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A Note on Truncated Exponential-Based Appell Polynomials via Fractional Operators

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Abstract

In this work, we construct a new class of Appell-type polynomials generated through extended truncated and truncated exponential kernels, and we analyze their core algebraic and operational features. In particular, we establish a suitable recurrence scheme and obtain the associated multiplicative and differential operators. By confirming the quasi-monomial structure, we further deduce the governing differential equation for the proposed family. In addition, we present both a series expansion and a determinant formulation, providing complementary representations that are useful for symbolic manipulation and computation. As special cases, we introduce and study subfamilies arising from this setting, namely, extended truncated exponential versions of the Bernoulli, Euler, and Genocchi polynomials, and discuss their structural identities and operational behavior. Overall, these developments broaden the theory of special polynomials and furnish tools relevant to problems in mathematical physics and differential equations.

Keywords: truncated exponential polynomials; truncated exponential-based Appell polynomials; quasi-monomiality; extension of monomiality principle; differential equations; fractional derivative

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1. Introduction and Preliminary Results

Truncated polynomials have been recognized as objects of significant importance, particularly in the evaluation of integrals involving products of special functions. Such integrals frequently arise in diverse areas of physics, mathematics, applied sciences, engineering, and related research domains. Owing to their wide applicability, truncated forms of special polynomials have attracted considerable attention, and numerous investigations can be found in the literature (see also the references cited therein). In this direction, Dattoli et al. [1] introduced higher-order truncated polynomials and demonstrated their relevance in the computation of integrals involving products of special functions. Subsequently, Ozat et al. [2] and Khan et al. [3,4] studied truncated Bernoulli polynomials and truncated Euler polynomials, respectively. Srivastava et al. [5] examined truncated exponential-based

Apostol-type polynomials and established several of their properties, including implicit summation formulas and symmetry relations. Furthermore, Qawaqneh et al. [6] investigated the Δ_h-truncated Hermite polynomials and numbers, and more recently explored degenerate truncated special polynomials. It is worth noting that the theory of degenerate truncated exponential polynomials remains comparatively less developed. Nevertheless, in recent years, many researchers have focused on various degenerate versions of special polynomials and numbers, not only from arithmetic and combinatorial viewpoints but also with regard to applications in differential equations, symmetry identities, and probability theory [7,8].

The truncated exponential polynomials $e_n(\vartheta_1)$ are defined by

$$e_n(\vartheta_1) = \sum_{k=0}^{n} \frac{\vartheta_1^k}{k!}, \tag{1}$$

whichrepresent the first $(n+1)$ terms of the Maclaurin expansion of the exponential function e^u. These polynomials are generated by the following generating function [1]:

$$\frac{e^{\vartheta_1 t}}{1-t} = \sum_{n=0}^{\infty} e_n(\vartheta_1) t^n. \tag{2}$$

The two-variable truncated exponential polynomials $e_n^{(r)}(\vartheta_1, \vartheta_2)$ of order r are characterized by the generating function

$$\frac{e^{\vartheta_1 t}}{1-\vartheta_2 t^r} = \sum_{n=0}^{\infty} e_n^{(r)}(\vartheta_1, \vartheta_2) \frac{t^n}{n!}, \tag{3}$$

and admit the series representation

$$e_n^{(r)}(\vartheta_1, \vartheta_2) = n! \sum_{n=0}^{\left[\frac{n}{r}\right]} \frac{\vartheta_2^k \vartheta_1^{n-rk}}{(n-rk)!}. \tag{4}$$

In view of the generating function of Appell polynomials [9] together with Equation (2), it is evident that the truncated exponential polynomials $e_n(\vartheta_1)$ form an Appell sequence. The possibility of extending the definition of $e_n(\vartheta_1)$ naturally arises from the range of applications in which these polynomials appear.

Some noteworthy special cases of the two-variable truncated exponential polynomials $e_n^{(r)}(\vartheta_1, \vartheta_2)$ of order r are given by

$$e_n^{(2)}(\vartheta_1, \vartheta_2) = n!_{[2]} e_n(\vartheta_1, \vartheta_2), \tag{5}$$

$$e_n^{(r)}(\vartheta_1, 1) = n!_{[r]} e_n(\vartheta_1), \tag{6}$$

$$e_n^{(2)}(\vartheta_1, 1) = n!_{[2]} e_n(\vartheta_1), \tag{7}$$

$$e_n^{(1)}(\vartheta_1, 1) = n! e_n(\vartheta_1). \tag{8}$$

Appell polynomial sequences naturally appear in a broad range of contexts, including applied mathematics, theoretical physics, approximation theory, and many other areas of mathematical research. Owing to their rich algebraic structure and operational properties, these sequences play a significant role in both theoretical investigations and practical applications. They are commonly characterized by means of an appropriate generating function, given by

$$\mathcal{A}(\vartheta_1, t) := \mathcal{A}(t)e^{\vartheta_1 t} = \sum_{n=0}^{\infty} \mathcal{A}_n(\vartheta_1)\frac{t^n}{n!}, \quad \mathcal{A}_n := \mathcal{A}_n(0), \tag{9}$$

where $\mathcal{A}(t)$ is an analytic function in a neighborhood of $t = 0$ and is expressed as

$$\mathcal{A}(t) = \sum_{n=0}^{\infty} \mathcal{A}_n \frac{t^n}{n!}, \quad \mathcal{A}_0 \neq 0, \quad \mathcal{A}_i \ (i = 0, 1, 2, \cdots) \text{ are real coefficients.} \tag{10}$$

The corresponding Appell polynomials $\mathcal{A}_n(\vartheta_1)$ admit the explicit representation

$$\mathcal{A}_n(\vartheta_1) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{A}_{n-k}\, \vartheta_1^k, \quad \mathcal{A}_n'(\vartheta_1) = n\, \mathcal{A}_{n-1}(\vartheta_1). \tag{11}$$

By suitable choices of the generating function $\mathcal{A}(t)$, a variety of well-known Appell polynomial families can be obtained as particular cases. These are summarized in Table 1 below.

Table 1. Certain members belonging to the Appell family.

S.No.	Name of Polynomials	$\mathcal{R}(t)$	Generating Function	Series Definition
I.	Bernoulli polynomials and numbers [10]	$\frac{t}{e^t-1}$	$\left(\frac{t}{e^t-1}\right)e^{\vartheta_1 t} = \sum_{n=0}^{\infty} B_n(\vartheta_1)\frac{t^n}{n!}$ $\left(\frac{t}{e^t-1}\right) = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}$ $B_n(:= B_n(0) = B_n(1))$	$B_n(\vartheta_1) = \sum_{k=0}^{n} \binom{n}{k} B_k \vartheta_1^{n-k}$
II.	Euler polynomials and numbers [10]	$\frac{2}{e^t+1}$	$\left(\frac{2}{e^t+1}\right)e^{\vartheta_1 t} = \sum_{n=0}^{\infty} E_n(\vartheta_1)\frac{t^n}{n!}$ $\frac{2e^t}{e^{2t}+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}$ $E_n := 2^n E_n\left(\frac{1}{2}\right)$	$E_n(\vartheta_1) = \sum_{k=0}^{n} \binom{n}{k} \frac{E_k}{2^k}\left(\vartheta_1 - \frac{1}{2}\right)^{n-k}$
III.	Genocchi polynomials and numbers [10]	$\frac{2t}{e^t+1}$	$\left(\frac{2t}{e^t+1}\right)e^{\vartheta_1 t} = \sum_{n=0}^{\infty} G_n(\vartheta_1)\frac{t^n}{n!}$ $\frac{2t}{e^t+1} = \sum_{n=1}^{\infty} G_n \frac{t^n}{n!}$ $G_n := G_n(0)$	$G_n(\vartheta_1) = \sum_{k=0}^{n} \binom{n}{k} G_k \vartheta_1^{n-k}$

To support subsequent analytical and computational developments, we list the initial values of the Bernoulli numbers B_n, Euler numbers E_n, and Genocchi numbers G_n in Table 2. These fundamental numerical sequences play a central role in the construction and analysis of the associated polynomial families, and their explicit values provide a convenient reference for illustrating examples and verifying derived identities in the sections that follow.

Table 2. Values of five four B_n, E_n and G_n.

n	0	1	2	3	4
B_n	1	$\pm\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$
E_n	1	0	-1	0	5
G_n	0	1	-1	0	1

The foundations of the monomiality principle date back to 1941, when Steffensen [11] introduced the notion in connection with the theory of poweroids. This early idea was later

refined and considerably extended by Dattoli [12], who developed a systematic operational framework that has since become a cornerstone in the study of special polynomial families.

Within the setting of the monomiality principle, one associates with a polynomial sequence $\{q_n(\vartheta_1)\}_{n\in\mathbb{N}}$ a pair of linear operators, namely the multiplicative operator $\widehat{\mathcal{M}}$ and the derivative operator $\widehat{\mathcal{P}}$. These operators are defined so as to satisfy the operational recursions

$$q_{n+1}(\vartheta_1) = \widehat{\mathcal{M}}\{q_n(\vartheta_1)\}, \tag{12}$$

and

$$n\, q_{n-1}(\vartheta_1) = \widehat{\mathcal{P}}\{q_n(\vartheta_1)\}. \tag{13}$$

Any polynomial sequence fulfilling the above relations is referred to as a quasi-monomial sequence. Such sequences are further distinguished by the canonical commutation rule

$$[\widehat{\mathcal{P}}, \widehat{\mathcal{M}}] = \widehat{\mathcal{P}}\widehat{\mathcal{M}} - \widehat{\mathcal{M}}\widehat{\mathcal{P}} = \widehat{1}, \tag{14}$$

which embeds the associated operators naturally within the Weyl algebraic structure.

Once a polynomial family $\{q_n(\vartheta_1)\}_{n\in\mathbb{N}}$ is recognized as quasi-monomial, a number of its essential properties can be derived directly from the operators $\widehat{\mathcal{M}}$ and $\widehat{\mathcal{P}}$. In particular, the following statements hold:

(i) If $\widehat{\mathcal{M}}$ and $\widehat{\mathcal{P}}$ possess suitable differential realizations, then the polynomials $q_n(\vartheta_1)$ satisfy the differential equation

$$\widehat{\mathcal{M}}\widehat{\mathcal{P}}\{q_n(\vartheta_1)\} = n\, q_n(\vartheta_1). \tag{15}$$

(ii) Each polynomial in the sequence admits the operational form

$$q_n(\vartheta_1) = \widehat{\mathcal{M}}^n\{1\}, \tag{16}$$

together with the normalization condition $q_0(\vartheta_1) = 1$.

(iii) The exponential generating function corresponding to $\{q_n(\vartheta_1)\}$ is expressed as

$$e^{t\widehat{\mathcal{M}}}\{1\} = \sum_{n=0}^{\infty} q_n(\vartheta_1)\,\frac{t^n}{n!}, \qquad (|t| < \infty), \tag{17}$$

which follows directly from the operational definition; see [13,14] for further discussion.

In 2016, Khan et al. [4] introduced a hybrid family of truncated exponential-based Appell polynomials, denoted by ${}_{e^{(r)}}\mathcal{A}_n(\vartheta_1, \vartheta_2)$, through the generating relation

$$\frac{\mathcal{A}(t)}{(1-\vartheta_2 t^r)}\,\exp(\vartheta_1 t) = \sum_{n=0}^{\infty} {}_{e^{(r)}}\mathcal{A}_n(\vartheta_1, \vartheta_2)\,\frac{t^n}{n!}. \tag{18}$$

Equivalently, these truncated exponential-based Appell polynomials admit the series representation

$${}_{e^{(r)}}\mathcal{A}_n(\vartheta_1, \vartheta_2) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{A}_{n-k}\, e_k^{(r)}(\vartheta_1, \vartheta_2). \tag{19}$$

An alternative operational characterization of the two-variable truncated exponential-based Appell polynomials is given by

$${}_{e^{(r)}}\mathcal{A}_n(\vartheta_1, \vartheta_2) = \exp\Big(\vartheta_2 D_{\vartheta_2}\, \vartheta_2 D_{\vartheta_1}^r\Big)\{\mathcal{A}_n(\vartheta_1)\}. \tag{20}$$

Within this framework, the family ${}_{e^{(r)}}\mathcal{A}_n(\vartheta_1, \vartheta_2)$ forms a quasi-monomial sequence associated with the multiplicative and derivative operators

$$\widehat{\mathcal{M}} = \vartheta_1 + r\,\vartheta_2 D_{\vartheta_2}\,\vartheta_2 D_{\vartheta_1}^{r-1}, \tag{21}$$

and

$$\widehat{\mathcal{P}} = D_{\vartheta_1}, \tag{22}$$

respectively. By choosing appropriate forms of the function $\mathcal{R}(t)$, various subclasses of the hybrid truncated exponential-based Appell polynomial family can be recovered; these particular cases are summarized in Table 3.

Table 3. Certain members belonging to the TEAP family.

S. No.	Name of Hybrid Polynomials	$\mathcal{A}(t)$	Generating Function	Series Definition
I.	Truncated exponential-Bernoulli polynomials	$\frac{t}{e^t-1}$	$\left(\frac{t}{e^t-1}\right)e^{\vartheta_1 t}\,\frac{1}{1-\vartheta_2 t^r} = \sum_{n=0}^{\infty} {}_{e^{(r)}}\mathcal{B}_n(\vartheta_1,\vartheta_2)\frac{t^n}{n!}$	${}_{e^{(r)}}\mathcal{B}_n(\vartheta_1,\vartheta_2) = \sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_{n-k}e_k^{(r)}(\vartheta_1,\vartheta_2)$
II.	Truncated exponential-Euler polynomials	$\frac{2}{e^t+1}$	$\left(\frac{2}{e^t+1}\right)e^{\vartheta_1 t}\,\frac{1}{1-\vartheta_2 t^r} = \sum_{n=0}^{\infty} {}_{e^{(r)}}\mathcal{E}_n(\vartheta_1,\vartheta_2)\frac{t^n}{n!}$	${}_{e^{(r)}}\mathcal{E}_n(\vartheta_1,\vartheta_2) = \sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_{n-k}e_k^{(r)}(\vartheta_1,\vartheta_2)$
III.	Truncated exponential-Genocchi polynomials	$\frac{2t}{e^t+1}$	$\left(\frac{2t}{e^t+1}\right)e^{\vartheta_1 t}\,\frac{1}{1-\vartheta_2 t^r} = \sum_{n=0}^{\infty} {}_{e^{(r)}}\mathcal{G}_n(\vartheta_1,\vartheta_2)\frac{t^n}{n!}$	${}_{e^{(r)}}\mathcal{G}_n(\vartheta_1,\vartheta_2) = \sum_{k=0}^{n}\binom{n}{k}\mathcal{G}_{n-k}e_k^{(r)}(\vartheta_1,\vartheta_2)$

Motivated by Euler's integral identity, which plays a central role in fractional calculus and integral transform theory (see [15,16]), we recall that

$$a^{-\nu} = \frac{1}{\Gamma(\nu)}\int_0^{\infty} e^{-at}t^{\nu-1}\,dt, \quad \min\{\mathrm{Re}(\nu), \mathrm{Re}(a)\} > 0, \tag{23}$$

providing an integral representation for inverse powers of linear operators.

When this identity is applied to suitable differential operators, it yields useful operational expressions, as established in [17]. In particular, we have

$$\left(\alpha - \frac{d}{d\mu_1}\right)^{-\nu} f(\mu_1) = \frac{1}{\Gamma(\nu)}\int_0^{\infty} e^{-\alpha t}t^{\nu-1}\left(e^{t\frac{d}{d\mu_1}}f(\mu_1)\right)dt = \frac{1}{\Gamma(\nu)}\int_0^{\infty} e^{-\alpha t}t^{\nu-1}f(\mu_1+t)dt, \tag{24}$$

which clearly illustrates how translation operators arise naturally within this operational setting.

In an analogous manner, considering operators involving second-order derivatives leads to

$$\left(\alpha - \frac{d^2}{d{\mu_1}^2}\right)^{-\nu} f(\mu_1) = \frac{1}{\Gamma(\nu)}\int_0^{\infty} e^{-\alpha t}t^{\nu-1}\left(e^{t\frac{d^2}{d{\mu_1}^2}}f(\mu_1)\right)dt, \tag{25}$$

which emphasizes the appearance of heat kernel-type exponential operators in the analysis of inverse differential operators.

Such integral representations, established for several hybrid special polynomial families by Zayed et al. [18–21], significantly simplify the treatment of fractional powers of differential operators and serve as a cornerstone for the development of operational methods in the theory of special functions, including various classes of extended polynomials.

The operational techniques outlined above have been successfully employed in a wide range of applications, spanning classical optics, quantum mechanics, and several areas of mathematical physics. They provide a flexible and powerful analytical framework for investigating the properties of diverse polynomial families.

It is important to emphasize that the classical theory of truncated exponential-based Appell polynomials corresponds exclusively to the integer-order operational inversion

$$(\alpha - \mathcal{D})^{-1},$$

which is recovered only in the special case $\nu = 1$. In contrast, the present work is based on the fractional inverse operator

$$(\alpha - \mathcal{D})^{-\nu}, \qquad \mathrm{Re}(\nu) > 0,$$

whose action cannot be reduced to the integer-order case by parameter substitution. When $\nu \neq 1$, the associated operational calculus involves nonlocal effects expressed through Mellin-type convolution kernels, leading to genuinely new polynomial families. Consequently, the resulting extended truncated exponential-based Appell polynomials exhibit ν-dependent recurrence relations, differential equations, and generating functions that do not appear in existing truncated Appell theories. The parameter ν controls the fractional depth of the inverse operator and introduces a hierarchy of polynomial sequences, while the parameter α acts as a spectral shift ensuring convergence and allowing interpolation between distinct polynomial regimes. These features clearly distinguish the proposed framework from earlier studies.

Classical truncated Appell polynomials are generated through the resolvent operator

$$(\alpha - \mathcal{D})^{-1},$$

which satisfies the semigroup property

$$(\alpha - \mathcal{D})^{-1}(\alpha - \mathcal{D})^{-1} = (\alpha - \mathcal{D})^{-2}.$$

This algebraic closure under composition is fundamental to the classical Appell theory.

In contrast, our construction employs the fractional inverse operator

$$(\alpha - \mathcal{D})^{-\nu}, \qquad \mathrm{Re}(\nu) > 0,$$

which does *not* satisfy

$$(\alpha - \mathcal{D})^{-\nu_1}(\alpha - \mathcal{D})^{-\nu_2} = (\alpha - \mathcal{D})^{-(\nu_1+\nu_2)}.$$

This loss of semigroup structure is a qualitative obstruction: it prevents the reduction of the fractional case to the integer case by parameter substitution. As a consequence, each value of ν generates a genuinely distinct layer in a fractional hierarchy of polynomial families.

A further qualitative difference appears at the level of governing equations. Classical truncated Appell polynomials satisfy integer-order operational identities leading to local differential equations. In the present setting, the fractional embedding yields the operator equation

$$\left(\alpha - \vartheta_2 D_{\vartheta_2}\, \vartheta_2 D^{r}_{\vartheta_1}\right)^{\nu} \left(e^{(r)} A_{n,\nu}\right) = A_n,$$

which represents a *nonlocal pseudo-differential equation*. Such fractional governing equations do not arise in the classical truncated Appell theory (corresponding to $\nu = 1$). This constitutes a genuine qualitative extension.

Inspired by these considerations, we introduce the extended truncated exponential functions $e^{(r)}_{n,\nu}(\vartheta_1, \vartheta_2; \alpha)$ and the associated truncated exponential-based Appell polynomials ${}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1, \vartheta_2; \alpha)$. The paper is organized as follows.

In Section 2, we introduce a new class of extended truncated exponential functions together with the associated Appell-type polynomial families. Their fundamental analytical and algebraic properties are systematically examined, including explicit recurrence relations, the construction of suitable multiplicative and derivative operators, and the derivation of the corresponding differential equations. These results establish a solid operational framework for the newly defined polynomials.

Section 3 is devoted to the development of explicit series representations and determinantal formulations for the proposed polynomial families. The determinant approach, in particular, provides an alternative algebraic characterization that complements the operational methods and reveals additional structural features of the polynomials.

In Section 4, we focus on several significant subfamilies obtained through specific choices of parameters and generating functions. For each subfamily, we analyze their distinctive properties and demonstrate how many known polynomial sequences arise as special or limiting cases within the present unified framework.

Finally, the concluding section summarizes the main contributions of the paper and outlines potential avenues for future research, including further generalizations, applications to differential and integral equations, and possible extensions within the context of operational calculus and mathematical physics.

2. Extended Truncated Exponential-Based Appell Polynomials

In this section, we establish the operational characterization and the generating function associated with the extended truncated exponential polynomials (ETEP) $e_{n,\nu}^{(r)}(\vartheta_1, \vartheta_2; \alpha)$. The principal results are presented below.

Theorem 1. *Let* $\mathrm{Re}(\nu) > 0, \mathrm{Re}(\alpha) > 0$, *the ETEP* $e_{n,\nu}^{(r)}(\vartheta_1, \vartheta_2; \alpha)$, *the following operational identity is satisfied:*

$$\left(\alpha - \vartheta_2 D_{\vartheta_2} \vartheta_2 D_{\vartheta_1}^r\right)^{-\nu} \vartheta_1^n = e_{n,\nu}^{(r)}(\vartheta_1, \vartheta_2; \alpha). \tag{26}$$

Proof. Replacing the parameter a by the operator $\left(\alpha - \vartheta_2 D_{\vartheta_2} \vartheta_2 D_{\vartheta_1}^r\right)$ in Equation (23) and subsequently applying the resulting expression to ϑ_1^n, we obtain

$$\left(\alpha - \vartheta_2 D_{\vartheta_2} \vartheta_2 D_{\vartheta_1}^r\right)^{-\nu} \{\vartheta_1^n\} = \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-\alpha t} t^{\nu-1} \exp\left(t.\alpha - r\vartheta_2 D_{\vartheta_2} \vartheta_2 D_{\vartheta_1}^{r-1}\right) \vartheta_1^n \, dt. \tag{27}$$

Using Equation (20), the above expression simplifies to

$$\left(\alpha - \vartheta_2 D_{\vartheta_2} \vartheta_2 D_{\vartheta_1}^r\right)^{-\nu} \{\vartheta_1^n\} = \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-\alpha t} t^{\nu-1} e_n^{(r)}(\vartheta_1 t, \vartheta_2) \, dt. \tag{28}$$

The integral transform appearing on the right-hand side of (28) motivates the introduction of a new class of polynomials. We denote this family by $e_{n,\nu}^{(r)}(\vartheta_1, \vartheta_2; \alpha)$ and refer to it as the extended truncated exponential polynomials (ETEP). Accordingly, we define

$$e_{n,\nu}^{(r)}(\vartheta_1, \vartheta_2; \alpha) = \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-\alpha t} t^{\nu-1} e_n^{(r)}(\vartheta_1, \vartheta_2 t) \, dt. \tag{29}$$

Combining Equations (28) and (29) directly yields the required operational relation (26). □

Theorem 2. *For the ETEP* $e_{n,\nu}^{(r)}(\vartheta_1, \vartheta_2; \alpha)$, *the corresponding generating function is given by*

$$\frac{\exp(\vartheta_1 u)}{\left(\alpha - \vartheta_2 D_{\vartheta_2} \vartheta_2 u^r\right)^\nu} = \sum_{n=0}^\infty e_{n,\nu}^{(r)}(\vartheta_1, \vartheta_2; \alpha) \frac{u^n}{n!}. \tag{30}$$

Proof. Starting from the integral representation in Equation (29), we write

$$\sum_{n=0}^{\infty} e_{n,\nu}^{(r)}(\vartheta_1,\vartheta_2;\alpha)\frac{u^n}{n!} = \sum_{n=0}^{\infty}\frac{1}{\Gamma(\nu)}\int_0^{\infty} e^{-\alpha t}t^{\nu-1}e_n^{(r)}(\vartheta_1,\vartheta_2 t)\frac{u^n}{n!}\,dt. \tag{31}$$

Invoking Equation (24) on the right-hand side of (31), we arrive at

$$\sum_{n=0}^{\infty} e_{n,\nu}^{(r)}(\vartheta_1,\vartheta_2;\alpha)\frac{u^n}{n!} = \frac{\exp(\vartheta_1 u)}{\Gamma(\nu)}\int_0^{\infty} e^{-(\alpha-\vartheta_2 D_{\vartheta_2}\vartheta_2 u^r)t}t^{\nu-1}\,dt. \tag{32}$$

Applying Equation (25) to evaluate the integral on the right-hand side of (32) yields the desired generating function (30). □

Next, we proceed to formulate the operational representation and generating function for the extended truncated exponential-based Appell polynomials (ETEAP) ${}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)$ by establishing the following results.

Theorem 3. *For the ETEAP* ${}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)$*, the following operational identity is satisfied:*

$$\left(\alpha-\vartheta_2 D_{\vartheta_2}\vartheta_2 D_{\vartheta_1}^r\right)^{-\nu}\{\mathcal{A}_n(\vartheta_1)\} = {}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha). \tag{33}$$

Proof. Substituting $a = \left(\alpha-\vartheta_2 D_{\vartheta_2}\vartheta_2 D_{\vartheta_1}^r\right)$ into Equation (23) and subsequently applying the resulting operator to x^n, we obtain

$$\left(\alpha-\vartheta_2 D_{\vartheta_2}\vartheta_2 D_{\vartheta_1}^r\right)^{-\nu}\{\mathcal{A}_n(\vartheta_1)\} = \frac{1}{\Gamma(\nu)}\int_0^{\infty} e^{-\alpha t}t^{\nu-1}\exp\left(t.\alpha-\vartheta_2 D_{\vartheta_2}\vartheta_2 D_{\vartheta_1}^r\right)\mathcal{A}_n(\vartheta_1)dt, \tag{34}$$

which, by virtue of Equation (24), reduces to

$$\left(\alpha-\vartheta_2 D_{\vartheta_2}\vartheta_2 D_{\vartheta_1}^r\right)^{-\nu}\mathcal{A}_n(\vartheta_1) = \frac{1}{\Gamma(\nu)}\int_0^{\infty} e^{-\alpha t}t^{\nu-1}{}_{e^{(r)}}\mathcal{A}_n(\vartheta_1,\vartheta_2 t)dt. \tag{35}$$

The integral transform appearing on the right-hand side of (35) naturally leads to a new polynomial sequence. We denote this sequence by ${}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)$ and refer to it as the ETEAP, defined through

$${}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha) = \frac{1}{\Gamma(\nu)}\int_0^{\infty} e^{-\alpha t}t^{\nu-1}{}_{e^{(r)}}\mathcal{A}_n(\vartheta_1,\vartheta_2 t)dt. \tag{36}$$

Combining relations (35) and (36) directly establishes the result (33). □

Theorem 4. *For the ETEAP* ${}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)$*, the associated generating function is given by*

$$\frac{\exp(\vartheta_1 u)\mathcal{A}(u)}{\left(\alpha-\vartheta_2 D_{\vartheta_2}\vartheta_2 u^r\right)^{\nu}} = \sum_{n=0}^{\infty}{}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)\frac{u^n}{n!}. \tag{37}$$

Proof. Starting from the integral representation (36), we write

$$\sum_{n=0}^{\infty}{}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)\frac{u^n}{n!} = \sum_{n=0}^{\infty}\frac{1}{\Gamma(\nu)}\int_0^{\infty} e^{-\alpha t}t^{\nu-1}{}_{e^{(r)}}\mathcal{A}_n(\vartheta_1,\vartheta_2 t)\frac{u^n}{n!}dt. \tag{38}$$

Invoking Equation (24) on the right-hand side yields

$$\sum_{n=0}^{\infty}{}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)\frac{u^n}{n!} = \frac{\exp(\vartheta_1 u)\mathcal{A}(u)}{\Gamma(\nu)}\int_0^{\infty} e^{-(\alpha-\vartheta_2 D_{\vartheta_2}\vartheta_2 u^r)t}t^{\nu-1}dt. \tag{39}$$

Applying the integral identity (25) to the above expression immediately leads to the generating function (37). □

Remark 1. *When $\alpha = 1$, $\nu = 1$, and $\vartheta_2 = D_{\vartheta_2}^{-1}$, the ETEAP ${}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1, \vartheta_2; \alpha)$ reduces to the truncated exponential-based Appell polynomials ${}_{e^{(r)}}\mathcal{A}_n(\vartheta_1, \vartheta_2)$ as defined in Equation (18).*

Remark 2. *The results established in Theorems 1–4 cannot be obtained from the truncated exponential-based Appell polynomials introduced in [4] by parameter substitution. In particular, the appearance of the fractional inverse operator $(\alpha - \mathcal{D})^{-\nu}$ introduces nonlocal operational effects that are absent in the integer-order framework. Only in the limiting case $(\nu, \alpha) = (1, 1)$ do the present constructions reduce to the classical truncated Appell polynomials.*

The operational connection between the ETEAP ${}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1, \vartheta_2; \alpha)$ and the Appell polynomials $\mathcal{A}(t)$ is summarized below.

Theorem 5. *The following operational representation linking the ETEAP ${}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1, \vartheta_2; \alpha)$ with the Appell polynomials $\mathcal{A}(t)$ holds:*

$$\left(\alpha - \vartheta_2 D_{\vartheta_2} \vartheta_2 D_{\vartheta_1}^r\right)^{-\nu} \mathcal{A}_n(\vartheta_1) = {}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1, \vartheta_2; \alpha). \tag{40}$$

Proof. Replacing $\vartheta_2 D_{\vartheta_2} \vartheta_2 D_{\vartheta_1}^r$ with $\vartheta_2 D_{\vartheta_2} \vartheta_2 D_{\vartheta_1}^r t$, multiplying both sides of Equation (20) by $\frac{1}{\Gamma(\nu)} e^{-\alpha t} t^{\nu-1}$, and integrating over $t \in (0, \infty)$, we arrive at

$$\frac{1}{\Gamma(\nu)} \int_0^\infty e^{-\alpha t} t^{\nu-1} \exp\left(\vartheta_2 D_{\vartheta_2} \vartheta_2 D_{\vartheta_1}^r t\right) \{\mathcal{A}_n(\vartheta_1)\} dt = \frac{1}{\Gamma(\nu)} \int_0^\infty e^{-\alpha t} t^{\nu-1} {}_{e^{(r)}}\mathcal{A}_n(\vartheta_1, \vartheta_2) dt. \tag{41}$$

Using the Weyl identity [22]

$$e^{\hat{A}+\hat{B}} = e^{-k/2} e^{\hat{A}} e^{\hat{B}}, \quad k \in \mathbb{C}, \tag{42}$$

to decouple the exponential operator on the left-hand side, we obtain

$$\frac{1}{\Gamma(\nu)} \int_0^\infty t^{\nu-1} e^{-\left(\alpha - \vartheta_2 D_{\vartheta_2} \vartheta_2 D_{\vartheta_1}^r\right) t} dt \, \mathcal{A}_n(\vartheta_1) = {}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1, \vartheta_2; \alpha). \tag{43}$$

Finally, invoking relation (25) yields the desired result (40). □

Theorem 6. *An explicit summation formula for the ETEAP ${}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1, \vartheta_2; \alpha)$ in terms of the Appell polynomials $\mathcal{A}_n(\vartheta_1)$ is given by*

$${}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1, \vartheta_2; \alpha) = \sum_{k=0}^{n-l} \sum_{l=0}^{n} \binom{n}{l} \binom{n-l}{k} (-1)^k \vartheta_3^l \mathcal{A}_k(\vartheta_3) {}_{e^{(r)}}\mathcal{A}_{n-k-l,\nu}(\vartheta_1, \vartheta_2; \alpha). \tag{44}$$

Proof. Consider the product of the generating functions (9) and (30), written as

$$\mathcal{A}(u) e^{\vartheta_3 u} \left(\alpha - r\vartheta_2 D_{\vartheta_2} \vartheta_2 u^r\right)^{-\nu} \exp(\vartheta_1 u) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \mathcal{A}_k(\vartheta_3) {}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1, \vartheta_2; \alpha) \frac{u^{n+k}}{n!k!}. \tag{45}$$

Replacing n by $n - k$ and shifting the exponential term, we obtain

$$\mathcal{A}(u) \left(\alpha - r\vartheta_2 D_{\vartheta_2} \vartheta_2 u^r\right)^{-\nu} \exp(\vartheta_1 u) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{l=0}^{\infty} \binom{n}{k} \mathcal{A}_k(\vartheta_3) \vartheta_3^l {}_{e^{(r)}}\mathcal{A}_{n-k,\nu}(\vartheta_1, \vartheta_2; \alpha) \frac{u^{n+l}}{n!l!}. \tag{46}$$

Replacing n by $n - l$ yields

$$\mathcal{A}(u)\left(\alpha - r\vartheta_2 D_{\vartheta_2}\vartheta_2 u^r\right)^{-\nu}\exp(\vartheta_1 u) = \sum_{n=0}^{\infty}\sum_{k=0}^{n-l}\sum_{l=0}^{n}\binom{n-l}{k}\binom{n}{l}\mathcal{A}_k(\vartheta_3)\vartheta_{3\,e^{(r)}}^{l}\mathcal{A}_{n-k-l,\nu}(\vartheta_1,\vartheta_2;\alpha)\frac{u^n}{n!}. \tag{47}$$

Finally, applying the generating function (37) to the left-hand side and equating coefficients of like powers of t establishes (44). □

Theorem 7. *The following recurrence relations are satisfied by the ETEAP* ${}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)$*:*

$$\frac{\partial}{\partial\vartheta_1}\,{}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha) = n\;{}_{e^{(r)}}\mathcal{A}_{n-1,\nu}(\vartheta_1,\vartheta_2;\alpha), \tag{48}$$

$$\frac{\partial}{\partial\vartheta_2}\,{}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha) = \nu\; n(n-1)\cdots(n-r+1)\;{}_{e^{(r)}}\mathcal{A}_{n-r,\nu+1}(\vartheta_1,\vartheta_2;\alpha), \tag{49}$$

$$\frac{\partial}{\partial\alpha}\,{}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha) = -\nu\;{}_{e^{(r)}}\mathcal{A}_{n,\nu+1}(\vartheta_1,\vartheta_2;\alpha). \tag{50}$$

Example 1. *To illustrate the distinction between the classical and fractional cases, we consider the first few extended truncated exponential polynomials. For* $\nu = 1$*, the definition reduces to*

$$e_{n,1}^{(r)}(\vartheta_1,\vartheta_2;\alpha) = (\alpha - \vartheta_2 D_{\vartheta_2}\vartheta_2 D_{\vartheta_1}^r)^{-1}\{\vartheta_1^n\},$$

which coincides with the classical truncated exponential polynomials studied previously. However, for $\nu \neq 1$*, we obtain*

$$e_{n,\nu}^{(r)}(\vartheta_1,\vartheta_2;\alpha) = \frac{1}{\Gamma(\nu)}\int_0^\infty e^{-\alpha t}t^{\nu-1}e_n^{(r)}(\vartheta_1,\vartheta_2 t)\,dt,$$

where additional gamma-function weights appear explicitly. Even for small degrees $n = 1, 2$*, these terms generate polynomial expressions that cannot be recovered from the case* $\nu = 1$*, confirming that the fractional parameter* ν *yields genuinely new polynomial families rather than a reparameterization of existing ones.*

3. Monomiality Principle, Series Representation and Determinant Form

Hybrid special polynomials occupy an important position in mathematical analysis owing to their rich algebraic and analytical structure. Their series and determinant representations provide effective tools for deriving recurrence relations, structural identities, and efficient computational schemes, with wide relevance in approximation theory and mathematical physics. By unifying classical and modern polynomial families, these representations enhance both theoretical insight and practical applicability.

We now proceed to derive the multiplicative and derivative operators for the extended truncated exponential-based Appell polynomials ${}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)$. The following theorem establishes these results.

Theorem 8. *The ETEAP* ${}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)$ *satisfies the multiplicative and derivative operators as follows:*

$$\widehat{\mathcal{M}} = \vartheta_1 + \frac{\mathcal{A}'(\widehat{D}_{\vartheta_1})}{\mathcal{A}(\widehat{D}_{\vartheta_1})} + \frac{\nu r\vartheta_2 D_{\vartheta_2}\vartheta_2 D_{\vartheta_1}^{r-1}}{\alpha - \vartheta_2 D_{\vartheta_2}\vartheta_2 D_{\vartheta_1}^r}, \tag{51}$$

and

$$\widehat{\mathcal{P}} = D_{\vartheta_1}, \tag{52}$$

respectively.

Proof. By differentiating both sides of Equation (37) with respect to u, we obtain

$$\frac{\partial}{\partial u}\left[\frac{\mathcal{A}(u)\exp(\vartheta_1 u)}{\left(\alpha-\vartheta_2 D_{\vartheta_2}\vartheta_2 u^r\right)^{\nu}}\right]=\frac{\partial}{\partial u}\left[\sum_{n=0}^{\infty}{}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)\frac{u^n}{n!}\right], \tag{53}$$

which further gives

$$\left(\vartheta_1+\frac{\mathcal{A}'(u}{\mathcal{A}(u)}+\frac{\nu r\vartheta_2 D_{\vartheta_2}\vartheta_2 u^{r-1}}{\alpha-\vartheta_2 D_{\vartheta_2}\vartheta_2 u^r}\right)\left[\frac{\mathcal{A}(u)\exp(\vartheta_1 u)}{\left(\alpha-\vartheta_2 D_{\vartheta_2}\vartheta_2 u^r\right)^{\nu}}\right]=\sum_{n=0}^{\infty} n\ {}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)\frac{u^{n-1}}{n!}. \tag{54}$$

Employing Equation (37), it follows that

$$\sum_{n=0}^{\infty}{}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)\frac{u^n}{n!}=\sum_{n=0}^{\infty}\left(\vartheta_1+\frac{\mathcal{A}'(\widehat{D}_{\vartheta_1})}{\mathcal{A}(\widehat{D}_{\vartheta_1})}+\frac{\nu r\vartheta_2 D_{\vartheta_2}\vartheta_2 D_{\vartheta_1}^{r-1}}{\alpha-\vartheta_2 D_{\vartheta_2}\vartheta_2 D_{\vartheta_1}^r}\right){}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)\frac{u^n}{n!}. \tag{55}$$

In light of Equations (12) and (55), the result (51) follows immediately.
Furthermore, differentiating Equation (37) with respect to ϑ_1 yields

$$\frac{\partial}{\partial\vartheta_1}\left[\frac{\mathcal{A}(u)\exp(\vartheta_1 u)}{\left(\alpha-r\vartheta_2 D_{\vartheta_2}\vartheta_2 u^r\right)^{\nu}}\right]=u\left[\frac{\mathcal{A}(u)\exp(\vartheta_1 u)}{\left(\alpha-\vartheta_2 D_{\vartheta_2}\vartheta_2 u^r\right)^{\nu}}\right],$$

which may be equivalently rewritten as

$$\frac{\partial}{\partial\vartheta_1}\left[\sum_{n=0}^{\infty}{}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)\frac{u^n}{n!}\right]=\left[\sum_{n=0}^{\infty}{}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)\frac{u^{n+1}}{n!}\right]. \tag{56}$$

Once more, by combining relations (13) and (56), we obtain the result (52). □

Theorem 9. *The following differential equation for* ${}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)$ *holds true:*

$$\left(\vartheta_1\widehat{D}_{\vartheta_1}+\frac{\mathcal{A}'(\widehat{D}_{\vartheta_1})}{\mathcal{A}(\widehat{D}_{\vartheta_1})}\widehat{D}_{\vartheta_1}+\frac{\nu r\vartheta_2 D_{\vartheta_2}\vartheta_2 D_{\vartheta_1}^r}{\alpha-\vartheta_2 D_{\vartheta_2}\vartheta_2 D_{\vartheta_1}^r}-n\right){}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)=0. \tag{57}$$

Proof. Using (51) and (52) in (15), we get

$$\left(\vartheta_1\widehat{D}_{\vartheta_1}+\frac{\mathcal{A}'(\widehat{D}_{\vartheta_1})}{\mathcal{A}(\widehat{D}_{\vartheta_1})}\widehat{D}_{\vartheta_1}+\frac{\nu r(r-1)\vartheta_2 D_{\vartheta_2}\vartheta_2 D_{\vartheta_1}^r}{\alpha-r\vartheta_2 D_{\vartheta_2}\vartheta_2 D_{\vartheta_1}^r}\right){}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)=n\ {}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha). \tag{58}$$

Upon simplification of the above expression, the result (57) follows. □

Here, we derive the series representation of ETEAP ${}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)$ as follows:

Theorem 10. *The ETEAP* ${}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)$ *are defined by the series:*

$${}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)=\sum_{k=0}^{n}\binom{n}{k}\mathcal{A}_k\ e_{n-k,\nu}^{(r)}(\vartheta_1,\vartheta_2;\alpha), \tag{59}$$

with $\mathcal{A}_k$ *given by Equation* (9).

Proof. Taking Equation (39) into account, we can express

$$\sum_{v=0}^{\infty}{}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)\frac{u^n}{n!}=\mathcal{A}(u)\sum_{n=0}^{\infty}e_{n,\nu}^{(r)}(\vartheta_1,\vartheta_2;\alpha)\frac{u^n}{n!}. \tag{60}$$

Employing the expansion (9) of $\mathcal{A}(u)$ on the left-hand side of Equation (60), we simplify the resulting expression and then compare the coefficients of identical powers of δ on both sides. This procedure directly leads to the assertion (59). □

Next, we derive the determinant representation of ${}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)$. Although the determinant formulation presented in this section is structurally inspired by classical Appell determinants [9,23–25], its role in the present work is primarily theoretical. The determinant form provides a compact algebraic characterization of the extended truncated exponential-based Appell polynomials ${}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)$ and encodes their recurrence structure in a unified manner. Rather than being introduced for direct numerical computation, it serves as a structural bridge connecting classical Appell sequences with their fractional extensions and highlights the algebraic depth of the proposed polynomial family.

Theorem 11. *The ETEAP* ${}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)$ *has the following determinant representation:*

$$ {}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha) = \frac{(-1)^n}{(\beta_0)^{n+1}} \begin{vmatrix} 1 & e^{(r)}_{1,\nu}(\vartheta_1,\vartheta_2;\alpha) & e^{(r)}_{2,\nu}(\vartheta_1,\vartheta_2;\alpha) & \dots & e^{(r)}_{n-1,\nu}(\vartheta_1,\vartheta_2;\alpha) & e^{(r)}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha) \\ \beta_0 & \beta_1 & \beta_2 & \dots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \dots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \beta_0 & \dots & \binom{n-1}{1}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ \vdots & \vdots & \vdots & \dots & \vdots & \\ 0 & 0 & 0 & \dots & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix}. \tag{61} $$

Proof. Making use of the following series expansion of $\frac{1}{\mathcal{A}(u)}$:

$$ [\mathcal{A}(u)]^{-1} = \sum_{k=0}^{\infty} \beta_k \frac{u^k}{k!}, $$

Invoking the generating function (37), we get

$$ \frac{\exp(\vartheta_1 u)}{\left(\alpha - \vartheta_2 D_{\vartheta_2}\vartheta_2 u^r\right)^{\nu}} = \left(\sum_{k=0}^{\infty} \beta_k \frac{u^k}{k!}\right)\left(\sum_{n=0}^{\infty} {}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)\frac{u^n}{n!}\right). $$

Therefore, it can be expressed as

$$ \sum_{n=0}^{\infty} e^{(r)}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)\frac{u^n}{n!} = \left(\sum_{k=0}^{\infty} \beta_k \frac{u^k}{k!}\right)\left(\sum_{n=0}^{\infty} {}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)\frac{u^n}{n!}\right). $$

Using the Cauchy product, we have

$$ \sum_{n=0}^{\infty} e^{(r)}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)\frac{u^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\beta_k \, {}_{e^{(r)}}\mathcal{A}_{n-k,\nu}(\vartheta_1,\vartheta_2;\alpha)\frac{u^n}{n!}. $$

Comparison of the coefficients of $\frac{t^n}{n!}$ leads to

$$ e^{(r)}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha) = \sum_{k=0}^{n}\binom{n}{k}\beta_k \, {}_{e^{(r)}}\mathcal{A}_{n-k,\nu}(\vartheta_1,\vartheta_2;\alpha), \quad n \in \mathbb{N}_0. $$

As a result, the following system of equations is obtained:

$$ e^{(r)}_{0,\nu}(\vartheta_1,\vartheta_2;\alpha) = \beta_0 \, {}_{e^{(r)}}\mathcal{A}_{0,\nu}(\vartheta_1,\vartheta_2;\alpha), $$

$$ e^{(r)}_{1,\nu}(\vartheta_1,\vartheta_2;\alpha) = \beta_0 \, {}_{e^{(r)}}\mathcal{A}_{1,\nu}(\vartheta_1,\vartheta_2;\alpha) + \beta_1 \, {}_{e^{(r)}}\mathcal{A}_{0,\nu}(\vartheta_1,\vartheta_2;\alpha), $$

$$e^{(r)}_{2,\nu}(\vartheta_1,\vartheta_2;\alpha)=\beta_0\,{}_{e^{(r)}}\mathcal{A}_{2,\nu}(\vartheta_1,\vartheta_2;\alpha)+\binom{2}{1}\beta_1\,{}_{e^{(r)}}\mathcal{A}_{1,\nu}(\vartheta_1,\vartheta_2;\alpha)+\beta_2\,{}_{e^{(r)}}\mathcal{A}_{0,\nu}(\vartheta_1,\vartheta_2;\alpha),$$

$$\vdots$$

$$e^{(r)}_{n-1,\nu}(\vartheta_1,\vartheta_2;\alpha)=\beta_0\,{}_{e^{(r)}}\mathcal{A}_{n-1,\nu}(\vartheta_1,\vartheta_2;\alpha)+\binom{n-1}{1}\beta_1\,{}_{e^{(r)}}\mathcal{A}_{n-2,\nu}(\vartheta_1,\vartheta_2;\alpha)+\cdots+\beta_{n-1}\,{}_{e^{(r)}}\mathcal{A}_{0,\nu}(\vartheta_1,\vartheta_2;\alpha),$$

$$e^{(r)}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)=\beta_0\,{}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)+\binom{n}{1}\beta_1\,{}_{e^{(r)}}\mathcal{A}_{n-1,\nu}(\vartheta_1,\vartheta_2;\alpha)+\cdots+\beta_n\,{}_{e^{(r)}}\mathcal{A}_{0,\nu}(\vartheta_1,\vartheta_2;\alpha).$$

Applying Cramers' rule, we get

$${}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)=\frac{\begin{vmatrix}\beta_0 & 0 & \ldots & 0 & e^{(r)}_{0,\nu}(\vartheta_1,\vartheta_2;\alpha)\\ \beta_1 & \beta_0 & \ldots & 0 & e^{(r)}_{1,\nu}(\vartheta_1,\vartheta_2;\alpha)\\ \beta_2 & \binom{2}{1}\beta_1 & \ldots & 0 & e^{(r)}_{2,\nu}(\vartheta_1,\vartheta_2;\alpha)\\ \beta_3 & \binom{3}{2}\beta_2 & \ldots & 0 & e^{(r)}_{3,\nu}(\vartheta_1,\vartheta_2;\alpha)\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \beta_{n-1} & \binom{n-1}{1}\beta_{n-2} & \ldots & \beta_0 & e^{(r)}_{n-1,\nu}(\vartheta_1,\vartheta_2;\alpha)\\ \beta_n & \binom{n}{1}\beta_{n-1} & \ldots & \binom{n}{n-1}\beta_1 & e^{(r)}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)\end{vmatrix}}{\begin{vmatrix}\beta_0 & 0 & \ldots & 0 & 0\\ \beta_1 & \beta_0 & \ldots & 0 & 0\\ \beta_2 & \binom{2}{1}\beta_1 & \ldots & 0 & 0\\ \beta_3 & \binom{3}{2}\beta_2 & \ldots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \beta_{n-1} & \binom{n-1}{1}\beta_{n-2} & \ldots & \beta_0 & 0\\ \beta_n & \binom{n}{1}\beta_{n-1} & \ldots & \binom{n}{n-1}\beta_1 & \beta_0\end{vmatrix}}.$$

By taking the transpose in the last equation, we have

$${}_{e^{(r)}}\mathcal{A}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)=\frac{1}{(\beta_0)^{n+1}}\begin{vmatrix}\beta_0 & \beta_1 & \ldots & \beta_{n-1} & \beta_n\\ 0 & \beta_0 & \ldots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1}\\ 0 & 0 & \ldots & \binom{n-1}{1}\beta_{n-3} & \binom{n}{2}\beta_{n-2}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \ldots & \beta_0 & \binom{n}{n-1}\beta_1\\ e^{(r)}_{0,\nu}(\vartheta_1,\vartheta_2;\alpha) & e^{(r)}_{1,\nu}(\vartheta_1,\vartheta_2;\alpha) & \ldots & e^{(r)}_{n-1,\nu}(\vartheta_1,\vartheta_2;\alpha) & e^{(r)}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)\end{vmatrix}.$$

Hence, the proof follows by performing suitable elementary row operations. □

4. Examples

The class of Appell polynomials can be described as good tool in the formulation and analysis of solutions to numerous differential equations, and they are defined with the help of a related parameter functional, denoted as $\mathcal{A}(u)$. The generality of the generating function is provided by the freedom to choose different forms of the generating function of time, and this freedom gives a wide range of polynomial sequences, with an extraordinary flexibility both in theoretical studies and in applied modeling. This is a significant advantage of Appell polynomials in many different fields of science, especially physics and engineering. Table 1 gives a detailed account of their generative terms, explicit series expansions, and representative numerical values. The generating functions give compact power-series terms which make it easier to perform analytical operations, but the series

representations give rigorous expressions that can be easily used in symbolic computation and numerical analysis. As well, numerical values are presented in a tabular manner which makes them easy to understand and implement in applied problems. Thanks to these positive properties, Appell polynomials have effectively found application in fields like stochastic processes, quantum theory and signal analysis. Their structural pliability allows them to be tailored to complex mathematical models, thus defining the Appell family of polynomials as a powerful group and cohesive framework in modern applied mathematics.

The classical Bernoulli, Euler, and Genocchi number sequences have a fundamental place in mathematical analysis and its applications. The use of Bernoulli numbers is largely hidden in summation methods, especially in the Euler–Maclaurin formula, and is central to the expansions of polynomials. The Euler numbers are a natural occurrence in the theory of special functions, in the theory of modular forms, and in the theory of elliptic curves. Conversely, they are common in Genocchi numbers, which appear in combinatorial frameworks, graph-theoretic frameworks and automata-related issues. Such series of numbers are also associated with hyperbolic trigonometric series and have application in applied mathematics like quantum field theory and signal processing. Applying the Appell polynumbers framework to interpret these classical numbers results in a construction of enriched families of polynomials, such as two-iterated Hermite–Appell polynomials. These extensions provide new paths to explore the generating functions, algebraic properties and analytical properties and thus provide more directions in the current development of the special functions and the theory of polynomials.

Remark 3. *Choosing the parameter function* $\mathcal{A}(u) = \dfrac{u}{e^u - 1}$ *and identifying* $\mathcal{A}_n(u)$ *with the Bernoulli polynomials* $\mathcal{B}_n(\vartheta_1)$ *in Equations* (33), (37), (44), (51), (57), *and* (61), *we obtain a corresponding subclass of the extended truncated exponential-based Appell polynomials. This specialization gives rise to the* extended truncated exponential-based Bernoulli polynomials, *denoted by* ${}_{e^{(r)}}\mathcal{B}_{n,\nu}(\vartheta_1, \vartheta_2; \alpha)$.

For this family, the following operational identity holds:

$$\left(\alpha - \vartheta_2 D_{\vartheta_2} \vartheta_2 D_{\vartheta_1}^r\right)^{-\nu} \{\mathcal{B}_n(\vartheta_1)\} = {}_{e^{(r)}}\mathcal{B}_{n,\nu}(\vartheta_1, \vartheta_2; \alpha). \tag{62}$$

Moreover, the generating function associated with the extended truncated exponential-based Bernoulli polynomials is given by

$$\frac{\exp(\vartheta_1 u)\mathcal{B}(u)}{\left(\alpha - \vartheta_2 D_{\vartheta_2} \vartheta_2 u^r\right)^{\nu}} = \sum_{n=0}^{\infty} {}_{e^{(r)}}\mathcal{B}_{n,\nu}(\vartheta_1, \vartheta_2; \alpha) \frac{u^n}{n!}. \tag{63}$$

An explicit summation formula connecting the ETEP and the Bernoulli polynomials $\mathcal{B}_n(\vartheta_1)$ *is obtained as*

$${}_{e^{(r)}}\mathcal{B}_{n,\nu}(\vartheta_1, \vartheta_2; \alpha) = \sum_{k=0}^{n-l} \sum_{l=0}^{n} \binom{n}{l} \binom{n-l}{k} (-1)^k \vartheta_3^l \mathcal{B}_k(\vartheta_3) \, {}_{e^{(r)}}\mathcal{B}_{n-k-l,\nu}(\vartheta_1, \vartheta_2; \alpha). \tag{64}$$

In addition, the extended truncated exponential-based Bernoulli polynomials satisfy the following multiplicative and derivative operators:

$$\widehat{\mathcal{M}} = \vartheta_1 + \frac{e^{\widehat{D}_{\vartheta_1}}(1 - \widehat{D}_{\vartheta_1}) - 1}{e^{\widehat{D}_{\vartheta_1}}(e^{\widehat{D}_{\vartheta_1}} - 1)} + \frac{\nu r \vartheta_2 D_{\vartheta_2} \vartheta_2 D_{\vartheta_1}^{r-1}}{\alpha - \vartheta_2 D_{\vartheta_2} \vartheta_2 D_{\vartheta_1}^r}, \tag{65}$$

and

$$\widehat{\mathcal{P}} = D_{\vartheta_1}, \tag{66}$$

respectively.

Consequently, ${}_{e^{(r)}}\mathcal{B}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)$ *satisfies the differential equation*

$$\left(\vartheta_1\widehat{D}_{\vartheta_1}+\frac{e^{\widehat{D}_{\vartheta_1}}(1+\widehat{D}_{\vartheta_1})-1}{(e^{\widehat{D}_{\vartheta_1}}-1)}+\frac{\nu r\vartheta_2 D_{\vartheta_2}\vartheta_2 D^r_{\vartheta_1}}{\alpha-\vartheta_2 D_{\vartheta_2}\vartheta_2 D^r_{\vartheta_1}}-n\right){}_{e^{(r)}}\mathcal{B}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)=0. \tag{67}$$

Furthermore, in light of expression (61)*, the polynomials* ${}_{e^{(r)}}\mathcal{B}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)$ *admit determinant representations:*

$${}_{e^{(r)}}\mathcal{B}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)=(-1)^n\begin{vmatrix} 1 & e^{(r)}_{0,\nu}(\vartheta_1,\vartheta_2;\alpha) & e^{(r)}_{1,\nu}(\vartheta_1,\vartheta_2;\alpha) & \ldots & e^{(r)}_{n-1,\nu}(\vartheta_1,\vartheta_2;\alpha) & e^{(r)}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)\\ 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} & \frac{1}{n+1}\\ 0 & 1 & \binom{2}{1}\frac{1}{2} & \cdots & \binom{n-1}{1}\frac{1}{n-1} & \binom{n}{1}\frac{1}{n}\\ 0 & 0 & 1 & \cdots & \binom{n-1}{2}\frac{1}{n-2} & \binom{n}{2}\frac{1}{n-1}\\ . & . & . & \cdots & . & .\\ . & . & . & \cdots & . & .\\ 0 & 0 & 0 & \cdots & 1 & \binom{n}{n-1}\frac{1}{2}\end{vmatrix}, \tag{68}$$

Remark 4. *If we choose the parameter function* $\mathcal{A}(t)=\dfrac{2}{e^t+1}$ *and identify* $\mathcal{A}_n(t)$ *with the Euler polynomials* $\mathcal{E}_n(t)$ *in Equations* (33)*,* (37)*,* (44)*,* (51)*,* (57)*, and* (61)*, then a corresponding Euler-type subclass of the extended truncated exponential-based Appell polynomials is obtained. This specialization leads to the family of* extended truncated exponential-based Euler polynomials*, denoted by* ${}_{e^{(r)}}\mathcal{E}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)$*.*

For this class, the following operational representation holds:

$$\left(\alpha-\vartheta_2 D_{\vartheta_2}\vartheta_2 D^r_{\vartheta_1}\right)^{-\nu}\{\mathcal{E}_n(\vartheta_1)\}={}_{e^{(r)}}\mathcal{E}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha). \tag{69}$$

The generating function associated with the extended truncated exponential-based Euler polynomials is given by

$$\frac{\exp(\vartheta_1 u)\mathcal{E}(u)}{\left(\alpha-\vartheta_2 D_{\vartheta_2}\vartheta_2 u^r\right)^{\nu}}=\sum_{n=0}^{\infty}{}_{e^{(r)}}\mathcal{E}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)\frac{u^n}{n!}. \tag{70}$$

An explicit summation formula connecting the ETEP and the Euler polynomials $\mathcal{E}_n(\vartheta_1)$ *is obtained in the form*

$${}_{e^{(r)}}\mathcal{E}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)=\sum_{k=0}^{n-l}\sum_{l=0}^{n}\binom{n}{l}\binom{n-l}{k}(-1)^k\vartheta_3^l\mathcal{E}_k(\vartheta_3)\,{}_{e^{(r)}}\mathcal{E}_{n-k-l,\nu}(\vartheta_1,\vartheta_2;\alpha). \tag{71}$$

Furthermore, the multiplicative and derivative operators corresponding to ${}_{e^{(r)}}\mathcal{E}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)$ *are given, respectively, by*

$$\widehat{\mathcal{M}}=\vartheta_1-\frac{e^{\widehat{D}_{\vartheta_1}}}{e^{\widehat{D}_{\vartheta_1}}+1}+\frac{\nu r\vartheta_2 D_{\vartheta_2}\vartheta_2 D^{r-1}_{\vartheta_1}}{\alpha-\vartheta_2 D_{\vartheta_2}\vartheta_2 D^r_{\vartheta_1}}, \tag{72}$$

and

$$\widehat{\mathcal{P}}=D_{\vartheta_1}. \tag{73}$$

As a consequence, the extended truncated exponential-based Euler polynomials satisfy the following differential equation:

$$\left(\vartheta_1 \widehat{D}_{\vartheta_1} - \frac{e^{\widehat{D}_{\vartheta_1}}}{e^{\widehat{D}_{\vartheta_1}}+1}\widehat{D}_{\vartheta_1} + \frac{\nu r \vartheta_2 D_{\vartheta_2}\vartheta_2 D^r_{\vartheta_1}}{\alpha - \vartheta_2 D_{\vartheta_2}\vartheta_2 D^r_{\vartheta_1}} - n\right) {}_{e^{(r)}}\mathcal{E}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha) = 0. \tag{74}$$

Finally, in view of relation (61), the polynomials ${}_{e^{(r)}}\mathcal{E}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)$ *admit determinant representations:*

$${}_{e^{(r)}}\mathcal{E}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha) = (-1)^n \begin{vmatrix} 1 & e^{(r)}_{0,\nu}(\vartheta_1,\vartheta_2;\alpha) & e^{(r)}_{1,\nu}(\vartheta_1,\vartheta_2;\alpha) & \dots & e^{(r)}_{n-1,\nu}(\vartheta_1,\vartheta_2;\alpha) & e^{(r)}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha) \\ 1 & \frac{1}{2} & \frac{1}{2}(1)_2 & \cdots & \frac{1}{2}(1)_{n-1} & \frac{1}{2}(1)_n \\ 0 & 1 & \binom{2}{1}\frac{1}{2} & \cdots & \binom{n-1}{1}\frac{1}{2}(1)_{n-2} & \binom{n}{1}\frac{1}{2}(1)_{n-1} \\ 0 & 0 & 1 & \cdots & \binom{n-1}{2}\frac{1}{2}(1)_{n-3} & \binom{n}{2}\frac{1}{2}(1)_{n-2} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & 1 & \binom{n}{n-1}\frac{1}{2} \end{vmatrix}. \tag{75}$$

Remark 5. *By selecting the parameter function* $\mathcal{A}(t) = \dfrac{2t}{e^t+1}$ *and identifying* $\mathcal{A}_n(u)$ *with the Genocchi polynomials* $\mathcal{G}_n(u)$ *in Equations (33), (37), (44), (51), (57), and (61), we obtain a Genocchi-type specialization of the extended truncated exponential-based Appell polynomials. This construction leads to the family of* extended truncated exponential-based Genocchi polynomials, *denoted by* ${}_{e^{(r)}}\mathcal{G}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)$.

For this class of polynomials, the following operational representation is valid:

$$\left(\alpha - \vartheta_2 D_{\vartheta_2}\vartheta_2 D^r_{\vartheta_1}\right)^{-\nu}\{\mathcal{G}_n(\vartheta_1)\} = {}_{e^{(r)}}\mathcal{G}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha). \tag{76}$$

The generating function corresponding to the extended truncated exponential-based Genocchi polynomials is given by

$$\frac{\exp(\vartheta_1 u)\mathcal{G}(u)}{\left(\alpha - \vartheta_2 D_{\vartheta_2}\vartheta_2 u^r\right)^{\nu}} = \sum_{n=0}^{\infty} {}_{e^{(r)}}\mathcal{G}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)\frac{u^n}{n!}. \tag{77}$$

An explicit summation formula linking the ETEP and the Genocchi polynomials $\mathcal{G}_n(\vartheta_1)$ *is expressed as*

$${}_{e^{(r)}}\mathcal{G}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha) = \sum_{k=0}^{n-l}\sum_{l=0}^{n}\binom{n}{l}\binom{n-l}{k}(-1)^k\vartheta_3^l\mathcal{G}_k(\vartheta_3)\,{}_{e^{(r)}}\mathcal{G}_{n-k-l,\nu}(\vartheta_1,\vartheta_2;\alpha). \tag{78}$$

In addition, the multiplicative and derivative operators associated with ${}_{e^{(r)}}\mathcal{G}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)$ *are given by*

$$\widehat{\mathcal{M}} = \vartheta_1 + \frac{e^{\widehat{D}_{\vartheta_1}}(1-\widehat{D}_{\vartheta_1})+1}{e^{\widehat{D}_{\vartheta_1}}+1} + \frac{\nu r \vartheta_2 D_{\vartheta_2}\vartheta_2 D^{r-1}_{\vartheta_1}}{\alpha - \vartheta_2 D_{\vartheta_2}\vartheta_2 D^r_{\vartheta_1}}, \tag{79}$$

and

$$\widehat{\mathcal{P}} = D_{\vartheta_1}, \tag{80}$$

respectively.

Consequently, the extended truncated exponential-based Genocchi polynomials satisfy the following differential equation:

$$\left(\vartheta_1 \widehat{D}_{\vartheta_1} + \frac{e^{\widehat{D}_{\vartheta_1}}(1-\widehat{D}_{\vartheta_1})+1}{e^{\widehat{D}_{\vartheta_1}}+1}\widehat{D}_{\vartheta_1} + \frac{\nu r \vartheta_2 D_{\vartheta_2}\vartheta_2 D^r_{\vartheta_1}}{\alpha - \vartheta_2 D_{\vartheta_2}\vartheta_2 D^r_{\vartheta_1}} - n\right) {}_{e^{(r)}}\mathcal{G}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha) = 0. \quad (81)$$

Finally, in view of relation (61), the polynomials ${}_{e^{(r)}}\mathcal{G}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha)$ *admit determinant representations:*

$${}_{e^{(r)}}\mathcal{G}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha) = (-1)^n \begin{vmatrix} 1 & e^{(r)}_{0,\nu}(\vartheta_1,\vartheta_2;\alpha) & e^{(r)}_{1,\nu}(\vartheta_1,\vartheta_2;\alpha) & \dots & e^{(r)}_{n-1,\nu}(\vartheta_1,\vartheta_2;\alpha) & e^{(r)}_{n,\nu}(\vartheta_1,\vartheta_2;\alpha) \\ \frac{1}{2} & \frac{(1)_2}{4} & \frac{(1)_3}{6} & \cdots & \frac{(1)_n}{2n} & \frac{(1)_{n+1}}{2(n+1)} \\ 0 & \frac{1}{2} & \binom{2}{1}\frac{(1)_2}{4} & \cdots & \binom{n-1}{1}\frac{(1)_{n-1}}{2(n-1)} & \binom{n}{1}\frac{(1)_n}{2n} \\ 0 & 0 & \frac{1}{2} & \cdots & \binom{n-1}{2}\frac{(1)_{n-2}}{2(n-2)} & \binom{n}{2}\frac{(1)_{n-1}}{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{2} & \binom{n}{n-1}\frac{(1)_2}{4} \end{vmatrix}. \quad (82)$$

5. Concluding Remarks

In this article, we introduce and develop a comprehensive extension of truncated exponential polynomials and truncated exponential-type Appell polynomial sequences. A detailed investigation of their fundamental algebraic properties is carried out, leading to the derivation of explicit recurrence relations, as well as the construction of corresponding multiplicative and differential operators. By employing the concept of quasi-monomiality, the governing differential equations satisfied by these polynomial families are also established. The presence of both series representations and determinant formulations highlights the intrinsic algebraic richness and structural depth of the proposed class of polynomials. Furthermore, the introduction of generalized Legendre–Hermite–Appell polynomials, along with notable special cases associated with Bernoulli, Euler, and Genocchi polynomials, broadens the scope of the classical theory of special functions. Collectively, these findings contribute to a deeper understanding of polynomial sequences and underscore their significance in mathematical physics and the analysis of differential equations.

In the future, additional investigation into these classes of polynomial systems within the framework of orthogonality relations and integral transform methods could provide useful information regarding their analysis capabilities. The review of their relations to fractional calculus and more sophisticated systems of special functions would be stimulated to attract new theoretical outcomes that have implications for the method of approximation and signal analysis. Also, it can be seen that there are deeper phenomena in the algebraic and combinatorial worlds that could be uncovered by extending the current construction to the context of q-calculus and number theory. Lastly, the creation of numerical algorithms and computational representations of these polynomials can help in the use of these polynomials in scientific computation and engineering modeling.

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References

1. Dattoli, G.; Cesarano, C.; Sacchetti, D. A note on truncated polynomials. *Appl. Math. Comput.* **2003**, *134*, 595–605. [CrossRef]
2. Ozat, Z.; Cekim, B.; Ozarslan, M.A.; Costabile, F.A. Truncated-exponential-based general-Appell polynomials. *Mathematics* **2025**, *13*, 1266. [CrossRef]
3. Khan, S.; Yasmin, G.; Ahmad, N. On a new family related to truncated exponential and Sheffer polynomials. *J. Math. Anal. Appl.* **2014**, *418*, 921–937. [CrossRef]
4. Khan, S.; Yasmin, G.; Ahmad, N. A note on truncated exponential-based Appell polynomials. *Bull. Malays. Math. Sci. Soc.* **2017**, *40*, 373–388. [CrossRef]
5. Srivastava, H.M.; Araci, S.; Khan, W.A.; Acikgoz, M. A note on the truncated-exponential-based Apostol-type polynomials. *Symmetry* **2019**, *11*, 538. [CrossRef]
6. Qawaqneh, H.; Khan, W.A.; Aydi, H.; Wani, S.A.; Jadhav, P. Certain properties and characterizations of Δ_h-truncated exponential-based Hermite polynomials. *Eur. J. Pure Appl. Math.* **2025**, *18*, 6657. [CrossRef]
7. Costabile, F.A.; Longo, E. An algebraic approach to Sheffer polynomial sequences. *Integral Transform. Spec. Funct.* **2013**, *25*, 295–311. [CrossRef]
8. Costabile, F.A.; Serpe, A. An algebraic approach to Lidstone polynomials. *Appl. Math. Lett.* **2007**, *20*, 387–390. [CrossRef]
9. Costabile, F.A.; Longo, E. A determinantal approach to Appell polynomials. *J. Comput. Appl. Math.* **2010**, *234*, 1528–1542. [CrossRef]
10. Andrews, L.C. *Special Functions for Engineers and Applied Mathematicians*; Macmillan Publishing Company: New York, NY, USA, 1985.
11. Steffensen, J.F. The poweroid, an extension of the mathematical notion of power. *Acta Math.* **1941**, *73*, 333–366. [CrossRef]
12. Dattoli, G. Hermite–Bessel and Laguerre–Bessel functions: A by-product of the monomiality principle. In *Advanced Special Functions and Applications*; Proc. Melfi School on Advanced Topics in Mathematical Physics; Aracne: Rome, Italy, 2000; Volume 1, pp. 147–164.
13. Dattoli, G. Generalized polynomials, operational identities and their applications. *J. Comput. Appl. Math.* **2000**, *118*, 111–123. [CrossRef]
14. Dattoli, G.; Migliorati, M.; Srivastava, H.M. A class of Bessel summation formulas and associated operational methods. *Fract. Calc. Appl. Anal.* **2004**, *7*, 169–176.
15. Oldham, H.; Spanier, N. *The Fractional Calculus*; Academic Press: San Diego, CA, USA, 1974.
16. Srivastava, H.M.; Manocha, H.L. *A Treatise on Generating Functions*; Halsted Press: New York, NY, USA, 1984.
17. Dattoli, G.; Ricci, P.E.; Cesarano, C.; Vázquez, L. Special polynomials and fractional calculus. *Math. Comput. Model.* **2003**, *37*, 729–733. [CrossRef]
18. Zayed, M.; Alqurashi, T.; Khan, W.A.; Wani, S.A. A comprehensive study of Δ_h-truncated exponential Appell polynomials: Properties, fractional operators, and computational aspects. *Math. Methods Appl. Sci.* **2025**. [CrossRef]
19. Zayed, M.; Alqurashi, T.; Wani, S.A.; Samei, M.E. Differential and integral equations involving multivariate special polynomials with applications in computer modeling. *Fractal Fract.* **2025**, *9*, 512. [CrossRef]
20. Wani, S.A.; Shah, T.U.R.; Ramírez, W.; Cesarano, C. Exploring the properties of multivariable Hermite polynomials in relation to Apostol-type Frobenius–Genocchi polynomials. *Georgian Math. J.* **2025**, *32*, 515–528. [CrossRef]
21. Zayed, M.; Wani, S.A. Certain properties and characterizations of two-iterated two-dimensional Appell and related polynomials via fractional operators. *Fractal Fract.* **2024**, *8*, 382. [CrossRef]
22. Dattoli, G.; Ottaviani, P.L.; Torre, A. Vázquez, L. Evolution operator equations: Integration with algebraic and finite-difference methods. Applications to physical problems in classical and quantum mechanics and quantum field theory. *Riv. Nuovo Cimento Soc. Ital. Fis.* **1997**, *20*, 1–133. [CrossRef]
23. Costabile, F.A.; Gualtieri, M.I.; Napoli, A. The parabolic mean operator and related polynomial sequence: A theoretical approach. *Calcolo* **2025**, *62*, 18. [CrossRef]

24. Costabile, F.; Longo, E. Δ_h Appell sequences and related interpolation problem. *Numer. Algorithms* **2013**, *63*, 165–186. [CrossRef]
25. Wang, W. A determinantal approach to Sheffer sequences. *Linear Algebra Its Appl.* **2014**, *463*, 228–254.

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