_2 S_4^{-1} S_3 < 0$.

3. Problem Formulation

The main focus of this paper is on LPV systems expressed in the form given below:

$$x(k+1) = A(w(k))x(k) + Bu(k) + d(k), \quad (4)$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, $w \in \mathbb{R}^s$, and $d(k) \in \mathbb{R}^n$ are the state, control input, scheduling parameter, and perturbation vector at time k , respectively. The matrix $B \in \mathbb{R}^{n \times m}$ is the input dynamic and is assumed to be fixed, the function $A(w(k))$ is linearly dependent on the parameter w_k , and the expression lies in the following polytope:

$$A(w(k)) = \sum_{i=1}^s w_i A_i,$$

where the vertices of $A(w(k))$ are represented by the matrices $A_i \in \mathbb{R}^{n \times n}$, $i = 1, 2, \dots, s$, and the scheduling parameter w is located on the following polytope Ω :

$$\Omega = \left\{ w \in \mathbb{R}^s : w_i \geq 0, \sum_{i=1}^s w_i = 1 \right\}. \quad (5)$$

The state $x(k)$ is confined in the set $\mathcal{X}$, where the set $\mathcal{X}$ is a ellipsoidal C-set or polyhedral C-set; the input $u(k)$ is constrained in the set $\mathcal{U}$, where the set $\mathcal{U}$ is a polyhedral C-set.

$$\mathcal{U} = \{u \in \mathbb{R}^m : Uu \leq \mathbf{1}\}. \quad (6)$$

The perturbation term $d(k)$ is defined in the convex combination of the vertices d^k ($k = 1, 2, \dots, n_d$) of the polyhedral C-set $\mathcal{D}$, which is defined as follows:

$$\mathcal{D} = \left\{ \sum_{k=1}^{n_d} \alpha_k d^k : \mathbf{1}^\top \alpha = 1, \alpha \geq 0 \right\}. \quad (7)$$

The gain-scheduling controller is commonly formulated in the following form:

$$u(k) = K(w(k))x(k), \quad (8)$$

where $K(w(k))$ is defined as a polytope, which takes the following specific form:

$$K(w(k)) = \sum_{i=1}^s w_i K_i, \quad (9)$$

and the matrices $K_i \in \mathbb{R}^{m \times n}$ ($i = 1, 2, \dots, s$) are the vertices of the polytope $K(w(k))$.

We assume that the T sample datasets generated from LPV system (4) are available, and the collected data are arranged in the following matrices:

$$\begin{aligned} X_{1,T} &= [x(1) \ x(2) \ \cdots \ x(T)] \in \mathbb{R}^{n \times T}, \\ X_{0,T} &= [x(0) \ x(1) \ \cdots \ x(T-1)] \in \mathbb{R}^{n \times T}, \\ U_{0,T} &= [u(0) \ u(1) \ \cdots \ u(T-1)] \in \mathbb{R}^{m \times T}, \\ W_{0,T} &= [w(0) \ w(1) \ \cdots \ w(T-1)] \in \mathbb{R}^{s \times T}, \\ \mathcal{D}_{0,T} &= [d(0) \ d(1) \ \cdots \ d(T-1)] \in \mathbb{R}^{n \times T}, \\ X_W &= W_{0,T} \odot X_{0,T} \in \mathbb{R}^{ns \times T}. \end{aligned} \quad (10)$$

Assumption 1. Assume that the number of operating modes s is known.

Assumption 2. The matrix X_W has full row rank, i.e., $\text{rank}(X_W) = ns$ and $T \geq ns + 1$.

Remark 2. Existing data-driven control methodologies typically require strict persistence of excitation conditions on input-output data for controller synthesis, as demonstrated in [36–39]. However, recent developments have proposed relaxed requirements for data quality. Specifically, ref. [29] establishes a novel rank condition requiring only that the matrix has full row rank, enabling direct controller design from experimental data without system identification. Importantly, the proposed data-driven framework demonstrates the ability to synthesize safety-critical controllers directly from datasets that fail to satisfy persistent excitation conditions.

There are two problems of interest in this paper:

Problem 1:

Given a polyhedral C-set $\mathcal{S}(H, h)$ as in (2), we seek to synthesize a parameter-dependent state feedback control law $u(k) = K(w(k))x(k)$, ensuring that $\mathcal{S}$ remains λ -contractive for the LPV system (4) with bounded disturbances.

Problem 2:

For a given ellipsoid C-set $\mathcal{E}(P, 1)$ in (3), and a polyhedral convex set $\mathcal{U}$ as in (6), find the controller $u(k) = K(w(k))x(k)$ such that $\mathcal{E}(P, 1)$ is λ -contractive in the LPV system (4) without disturbance vectors $d(k)$, and the control input $u(k)$ is included in $\mathcal{U}$.

4. Solve Problem 1 with Data-Based Approach

The LPV system (4) is sorted out to obtain the following equation:

$$\begin{aligned} x(k+1) &= A(w(k))x(k) + Bu(k) + d(k) \\ &= \sum_{i=1}^s w_i A_i x(k) + Bu(k) + d(k) \\ &= A_{1,s}(w(k) \otimes x(k)) + Bu(k) + d(k), \end{aligned} \quad (11)$$

where $A_{1,s} = [A_1 \ A_2 \ \cdots \ A_s]$.

Theorem 1 (Model-based approach [34]). *The polyhedral $\mathcal{S} = \{x \in \mathbb{R}^n : Hx \leq h\}$ is λ -contractive on the above system (4) if and only if there exist matrices $K_i \in \mathbb{R}^{m \times n}$ ($i = 1, 2, \dots, s$) such that the following holds:*

$$H((A_i + BK_i)s + \tilde{d}) \leq \lambda h, \quad \forall s \in \text{vert}(\mathcal{S}), \quad \forall \tilde{d} \in \text{vert}(\mathcal{D}), \quad i = 1, 2, \dots, s.$$

Theorem 2 (Data-based approach). *Under the condition that both Assumption 1 and Assumption 2 are satisfied, the polyhedral $\mathcal{S} = \{x \in \mathbb{R}^n : Hx \leq h\}$ is λ -contractive with respect to system (4) if there exists a matrix $G_{1,s} \in \mathbb{R}^{T \times ns}$ such that*

$$\begin{aligned} H([(X_{1,T} - \delta_{jk})G_{1,s}D_i]s + \tilde{d}) &\leq \lambda h, \\ \forall s \in \text{vert}(\mathcal{S}), \quad \forall \tilde{d} \in \text{vert}(\mathcal{D}), \quad i &= 1, 2, \dots, s, \quad j = 1, 2, \dots, T, \quad k = 1, 2, \dots, n_d. \end{aligned}$$

$$\text{where } \delta_{jk} = \begin{bmatrix} \underbrace{0}_{1\text{-st}} & \cdots & \underbrace{Td^{(k)}}_{j\text{-th}} & \cdots & \underbrace{0}_{T\text{-th column}} \end{bmatrix} \in \mathbb{R}^{n \times T}, \quad D_i = \begin{bmatrix} 0_n & 0_n & \cdots & \underbrace{I_n}_{i\text{-th } n \times n \text{ matrix}} \\ 0_n & \cdots & 0_n \end{bmatrix}^T \in \mathbb{R}^{ns \times n}.$$

Proof. If X_W is full row rank, then there exists a right inverse matrix $G_{1,s}$ such that the following equality holds:

$$X_W G_{1,s} = I, \quad (12)$$

and since $T \geq ns + 1$, this right inverse matrix is not unique. $\square$

Based on the system in (4), the collected data samples in (10) satisfy the following equality:

$$X_{1,T} = A_{1,s}X_W + BU_{0,T} + \mathcal{D}_{0,T}, \quad (13)$$

and multiplying both sides of Equation (13) by $G_{1,s}$ yields

$$\begin{aligned} X_{1,T}G_{1,s} &= A_{1,s}X_W G_{1,s} + BU_{0,T}G_{1,s} + \mathcal{D}_{0,T}G_{1,s} \\ &= A_{1,s} + BK_{1,s} + \mathcal{D}_{0,T}G_{1,s}. \end{aligned} \quad (14)$$

Let $K_{1,s} = [K_1, K_2, \dots, K_s]$ with $K_i \in \mathbb{R}^{m \times n}$ and $G_{1,s}$ satisfy the following equation

$$\begin{pmatrix} K_{1,s} \\ I \end{pmatrix} = \begin{pmatrix} U_{0,T} \\ X_W \end{pmatrix} G_{1,s}. \quad (15)$$

Rearranging Equation (14), the following can be obtained:

$$(X_{1,T} - \mathcal{D}_{0,T})G_{1,s} = A_{1,s} + BK_{1,s}, \quad (16)$$

then

$$\begin{aligned} A_i + BK_i &= (A_{1,s} + BK_{1,s})D_i \\ &= (X_{1,T} - \mathcal{D}_{0,T})G_{1,s}D_i. \end{aligned} \quad (17)$$

Therefore, substituting $A_i + BK_i$ by $(X_{1,T} - \mathcal{D}_{0,T})G_{1,s}D_i$ in (17), the inequality $H((A_i + BK_i)s + \tilde{d}) \leq \lambda h$ becomes $H([(X_{1,T} - \mathcal{D}_{0,T})G_{1,s}D_i]s + \tilde{d}) \leq \lambda h$. We can derive the following data-based condition corresponding to Theorem 1, i.e., if Assumption 1 and Assumption 2 hold, then the polyhedral C-set $\mathcal{S} = \{x \in \mathbb{R}^n : Hx \leq h\}$ is λ -contractive with respect to system (4) if and only if there exists a matrix $G_{1,s}$ such that the following condition is satisfied:

$$H([(X_{1,T} - \mathcal{D}_{0,T})G_{1,s}D_i]s + \tilde{d}) \leq \lambda h, \quad \forall s \in \text{vert}(\mathcal{S}), \quad \forall \tilde{d} \in \text{vert}(\mathcal{D}), \quad i = 1, 2, \dots, s. \quad (18)$$

Since the disturbance sequence $\mathcal{D}_{0,T}$ is unknown, a conservative way to address this issue is to impose the formula $H([(X_{1,T} - \mathcal{D}_{0,T})G_{1,s}D_i]s + \tilde{d}) \leq \lambda h$, which must hold for all possible disturbance sequences $d(0), d(1), \dots, d(T-1)$ as long as each $d(0), d(1), \dots, d(T-1)$ belongs to the set $\mathcal{D}_{0,T}$; thus, we adopt the methodology proposed in [23] to address this issue by replacing $\mathcal{D}_{0,T}$ with δ_{jk} , where δ_{jk} is defined as follows:

$$\delta_{jk} = \left[\underbrace{0}_{1\text{-st}} \mid \dots \mid \underbrace{Td^{(k)}}_{j\text{-th}} \mid \dots \mid \underbrace{0}_{T\text{-th column}} \right], \quad j = 1, 2, \dots, T, \quad k = 1, 2, \dots, n_d. \quad (19)$$

Because $\mathcal{D}_{0,T}$ belongs to the polyhedral C-set $\mathcal{D}$, $\mathcal{D}_{0,T}$ can be rewritten as follows:

$$\mathcal{D}_{0,T} = \left[\sum_{k=1}^{n_d} \alpha_{1,k} d^{(k)} \mid \dots \mid \sum_{k=1}^{n_d} \alpha_{j,k} d^{(k)} \mid \dots \mid \sum_{k=1}^{n_d} \alpha_{T,k} d^{(k)} \right], \quad j = 1, 2, \dots, T, \quad k = 1, 2, \dots, n_d, \quad (20)$$

where the vectors $\alpha_1, \alpha_2, \dots, \alpha_T$ satisfy $\mathbf{1}^\top \alpha_1 = 1$ and $\alpha_1 \geq 0$, $\mathbf{1}^\top \alpha_2 = 1$ and $\alpha_2 \geq 0, \dots, \mathbf{1}^\top \alpha_T = 1$ and $\alpha_T \geq 0$.

Consider the following Inequality (21):

$$H([(X_{1,T} - \delta_{jk})G_{1,s}D_i]s + \tilde{d}) \leq \lambda h. \quad (21)$$

For such s and $\tilde{d}$ in (21), the following condition is satisfied for all j and k :

$$\frac{\alpha_{j,k}}{T} HX_{1,T}G_{1,s}D_i s - \frac{\alpha_{j,k}}{T} H\delta_{jk}G_{1,s}D_i s + \frac{\alpha_{j,k}}{T} H\tilde{d} \leq \frac{\alpha_{j,k}}{T} \lambda h, \quad (22)$$

and summing over all k yields

$$\frac{1}{T} HX_{1,T}G_{1,s}D_i s - \frac{1}{T} H \left[0 \mid \dots \mid \sum_{k=1}^{n_d} \alpha_{j,k} Td^{(k)} \mid \dots \mid 0 \right] G_{1,s}D_i s + \frac{1}{T} H\tilde{d} \leq \frac{1}{T} \lambda h. \quad (23)$$

Taking the sum over all j , we obtain the following inequality (24):

$$HX_{1,T}G_{1,s}D_i s - H \left[\sum_{k=1}^{n_d} \alpha_{1,k} d^{(k)} \mid \dots \mid \sum_{k=1}^{n_d} \alpha_{j,k} d^{(k)} \mid \dots \mid \sum_{k=1}^{n_d} \alpha_{T,k} d^{(k)} \right] G_{1,s}D_i s + H\tilde{d} \leq \lambda h. \quad (24)$$

Through further rearrangement, we obtain the following:

$$HX_{1,T}G_{1,s}D_i s - H\mathcal{D}_{0,T}G_{1,s}D_i s + H\tilde{d} = H([(X_{1,T} - \mathcal{D}_{0,T})G_{1,s}D_i]s + \tilde{d}) \leq \lambda h. \quad (25)$$

Hence, Inequality (21) implies Inequality (18). Consequently, it follows that (21) is a sufficient condition for (18).

Then, if there exists the matrix $G_{1,s}$ such that $H([(X_{1,T} - \delta_{jk})G_{1,s}]s + \tilde{d}) \leq \lambda h, \forall s \in \text{vert}(\mathcal{S}), \forall \tilde{d} \in \text{vert}(\mathcal{D})$, the polyhedral C-set $\mathcal{S} = \{x \in \mathbb{R}^n : Hx \leq h\}$ on the above system (4) is therefore λ -contractive. Thus, the proof is completed.

The following Theorem 3 provides the solution to Problem 1 under control input constraints:

Theorem 3. Consider the generated data in (10) under Assumptions 1 and 2, with $u \in \mathcal{U}$. Then, Problem 1 is solved if there exists a matrix $G_{1,s} \in \mathbb{R}^{T \times ns}$ such that

$$H([(X_{1,T} - \delta_{jk})G_{1,s}D_i]s + \tilde{d}) \leq \lambda h, \quad (26a)$$

$$\forall s \in \text{vert}(\mathcal{S}), \forall \tilde{d} \in \text{vert}(\mathcal{D}), i = 1, 2, \dots, s, j = 1, 2, \dots, T, k = 1, 2, \dots, n_d, \\ UU_0G_{1,s}D_i \leq \mathbf{1}, \forall s \in \text{vert}(\mathcal{S}), i = 1, 2, \dots, s \quad (26b)$$

Then, we get $K_{1,s} = U_0G_{1,s}$ and the control gains that solve the problem are obtained as $K_i = K_{1,s}D_i = U_0G_{1,s}D_i$.

Proof. The proof of Theorem 3 is similar to that of Theorem 2 and is omitted here for the sake of brevity. $\square$

5. Solve Problem 2 with Data-Based Approach

In the next subsection, consider the system (4) in the absence of the external disturbance $d(k)$; that is,

$$x(k+1) = A(w(k))x(k) + Bu(k). \quad (27)$$

The data samples obtained from the LPV system (27) are given below:

$$\begin{aligned} X_1 &= [x(1) \ x(2) \ \dots \ x(T)] \in \mathbb{R}^{n \times T}, \\ X_0 &= [x(0) \ x(1) \ \dots \ x(T-1)] \in \mathbb{R}^{n \times T}, \\ U_0 &= [u(0) \ u(1) \ \dots \ u(T-1)] \in \mathbb{R}^{m \times T}, \\ W_0 &= [w(0) \ w(1) \ \dots \ w(T-1)] \in \mathbb{R}^{s \times T}, \\ X_W &= W_0 \odot X_0 \in \mathbb{R}^{ns \times T}. \end{aligned} \quad (28)$$

The control law design ensures that the ellipsoid (3) satisfies λ -contractivity conditions under the system (27), i.e., $x(k) \in \mathcal{E} \Rightarrow x(k+1) \in \lambda\mathcal{E}$.

Case 1: When $x_0 = 0$, the ellipsoid becomes $\mathcal{E}(P, 1) = \{x \in \mathbb{R}^n : \sqrt{x^\top Px} \leq 1\}$, which is clearly an ellipsoidal C-set. As established in the work of Modares et al., the necessary and sufficient condition for λ -contractivity of an ellipsoidal C-set in system (27) with state-feedback controller (8) is given by

$$(A_i + BK_i)^\top P(A_i + BK_i) \preceq \lambda^2 P, \quad i = 1, 2, \dots, s. \quad (29)$$

By applying the Schur complement, we can equivalently derive the following expression:

$$\begin{bmatrix} \lambda^2 P & A_i + BK_i \\ (A_i + BK_i)^\top & P^{-1} \end{bmatrix} \succeq 0. \quad (30)$$

Furthermore, the above LMI (30) can be rewritten as follows:

$$\begin{bmatrix} \lambda^2 P & (A_{1,s} + BK_{1,s})D_i \\ D_i^\top (A_{1,s} + BK_{1,s})^\top & P^{-1} \end{bmatrix} \succeq 0. \quad (31)$$

Case 2: If $x_0 \neq 0$, it must be a fixed point for all vertex systems $x(k+1) = (A_i + BK_i)x(k)$, i.e., $(A_i + BK_i)x_0 = x_0$, $i = 1, 2, \dots, s$.

Given the equilibrium condition $(A_i + BK_i)x_0 = x_0$, λ -contractivity of $\mathcal{E}(x_0, P, 1)$ in system (27) is equivalent to

$$((A_i + BK_i)x(k) - x_0)^\top P((A_i + BK_i)x(k) - x_0) \leq \lambda^2(x(k) - x_0)^\top P(x(k) - x_0), \quad (32)$$

and substituting the equality $(A_i + BK_i)x_0 = x_0$ in (32), we obtain the following inequality:

$$(x(k) - x_0)^\top (A_i + BK_i)^\top P(A_i + BK_i)(x(k) - x_0) \leq \lambda^2(x(k) - x_0)^\top P(x(k) - x_0), \quad (33)$$

which leads to

$$(A_i + BK_i)^\top P(A_i + BK_i) \preceq \lambda^2 P, \quad (34)$$

and application of the Schur complement followed by further organizing yields (31).

The following Theorem 4 provides the solution to Problem 2 in Case 1 under the constraints of the control input.

Theorem 4. Suppose both Assumptions 1 and 2 hold; let an ellipsoid be $\mathcal{E}(P, 1)$ and the input u constrained in the polyhedral convex set $\mathcal{U} = \{u \in \mathbb{R}^m : Uu \leq \bar{\mathbf{1}}\}$ (which includes the origin as an interior point); then, using the collected data in (28), if there exists a matrix $G_{1,s}$ such that (35)–(37) are satisfied, then Problem 2 is solved.

$$\begin{bmatrix} \lambda^2 P & X_1 G_{1,s} D_i \\ (X_1 G_{1,s} D_i)^\top & P^{-1} \end{bmatrix} \succeq 0, \quad i = 1, 2, \dots, s \quad (35)$$

$$X_W G_{K_{1,s}} = I, \quad (36)$$

$$\begin{bmatrix} 1 & (UU_0 G_{1,s} D_i P^{-\frac{1}{2}})_l^\top \\ (UU_0 G_{1,s} D_i P^{-\frac{1}{2}})_l & I \end{bmatrix} \succeq 0, \quad i = 1, 2, \dots, s, \quad l = 1, 2, \dots, n \quad (37)$$

Proof. Since Assumption 2 holds, there must exist a matrix $G_{1,s}$ such that the following equality holds:

$$X_W G_{1,s} = I. \quad (38)$$

Based on the data collected in (28), it can be seen that the system (27) can be rewritten as follows:

$$X_1 = A_{1,s} X_W + B U_0. \quad (39)$$

Multiplying both sides of Equation (39) by the matrix $G_{1,s}$ yields the following equation:

$$X_1 G_{1,s} = A_{1,s} + B U_0 G_{1,s}. \quad (40)$$

Let $K_{1,s} = U_0 G_{1,s}$; the Equation (40) holds:

$$X_1 G_{1,s} = A_{1,s} + B K_{1,s}. \quad (41)$$

The ellipsoid $\mathcal{E}(P, 1) = \{x \in \mathbb{R}^n : \sqrt{x^\top P x} \leq 1\}$ is λ -contractive under the LPV system (27) with state-feedback control law (8) if and only if

$$D_i^\top (A_{1,s} + B K_{1,s})^\top P (A_{1,s} + B K_{1,s}) D_i \preceq \lambda^2 P, \quad i = 1, 2, \dots, s \quad (42)$$

Through combination with Equation (41) and application of the Schur complement, we derive (35).

We here provide the proof of (37): Since $u \in \mathcal{U} = \{u \in \mathbb{R}^m : Uu \leq \bar{1}\}$, the control input $u(t)$ satisfies the constraint $UU_0G_{1,s}D_i x \leq \bar{1}$, where $K_i = K_{1,s}D_i = U_0G_{1,s}D_i$. Let $x = P^{-\frac{1}{2}}v$, where $\|v\|_2 \leq 1$. By substituting $x = P^{-\frac{1}{2}}v$ (where $x \in \mathcal{E}(P, 1)$) into $UU_0G_{1,s}D_i x \leq \bar{1}$, we obtain $UU_0G_{1,s}D_i P^{-\frac{1}{2}}v \leq \bar{1}$, $\|v\|_2 \leq 1$.

The inequality $UU_0G_{1,s}D_i P^{-\frac{1}{2}}v \leq \bar{1}$ must hold for all $\|v\|_2 \leq 1$, which is equivalent to the following $\|(UU_0G_{1,s}D_i P^{-\frac{1}{2}})_l\|_2 \leq 1$, $l = 1, 2, \dots, n$, where $(UU_0G_{1,s}D_i P^{-\frac{1}{2}})_l$ denotes the l -th row of $UU_0G_{1,s}D_i P^{-\frac{1}{2}}$. We now prove the validity of this equivalent condition.

$\Leftarrow$:

For any $\|v\|_2 \leq 1$, we have $|(UU_0G_{1,s}D_i P^{-\frac{1}{2}})_l v| \leq \|(UU_0G_{1,s}D_i P^{-\frac{1}{2}})_l\|_2 \|v\|_2 \leq \|(UU_0G_{1,s}D_i P^{-\frac{1}{2}})_l\|_2$. That is, if $\|(UU_0G_{1,s}D_i P^{-\frac{1}{2}})_l\|_2 \leq 1$, then $|(UU_0G_{1,s}D_i P^{-\frac{1}{2}})_l v| \leq 1$ holds for all $\|v\|_2 \leq 1$.

$\Rightarrow$: Next, we prove that if $|(UU_0G_{1,s}D_i P^{-\frac{1}{2}})_l v| \leq 1$ holds for all $\|v\|_2 \leq 1$, then $\|(UU_0G_{1,s}D_i P^{-\frac{1}{2}})_l\|_2 \leq 1$.

The proof can be completed by contradiction:

Suppose there exists a row $\|(UU_{0,T}G_{1,s}D_i P^{-\frac{1}{2}})_k\|_2 > 1$; then, for $v = \frac{(UU_{0,T}G_{1,s}D_i P^{-\frac{1}{2}})_k^\top}{\|(UU_{0,T}G_{1,s}D_i P^{-\frac{1}{2}})_k\|_2}$ ($\|v\|_2 = 1$), we have

$$(UU_{0,T}G_{1,s}D_i P^{-\frac{1}{2}})_k v = \|(UU_{0,T}G_{1,s}D_i P^{-\frac{1}{2}})_k^\top\|_2 = \|(UU_{0,T}G_{1,s}D_i P^{-\frac{1}{2}})_k\|_2 > 1,$$

which contradicts the given condition.

Using the Schur complement Lemma 1, the condition $\|(UU_{0,T}G_{1,s}D_i P^{-\frac{1}{2}})_l\|_2 \leq 1$ is equivalent to the following (43):

$$\begin{bmatrix} I & (UU_{0,T}G_{1,s}D_i P^{-\frac{1}{2}})_l \\ (UU_{0,T}G_{1,s}D_i P^{-\frac{1}{2}})_l^\top & 1 \end{bmatrix} \succeq 0. \quad (43)$$

□

6. Numerical Example

To verify the correctness of the conclusions and the feasibility of the methods presented in this brief paper, we constructed two different numerical examples. Two numerical examples for LPV systems with different constraints are provided below.

Example 1. Consider the following LPV system without control input constraints:

$$x(k+1) = A(w(k))x(k) + Bu(k) + d(k), \quad (44)$$

where the matrices A_1 , A_2 , and B used to generate the data samples in Figure 1 are given as follows:

$$A_1 = \begin{bmatrix} 1.2 & 0 \\ 0 & 0.3 \end{bmatrix}, A_2 = \begin{bmatrix} -1.2 & 0 \\ 3.0 & 0.4 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Let $U_0 = [0.630 \ 0.812 \ -0.750 \ 0.830 \ 0.265]$, $x_0 = [1; 1]$, and the polyhedral C -set $S(H, h) = \{x \in \mathbb{R}^n : Hx \leq h\}$ is given by the following:

$$H = \begin{bmatrix} 1/5 & 2/5 \\ -1/5 & -2/5 \\ -3/20 & 1/5 \\ 3/20 & -1/5 \end{bmatrix}, h = \begin{bmatrix} 2 \\ 2 \\ 1.5 \\ 1.5 \end{bmatrix}, \text{ the contraction rate is taken as } \lambda = 0.84. \text{ The}$$

vertices of polyhedron $\mathcal{D}$ are selected as the following vectors: $d_1 = [0.1; 0.1]$, $d_2 = [-0.1; -0.1]$, $d_3 = [-0.1; 0.1]$, and $d_4 = [0.1; -0.1]$.

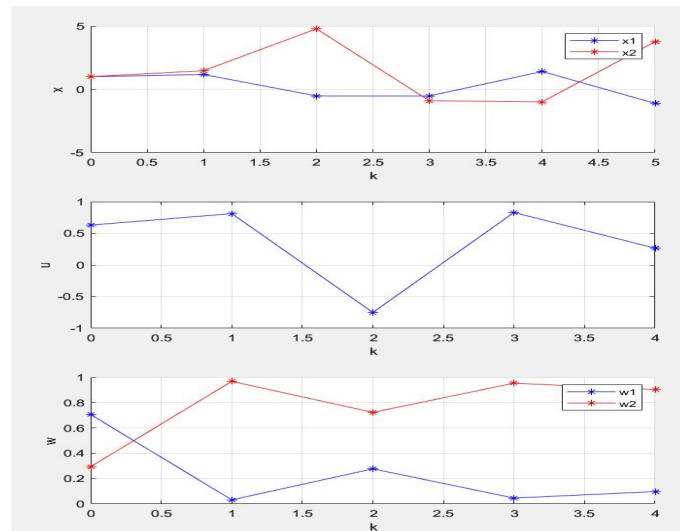


Figure 1. State sequences of data, input sequences of data, and gain-scheduling sequences of data as in (10) with $T = 5$.

By applying Theorem 2, we obtain the solution $G_{1,2}$ to the linear optimization problem, which yields the control gain $K_{1,2}$ in Equation (9) as follows:

$$K_{1,2} = U_0 G_{1,2} = [-0.0012 \quad -0.0287 \quad -0.0012 \quad -0.0287].$$

Figure 2 demonstrates that the system states evolve within the safe set at all times, validating the provided safety guarantee. Furthermore, Figure 3 shows that the control signals satisfy the requirements of the given control input constraint set $\mathcal{U}$. Therefore, this example serves to demonstrate the feasibility of our approach to Problem 1.

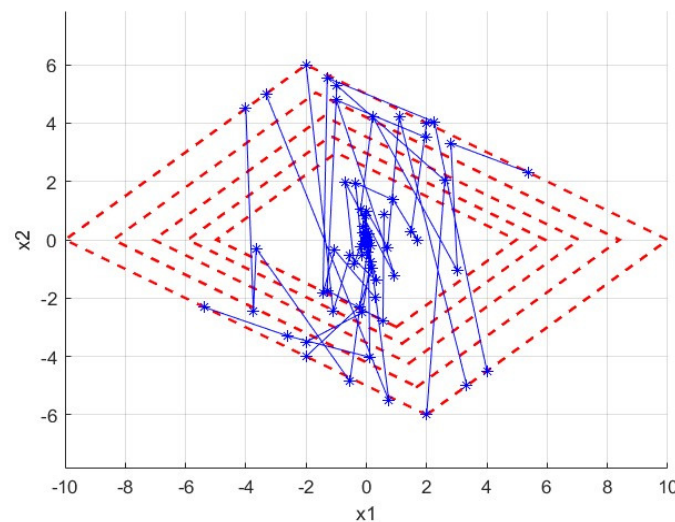


Figure 2. The set $\mathcal{S}(H, h)$ and the solutions generated by the gain-scheduling state feedback control law. The sets $\mathcal{S}$, $\lambda\mathcal{S}$, $\lambda^2\mathcal{S}$, $\lambda^3\mathcal{S}$, and $\lambda^4\mathcal{S}$ are depicted by red dotted lines.

Example 2. Consider the following LPV system:

$$x(k+1) = A(w(k))x(k) + Bu(k), \quad (45)$$

where $A_1 = \begin{bmatrix} 1 & 2/3 \\ -1/3 & 0.1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 4/5 & 2/5 \\ -2/5 & 6/5 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and the set $\mathcal{U}$ specifies the condition $-8 \leq u \leq 8$, i.e., $\mathcal{U} = \begin{bmatrix} -1/8 \\ 1/8 \end{bmatrix}$. System (45) is defined by an ellipsoidal C-set with the following parameters:

$$\mathcal{E}(P, 1) = \{x \in \mathbb{R}^n : \sqrt{x^\top P x} \leq 1\}, \quad (46)$$

where $P = \begin{bmatrix} 1/2 & -1/4 \\ -1/4 & 1/2 \end{bmatrix}$, and the contraction rate is taken as $\lambda = 0.84$. Theorem 4 directly yields the result that

$$K_{1,2} = U_0 G_{1,2} = [-0.1254 \quad 0.0845 \quad -0.3526 \quad -1.3222].$$

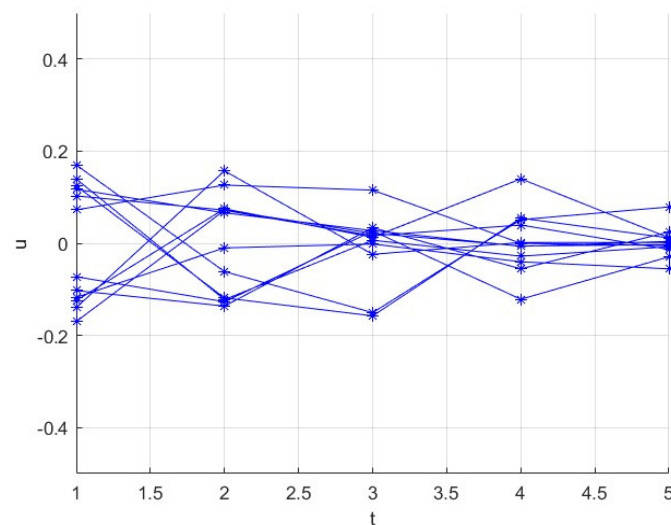


Figure 3. Control signal u corresponding to the solutions for different initial conditions depicted in Figure 2.

Example 2 serves to validate our approach to Problem 2 by verifying that the obtained ellipsoidal C-set can be rendered safe and that all control inputs satisfy the associated constraints for system with ellipsoidal control constraints. Figure 4 demonstrates that the system states evolve within the safe set at all times, validating the provided safety guarantee. Furthermore, Figure 5 shows that the control signals satisfy the requirements of the given control input constraint set $\mathcal{U}$. Therefore, example 2 serves to demonstrate the feasibility of our approach to Problem 2.

The results of the two numerical examples indicate that, regardless of whether the LPV system has control constraints, external disturbances, or whether the constraints are polytopic or ellipsoidal, the control gain K can be effectively obtained according to our Theorem 2, ensuring the stability and safety invariance of the LPV system. This demonstrates that the method proposed in this brief paper is correct and feasible.

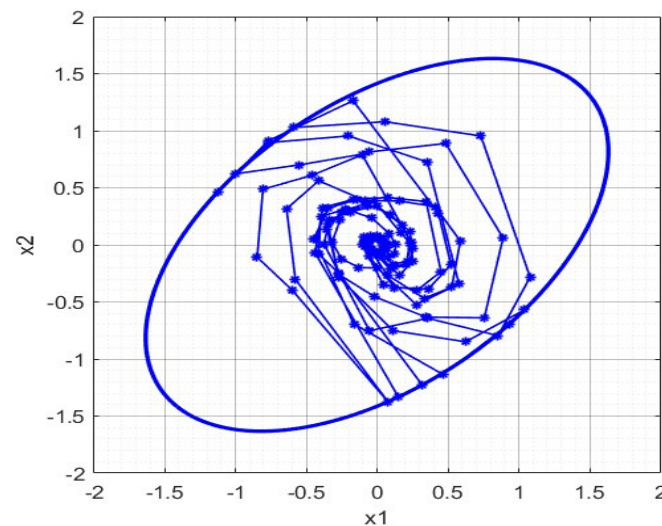


Figure 4. The safe set $\mathcal{E}(P, 1)$, given as in (46), with $U = \begin{bmatrix} -1/8 \\ 1/8 \end{bmatrix}$ and $\lambda = 0.84$. Solutions arising from the state feedback law $u(k) = K(w(k))x(k)$ (see (8)) designed.

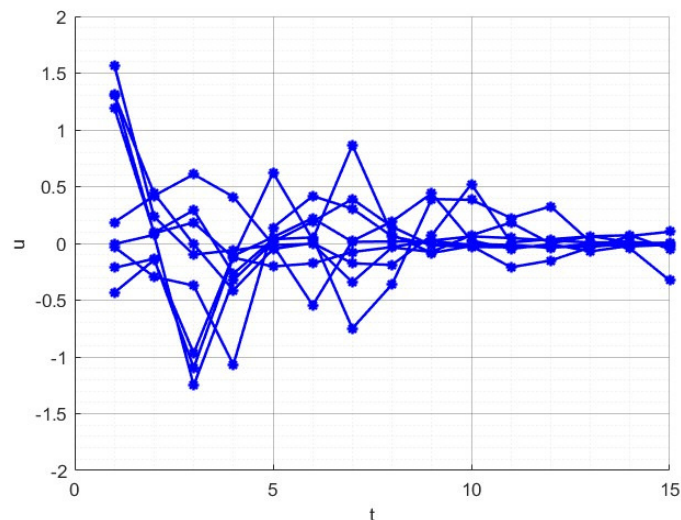


Figure 5. Control signal u corresponding to the solutions for different initial conditions depicted in Figure 4. The control signal satisfies the constraints given by U .

7. Conclusions

This paper focuses on the safety controller design for LPV systems characterized by a constant input matrix (i.e., the B-matrix). Building upon the data-driven framework established in [29], the present work extends the research along two specific avenues: First, it investigates the synthesis of robust safety controllers for polyhedral safe sets subject to external bounded disturbances. Second, it addresses the controller synthesis problem for ellipsoidal safe sets under input constraints, ensuring the feasibility of the control actions. In the context of ellipsoidal safe sets, the incorporation of input constraints inherently gives rise to nonlinear inequalities. To tackle this challenge, we propose a systematic transformation method that converts these nonlinear constraints into an equivalent formulation of LMIs. This reformulation enables the effective resolution of the problem using SDP tools. Current methods primarily focus on polyhedral and ellipsoidal safe sets. Future research should prioritize expanding in the following two directions: (1) designing safety controllers for non-convex sets, and (2) developing scalable computational frameworks and efficient dimensionality reduction techniques for safety control in high-dimensional systems.

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