axioms

MDPI

Article

Multivalued Fixed Point Results and Homotopy Theorem in Hausdorff-Type Fuzzy Partial Metric Spaces

Ilyas Khan[1,*], Talha Anwar [2,3] and Osman Abdalla Adam Osman [4,*]

[1] Abdus Salam School of Mathematical Sciences, Government College University, Lahore 54600, Pakistan
[2] Department of Computer Engineering, Biruni University, 34010 Istanbul, Turkey; talha.anwar@biruni.edu.tr
[3] Faculty of Engineering, Shinawatra University, Bangkok 10900, Thailand
[4] Department of Mathematics, College of Sciences, Qassim University, Buraydah 51452, Saudi Arabia
* Correspondence: ilyas.khan_22@sms.edu.pk (I.K.); o.osman@qu.edu.sa (O.A.A.O.)

Abstract

We study multivalued fixed-point problems in fuzzy partial metric spaces (FPMSs). Under the additional normalization condition $\Psi_P(x, x, t) = 1$ for all $x \in \Omega$ and $t > 0$, we establish a Hausdorff-type construction on the family of nonempty compact subsets and investigate its fundamental properties. Using this framework, we prove a fixed-point theorem (FPT) for multivalued fuzzy η-contractive mappings under suitable completeness and contractivity assumptions. A separate example illustrates a particular non-normalized fuzzy partial metric and its induced Hausdorff-type function, without asserting a general construction for arbitrary non-unit self-distances. We also establish a homotopy FPT for a parameterized family of multivalued mappings, showing that fixed-point (FP) existence is preserved between the endpoints of the parameter interval. By advancing research in fuzzy FP theory, this study contributes to the mathematical research and innovation associated with Industry, Innovation and Infrastructure.

Keywords: Hausdorff fuzzy partial metric space; Hausdorff fuzzy partial metric; fuzzy partial contraction; homotopy theorem; iterative methods; multivalued fixed point results

MSC: Primary: 54H25 Secondary: 47H10, 54C60, 54E50.

1. Introduction

An essential extension of a metric space (MS) is a partial metric space (PMS), where the distance of a point from itself does not need to be zero. It has been demonstrated that a topology τ_{Ψ_p} on Ω, which is 0, can be produced from a partial metric Ψ_P. Similarly, given a partial metric Ψ_P, there is always a consistent metric s.t. a sequence $\{\zeta_n\}$ is Cauchy in the PMS (Ω, Ψ_P) if and only if $\{\zeta_n\}$ is Cauchy in the MS (Ω, d_P) [1]. FP results in PMSs have been examined by numerous authors (see [2–4] and references therein). The foundation of fuzzy mathematics was laid by Zadeh [5] in 1965. FP theory is still a staple of nonlinear functional analysis, providing invaluable insight into the solution of equations in mathematics, physics, economics, and engineering. Since the seminal contraction principle of Banach, the discipline has developed further in abstract spaces to cover more complicated systems. Among them, fuzzy metric spaces, introduced by Kramosil et al. [6] and developed by George et al. [7], represent the uncertainty that is present in real-world phenomena by approximating the concept of nearness through fuzzy sets. At the same time, PMSs are the generalization of classical metrics, which is that they

Check for updates

Academic Editor: Juan José Miñana

Received: 12 October 2026
Revised: 15 November 2026
Accepted: 17 November 2026
Published: 18 November 2026

Copyright: © 2026 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license.

can have non-zero self-distances, which are suited to fields such as computer science where self-similarity is important. Their combination into FPMSs forms a powerful model to deal concurrently with imprecision and non-zero self-distances, but their theory of FP is not properly studied. Alternatively, Kramosil and Michalek [6] established the notion of fuzzy metric space (FMS), a generalization of MS in the fuzzy context, based on the idea of probabilistic MS. Subsequently, George et al. [7] introduced a new notion of FMS, which we refer to as GV-FMS, by modifying the notion of FMS in the sense of Kramosil et al. [6]. It has been demonstrated that a GV-fuzzy metric M can provide a Hausdorff topology τ_M on Ω. In GV-FMSs, numerous authors have constructed different fuzzy contractive mappings and derived numerous FPTs [8–14].

Markin [15] started looking into FPs for nonexpansive maps and multivalued contractions using the Pompeiu–Hausdorff metric in 1973. Heilpern [16] started studying fuzzy multivalued maps in 1981 and discovered a fuzzy cousin of Nadler's FPT [17]. The existence of FPs in fuzzy multivalued maps was later examined by numerous authors (see [18–23]). The notion of a partial Hausdorff metric was first presented by Hassen Aydi at el. [24], who also started researching FP theory for multi-valued mappings on PMS using the partial Hausdorff metric and proved an analog of the well-known Nadler's FPT.

Recent studies have further demonstrated the use of fuzzy and fractional-order methods in nonlinear control, synchronization, and the analysis of dynamical systems. Mohammadzadeh et al. [25] developed a fractional-order type-3 fuzzy control method for autonomous vehicle path tracking, while Liu et al. [26] and Jin et al. [27] investigated fuzzy-type neural network approaches to the synchronization of chaotic systems. Related developments include the study of existence, uniqueness, and continuous dependence for fractional difference equations by Li et al. [28], as well as Datko-type stability results for dynamical systems by Yue and Wang [29]. Recent advances in nonlinear control have also addressed tracking performance and fault tolerance in networked dynamical systems subject to communication constraints and denial-of-service attacks [30]. These studies illustrate the continuing development of mathematical methods for addressing nonlinear behavior, uncertainty, and stability. Motivated by this broader progress, the present work focuses on multivalued fixed-point theory in fuzzy partial metric spaces, where fuzziness and nonzero self-distances are incorporated into a common framework.

In this paper, we create a FP framework for multivalued mappings in FPMSs. First, we define the fundamental features required for set-valued contractions and present a Hausdorff-type fuzzy partial metric on the family of compact nonempty subsets. Specifically, we establish the relationship between the induced set-valued structure and the underlying fuzzy partial metric and show the relevant partial Hausdorff inequalities. Based on this, we adapt the Nadler-type method to the current context by establishing a multivalued fuzzy (η)-contractive FPT in 1-complete FPMSs. The presence of a FP and the assumptions of the theorem are demonstrated with an example. Lastly, we examine a parameterized multivalued mapping and demonstrate that the set of parameters for which a FP occurs is both open and closed in the parameter interval in order to prove a homotopy FPT. As a result, the study offers a continuation theory for the FPs of multivalued fuzzy (η)-contractions in FPMSs as well as an existence result for these contractions.

2. Preliminaries

Let (Ω, Φ) be a MS and let $\mathscr{C}(\Omega)$ denote the family of all compact nonempty subsets of Ω. For $\mathfrak{L}, \mathfrak{M} \in \mathscr{C}(\Omega)$, define

$$\mathcal{H}(\mathfrak{L}, \mathfrak{M}) := \max\left\{\sup_{a\in\mathfrak{L}} \Phi(a, \mathfrak{M}),\ \sup_{b\in\mathfrak{M}} \Phi(b, \mathfrak{L})\right\},$$

https://doi.org/10.3390/axioms15110864

where

$$\Phi(x_0, \mathfrak{L}) := \inf\{\, \Phi(x_0, a) : a \in \mathfrak{L} \,\}$$

denotes the distance from a point $x_0 \in \Omega$ to the set $\mathfrak{L}$. It is well known that H defines a metric on $\mathscr{C}(\Omega)$ called the Hausdorff metric induced by the metric Φ.

Definition 1. *Let Ω be a nonempty set. An element $x_0 \in \Omega$ is called a* FP *of a multivalued mapping $T\colon \Omega \to 2^{\Omega}$ if*

$$x_0 \in T(x_0),$$

where 2^{Ω} denotes the family of all nonempty subsets of Ω.

Remember that a multivalued mapping $T\colon \Omega \to \mathscr{C}(\Omega)$ is considered a contractionif there is a constant $\mathfrak{d} \in [0, 1)$ s.t.

$$H\big(T(x_0), T(y_0)\big) \leq \mathfrak{d}\Phi(x_0, y_0),$$

$\forall\, x_0, y_0 \in \Omega$, where H represents the Hausdorff metric on $\mathscr{C}(\Omega)$.

Nadler started the study of FPs for multivalued contractions using the Hausdorff metric and proved the following key finding.

Theorem 1. *Let (Ω, Φ) be a complete MS and let $T : \Omega \to \mathscr{C}(\Omega)$ be a contraction mapping. Then there exists $x_0 \in \Omega$ s.t. [17].*

$$x_0 \in T(x_0).$$

Definition 2. *Let Ω be a nonempty set. A function $\Psi : \Omega \times \Omega \to \mathbb{R}$ is called a partial metricon Ω if, $\forall\, x_0, y_0, z_0 \in \Omega$, the following properties hold:*

(PM1) $x_0 = y_0 \iff \Psi(x_0, x_0) = \Psi(x_0, y_0) = \Psi(y_0, y_0)$;
(PM2) $\Psi(x_0, x_0) \geq 0$ *and* $\Psi(x_0, x_0) \leq \Psi(x_0, y_0)$;
(PM3) $\Psi(x_0, y_0) = \Psi(y_0, x_0)$;
(PM4) $\Psi(x_0, z_0) \leq \Psi(x_0, y_0) + \Psi(y_0, z_0) - \Psi(y_0, y_0)$.

The pair (Ω, Ψ) is then called a PMS.

Every MS (Ω, Φ) is a particular case of a PMS, since the metric Φ satisfies the additional requirement

$$\Phi(x_0, x_0) = 0 \quad \forall\, x_0 \in \Omega.$$

Definition 3. *A triangular norm, also known as a t-norm, is a binary operation $*\colon [0, 1] \times [0, 1] \to [0, 1]$ s.t., $\forall\, x_0, y_0, z_0 \in [0, 1]$, the following requirements are met:*

(T1) $x_0 * y_0 = y_0 * x_0$;
(T2) $x_0 * (y_0 * z_0) = (x_0 * y_0) * z_0$;
(T3) $x_0 * y_0 \geq x_0 * z_0$ *whenever* $y_0 \geq z_0$;
(T4) $x_0 * 1 = x_0$.

Furthermore, $*$ is referred to as a continuous t-norm (Ct-n) if it is continuous on $[0, 1] \times [0, 1]$ with respect to the standard topology.

Using Ct-n, one can introduce the concept of a FMS originally proposed by Kramosil and Michalek. This notion is now commonly employed in the form established by Grabiec, which provides an equivalent and widely accepted formulation on a nonempty set Ω.

Definition 4. *A FMS is an ordered triple $(\Omega, D, *)$, where Ω is a nonempty set, $*$ is a Ct-n, and D is a fuzzy set on $\Omega \times \Omega \times [0,\infty)$ satisfying, $\forall\ x_0, y_0, z_0 \in \Omega$ and $s, t \in (0,\infty)$, the following conditions:*

(KM1) $D(x_0, y_0, 0) = 0$;
(KM2) $D(x_0, y_0, t) = 1\ \forall\ t > 0 \iff x_0 = y_0$;
(KM3) $D(x_0, y_0, t) = D(y_0, x_0, t)$;
(KM4) $D(x_0, z_0, t+s) \geq D(x_0, y_0, t) * D(y_0, z_0, s)$;
(KM5) for each $x_0, y_0 \in \Omega$, the mapping $D_{x_0,y_0} : (0,\infty) \to [0,1]$, is left–continuous.

The triple $(\Omega, D, *)$ is then called a FMS.

Proposition 1. *Let $*$ denote a Ct-n. The corresponding $*$–residuum operator $\to_*$ is uniquely defined by*

$$x_0 \to_* y_0 = \begin{cases} 1, & \text{if } x_0 \leq y_0, \\ \max\{\, z_0 \in [0,1] : x_0 * z_0 = y_0 \,\}, & \text{if } x_0 > y_0, \end{cases} \tag{1}$$

$\forall\ x_0, y_0 \in [0,1]$.

For a detailed discussion of the residuum operator, we refer to [31]. In particular, the following result will be used later.

Proposition 2. *Let $*$ be a Ct-n and let $\wedge$ denote the minimum t-norm. Then*

$$x_0 \wedge y_0 = x_0 * (x_0 \to_* y_0), \tag{2}$$

$\forall\ x_0, y_0 \in [0,1]$.

As a direct consequence of the above propositions, we obtain the following corollary.

Corollary 1. *Let $*$ denote a Ct-n. Then*

$$x_0 \to_* y_0 \geq y_0,$$

$\forall\ x_0, y_0 \in [0,1]$.

Definition 5. *In view of the preceding notions, we now introduce the concept of a FPMS. An ordered triple $(\Omega, \Psi_P, *)$ is called a FPMS if Ω is a nonempty set, $*$ is a Ct-n, and Ψ_P is a fuzzy set on $\Omega \times \Omega \times [0,\infty)$ s.t., $\forall\ x_0, y_0, z_0 \in \Omega$ and $s, t \in (0,\infty)$, the following conditions are satisfied:*

(FPMS0) $\Psi_P(x_0, y_0, 0) = 0$;
(FPMS1) $\Psi_P(x_0, y_0, t) \leq \Psi_P(x_0, x_0, t)$;
(FPMS2) $\Psi_P(x_0, y_0, t) = \Psi_P(x_0, x_0, t) = \Psi_P(y_0, y_0, t)\ \forall\ t > 0 \iff x_0 = y_0$;
(FPMS3) $\Psi_P(x_0, y_0, t) = \Psi_P(y_0, x_0, t)$;
(FPMS4)

$$\begin{aligned} \Psi_P(x_0, x_0, t+s) \to_* \Psi_P(x_0, z_0, t+s) \geq \big(\Psi_P(x_0, x_0, t) \to_* \Psi_P(x_0, y_0, t)\big) \\ * \big(\Psi_P(y_0, y_0, s) \to_* \Psi_P(y_0, z_0, s)\big); \end{aligned}$$

(FPMS5) for each $x_0, y_0 \in \Omega$, the mapping $\Psi_{P_{x_0,y_0}} : (0,\infty) \to [0,1]$, is left–continuous.

The triple $(\Omega, \Psi_P, *)$ is called a FPMS.

Remark 1. *The triangle axiom (FPMS4) differs from the classical fuzzy metric triangle inequality because the self-values $\Psi_P(x, x, t)$ in a fuzzy partial metric need not be equal to* 1. *Consequently,*

(FPMS4) involves the operation $\to_*$ to compare the value between two points with the corresponding self-value. In contrast, the classical fuzzy metric triangle inequality is expressed directly in terms of the fuzzy metric values:

$$M(x,z,t+s) \geq M(x,y,t) * M(y,z,s).$$

When $\Psi_P(x,x,t) = 1$ for every $x \in \Omega$ and $t > 0$, and the operation $\to_*$ satisfies $1 \to_* a = a$, the terms involving $\to_*$ reduce to the corresponding fuzzy metric values. In this case, (FPMS4) takes the form of the classical fuzzy metric triangle inequality. Thus, (FPMS4) accommodates the possibility of self-values different from 1. Its use in subsequent proofs requires the relevant inequalities to be derived explicitly from this axiom and the properties of $*$ and $\to_*$.

Definition 6. *(i) In an FPMS $(\Omega, \Psi_P, *)$, a sequence $\{a_n\}$ is called Ψ_P-Cauchy (or simply Cauchy) if, for every $t > 0$,*

$$\lim_{n,m\to\infty} [\Psi_P(a_n,a_n,t) \to_* \Psi_P(a_n,a_m,t)] = 1$$

and

$$\lim_{n,m\to\infty} [\Psi_P(a_m,a_m,t) \to_* \Psi_P(a_m,a_n,t)] = 1.$$

A sequence $\{a_n\}$ in Ω is called 1-Ψ_P-Cauchy (1-Cauchy) if $\lim_{n,m\to\infty} \Psi_P(a_n,a_m,t) = 1$ for every $t > 0$.

(ii) A sequence $\{a_n\}$ in an FPMS $(\Omega, \Psi_P, *)$ is said to converge to $x_0 \in \Omega$ with respect to Ψ_P if, for every $t > 0$,

$$\lim_{n\to\infty} [\Psi_P(x_0,x_0,t) \to_* \Psi_P(x_0,a_n,t)] = 1$$

and

$$\lim_{n\to\infty} [\Psi_P(a_n,a_n,t) \to_* \Psi_P(a_n,x_0,t)] = 1.$$

It is called 1-convergent to x_0 if $\lim_{n\to\infty} \Psi_P(x_0,a_n,t) = 1$ for every $t > 0$.

(iii) An FPMS $(\Omega, \Psi_P, *)$ is called complete if every Cauchy sequence in Ω converges with respect to Ψ_P. It is called 1-complete if every 1-Cauchy sequence in Ω is 1-convergent.

Remark 2. *Every (1)-Cauchy sequence is a (P)-Cauchy sequence, whereas the converse does not hold in general. We adopt 1-completeness in the main theorems because the iterative sequences constructed in the proofs are shown to be (1)-Cauchy, and 1-completeness ensures their convergence in the corresponding sense. Ordinary completeness alone does not guarantee this conclusion unless an additional relationship between the two notions of convergence is established.*

3. Main Results

Let $(\Omega, \Psi_P, *)$ be an FPMS, where the function Ψ_P induces a natural topology on Ω through open fuzzy balls and convergence is governed by the behavior of $\Psi_P(x_0,y_0,t)$ relative to self-distances. This topology generalizes the classical metric topology and preserves the essential analytical properties required for FP theory.

Denote by $\mathscr{C}(\Omega)$ the collection of all compact nonempty subsets of Ω with respect to the topology induced by Ψ_P. The class $\mathscr{C}(\Omega)$ provides an appropriate setting for the study of multivalued mappings and their contractive properties in FPMSs.

For subsets $\mathfrak{L}, \mathfrak{M} \in \mathscr{C}(\Omega)$, a Hausdorff-type fuzzy partial metric can be introduced by measuring the mutual fuzzy distances between points of one set and the other with respect to Ψ_P. This construction preserves the essential topological features induced by the fuzzy partial metric and enables the extension of contractive conditions from single-valued

mappings to multivalued operators. Consequently, this setting provides the appropriate framework for establishing FP results for multivalued mappings in FPMSs.

Theorem 2. *Let $(\Omega, \Psi_P, *)$ be a FPMS satisfying*

$$\Psi_P(x_0, x_0, t) = 1 \quad \text{for all } x_0 \in \Omega,\ t > 0.$$

Assume that Ψ_P is jointly continuous on $\Omega \times \Omega \times (0, \infty)$ with respect to the topology in which compactness is considered. Let $\mathscr{C}(\Omega)$ denote the family of all nonempty compact subsets of Ω. For $\mathfrak{L}, \mathfrak{M} \in \mathscr{C}(\Omega)$ and $t > 0$, define

$$\Psi_P(x_0, \mathfrak{M}, t) = \sup_{y_0 \in \mathfrak{M}} \Psi_P(x_0, y_0, t)$$

,

$$\delta(\mathfrak{L}, \mathfrak{M}, t) = \inf_{x_0 \in \mathfrak{L}} \Psi_P(x_0, \mathfrak{M}, t)$$

and

$$\mathcal{H}_P(\mathfrak{L}, \mathfrak{M}, t) = \min\{\delta(\mathfrak{L}, \mathfrak{M}, t), \delta(\mathfrak{M}, \mathfrak{L}, t)\}.$$

Then $(\mathscr{C}(\Omega), \mathcal{H}_P, *)$ is a Hausdorff-type FPMS.

Proof. We verify the defining properties of a FPMS.

(FPMS0). By definition,

$$\mathcal{H}_P(\mathfrak{L}, \mathfrak{M}, 0) = 0 \ \forall\ \mathfrak{L}, \mathfrak{M} \in \mathscr{C}(\Omega).$$

(FPMS1). Let $\mathfrak{L}, \mathfrak{M} \in \mathscr{C}(\Omega)$ and $t > 0$. Since $\Psi_P(x_0, x_0, t) = 1$ for every $x_0 \in \Omega$, we have

$$\Psi_P(x_0, \mathfrak{L}, t) = 1 \qquad (x_0 \in \mathfrak{L}).$$

Consequently,

$$\delta(\mathfrak{L}, \mathfrak{L}, t) = 1 \quad \text{and} \quad \mathcal{H}_P(\mathfrak{L}, \mathfrak{L}, t) = 1.$$

As $\mathcal{H}_P(\mathfrak{L}, \mathfrak{M}, t) \in [0, 1]$, it follows that

$$\mathcal{H}_P(\mathfrak{L}, \mathfrak{M}, t) \leq \mathcal{H}_P(\mathfrak{L}, \mathfrak{L}, t).$$

(FPMS2). Suppose first that $\mathfrak{L} = \mathfrak{M}$. Then

$$\mathcal{H}_P(\mathfrak{L}, \mathfrak{M}, t) = \mathcal{H}_P(\mathfrak{L}, \mathfrak{L}, t) = \mathcal{H}_P(\mathfrak{M}, \mathfrak{M}, t) = 1$$

for every $t > 0$. Conversely, suppose that

$$\mathcal{H}_P(\mathfrak{L}, \mathfrak{M}, t) = \mathcal{H}_P(\mathfrak{L}, \mathfrak{L}, t) = \mathcal{H}_P(\mathfrak{M}, \mathfrak{M}, t)$$

for every $t > 0$. Since the self-distances equal 1, we obtain $\mathcal{H}_P(\mathfrak{L}, \mathfrak{M}, t) = 1$. Hence

$$\delta(\mathfrak{L}, \mathfrak{M}, t) = \delta(\mathfrak{M}, \mathfrak{L}, t) = 1.$$

For any $x_0 \in \mathfrak{L}$, the equality $\delta(\mathfrak{L}, \mathfrak{M}, t) = 1$ for every $t > 0$ implies

$$\sup_{y_0 \in \mathfrak{M}} \Psi_P(x_0, y_0, t) = 1, \qquad t > 0.$$

By compactness of $\mathfrak{M}$ and continuity of Ψ_P, the supremum is attained. Hence, for each $t > 0$, the set

$$K_t = \{y_0 \in \mathfrak{M} : \Psi_P(x_0, y_0, t) = 1\}$$

is nonempty and closed in $\mathfrak{M}$. Moreover, by (FPMS4) and the normalization $\Psi_P(y_0, y_0, s) = 1$, we have

$$\Psi_P(x_0, y_0, t + s) \geq \Psi_P(x_0, y_0, t), \qquad s, t > 0.$$

Thus, $K_{t_1} \subseteq K_{t_2}$ whenever $0 < t_1 < t_2$. Consequently, the family $\{K_t : t > 0\}$ has the finite-intersection property. Since $\mathfrak{M}$ is compact, we obtain

$$\bigcap_{t>0} K_t \neq \varnothing.$$

Therefore, there exists a single $y_0 \in \mathfrak{M}$ such that

$$\Psi_P(x_0, y_0, t) = 1, \qquad \text{for every } t > 0.$$

Since $\Psi_P(x_0, x_0, t) = \Psi_P(y_0, y_0, t) = 1$ for every $t > 0$, axiom (FPMS2) yields $x_0 = y_0$. Hence $\mathfrak{L} \subseteq \mathfrak{M}$. Interchanging $\mathfrak{L}$ and $\mathfrak{M}$ gives $\mathfrak{M} \subseteq \mathfrak{L}$, and therefore $\mathfrak{L} = \mathfrak{M}$.

(FPMS3). By the definition of $\mathcal{H}_P$,

$$\mathcal{H}_P(\mathfrak{L}, \mathfrak{M}, t) = \mathcal{H}_P(\mathfrak{M}, \mathfrak{L}, t).$$

Thus $\mathcal{H}_P$ is symmetric.

(FPMS4). Since $\Psi_P(x_0, x_0, t) = 1$, the axiom (FPMS4) in Ω reduces to

$$\Psi_P(x_0, z_0, t + s) \geq \Psi_P(x_0, y_0, t) * \Psi_P(y_0, z_0, s)$$

for all $x_0, y_0, z_0 \in \Omega$ and $t, s > 0$, where we use $1 \to_* a = a$. Fix $\mathfrak{L}, \mathfrak{M}, \mathfrak{N} \in \mathscr{C}(\Omega)$ and $t, s > 0$. By compactness and continuity, the supremum in the definitions below are attained. For any $x_0 \in \mathfrak{L}$, choose $y_0 \in \mathfrak{M}$ such that

$$\Psi_P(x_0, y_0, t) = \Psi_P(x_0, \mathfrak{M}, t) \geq \delta(\mathfrak{L}, \mathfrak{M}, t) \geq \mathcal{H}_P(\mathfrak{L}, \mathfrak{M}, t).$$

Next, choose $z_0 \in \mathfrak{N}$ such that

$$\Psi_P(y_0, z_0, s) = \Psi_P(y_0, \mathfrak{N}, s) \geq \delta(\mathfrak{M}, \mathfrak{N}, s) \geq \mathcal{H}_P(\mathfrak{M}, \mathfrak{N}, s).$$

The preceding inequality in Ω and the monotonicity of the Ct-n yield

$$\Psi_P(x_0, z_0, t + s) \geq \mathcal{H}_P(\mathfrak{L}, \mathfrak{M}, t) * \mathcal{H}_P(\mathfrak{M}, \mathfrak{N}, s).$$

Taking the supremum over $z_0 \in \mathfrak{N}$ and then the infimum over $x_0 \in \mathfrak{L}$ gives

$$\delta(\mathfrak{L}, \mathfrak{N}, t + s) \geq \mathcal{H}_P(\mathfrak{L}, \mathfrak{M}, t) * \mathcal{H}_P(\mathfrak{M}, \mathfrak{N}, s).$$

Applying the same argument in the reverse direction gives

$$\delta(\mathfrak{N}, \mathfrak{L}, t + s) \geq \mathcal{H}_P(\mathfrak{L}, \mathfrak{M}, t) * \mathcal{H}_P(\mathfrak{M}, \mathfrak{N}, s).$$

Taking the minimum of these two inequalities, we conclude that

$$\mathcal{H}_P(\mathfrak{L}, \mathfrak{N}, t + s) \geq \mathcal{H}_P(\mathfrak{L}, \mathfrak{M}, t) * \mathcal{H}_P(\mathfrak{M}, \mathfrak{N}, s).$$

Since $\mathcal{H}_P(\mathfrak{L}, \mathfrak{L}, t+s) = 1$, the last inequality is equivalent to

$$\mathcal{H}_P(\mathfrak{L}, \mathfrak{L}, t+s) \to_* \mathcal{H}_P(\mathfrak{L}, \mathfrak{N}, t+s) \geq \big(\mathcal{H}_P(\mathfrak{L}, \mathfrak{L}, t) \to_* \mathcal{H}_P(\mathfrak{L}, \mathfrak{M}, t)\big) * \big(\mathcal{H}_P(\mathfrak{M}, \mathfrak{M}, s) \to_* \mathcal{H}_P(\mathfrak{M}, \mathfrak{N}, s)\big).$$

Thus (FPMS4) holds.

(FPMS5). For fixed $\mathfrak{L}, \mathfrak{M} \in \mathscr{C}(\Omega)$, joint continuity of Ψ_P implies continuity in t of its maximum over the compact set $\mathfrak{L} \times \mathfrak{M}$. In particular,

$$\delta(\mathfrak{L}, \mathfrak{M}, t) = \min_{x_0 \in \mathfrak{L}} \max_{y_0 \in \mathfrak{M}} \Psi_P(x_0, y_0, t)$$

is continuous in $t > 0$. The same holds for $\delta(\mathfrak{M}, \mathfrak{L}, t)$. Therefore $\mathcal{H}_P(\mathfrak{L}, \mathfrak{M}, t)$, being the minimum of these two continuous functions, is continuous on $(0, \infty)$. Hence it is left-continuous.

All the defining properties of a FPMS are satisfied. Therefore $(\mathscr{C}(\Omega), \mathcal{H}_P, *)$ is a Hausdorff-type FPMS. □

Remark 3. *The Hausdorff-type fuzzy partial metric construction established in Theorem 2 is considered under the additional normalization condition $\Psi_P(x, x, t) = 1$, $x \in \Omega$, $t > 0$. Although the general definition of a FPMS permits non-unit self-distances, the present construction and its subsequent applications are established under this normalized condition. Thus, the results obtained through Theorem 2 concern this particular subclass of FPMSs.*

Lemma 1. *Let $\mathfrak{M} \in \mathscr{C}(\Omega)$ and $x_0 \in \mathfrak{M}$. Then for every $t \geq 0$,*

$$\Psi_P(x_0, \mathfrak{M}, t) = \Psi_P(x_0, x_0, t).$$

Proof. Since $x_0 \in \mathfrak{M}$, taking $y_0 = x_0$ in the supremum gives $\Psi_P(x_0, \mathfrak{M}, t) \geq \Psi_P(x_0, x_0, t)$. In contrast, by (FPMS1), $\Psi_P(x_0, y_0, t) \leq \Psi_P(x_0, x_0, t)$ for every $y_0 \in \Omega$, hence in particular for every $y_0 \in \mathfrak{M}$; taking the supremum over $y_0 \in \mathfrak{M}$ yields $\Psi_P(x_0, \mathfrak{M}, t) \leq \Psi_P(x_0, x_0, t)$. Combining the two inequalities gives the claim. □

Remark 4. *Let $(\Omega, \Psi_P, *)$ be a FPMS and $\mathfrak{L}$ be any nonempty set in $(\Omega, \Psi_P, *)$, then*

$$a \in \bar{\mathfrak{L}} \iff \Psi_P(a, \mathfrak{L}, t) = \Psi_P(a, a, t),$$

where $\bar{\mathfrak{L}}$ represents $\mathfrak{L}$'s closure with regard to the fuzzy partial metric Ψ_P. Observe that if and only if $\mathfrak{L} = \bar{\mathfrak{L}}$, $\mathfrak{L}$ is closed in $(\Omega, \Psi_P, *)$.

We will now examine certain characteristics of mapping $\mathcal{H}_P : \mathscr{C}(\Omega) \times \mathscr{C}(\Omega) \times (0, \infty) \to [0, 1]$.

Proposition 3. *Let $(\Omega, \Psi_P, *)$ be a FPMS that satisfies $\Psi_P(x_0, x_0, t) = 1$ for all $x_0 \in \Omega$, $t > 0$. For any $\mathfrak{L}, \mathfrak{M}, \mathfrak{N} \in \mathscr{C}(\Omega)$, we have the following:*

(i) $\forall\ \mathfrak{L}, \mathfrak{M} \in \mathscr{C}(\Omega)$ *and* $t \geq 0$,

$$\delta(\mathfrak{L}, \mathfrak{M}, t) \leq \delta(\mathfrak{L}, \mathfrak{L}, t) = \inf_{x \in \mathfrak{L}} \Psi_P(x, x, t);$$

(ii) *If $\delta(\mathfrak{L}, \mathfrak{M}, t) = 1$ for every $t > 0$, then $\mathfrak{L} \subseteq \mathfrak{M}$.*

(iii) $\delta(\mathfrak{L}, \mathfrak{N}, t+s) \geq \delta(\mathfrak{L}, \mathfrak{M}, t) * \delta(\mathfrak{M}, \mathfrak{N}, s)$.

Proof. (i) For any $x_0 \in \mathfrak{L}$, axiom (FPMS1) gives

$$\Psi_P(x_0, y_0, t) \leq \Psi_P(x_0, x_0, t) \qquad \text{for every } y_0 \in \mathfrak{M}.$$

Taking the supremum over $y_0 \in \mathfrak{M}$ and then the infimum over $x_0 \in \mathfrak{L}$, we obtain

$$\delta(\mathfrak{L}, \mathfrak{M}, t) \leq \inf_{x_0 \in \mathfrak{L}} \Psi_P(x_0, x_0, t).$$

Moreover, since $x_0 \in \mathfrak{L}$ and (FPMS1) holds,

$$\sup_{y_0 \in \mathfrak{L}} \Psi_P(x_0, y_0, t) = \Psi_P(x_0, x_0, t).$$

Consequently,

$$\delta(\mathfrak{L}, \mathfrak{L}, t) = \inf_{x \in \mathfrak{L}} \Psi_P(x_0, x_0, t).$$

Combining these relations yields

$$\delta(\mathfrak{L}, \mathfrak{M}, t) \leq \delta(\mathfrak{L}, \mathfrak{L}, t) = \inf_{x \in \mathfrak{L}} \Psi_P(x_0, x_0, t).$$

(ii) Suppose that $\delta(\mathfrak{L}, \mathfrak{M}, t) = 1$. By definition $\inf_{x_0 \in \mathfrak{L}} \Psi_P(x_0, \mathfrak{M}, t) = 1 \; \forall \; x_0 \in \mathfrak{L}$. Since $0 \leq \Psi_P(x_0, \mathfrak{M}, t) \leq 1$, we obtain $\Psi_P(x_0, \mathfrak{M}, t) = 1$ for every $x_0 \in \mathfrak{L}$. Now Ψ_P is a FPMS, we have $\Psi_P(x_0, \mathfrak{M}, t) \leq \Psi_P(x_0, x_0, t)$. Hence $1 = \Psi_P(x_0, \mathfrak{M}, t) \leq \Psi_P(x_0, x_0, t) \leq 1$, so $\Psi_P(x_0, x_0, t) = 1$. Therefore $\Psi_P(x_0, \mathfrak{M}, t) = \Psi_P(x_0, x_0, t)$. By the closure characterization of FPMSs, $x_0 \in \overline{\mathfrak{M}}$. Since $\mathfrak{M}$ is closed, $\overline{\mathfrak{M}} = \mathfrak{M}$, and thus $x_0 \in \mathfrak{M}$. Hence $\mathfrak{L} \subseteq \mathfrak{M}$.
(iii) Fix $x_0 \in \mathfrak{L}$. By compactness of $\mathfrak{M}$ and continuity of Ψ_P, there exists $y_{x_0} \in \mathfrak{M}$ such that

$$\Psi_P(x_0, y_{x_0}, t) = \sup_{y_0 \in \mathfrak{M}} \Psi_P(x_0, y_0, t).$$

Hence

$$\Psi_P(x_0, y_{x_0}, t) \geq \delta(\mathfrak{L}, \mathfrak{M}, t).$$

Similarly, there exists $z_{x_0} \in \mathfrak{N}$ such that

$$\Psi_P(y_{x_0}, z_{x_0}, s) = \sup_{z_0 \in \mathfrak{N}} \Psi_P(y_{x_0}, z_0, s) \geq \delta(\mathfrak{M}, \mathfrak{N}, s).$$

Since $\Psi_P(x_0, x_0, u) = 1$ for every $u > 0$, (FPMS4) reduces to

$$\Psi_P(x_0, z_0, t + s) \geq \Psi_P(x_0, y_0, t) * \Psi_P(y_0, z_0, s).$$

Using the monotonicity of the t-norm, we obtain

$$\begin{aligned} \Psi_P(x_0, z_{x_0}, t + s) &\geq \Psi_P(x_0, y_{x_0}, t) * \Psi_P(y_{x_0}, z_{x_0}, s) \\ &\geq \delta(\mathfrak{L}, \mathfrak{M}, t) * \delta(\mathfrak{M}, \mathfrak{N}, s). \end{aligned}$$

Taking the supremum over $z_0 \in \mathfrak{N}$ and then the infimum over $x_0 \in \mathfrak{L}$ gives

$$\delta(\mathfrak{L}, \mathfrak{N}, t + s) \geq \delta(\mathfrak{L}, \mathfrak{M}, t) * \delta(\mathfrak{M}, \mathfrak{N}, s).$$

This completes the proof. □

Proposition 4. *Let $(\Omega, \Psi_P, *)$ be a FPMS that satisfies $\Psi_P(x_0, x_0, t) = 1$ for all $x_0 \in \Omega$, $t > 0$. For any $\mathfrak{L}, \mathfrak{M}, \mathfrak{N} \in \mathscr{C}(\Omega)$, we have*

$$\mathcal{H}_P(\mathfrak{L}, \mathfrak{L}, t) \geq \mathcal{H}_P(\mathfrak{L}, \mathfrak{M}, t)$$

Proof. From (ii) of Proposition 3, we have $\mathcal{H}_P(\mathfrak{L}, \mathfrak{L}, t) = \delta(\mathfrak{L}, \mathfrak{L}, t) \geq \delta(\mathfrak{L}, \mathfrak{M}, t) \geq \mathcal{H}_P(\mathfrak{L}, \mathfrak{M}, t)$. □

Corollary 2. *Let $(\Omega, \Psi_P, *)$ be a FPMS satisfying $\Psi_P(x, x, t) = 1$, $x \in \Omega$, $t > 0$. For $\mathfrak{L}, \mathfrak{M} \in \mathscr{C}(\Omega)$, if*

$$H_P(\mathfrak{L}, \mathfrak{M}, t) = 1 \quad \textit{for every } t > 0,$$

then $\mathfrak{L} = \mathfrak{M}$.

Proof. Let $\mathcal{H}_P(\mathfrak{L}, \mathfrak{M}, t) = 1$. By definition of $\mathcal{H}_P$, $\delta(\mathfrak{L}, \mathfrak{M}, t) = \delta(\mathfrak{M}, \mathfrak{L}, t) = 1$. Using (ii) of Proposition 3, we obtain $\mathfrak{L} \subseteq \mathfrak{M}$ and $\mathfrak{M} \subseteq \mathfrak{L}$. Thus, $\mathfrak{L} = \mathfrak{M}$. □

Remark 5. *Under the assumptions of Corollary 2, the converse also holds. Indeed, if $\mathfrak{L} = \mathfrak{M}$, then*

$$\mathcal{H}_P(\mathfrak{L}, \mathfrak{M}, t) = \mathcal{H}_P(\mathfrak{L}, \mathfrak{L}, t) = 1, \qquad t > 0.$$

Therefore, within the normalized setting of Corollary 2, we have

$$\mathfrak{L} = \mathfrak{M} \iff H_P(\mathfrak{L}, \mathfrak{M}, t) = 1 \quad \textit{for every } t > 0.$$

Example 1 concerns a non-normalized fuzzy partial metric and therefore lies outside the assumptions of Corollary 2.

Example 1. *Let $\Omega = [0, 1]$ and let $*$ be the product t-norm, defined by $a * b = ab$, $a, b \in [0, 1]$. Define $\Psi_P : \Omega \times \Omega \times [0, \infty) \to [0, 1]$ by*

$$\Psi_P(x_0, y_0, t) = \begin{cases} \dfrac{t}{t+1} \exp\left(-\dfrac{|x_0 - y_0|}{t+1}\right), & t > 0, \\ 0, & t = 0. \end{cases}$$

Then $\Psi_P(x_0, y_0, 0) = 0$ for all $(x_0, y_0) \in \Omega \times \Omega$, so FPMS0 is satisfied. Moreover, $(\Omega, \Psi_P, *)$ is a fuzzy partial metric space with non-unit self-distance

$$\Psi_P(x_0, x_0, t) = \frac{t}{t+1} < 1, \qquad t > 0.$$

Furthermore, by part (i) of Proposition 3,

$$\begin{aligned} \mathcal{H}_P(\Omega, \Omega, t) &= \delta(\Omega, \Omega, t) \\ &= \inf_{x_0 \in [0,1]} \Psi_P(x_0, \Omega, t) \\ &= \frac{t}{t+1} \neq 1, \qquad t > 0. \end{aligned}$$

Let $d_{\mathcal{H}}$ denote the usual Hausdorff metric on $\mathscr{C}([0, 1])$. Directly from the definition of Ψ_P, we obtain

$$\mathcal{H}_P(\mathfrak{A}, \mathfrak{B}, t) = \frac{t}{t+1} \exp\left(-\frac{d_{\mathcal{H}}(\mathfrak{A}, \mathfrak{B})}{t+1}\right), \qquad t > 0.$$

In particular,

$$\mathcal{H}_P(\mathfrak{A}, \mathfrak{A}, t) = \frac{t}{t+1} < 1.$$

The separation axiom (FPMS2) follows from the corresponding separation property of $d_{\mathcal{H}}$. Moreover, for the product t-norm, the residuum satisfies

$$\begin{aligned} \mathcal{H}_P(\mathfrak{A}, \mathfrak{A}, t+s) &\to_* \mathcal{H}_P(\mathfrak{A}, \mathfrak{C}, t+s) \\ &= \exp\left(-\frac{d_{\mathcal{H}}(\mathfrak{A}, \mathfrak{C})}{t+s+1}\right). \end{aligned}$$

By the triangle inequality for $d_{\mathcal{H}}$ and the fact that $t+s+1 \geq t+1$ and $t+s+1 \geq s+1$, we have

$$\frac{d_{\mathcal{H}}(\mathfrak{A}, \mathfrak{C})}{t+s+1} \leq \frac{d_{\mathcal{H}}(\mathfrak{A}, \mathfrak{B})}{t+1} + \frac{d_{\mathcal{H}}(\mathfrak{B}, \mathfrak{C})}{s+1}.$$

This establishes (FPMS4). The remaining axioms follow directly from the displayed formula and the properties of $d_{\mathcal{H}}$. Hence, $\mathcal{H}_P$ is a Hausdorff-type fuzzy partial metric on $\mathscr{C}([0,1])$ with non-unit self-distances.

Remark 6. *Any Hausdorff fuzzy metric can be shown to be a Hausdorff fuzzy partial metric. The opposite is false (see Example 1).*

3.1. Fixed Point of Multivalued Contraction Mapping

Definition 7. *Let $\mathfrak{E}$ be the class of all strictly decreasing functions $\eta : (0,1] \to [0,\infty)$ satisfying*

$$\eta(1) = 0, \qquad \lim_{r \to 0^+} \eta(r) = +\infty, \qquad \lim_{r \to 1^-} \eta(r) = 0,$$

and

$$\eta(r * s) \leq \eta(r) + \eta(s), \qquad \forall\, r, s \in (0,1].$$

Definition 8. *Let $(\Omega, \Psi_P, *)$ be a FPMS and $T : \Omega \to \mathscr{C}(\Omega)$ be a multivalued mapping. Let $\eta \in \mathfrak{E}$. The mapping T is said to be a multivalued fuzzy η-contractive mapping with respect to η if there exists $\mathfrak{d} \in (0,1)$ s.t.*

$$\eta(\mathcal{H}_P(Tx_0, Ty_0, t)) \leq \mathfrak{d}\, \eta(\Psi_P(x_0, y_0, t)) \tag{3}$$

$\forall\, x_0, y_0 \in \Omega$ and $t > 0$.

Theorem 3. *Let $(\Omega, \Psi_P, *)$ be a 1-complete FPMS and let $T : \Omega \to \mathscr{C}(\Omega)$ be a multivalued fuzzy η-contractive mapping. Presume that the subsequent conditions are met:*

(i) There exist $x_0 \in \Omega$ s.t. $\Psi_P(x_0, x_0, t) = 1\ \forall\, t > 0$ and $x_1 \in Tx_0$ s.t. $\bigwedge_{t>0} \Psi_P(x_0, x_1, t) > 0$.

(ii) For every $x, y \in \Omega$ and every $u \in Tx$, there exists $v \in Ty$ such that

$$\Psi_P(u, v, t) \geq \mathcal{H}_P(Tx, Ty, t) \quad \text{for every } t > 0.$$

Then T has a FP; that is, there exists $a^* \in \Omega$ s.t. $a^* \in Ta^*$.

Proof. By assumption (i), choose $x_1 \in Tx_0$ s.t.

$$\bigwedge_{t>0} \Psi_P(x_0, x_1, t) > 0.$$

Set $a_0 = x_0$ and $a_1 = x_1$. We construct a sequence $\{a_n\}_{n\geq 0}$ such that $a_{n+1} \in Ta_n$ for every $n \geq 0$. Suppose a_{n-1} and a_n have been chosen, where $n \geq 1$. Since $a_n \in Ta_{n-1}$, condition (ii), applied with $x = a_{n-1}$, $y = a_n$, and $u = a_n$, gives a single point $a_{n+1} \in Ta_n$ such that

$$\Psi_P(a_n, a_{n+1}, t) \geq \mathcal{H}_P(Ta_{n-1}, Ta_n, t) \quad \text{for every } t > 0.$$

By the contractive condition and the monotonicity of η,

$$\begin{aligned}\eta\big(\Psi_P(a_n, a_{n+1}, t)\big) &\leq \eta\big(\mathcal{H}_P(Ta_{n-1}, Ta_n, t)\big)\\ &\leq \mathfrak{d}\,\eta\big(\Psi_P(a_{n-1}, a_n, t)\big).\end{aligned}$$

Consequently, induction yields

$$\eta\big(\Psi_P(a_n, a_{n+1}, t)\big) \leq \mathfrak{d}^n \eta\big(\Psi_P(a_0, a_1, t)\big) \tag{4}$$

for every $n \geq 0$ and $t > 0$. Since $\Psi_P(x_0, x_0, t) = 1$, the preceding inequality with $x = y = x_0$ and the contractive condition imply

$$\eta(\Psi_P(x_1, x_1, t)) \leq \mathfrak{d}\,\eta(\Psi_P(x_0, x_0, t)) = \mathfrak{d}\,\eta(1) = 0.$$

Therefore $\Psi_P(x_1, x_1, t) = 1$. Repeating the same argument gives

$$\Psi_P(a_n, a_n, t) = 1 \qquad \forall\, n \in \mathbb{N},\ t > 0. \tag{5}$$

We next prove that $\{a_n\}$ is a 1-Cauchy sequence. Let $m > n$. Choose a strictly decreasing sequence $\{a_i\}_{i\in\mathbb{N}}$ of positive real numbers that satisfies

$$\sum_{i=1}^{\infty} a_i = 1.$$

For a fixed $t > 0$, put $t_i = a_i t$. Since $\Psi_P(a_n, a_n, t) = 1$, repeated application of (FPMS4) yields

$$\begin{aligned}\Psi_P(a_n, a_m, t) &= 1 \to_* \Psi_P(a_n, a_m, t)\\ &= \Psi_P(a_n, a_n, t) \to_* \Psi_P(a_n, a_m, t)\\ &\geq [\Psi_P(a_n, a_n, t_n) \to_* \Psi_P(a_n, a_{n+1}, t_n)]\\ &\quad * [\Psi_P(a_{n+1}, a_{n+1}, t_{n+1}) \to_* \Psi_P(a_{n+1}, a_{n+2}, t_{n+1})]\\ &\quad * \cdots * [\Psi_P(a_{m-1}, a_{m-1}, t_{m-1}) \to_* \Psi_P(a_{m-1}, a_m, t_{m-1})].\end{aligned}$$

Using (5), we have $\Psi_P(a_i, a_i, t_i) = 1$ and hence, $1 \to_* \Psi_P(a_i, a_{i+1}, t_i) = \Psi_P(a_i, a_{i+1}, t_i)$. Therefore,

$$\begin{aligned}\Psi_P(a_n, a_m, t) \geq \Psi_P(a_n, a_{n+1}, t_n) * \Psi_P(a_{n+1}, a_{n+2}, t_{n+1})\\ * \cdots * \Psi_P(a_{m-1}, a_m, t_{m-1}).\end{aligned} \tag{6}$$

Since η is decreasing, then repeated application of the $\mathfrak{E}$-property gives

$$\eta(\Psi_P(a_n, a_m, t)) \leq \sum_{i=n}^{m-1} \eta(\Psi_P(a_i, a_{i+1}, t_i)).$$

Using (4), we obtain

$$\eta(\Psi_P(a_n, a_m, t)) \leq \sum_{i=n}^{m-1} \mathfrak{d}^i \eta(\Psi_P(a_0, a_1, t_i)).$$

Since $\bigwedge_{t>0} \Psi_P(a_0, a_1, t) > 0$, the terms on the right are bounded above by a convergent geometric sequence. Thus

$$\lim_{n,m\to\infty} \eta(\Psi_P(a_n, a_m, t)) = 0.$$

Since $\eta(r) > 0$ for $r < 1$ and $\eta(1) = 0$, it follows that

$$\lim_{n,m\to\infty} \Psi_P(a_n, a_m, t) = 1.$$

Hence $\{a_n\}$ is a 1-Cauchy sequence.

By the 1-completeness of $(\Omega, \Psi_P, *)$, there exists $a^* \in \Omega$ s.t.

$$\lim_{n\to\infty} \Psi_P(a^*, a_n, t) = 1 \qquad \forall\, t > 0.$$

Consequently,

$$\Psi_P(a^*, a^*, t) = 1 \qquad \forall\, t > 0. \tag{7}$$

We now prove that $a^* \in Ta^*$. Fix $t > 0$. By (FPMS4), (7), and $t/2 + t/2 = t$, we have

$$\begin{aligned}
\Psi_P(a^*, Ta^*, t) = 1 \to_* \Psi_P(a^*, Ta^*, t) &\geq \left[\Psi_P\left(a^*, a^*, \frac{t}{2}\right) \to_* \Psi_P\left(a^*, a_n, \frac{t}{2}\right)\right] \\
&\quad * \left[\Psi_P\left(a_n, a_n, \frac{t}{2}\right) \to_* \Psi_P\left(a_n, Ta^*, \frac{t}{2}\right)\right] \\
&= \Psi_P\left(a^*, a_n, \frac{t}{2}\right) * \Psi_P\left(a_n, Ta^*, \frac{t}{2}\right).
\end{aligned} \tag{8}$$

On the other hand, by the definition of the Hausdorff fuzzy partial metric and the contraction condition,

$$\eta\left(\Psi_P\left(a_n, Ta^*, \frac{t}{2}\right)\right) \leq \eta\left(\mathcal{H}_P\left(Ta_{n-1}, Ta^*, \frac{t}{2}\right)\right)$$

and hence

$$\eta\left(\Psi_P\left(a_n, Ta^*, \frac{t}{2}\right)\right) \leq \mathfrak{d}\, \eta\left(\Psi_P\left(a_{n-1}, a^*, \frac{t}{2}\right)\right).$$

Since $\Psi_P\left(a_{n-1}, a^*, \frac{t}{2}\right) \longrightarrow 1$, we obtain $\lim_{n\to\infty} \Psi_P\left(a_n, Ta^*, \frac{t}{2}\right) = 1$. Together with $\lim_{n\to\infty} \Psi_P\left(a^*, a_n, \frac{t}{2}\right) = 1$, equation (8) gives

$$\Psi_P(a^*, Ta^*, t) = 1.$$

Since Ta^* is closed and

$$\Psi_P(a^*, a^*, t) = \Psi_P(a^*, Ta^*, t) = 1,$$

the closure characterization of FPMSs yields $a^* \in Ta^*$. Thus, a^* is a FP of T. □

Remark 7. *Condition (ii) is related to the point-selection property in Nadler's FPT for multivalued mappings. However, in the present fuzzy partial metric setting, it requires the selected point $v \in Ty$ to satisfy the stated inequality simultaneously for every $t > 0$. This uniformity is used in constructing the iterative sequence in the proof. The condition holds automatically when T is single-valued, viewed as a singleton-valued mapping, since*

$$\mathcal{H}_P(\{Tx\}, \{Ty\}, t) = \Psi_P(Tx, Ty, t).$$

It also holds for constant multivalued mappings $T(x) = A$, with $v = u$. Although condition (ii) is restrictive, weakening it would require additional assumptions or a corresponding modification of the proof.

Example 2. *Let $\Omega = [0, \infty)$ and let $\Psi(x, y) = \max\{x, y\}$. Define*

$$\Psi_P(x, y, t) = e^{-\frac{p(x,y)}{t+1}} = e^{-\frac{\max\{x,y\}}{t+1}}, \qquad t > 0.$$

Then $(\Omega, \Psi_P, *)$ is a complete FPMS, where $*$ is the minimum t-norm, that is,

$$a * b = \min\{a, b\}, \qquad a, b \in [0, 1].$$

Let $\eta : (0, 1] \to [0, \infty)$ be defined by

$$\eta(r) = \frac{1}{r} - 1.$$

Clearly, η is strictly decreasing and $\eta(1) = 0$. Define the multivalued mapping $T : \Omega \to \mathscr{C}(\Omega)$ by

$$T(x) = \{\lambda x\}, \qquad x \in \Omega,$$

where $\lambda \in (0, 1)$ is fixed. Since every $T(x)$ is a singleton, $T(x) \in \mathscr{C}(\Omega)$ for every $x \in \Omega$.

We show that T is a multivalued fuzzy η-contractive mapping with $k = \lambda$. For $x, y \in \Omega$, since $T(x) = \{\lambda x\}$ and $T(y) = \{\lambda y\}$, by the definition of the Hausdorff fuzzy partial metric, we have

$$H_P(Tx, Ty, t) = H_P(\{\lambda x\}, \{\lambda y\}, t) = \Psi_P(\lambda x, \lambda y, t).$$

Hence

$$H_P(Tx, Ty, t) = e^{-\frac{\lambda \max\{x,y\}}{t+1}}.$$

On the other hand,

$$\Psi_P(x, y, t) = e^{-\frac{\max\{x,y\}}{t+1}}.$$

Therefore,

$$\eta(H_P(Tx, Ty, t)) = e^{\frac{\lambda \max\{x,y\}}{t+1}} - 1 \text{ and } \eta(\Psi_P(x, y, t)) = e^{\frac{\max\{x,y\}}{t+1}} - 1.$$

Put

$$z = \frac{\max\{x, y\}}{t + 1} \geq 0.$$

Then

$$\eta(H_P(Tx, Ty, t)) = e^{\lambda z} - 1.$$

Since $0 < \lambda < 1$, the function

$$z \longmapsto \frac{e^{\lambda z} - 1}{e^z - 1}$$

is bounded above by λ for $z > 0$. Equivalently,

$$e^{\lambda z} - 1 \leq \lambda(e^z - 1), \qquad z \geq 0.$$

Consequently,

$$\eta(H_P(Tx, Ty, t)) \leq \lambda \eta(\Psi_P(x, y, t)).$$

Thus, T is a multivalued fuzzy η-contractive mapping with $\mathfrak{d} = \lambda \in (0,1)$. Next, we verify the auxiliary condition in the theorem. Let r, s, τ belong to the set of values of Ψ_P generated by the sequence $\{a_n\}$. If $\tau \geq r * s$, then, since $*$ is the minimum t-norm,

$$r * s = \min\{r, s\}.$$

Thus $\tau \geq \min\{r, s\}$. Since η is decreasing,

$$\eta(\tau) \leq \eta(\min\{r, s\}) = \max\{\eta(r), \eta(s)\} \leq \eta(r) + \eta(s).$$

Therefore, the theorem's condition (i) *is met. Finally, choose $x_0 = 0$. Then*

$$\Psi_P(x_0, x_0, t) = \Psi_P(0, 0, t) = 1 \qquad \forall\, t > 0.$$

Moreover, $T(0) = \{0\}$, and therefore

$$\bigwedge_{t>0} \Psi_P(x_0, Tx_0, t) = \bigwedge_{t>0} \Psi_P(0, 0, t) = 1 > 0.$$

Thus, all the hypotheses of the multivalued fuzzy η-contractive FPT are satisfied. Consequently, T has a FP. Indeed, $0 \in T(0)$, so $a^* = 0$ is a FP of T.

4. Homotopy Fixed-Point Theorem

We now investigate the stability of FPs for a parameterized family of multivalued mappings. The aim is to show that, under suitable contractive, parameter-continuity, boundary, and local stability conditions, the existence of a FP is preserved along the homotopy. More precisely, we consider a family $F(\cdot, \lambda)$, $\lambda \in [0,1]$, and establish that the existence of a FP at one endpoint of the parameter interval is equivalent to its existence at the other endpoint.

The following theorem gives the required homotopy result. Condition (a) prevents FPs from occurring on the boundary of the admissible region, while conditions (b) and (c) provide uniform contractivity and continuity with respect to the parameter. Conditions (d) and (e) ensure, respectively, the required structure of FPs and their local persistence under small variations of the parameter.

Theorem 4. *Let $(\Omega, \Psi_P, *)$ be a 1-complete FPMS, let $\mathfrak{L}$ be a nonempty open subset of Ω, and let $\mathfrak{N}$ be a closed subset of Ω s.t. $\mathfrak{L} \subseteq \mathfrak{N}$. Let*

$$F : \mathfrak{N} \times [0,1] \longrightarrow \mathscr{C}(\Omega)$$

be a multivalued mapping, where $\mathscr{C}(\Omega)$ denotes the family of all compact nonempty subsets of Ω. Assume that the requirements listed below are met.

(a) For every $x \in \mathfrak{N} \setminus \mathfrak{L}$ and every $\lambda \in [0,1]$, $x \notin F(x, \lambda)$.

(b) There exist $\eta \in \mathfrak{E}$ and $\mathfrak{d} \in (0,1)$ s.t.

$$\eta(\mathcal{H}_P(F(x,\lambda), F(y,\lambda), t)) \leq \mathfrak{d}\eta(\Psi_P(x, y, t))$$

$\forall\, x, y \in \mathfrak{N}$, $\lambda \in [0,1]$, and $t > 0$.

(c) There exists a continuous function $\varphi : [0,1] \to [0,\infty)$ s.t.

$$\eta(\mathcal{H}_P(F(x,\lambda), F(x,\mu), t)) \leq \mathfrak{d}|\varphi(\lambda) - \varphi(\mu)|$$

$\forall\, x \in \mathfrak{N}$, $\lambda, \mu \in [0,1]$, and $t > 0$.

(d) *If $x \in F(x,\lambda)$ for some $x \in \mathfrak{N}$ and $\lambda \in [0,1]$, then $F(x,\lambda) = \{x\}$.*

(e) *The Hausdorff-type fuzzy partial metric satisfies*

$$\eta(\mathcal{H}_P(\mathfrak{A},\mathfrak{C},t)) \leq \eta(\mathcal{H}_P(\mathfrak{A},\mathfrak{B},t)) + \eta(\mathcal{H}_P(\mathfrak{B},\mathfrak{C},t))$$

for all nonempty compact subsets $\mathfrak{A},\mathfrak{B},\mathfrak{C} \subseteq \Omega$ and every $t > 0$.

(f) *For every $\lambda_0 \in [0,1]$ and every FP $x_0 \in \mathfrak{L}$ of $F(\cdot,\lambda_0)$, there exist $t_0 > 0$ and $\varepsilon_0 \in (0,1)$ s.t.*

$$B_P(x_0,t_0,\varepsilon_0) = \{x \in \mathfrak{N} : \Psi_P(x_0,x,t_0) > 1-\varepsilon_0\} \subseteq \mathfrak{L},$$

and, for every λ sufficiently close to λ_0, the restriction

$$F(\cdot,\lambda) : B_P(x_0,t_0,\varepsilon_0) \to \mathscr{C}(B_P(x_0,t_0,\varepsilon_0))$$

satisfies the hypotheses of the multivalued fuzzy η-contractive FPT.

Then

$$F(\cdot,0) \text{ has a FP} \iff F(\cdot,1) \text{ has a FP.}$$

Proof. Define $Q = \{\lambda \in [0,1] : x \in F(x,\lambda) \text{ for some } x \in \mathfrak{L}\}$. We prove that Q is nonempty, open, and closed in $[0,1]$. Suppose first that $F(\cdot,0)$ has a FP x_0. By condition (a), this FP cannot belong to $\mathfrak{N} \setminus \mathfrak{L}$. Hence $x_0 \in \mathfrak{L}$, and therefore $0 \in Q$. Thus $Q \neq \varnothing$.

Case 1: Q is open.

Let $\lambda_0 \in Q$. Then there exists $x_0 \in \mathfrak{L}$ s.t. $x_0 \in F(x_0,\lambda_0)$. By condition (d),

$$F(x_0,\lambda_0) = \{x_0\}.$$

Since $\mathfrak{L}$ is open, there exist $t_0 > 0$ and $\varepsilon_0 \in (0,1)$ s.t. $B_P(x_0,t_0,\varepsilon_0) \subseteq \mathfrak{L}$. By condition (f), for every λ sufficiently close to λ_0, the mapping

$$F(\cdot,\lambda) : B_P(x_0,t_0,\varepsilon_0) \to \mathscr{C}(B_P(x_0,t_0,\varepsilon_0))$$

satisfies the hypotheses of the multivalued fuzzy η-contractive FPT. Consequently, for every λ sufficiently close to λ_0, there exists $x_\lambda \in B_P(x_0,t_0,\varepsilon_0) \subseteq \mathfrak{L}$ s.t. $x_\lambda \in F(x_\lambda,\lambda)$. Hence $\lambda \in Q$. Therefore, there exists a neighborhood U of λ_0 in $[0,1]$ s.t. $U \subseteq Q$. Thus Q is open in $[0,1]$.

Case 2: Q is closed.

Let $\{\lambda_n\} \subseteq Q$ be a sequence s.t. $\lambda_n \longrightarrow \lambda^* \in [0,1]$. For each n, choose $a_n \in \mathfrak{L}$ s.t. $a_n \in F(a_n,\lambda_n)$. By condition (d), $F(a_n,\lambda_n) = \{a_n\}$. For $m,n \in \mathbb{N}$, condition (d) gives

$$F(a_n,\lambda_n) = \{a_n\}, \qquad F(a_m,\lambda_m) = \{a_m\}.$$

Consequently,

$$\begin{aligned}\eta(\Psi_P(a_n,a_m,t)) &= \eta(\mathcal{H}_P(F(a_n,\lambda_n),F(a_m,\lambda_m),t)) \\ &\leq \eta(\mathcal{H}_P(F(a_n,\lambda_n),F(a_n,\lambda_m),t)) \\ &\quad + \eta(\mathcal{H}_P(F(a_n,\lambda_m),F(a_m,\lambda_m),t)) \\ &\leq \mathfrak{d}|\varphi(\lambda_n) - \varphi(\lambda_m)| + \mathfrak{d}\eta(\Psi_P(a_n,a_m,t)),\end{aligned}$$

where the first inequality follows from condition (e), and the last follows from conditions (b) and (c). Hence

$$\eta(\Psi_P(a_n,a_m,t)) \leq \frac{\mathfrak{d}}{1-\mathfrak{d}}|\varphi(\lambda_n) - \varphi(\lambda_m)|.$$

Since φ is continuous and $\lambda_n \to \lambda^*$, we have

$$|\varphi(\lambda_n) - \varphi(\lambda_m)| \longrightarrow 0 \qquad (n, m \to \infty).$$

Hence

$$\lim_{n,m\to\infty} \eta(\Psi_P(a_n, a_m, t)) = 0.$$

Given that $\eta(1) = 0$ and η is strictly decreasing, it follows that

$$\lim_{n,m\to\infty} \Psi_P(a_n, a_m, t) = 1 \qquad \text{for every } t > 0.$$

Thus $\{a_n\}$ is a 1-Cauchy sequence in Ω.

Since $(\Omega, \Psi_P, *)$ is 1-complete, there exists $a^* \in \Omega$ s.t.

$$\lim_{n\to\infty} \Psi_P(a^*, a_n, t) = 1 \qquad \text{for every } t > 0.$$

Since $F(a_n, \lambda_n) = \{a_n\}$, condition (e), followed by conditions (b) and (c), gives

$$\begin{aligned}
\eta(\Psi_P(a_n, F(a^*, \lambda^*), t)) &\leq \eta(\mathcal{H}_P(F(a_n, \lambda_n), F(a^*, \lambda^*), t)) \\
&\leq \eta(\mathcal{H}_P(F(a_n, \lambda_n), F(a_n, \lambda^*), t)) \\
&\quad + \eta(\mathcal{H}_P(F(a_n, \lambda^*), F(a^*, \lambda^*), t)) \\
&\leq \mathfrak{d}|\varphi(\lambda_n) - \varphi(\lambda^*)| + \mathfrak{d}\eta(\Psi_P(a_n, a^*, t)).
\end{aligned}$$

Since $\lambda_n \to \lambda^*$ and $\Psi_P(a_n, a^*, t) \to 1$, it follows that

$$\lim_{n\to\infty} \eta(\Psi_P(a_n, F(a^*, \lambda^*), t)) = 0.$$

Therefore,

$$\lim_{n\to\infty} \Psi_P(a_n, F(a^*, \lambda^*), t) = 1.$$

Combining this with $\Psi_P(a_n, a^*, t) \longrightarrow 1$ and the fuzzy partial metric triangle inequality gives

$$\Psi_P(a^*, F(a^*, \lambda^*), t) = 1 \qquad \text{for every } t > 0.$$

Hence $a^* \in F(a^*, \lambda^*)$. By condition (a), it follows that $a^* \in \mathfrak{L}$. Consequently, $\lambda^* \in Q$. Thus Q is closed in $[0, 1]$. Since $[0, 1]$ is connected, Q is nonempty, open, and closed. Therefore, $Q = [0, 1]$. In particular, $1 \in Q$. Hence $F(\cdot, 1)$ has a FP. The reverse implication follows by applying the same argument with the parameter transformation

$$\lambda \longmapsto 1 - \lambda.$$

Therefore,

$$F(\cdot, 0) \text{ has a FP} \iff F(\cdot, 1) \text{ has a FP.}$$

The proof is now complete. □

Remark 8. *The assumptions of the theorem serve different purposes in establishing the homotopy invariance of the existence of fixed-points. Condition (a) excludes the fixed points in $\mathfrak{N} \setminus \mathfrak{L}$, including the boundary points, so that the fixed points cannot occur at the boundary during homotopy. Condition (b) controls the variation of F with respect to the spatial variable through a uniform contractive estimate. Condition (c) controls its dependence on the parameter and ensures that the values of the multivalued mapping vary continuously in the prescribed fuzzy sense. Condition (d) requires every fixed point to be a singleton value of the mapping, which simplifies the fixed-*

point structure. Condition (e) provides the triangle inequality for the transformed Hausdorff-type fuzzy partial metric and is used when comparing the values of this quantity in the proof. Finally, condition (f) is a local stability assumption: near any fixed point, the mapping continues to satisfy the hypotheses of the multivalued FPT for parameters sufficiently close to the given parameter. It therefore ensures the local persistence of fixed points under parameter perturbations. Together, these conditions support the argument that fixed-point existence persists throughout the homotopy.

5. Conclusions

In this paper, we investigated multivalued FP problems in FPMSs. The Hausdorff-type construction and its fundamental properties were established under the additional normalization condition on the self-distances. We also presented a separate example with non-unit self-distances, for which the induced Hausdorff-type function was verified directly. This example illustrates a particular non-normalized case rather than a general hyperspace construction for arbitrary FPMSs. Using the established framework and the stated contractive assumptions, we obtained a FP result for multivalued fuzzy η-contractive mappings. We further proved a homotopy FPT for parameterized multivalued mappings, establishing the preservation of FP existence between the two endpoints. Further work may consider broader classes of non-normalized FPMSs and investigate conditions under which the hyperspace construction remains valid.

Author Contributions: Conceptualization, I.K., T.A.; methodology, I.K. and T.A.; software, T.A., and O.A.A.O.; validation, T.A., and O.A.A.O.; formal analysis, T.A.; investigation, I.K., T.A.; resources, I.K. and T.A.; data curation, I.K. and T.A.; writing—original draft preparation, I.K.; writing—review and editing, T.A.; visualization, I.K.; supervision, I.K.; project administration, O.A.A.O.; funding acquisition, O.A.A.O. All authors have read and agreed to the published version of the manuscript.

Funding: The researchers would like to thank the Deanship of Graduate Studies and Scientific Research at Qassim University (https://www.qu.edu.sa) for the financial support (QU-APC- 2026).

Data Availability Statement: Data will be available on demand from the corresponding author.

Acknowledgments: The authors gratefully acknowledge Qassim University, Saudi Arabia, for its support of this research.

Conflicts of Interest: The authors affirm that they do not have any competing interests.

Abbreviations

The following abbreviations are used in this manuscript:

FP	Fixed point
PMS	Partial metric space
FPMS	Fuzzy partial metric space

References

1. Matthews, S.G. Partial metric topology. *Ann. N. Y. Acad. Sci.* **1994**, *728*, 183–197.
2. Altun, I.; Erduran, A. Fixed point theorems for monotone mappings on partial metric spaces. *Fixed Point Theory Appl.* **2010**, *2011*, 508730.
3. Altun, I.; Sola, F.; Simsek, H. Generalized contractions on partial metric spaces. *Topol. Appl.* **2010**, *157*, 2778–2785. Correction in *Topology Appl.* **2010**, *157*, 2778–2785.
4. Romaguera, S. Fixed point theorems for generalized contractions on partial metric spaces. *Topol. Its Appl.* **2012**, *159*, 194–199.
5. Zadeh, L.A. Fuzzy sets. *Inf. Control* **1965**, *8*, 338–353.
6. Kramosil, I.; Michálek, J. Fuzzy metrics and statistical metric spaces. *Kybernetika* **1975**, *11*, 336–344.
7. George, A.; Veeramani, P. On some results in fuzzy metric spaces. *Fuzzy Sets Syst.* **1994**, *64*, 395–399.
8. Grabiec, M. Fixed points in fuzzy metric spaces. *Fuzzy Sets Syst.* **1988**, *27*, 385–389.
9. Gregori, V.; Miñana, J.J. Some remarks on fuzzy contractive mappings. *Fuzzy Sets Syst.* **2014**, *251*, 101–103.

10. Gregori, V.; Miñana, J.J. On fuzzy Ψ-contractive sequences and fixed point theorems. *Fuzzy Sets Syst.* **2016**, *300*, 93–101.
11. Gregori, V.; Sapena, A. On fixed-point theorems in fuzzy metric spaces. *Fuzzy Sets Syst.* **2002**, *125*, 245–252.
12. Miheţ, D. Fuzzy Ψ-contractive mappings in non-Archimedean fuzzy metric spaces. *Fuzzy Sets Syst.* **2008**, *159*, 739–744.
13. Miheţ, D. A note on fuzzy contractive mappings in fuzzy metric spaces. *Fuzzy Sets Syst.* **2014**, *251*, 83–91. Correction in *Fuzzy Sets and Systems* **2008**, *159*, 739–744.
14. Wardowski, D. Fuzzy contractive mappings and fixed points in fuzzy metric spaces. *Fuzzy Sets Syst.* **2013**, *222*, 108–114.
15. Markin, J.T. Continuous dependence of fixed point sets. *Proc. Am. Math. Soc.* **1973**, *38*, 545–547.
16. Heilpern, S. Fuzzy mappings and fixed point theorem. *J. Math. Anal. Appl.* **1981**, *83*, 566–569.
17. Nadler, S.B., Jr. Multi-Valued Contraction Mappings. *Pac. J. Math.* **1969**, *30*, 475–488.
18. Azam, A.; Arshad, M.; Vetro, P. On a pair of fuzzy Ψ-contractive mappings. *Math. Comput. Model.* **2010**, *52*, 207–214.
19. Azam, A.; Beg, I. Common fixed points of fuzzy maps. *Math. Comput. Model.* **2009**, *49*, 1331–1336.
20. Bose, R.K.; Sahani, D. Fuzzy mappings and fixed point theorems. *Fuzzy Sets Syst.* **1987**, *21*, 53–58.
21. Mohammed, S.S. On fuzzy soft set-valued maps with application. *J. Niger. Soc. Phys. Sci.* **2020**, 26–35.
22. Shagari, M.S.; Azam, A. Fixed point theorems of fuzzy set-valued maps with applications. **2020**, *9*, 68–86.
23. Qiu, D.; Shu, L. Supremum metric on the space of fuzzy sets and common fixed point theorems for fuzzy mappings. *Inf. Sci.* **2008**, *178*, 3595–3604.
24. Aydi, H.; Abbas, M.; Vetro, C. Partial Hausdorff metric and Nadler's fixed point theorem on partial metric spaces. *Topol. Its Appl.* **2012**, *159*, 3234–3242.
25. Mohammadzadeh, A.; Taghavifar, H.; Zhang, C.; Alattas, K.A.; Liu, J.; Vu, M.T. A non-linear fractional-order type-3 fuzzy control for enhanced path-tracking performance of autonomous cars. *IET Control Theory Appl.* **2024**, *18*, 40–54.
26. Liu, X.; Zhao, L.; Jin, J. A noise-tolerant fuzzy-type zeroing neural network for robust synchronization of chaotic systems. *Concurr. Comput. Pract. Exp.* **2024**, *36*, e8218.
27. Jin, J.; Chen, W.; Ouyang, A.; Yu, F.; Liu, H. A time-varying fuzzy parameter zeroing neural network for the synchronization of chaotic systems. *IEEE Trans. Emerg. Top. Comput. Intell.* **2023**, *8*, 364–376.
28. Li, X.; Tian, H.; Zhang, P.; Liu, X. Existence, uniqueness, and continuous dependence on initial/final values for liouville–caputo fractional difference equations. *Fractal Fract.* **2026**, *10*, 504.
29. Yue, T.; Wang, T. New Datko-type results for the uniform exponential instability in mean of cocycles. *Dyn. Syst.* **2026**, *41*, 128–134.
30. Yao, X.; Li, X.; Liu, Z.; Gao, H.; Li, J.; Liu, Y. Asymptotical fault-tolerant time-varying tracking control of networked mobile flexible manipulators under prescribed performances and DoS attacks. *IEEE Trans. Autom. Control* **2025**, *71*, 20–35.
31. Klement, E.P.; Mesiar, R.; Pap, E. *Triangular Norms*. In *Trends in Logic*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 2000; Volume 8.

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.