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Autocovariance Functions of Binary Sequences Obtained by Switching Two Binary Sequences with a Binary Control Sequence

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Abstract

This study analyzes the autocovariance function of a binary sequence generated by switching between two stationary binary source sequences according to a third binary control sequence. Assuming that the three component sequences are mutually statistically independent, we derive a closed-form expression for the autocovariance function of the resulting sequence in terms of the mean values and autocovariance functions of the source and control sequences. The derived expression is decomposed into three contributions, which clarify the respective roles of the component statistics and lead to several representative special cases. As a concrete realization, chaos-based binary sequences generated by one-dimensional piecewise-linear maps are employed because their mean values and autocovariance functions can be controlled analytically. Numerical experiments are conducted for various combinations of positively correlated, sign-alternating, and zero-autocovariance component sequences. The numerical autocovariance functions show good agreement with the theoretical predictions in all examined cases, as supported by quantitative error assessments based on repeated simulations. The proposed analysis is not restricted to chaos-based sequences and is applicable to arbitrary stationary binary sequences satisfying the mutual-independence assumption.

Keywords: binary sequence; autocovariance function; switching; stationary process; chaotic map

MSC: 60G10; 60G12; 37E05

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1. Introduction

Binary stochastic sequences are fundamental objects in the study of discrete stochastic processes and arise in a wide range of applications. Their temporal dependence can be characterized by quantities such as transition probabilities, autocorrelation functions, and autocovariance functions. Binary time series and their statistical modeling have been studied from various viewpoints, including classical binary time-series models [1,2], the estimation of autocorrelation properties [3], and more general models of dependent binary processes [4,5].

Another important research direction concerns the construction of binary sequences with prescribed or controllable correlation properties. Usatenko and Yampol'skii analyzed binary N-step Markov chains exhibiting long-range correlations [6]. Izrailev et al. proposed a method for generating correlated binary sequences from white noise [7], and an iterative

construction method was subsequently developed in [8]. More recently, dependent stationary binary sequences with prescribed autocovariance structures have been investigated using renewal-process-based models [9]. These studies demonstrate the importance of characterizing and controlling temporal dependence in binary sequences.

Chaotic dynamical systems provide another approach to generating binary sequences with analytically controllable statistical properties. In particular, binary sequences obtained by transforming real-valued chaotic orbits through binary-valued functions can have analytically tractable mean values and autocovariance functions. Piecewise-linear chaotic maps have been used to generate binary sequences with tunable exponential correlation characteristics [10], and related properties of Markov binary sequences generated by chaotic maps have also been investigated [11]. Mathematical foundations concerning invariant measures, ergodicity, and statistical properties of chaotic dynamical systems can be found in [12].

Switching or selecting between binary sequences is another way of constructing new binary sequences. For example, the Geffe generator employs a binary control signal to select between two binary sequences generated by linear feedback shift registers [13]. A similar switching structure is also used in stochastic computing, where a multiplexer selects between two stochastic bit streams to implement scaled addition [14,15]. Related switching mechanisms appear in regime-switching time-series models, for which the statistical properties, including autocovariance structures, have been investigated [16]. Although these models have different objectives and assumptions, they illustrate the importance of understanding the statistical properties of sequences generated through switching operations. In particular, the temporal dependence of an output binary sequence is relevant when evaluating statistical properties such as the variance of its finite-length sample mean.

Several related methods have also been investigated in chaos-based binary sequence generation. Tsuneda and Fujikawa [17] studied binary sequences generated by switching between two chaotic binary sequences according to the preceding output bit. In their method, the source sequences are advanced according to the switching history, and the autocorrelation properties of the resulting sequences were investigated numerically. Tsuneda [18] considered a different construction in which two binary-valued functions are alternately applied to a common chaotic orbit. The autocorrelation functions of the resulting sequences were theoretically expressed in terms of the autocorrelation and cross-correlation functions associated with the two binary functions. In contrast to these methods, the present study considers synchronous switching between two stationary binary source sequences using a third, statistically independent binary control sequence. The resulting autocovariance function is characterized in terms of the mean values and autocovariance functions of the three component sequences.

The purpose of this study is to establish an explicit relationship between the statistical properties of the component sequences and the autocovariance function of the switched binary sequence. Under the assumption that the three component sequences are stationary and mutually statistically independent, a closed-form expression is derived using standard covariance calculations. The significance of this result lies in its applicability to arbitrary stationary binary sequences satisfying these assumptions, rather than to a particular sequence-generation mechanism. Furthermore, the derived expression can be decomposed into contributions that clarify how the autocovariance functions and mean values of the source and control sequences affect the output autocovariance. This characterization provides a theoretical basis for analyzing and potentially designing switched binary sequences with various autocovariance characteristics.

To illustrate the theoretical results, we employ chaos-based binary sequences generated by one-dimensional piecewise-linear maps. These sequences provide analytically tractable mean values and autocovariance functions and are used as specific realizations of the general framework. By combining positively correlated, sign-alternating, and zero-autocovariance binary sequences, numerical experiments are performed to demonstrate the variety of autocovariance characteristics obtainable through the switching operation and to confirm the theoretical predictions.

The remainder of this paper is organized as follows. Section 2 derives the autocovariance function of the switched binary sequence and discusses its decomposition and several special cases. Section 3 introduces chaos-based binary sequences and the piecewise-linear maps used for their realization. Section 4 presents numerical experiments and compares the theoretical and numerical autocovariance functions. Section 5 discusses the implications and limitations of the theoretical results, and Section 6 concludes the paper.

2. Autocovariance Analysis of Switched Binary Sequences

2.1. Binary Switching Model

Let $\{X_n\}$, $\{Y_n\}$, and $\{Z_n\}$ be three stationary binary sequences taking values in $\{0,1\}$. We consider a new binary sequence $\{B_n\}$ defined by

$$B_n = X_n Z_n + Y_n(1 - Z_n), \quad n = 0, 1, 2, \cdots. \tag{1}$$

Thus, $B_n = X_n$ when $Z_n = 1$, whereas $B_n = Y_n$ when $Z_n = 0$. Therefore, $\{Z_n\}$ acts as a binary control sequence that switches between the two source sequences $\{X_n\}$ and $\{Y_n\}$. Throughout this section, the three sequences $\{X_n\}$, $\{Y_n\}$, and $\{Z_n\}$ are assumed to be mutually statistically independent. This assumption refers to independence among the three stochastic processes, rather than merely among the variables at the same time index. Temporal dependence within each component sequence is allowed. Stationarity is understood in the weak (second-order) sense, meaning that the mean values are time-invariant and the autocovariance functions depend only on the time lag. Strict stationarity is not required for Theorem 1. Under these assumptions, $\{B_n\}$ is also weakly stationary. Owing to stationarity, their means are independent of n. For notational simplicity, we write

$$\left.\begin{aligned} \mu_X &= E[X_n], \\ \mu_Y &= E[Y_n], \\ \mu_Z &= E[Z_n]. \end{aligned}\right\} \tag{2}$$

The mean of B_n is then given by

$$\begin{aligned} \mu_B &= E[B_n] \\ &= E[X_n]E[Z_n] + E[Y_n](1 - E[Z_n]) \\ &= \mu_X \mu_Z + \mu_Y(1 - \mu_Z). \end{aligned} \tag{3}$$

For a stationary sequence $\{U_n\}$, its autocovariance function is defined as

$$\begin{aligned} C_U(\ell) &= E[(U_n - \mu_U)(U_{n+\ell} - \mu_U)] \\ &= E[U_n U_{n+\ell}] - \mu_U^2, \quad \ell = 0, 1, 2, \cdots. \end{aligned} \tag{4}$$

Accordingly, the autocovariance functions of X_n, Y_n, Z_n, and B_n are denoted by $C_X(\ell)$, $C_Y(\ell)$, $C_Z(\ell)$, and $C_B(\ell)$, respectively.

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2.2. Main Theorem

The following theorem gives the autocovariance function of the switched binary sequence $\{B_n\}$ in terms of the statistical properties of the three component sequences.

Theorem 1. *Let $\{X_n\}$, $\{Y_n\}$, and $\{Z_n\}$ be mutually independent weakly stationary binary sequences, and let $\{B_n\}$ be defined by (1). Then, the autocovariance function of $\{B_n\}$ is given by*

$$C_B(\ell) = (C_X(\ell) + C_Y(\ell))(C_Z(\ell) + \mu_Z^2) + (\mu_X - \mu_Y)^2 C_Z(\ell) + (1 - 2\mu_Z)C_Y(\ell) \tag{5}$$

for $\ell = 0, 1, 2, \cdots$.

Proof of Theorem 1. From (1), we have

$$B_n B_{n+\ell} = (X_n Z_n + Y_n(1 - Z_n))(X_{n+\ell} Z_{n+\ell} + Y_{n+\ell}(1 - Z_{n+\ell})). \tag{6}$$

Expanding the right-hand side gives

$$\begin{aligned} B_n B_{n+\ell} =& X_n X_{n+\ell} Z_n Z_{n+\ell} + X_n Y_{n+\ell} Z_n (1 - Z_{n+\ell}) \\ &+ Y_n X_{n+\ell} (1 - Z_n) Z_{n+\ell} + Y_n Y_{n+\ell} (1 - Z_n)(1 - Z_{n+\ell}). \end{aligned} \tag{7}$$

Using the mutual independence of the three sequences and their stationarity, we obtain

$$\begin{aligned} E[B_n B_{n+\ell}] =& E[X_n X_{n+\ell}] E[Z_n Z_{n+\ell}] + 2\mu_X \mu_Y (\mu_Z - E[Z_n Z_{n+\ell}]) \\ &+ E[Y_n Y_{n+\ell}](1 - 2\mu_Z + E[Z_n Z_{n+\ell}]). \end{aligned} \tag{8}$$

From the definitions of the autocovariance functions,

$$E[X_n X_{n+\ell}] = C_X(\ell) + \mu_X^2, \tag{9}$$

$$E[Y_n Y_{n+\ell}] = C_Y(\ell) + \mu_Y^2, \tag{10}$$

$$E[Z_n Z_{n+\ell}] = C_Z(\ell) + \mu_Z^2. \tag{11}$$

Substituting (9)–(11) into (8) and subtracting

$$\mu_B^2 = [\mu_X \mu_Z + \mu_Y(1 - \mu_Z)]^2, \tag{12}$$

we obtain

$$C_B(\ell) = \mu_Z^2 C_X(\ell) + (1 - \mu_Z)^2 C_Y(\ell) + (C_X(\ell) + C_Y(\ell) + (\mu_X - \mu_Y)^2) C_Z(\ell), \tag{13}$$

which is equivalent to (5). The derivation follows from standard covariance identities for products of independent random processes, without imposing any specific functional form on their autocovariance functions. □

2.3. Decomposition and Special Cases

The equivalent expression (13) provides a useful interpretation of how the switching operation affects the temporal dependence of the output sequence. The first two terms represent the autocovariance functions of the source sequences weighted by the squared source-selection probabilities. The remaining term describes the additional contribution associated with the temporal dependence of the control sequence. Thus, the control sequence affects the output autocovariance through both its mean value and its autocovariance function. Furthermore, the difference between the mean values of the

source sequences determines part of the contribution associated with the autocovariance of the control sequence.

To examine these effects in more detail, we decompose (5) into three contributions as follows:

$$C_B(\ell) = P_1(\ell) + P_2(\ell) + P_3(\ell), \tag{14}$$

where

$$\left.\begin{aligned} P_1(\ell) &= (C_X(\ell) + C_Y(\ell))(C_Z(\ell) + \mu_Z^2), \\ P_2(\ell) &= (\mu_X - \mu_Y)^2 C_Z(\ell), \\ P_3(\ell) &= (1 - 2\mu_Z) C_Y(\ell). \end{aligned}\right\} \tag{15}$$

Note that the individual contributions $P_1(\ell)$, $P_2(\ell)$, and $P_3(\ell)$ are not necessarily nonnegative, particularly at nonzero lags. Furthermore, the decomposition is not unique, as illustrated by the equivalent expression (13), in which the terms are grouped differently. The present decomposition is adopted to clarify the separate effects of the mean values and autocovariance functions of the component sequences. The first term, $P_1(\ell)$, represents the combined contribution of the autocovariance functions of the two source sequences and the control sequence. The second term, $P_2(\ell)$, arises from the difference between the mean values of the two source sequences. In particular,

$$P_2(\ell) = 0 \tag{16}$$

if either

$$\mu_X = \mu_Y \tag{17}$$

or

$$C_Z(\ell) = 0. \tag{18}$$

Consequently, $P_2(\ell)$ is nonzero if and only if both $\mu_X \neq \mu_Y$ and $C_Z(\ell) \neq 0$. Thus, a difference between the mean values of the two source sequences contributes through $P_2(\ell)$ only when the control sequence has nonzero autocovariance at the corresponding lag. Similarly, the third term, $P_3(\ell)$, vanishes if

$$\mu_Z = \frac{1}{2} \tag{19}$$

or

$$C_Y(\ell) = 0. \tag{20}$$

Therefore, $P_3(\ell)$ is nonzero if and only if both $\mu_Z \neq 1/2$ and $C_Y(\ell) \neq 0$. In particular, when the control sequence is balanced, $P_3(\ell)$ vanishes independently of the autocovariance function of the source sequence $\{Y_n\}$. Based on these conditions, we consider the following four representative situations at nonzero lags ($\ell \geq 1$):

1. $P_2(\ell) = P_3(\ell) = 0$;
2. $P_2(\ell) \neq 0$ and $P_3(\ell) = 0$;
3. $P_2(\ell) = 0$ and $P_3(\ell) \neq 0$;
4. $P_1(\ell)$, $P_2(\ell)$, and $P_3(\ell)$ are all nonzero.

These cases will be used later to demonstrate how different autocovariance characteristics of the output sequence can be obtained by appropriately selecting the statistical properties of the three binary sequences.

Two simple consequences of (13) further illustrate the potential for controlling the autocovariance characteristics of the output sequence. First, if the two source sequences have zero autocovariance at all nonzero lags, the output autocovariance reduces to $C_B(\ell) = (\mu_X - \mu_Y)^2 C_Z(\ell)$ for $\ell \geq 1$. Thus, the autocovariance characteristics of the

control sequence can be reflected in the output sequence through the difference between the source mean values, even when both source sequences have zero autocovariance at nonzero lags. Conversely, if the control sequence has zero autocovariance at nonzero lags, the output autocovariance becomes $C_B(\ell) = \mu_Z^2 C_X(\ell) + (1-\mu_Z)^2 C_Y(\ell)$. In this case, the output autocovariance is a weighted combination of the source autocovariance functions. These relationships provide simple examples of how different autocovariance characteristics can be obtained by selecting the statistical properties of the component sequences.

3. Chaos-Based Realization of the Binary Sequences

3.1. Chaotic Binary Sequences

The theoretical results derived in Section 2 are applicable to arbitrary stationary binary sequences satisfying the mutual-independence assumption. In this section, as a specific realization of such sequences, we consider binary sequences generated from one-dimensional chaotic maps. Consider the one-dimensional nonlinear difference equation

$$x_{n+1} = \tau(x_n), \quad n = 0, 1, 2, \cdots, \tag{21}$$

where $\tau : I \to I$ is a map defined on an interval I. For appropriate choices of τ, the orbit $\{x_n\}_{n=0}^{\infty} = \{\tau^n(x_0)\}_{n=0}^{\infty}$ exhibits chaotic behavior, where τ^n denotes the n-fold iteration of τ.

Suppose that τ is ergodic and has an invariant density $f^*(x)$. Let the initial state x_0 be a random variable distributed according to the invariant density $f^*(x)$. Then, the binary stochastic process

$$S_n = G(\tau^n(x_0)), \quad n = 0, 1, 2, \cdots. \tag{22}$$

is stationary, where $G : I \to \{0, 1\}$ is a binary-valued function. Its mean and autocovariance function are given by

$$E[G] = \int_I G(x) f^*(x)\, dx, \tag{23}$$

$$C(\ell; G) = \int_I (G(x) - E[G])(G(\tau^\ell(x)) - E[G]) f^*(x) dx, \quad \ell = 0, 1, 2, \cdots. \tag{24}$$

Here, $f^*(x)$ represents the probability density of the real-valued chaotic sequence generated by the map under its invariant measure.

For certain classes of one-dimensional chaotic maps and binary functions, binary sequences having analytically tractable autocovariance functions can be generated. In particular, sequences with exponentially decaying autocovariance functions of the form

$$C(\ell; G) = \frac{E[G] - E[G]^2}{a^\ell}, \quad |a| > 1, \tag{25}$$

as well as sequences with

$$C(\ell; G) = (E[G] - E[G]^2)\delta(\ell), \tag{26}$$

can be constructed. Here, $\delta(\ell)$ denotes the Kronecker delta,

$$\delta(\ell) = \begin{cases} 1, & \ell = 0, \\ 0, & \ell \geq 1. \end{cases} \tag{27}$$

For $a > 1$, (25) gives a positive exponentially decaying autocovariance, whereas for $a < -1$, its sign alternates with the lag. Equation (26), on the other hand, corresponds to zero autocovariance at every nonzero lag.

To realize the three binary sequences introduced in Section 2, we employ three chaotic binary sequences generated independently as

$$\left.\begin{aligned} X_n &= G_1(\tau_1^n(x)), \\ Y_n &= G_2(\tau_2^n(y)), \\ Z_n &= G_3(\tau_3^n(z)), \end{aligned}\right\} \tag{28}$$

where G_1, G_2, and G_3 are binary-valued functions, and τ_1, τ_2, and τ_3 may in general be different chaotic maps. The initial states x, y, and z are assumed to be mutually independent random variables distributed according to the invariant measures of the corresponding maps. Since each sequence is determined only by its corresponding initial value and map, this construction satisfies the mutual-independence assumption used in Section 2. The output sequence B_n is then generated from X_n, Y_n, and Z_n according to (1).

3.2. Piecewise-Linear Chaotic Maps and Binary Functions

For the numerical realization, we employ the following three piecewise-linear chaotic maps defined on $I = [0, 1]$. First, for $a > 1$, define

$$\tau_P(x, t, a) = \begin{cases} \dfrac{x}{c_1}, & 0 \le x < c_1, \\ a(x - c_1), & c_1 \le x < c_2, \\ \dfrac{x - c_2}{1 - c_2}, & c_2 \le x \le 1, \end{cases} \tag{29}$$

where

$$c_1 = t(1 - a^{-1}), \quad c_2 = t + (1 - t)a^{-1}, \quad 0 < t < 1. \tag{30}$$

For $a > 1$ and $0 < t < 1$, these definitions ensure that $0 < c_1 < t < c_2 < 1$, with $c_2 - c_1 = 1/a$.

For $a < -1$, we use

$$\tau_N(x, t, a) = \begin{cases} \dfrac{x}{c_1}, & 0 \le x < c_1, \\ a(x - c_2), & c_1 \le x < c_2, \\ \dfrac{x - c_2}{1 - c_2}, & c_2 \le x \le 1, \end{cases} \tag{31}$$

where

$$c_1 = t + (1 - t)a^{-1}, \qquad c_2 = t(1 - a^{-1}). \tag{32}$$

For τ_N, the three intervals are properly defined when $0 < c_1 < t < c_2 < 1$. From (32), this condition is equivalent to $1/(1 - a) < t < -a/(1 - a)$ for $a < -1$.

The third map is the skew tent map

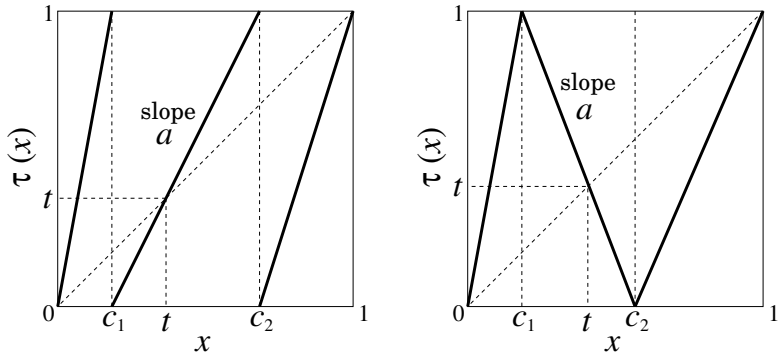
$$\tau_T(x, t) = \begin{cases} \dfrac{x}{t}, & 0 \le x < t, \\ \dfrac{1 - x}{1 - t}, & t \le x \le 1, \end{cases} \qquad 0 < t < 1. \tag{33}$$

The maps τ_P, τ_N, and τ_T are illustrated in Figures 1 and 2. These maps satisfy the fully-stretching condition, meaning that each monotonic branch maps its domain onto the entire interval $I = [0, 1]$, apart from possible endpoint conventions [10,11]. For τ_P and τ_N, the widths of the three branches are c_1, $1/|a|$, and $1 - c_2$, whose sum equals one. Therefore, the sum of the reciprocals of the absolute slopes is also one. Similarly, for the skew tent map

τ_T, this sum is $t + (1 - t) = 1$. Consequently, the Frobenius-Perron equation shows that all three maps have the uniform invariant density

$$f^*(x) = 1, \quad 0 \le x \le 1, \tag{34}$$

independently of the parameters a and t within their respective ranges. Furthermore, these maps are piecewise expanding and fully stretching, with the absolute slope of each branch greater than one. Consequently, they are mixing and hence ergodic with respect to the uniform invariant measure [12]. When the initial states are independently distributed according to this measure, the resulting binary stochastic processes are stationary and ergodic.



(**a**) $\tau_P(x, t, a)$ $(a > 1)$ (**b**) $\tau_N(x, t, a)$ $(a < -1)$

Figure 1. Piecewise-linear maps with three sections [10].

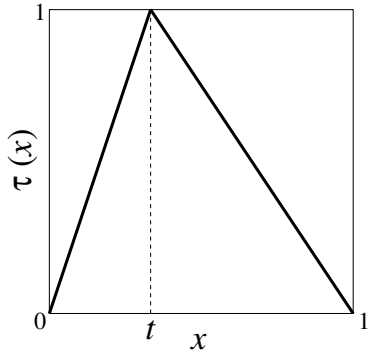


Figure 2. Skew tent map $\tau_T(x, t)$.

To convert the real-valued chaotic sequences into binary sequences, we use the threshold function

$$\Theta_d(x) = \begin{cases} 1, & x \ge d, \\ 0, & x < d. \end{cases} \tag{35}$$

In the following, the threshold is set to $d = t$. Since the invariant density is uniform, the mean of the resulting binary sequence is

$$E[\Theta_t] = \int_0^1 \Theta_t(x)\, dx = 1 - t. \tag{36}$$

Thus, the parameter t directly determines the mean value of the binary sequence. For τ_P and τ_N, the centered threshold function $h_t(x) = \Theta_t(x) - (1 - t)$ is an eigenfunction of the corresponding Frobenius-Perron operator P_τ with eigenvalue a^{-1} [10]. Therefore, the au-

tocovariance function can be expressed as $C(\ell) = \int_0^1 h_t(x)(P_\tau^\ell h_t)(x)dx = a^{-\ell}\int_0^1 h_t(x)^2 dx$, yielding

$$C(\ell) = \frac{t(1-t)}{a^\ell}, \quad \ell = 0, 1, 2, \cdots. \tag{37}$$

Here, the factor $t(1-t)$ represents the variance of the binary symbol, since $E[\Theta_t] = 1-t$ and hence

$$E[\Theta_t] - E[\Theta_t]^2 = t(1-t). \tag{38}$$

Consequently, τ_P generates a positive exponentially decaying autocovariance, whereas τ_N generates an exponentially decaying autocovariance with alternating signs. For the skew tent map τ_T, the threshold partition generates independent binary symbols under the uniform invariant measure. Thus, the resulting binary sequence has the autocovariance function

$$C(\ell) = t(1-t)\delta(\ell), \tag{39}$$

and therefore has zero autocovariance for all nonzero lags.

By independently selecting one of these maps and its parameters for each of X_n, Y_n, and Z_n, their mean values and autocovariance characteristics can be varied systematically. This makes it possible to examine the individual contributions identified in Section 2 and to construct switched binary sequences exhibiting a variety of autocovariance characteristics. The numerical examples presented in the next section use these chaotic binary sequences to verify the theoretical autocovariance function derived in Theorem 1.

4. Numerical Experiments

4.1. Experimental Conditions

In this section, numerical experiments are performed using the chaotic binary sequences introduced in Section 3 to verify the theoretical autocovariance function derived in Section 2. The three binary sequences X_n, Y_n, and Z_n are generated independently using the chaotic maps τ_P, τ_N, and τ_T, together with the threshold function Θ_t. The parameter values are selected so that the mean of the switched binary sequence satisfies

$$E[B_n] = \mu_B = \frac{1}{2}. \tag{40}$$

From (3), this condition is expressed as

$$\mu_X\mu_Z + \mu_Y(1-\mu_Z) = \frac{1}{2}. \tag{41}$$

To examine the individual contributions $P_1(\ell)$, $P_2(\ell)$, and $P_3(\ell)$ in (14), the experiments are classified into the following four cases. In the following classification, the conditions on $P_1(\ell)$, $P_2(\ell)$, and $P_3(\ell)$ refer to nonzero lags, i.e., $\ell \geq 1$, unless otherwise stated.

Case 1: $P_2(\ell) = P_3(\ell) = 0$ for $\ell \geq 1$

In Case 1, both the second and third terms in (14) vanish. Four different conditions are considered:

$$\begin{aligned}
&\text{Case 1-1:} \quad \mu_X = \mu_Y, \quad C_Y(\ell) = 0 \quad (\ell \geq 1),\\
&\text{Case 1-2:} \quad \mu_X = \mu_Y, \quad \mu_Z = \tfrac{1}{2},\\
&\text{Case 1-3:} \quad C_Y(\ell) = C_Z(\ell) = 0 \quad (\ell \geq 1),\\
&\text{Case 1-4:} \quad \mu_Z = \tfrac{1}{2}, \quad C_Z(\ell) = 0 \quad (\ell \geq 1).
\end{aligned} \tag{42}$$

For these four conditions, the autocovariance function of B_n reduces, respectively, to

$$\text{Case 1-1: } C_B(\ell) = C_X(\ell)(C_Z(\ell) + \mu_Z^2), \tag{43}$$
$$\text{Case 1-2: } C_B(\ell) = (C_X(\ell) + C_Y(\ell))(C_Z(\ell) + \tfrac{1}{4}), \tag{44}$$
$$\text{Case 1-3: } C_B(\ell) = \mu_Z^2 C_X(\ell), \tag{45}$$
$$\text{Case 1-4: } C_B(\ell) = \tfrac{1}{4}(C_X(\ell) + C_Y(\ell)), \tag{46}$$

for $\ell \geq 1$. These cases correspond to situations in which the autocovariance of the output sequence is determined solely by $P_1(\ell)$.

Case 2: $P_3(\ell) = 0$ for $\ell \geq 1$

In Case 2, $P_3(\ell)$ vanishes while $P_2(\ell)$ remains nonzero. The following two conditions are considered:

$$\begin{aligned} &\text{Case 2-1:} \quad C_Y(\ell) = 0, \quad \mu_X \neq \mu_Y \quad (\ell \geq 1), \\ &\text{Case 2-2:} \quad \mu_Z = \tfrac{1}{2}, \quad \mu_X \neq \mu_Y. \end{aligned} \tag{47}$$

In both cases, $C_Z(\ell) \neq 0$ for $\ell \geq 1$, as required for $P_2(\ell) \neq 0$. The corresponding autocovariance functions are

$$\text{Case 2-1: } C_B(\ell) = C_X(\ell)(C_Z(\ell) + \mu_Z^2) + (\mu_X - \mu_Y)^2 C_Z(\ell), \tag{48}$$
$$\text{Case 2-2: } C_B(\ell) = (C_X(\ell) + C_Y(\ell))(C_Z(\ell) + \tfrac{1}{4}) + (\mu_X - \mu_Y)^2 C_Z(\ell). \tag{49}$$

Thus, Case 2 illustrates the additional contribution of the autocovariance of the control sequence caused by the difference between the mean values of the two source sequences.

Case 3: $P_2(\ell) = 0$ for $\ell \geq 1$

In Case 3, $P_2(\ell)$ vanishes while $P_3(\ell)$ remains nonzero. The following two conditions are considered:

$$\begin{aligned} &\text{Case 3-1:} \quad \mu_X = \mu_Y, \quad \mu_Z \neq \frac{1}{2}, \\ &\text{Case 3-2:} \quad C_Z(\ell) = 0, \quad \mu_Z \neq \frac{1}{2} \quad (\ell \geq 1). \end{aligned} \tag{50}$$

In both cases, $C_Y(\ell) \neq 0$ for $\ell \geq 1$, as required for $P_3(\ell) \neq 0$. The corresponding autocovariance functions are

$$\text{Case 3-1: } C_B(\ell) = (C_X(\ell) + C_Y(\ell))(C_Z(\ell) + \mu_Z^2) + (1 - 2\mu_Z)C_Y(\ell), \tag{51}$$
$$\text{Case 3-2: } C_B(\ell) = \mu_Z^2(C_X(\ell) + C_Y(\ell)) + (1 - 2\mu_Z)C_Y(\ell), \tag{52}$$

respectively. Case 3 therefore demonstrates the effect of an unbalanced control sequence, i.e., $\mu_Z \neq \frac{1}{2}$, on the autocovariance of the output sequence.

Case 4: General Case

Finally, Case 4 considers the general situation in which

$$P_1(\ell) \neq 0, \quad P_2(\ell) \neq 0, \quad P_3(\ell) \neq 0. \tag{53}$$

In this case, no simplifying condition is imposed on (5), and all three contributions determine the autocovariance function of B_n.

The chaotic maps and parameter values used for X_n, Y_n, and Z_n in all experiments are summarized in Table 1. All parameter settings for τ_P and τ_N listed in Table 1 satisfy the corresponding parameter restrictions specified in Section 3.2, ensuring that the three branches of each map satisfy the *fully-stretching condition*. For each condition in Cases 1–3, two different parameter sets, denoted by "a" and "b", are employed. Two parameter sets are also considered for Case 4. Thus, a total of 18 parameter settings are examined.

Table 1. Parameters in numerical experiments.

Case	X_n		Y_n		Z_n	
	$E[X_n]$	Map τ_1	$E[Y_n]$	Map τ_2	$E[Z_n]$	Map τ_3
(1-1a)	0.5	$\tau_P(x, 0.5, 1.5)$	0.5	$\tau_T(y, 0.5)$	0.8	$\tau_P(z, 0.2, 1.5)$
(1-1b)	0.5	$\tau_N(x, 0.5, -1.5)$	0.5	$\tau_T(y, 0.5)$	0.5	$\tau_N(z, 0.5, -3)$
(1-2a)	0.5	$\tau_P(x, 0.5, 1.5)$	0.5	$\tau_P(y, 0.5, 1.2)$	0.5	$\tau_P(z, 0.5, 1.3)$
(1-2b)	0.5	$\tau_N(x, 0.5, -1.3)$	0.5	$\tau_P(y, 0.5, 1.5)$	0.5	$\tau_N(z, 0.5, -1.4)$
(1-3a)	0.5	$\tau_P(x, 0.5, 1.5)$	0.5	$\tau_T(y, 0.5)$	0.7	$\tau_T(z, 0.3)$
(1-3b)	0.5	$\tau_N(x, 0.5, -1.5)$	0.5	$\tau_T(y, 0.5)$	0.7	$\tau_T(z, 0.3)$
(1-4a)	0.5	$\tau_P(x, 0.5, 2.5)$	0.5	$\tau_N(y, 0.5, -1.5)$	0.5	$\tau_T(z, 0.5)$
(1-4b)	0.5	$\tau_N(x, 0.5, -2.5)$	0.5	$\tau_P(y, 0.5, 1.5)$	0.5	$\tau_T(z, 0.5)$
(2-1a)	0.6	$\tau_P(x, 0.4, 1.2)$	0.2	$\tau_T(y, 0.8)$	0.75	$\tau_P(z, 0.25, 1.5)$
(2-1b)	0.6	$\tau_P(x, 0.4, 1.2)$	0.2	$\tau_T(y, 0.8)$	0.75	$\tau_N(z, 0.25, -3.5)$
(2-2a)	0.6	$\tau_P(x, 0.4, 1.2)$	0.4	$\tau_P(y, 0.6, 1.3)$	0.5	$\tau_N(z, 0.5, -1.5)$
(2-2b)	0.6	$\tau_P(x, 0.4, 2.5)$	0.4	$\tau_N(y, 0.6, -2.5)$	0.5	$\tau_P(z, 0.5, 1.5)$
(3-1a)	0.5	$\tau_P(x, 0.5, 1.2)$	0.5	$\tau_N(y, 0.5, -1.5)$	0.2	$\tau_P(z, 0.8, 1.5)$
(3-1b)	0.5	$\tau_P(x, 0.5, 1.5)$	0.5	$\tau_N(y, 0.5, -1.5)$	0.25	$\tau_N(z, 0.75, -3.5)$
(3-2a)	0.5	$\tau_P(x, 0.5, 1.5)$	0.5	$\tau_P(y, 0.5, 1.2)$	0.4	$\tau_T(z, 0.6)$
(3-2b)	0.5	$\tau_N(x, 0.5, -1.5)$	0.5	$\tau_N(y, 0.5, -1.3)$	0.4	$\tau_T(z, 0.6)$
(4a)	0.6	$\tau_P(x, 0.4, 1.3)$	0.2	$\tau_P(y, 0.8, 1.2)$	0.75	$\tau_P(z, 0.25, 1.5)$
(4b)	0.6	$\tau_P(x, 0.4, 1.5)$	0.2	$\tau_N(y, 0.8, -4.5)$	0.75	$\tau_N(z, 0.25, -3.5)$

4.2. Numerical Results

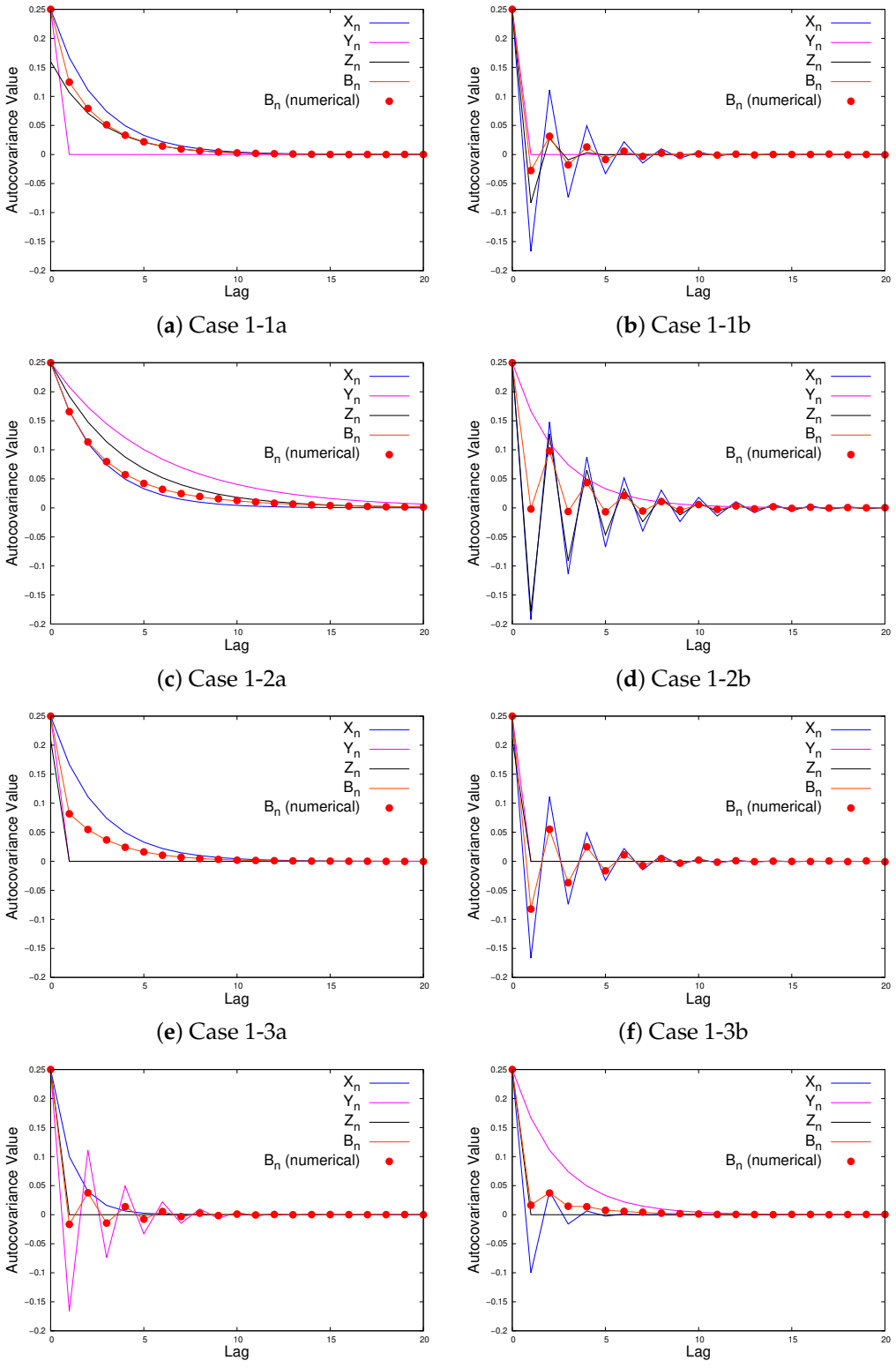
The numerical autocovariance function of B_n was evaluated as

$$\widehat{C}_B(\ell) = \frac{1}{N} \sum_{n=0}^{N-1} (B_n - \mu_B)(B_{n+\ell} - \mu_B), \quad \ell = 0, 1, \ldots, L, \tag{54}$$

where N denotes the number of samples used for the autocovariance calculation and L is the maximum lag. In (54), the theoretical mean $\mu_B = 1/2$, rather than the sample mean, is used because its value is known exactly from (40) for all parameter settings. The conventional sample autocovariance uses an estimated sample mean, which introduces an additional source of sampling variability. By using the known theoretical mean, we can directly assess the discrepancy between the theoretical and numerical autocovariance functions without the effect of mean estimation. For the ergodic sequences considered here, the sample mean converges to the theoretical mean as the sequence length increases, so the two estimators are asymptotically equivalent. Since $B_{n+\ell}$ is required in the above calculation, the binary sequence was generated up to B_{N-1+L}. For the numerical results presented in Figures 3–6, $N = 10^6$ and $L = 20$ were used. The dependence of the numerical error on N is examined separately in Section 4.3.

Figures 3–6 show the autocovariance functions obtained under the conditions described above. In each figure, the theoretical autocovariance functions of the three component sequences, $C_X(\ell)$, $C_Y(\ell)$, and $C_Z(\ell)$, together with the theoretical autocovariance function $C_B(\ell)$ calculated from (5), are plotted. The numerical values obtained from the generated sequence B_n are also shown for comparison. Figure 3 presents the results for Case 1, in which both $P_2(\ell)$ and $P_3(\ell)$ vanish for nonzero lags. The eight parameter settings

corresponding to Cases 1-1 through 1-4 demonstrate that even when only $P_1(\ell)$ contributes, a variety of autocovariance characteristics can be obtained by combining source and control sequences having different autocovariance functions. In particular, both monotonically decaying and sign-alternating autocovariance characteristics are observed depending on the chaotic maps selected for the component sequences.



(**a**) Case 1-1a (**b**) Case 1-1b

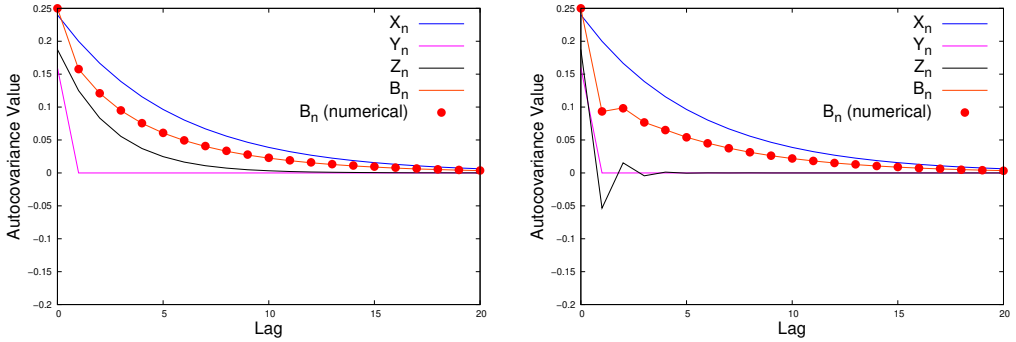
(**c**) Case 1-2a (**d**) Case 1-2b

(**e**) Case 1-3a (**f**) Case 1-3b

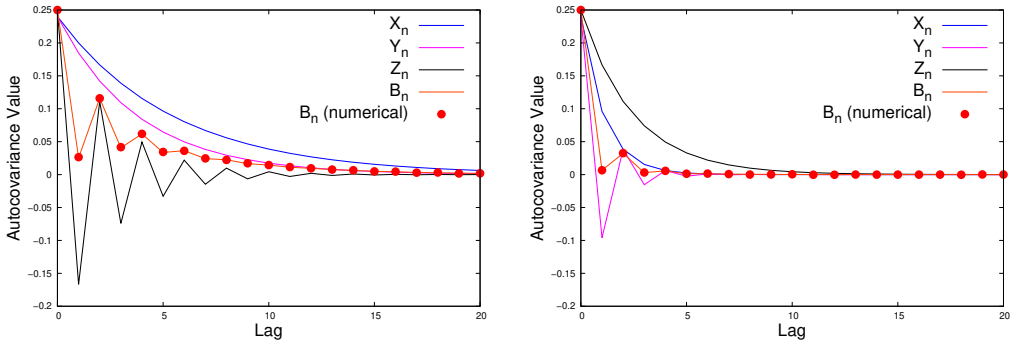
(**g**) Case 1-4a (**h**) Case 1-4b

Figure 3. Autocovariance functions in Case 1.

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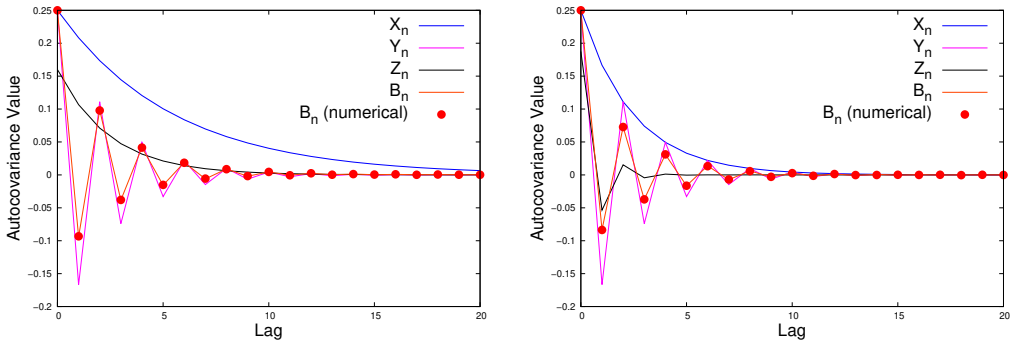


(**a**) Case 2-1a (**b**) Case 2-1b

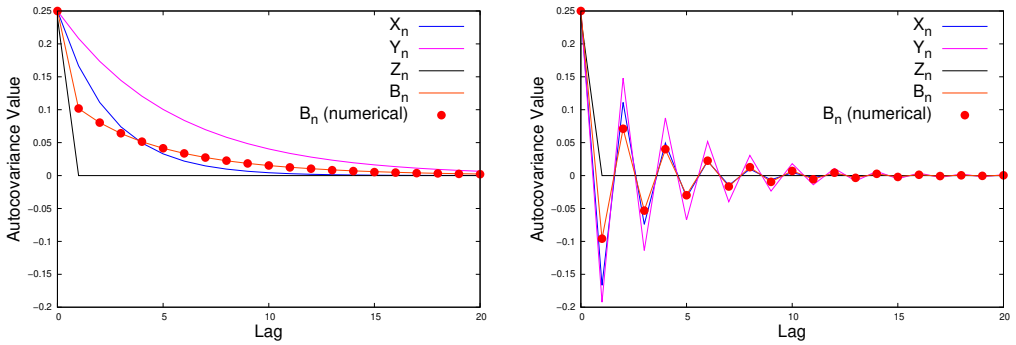


(**c**) Case 2-2a (**d**) Case 2-2b

Figure 4. Autocovariance functions in Case 2.



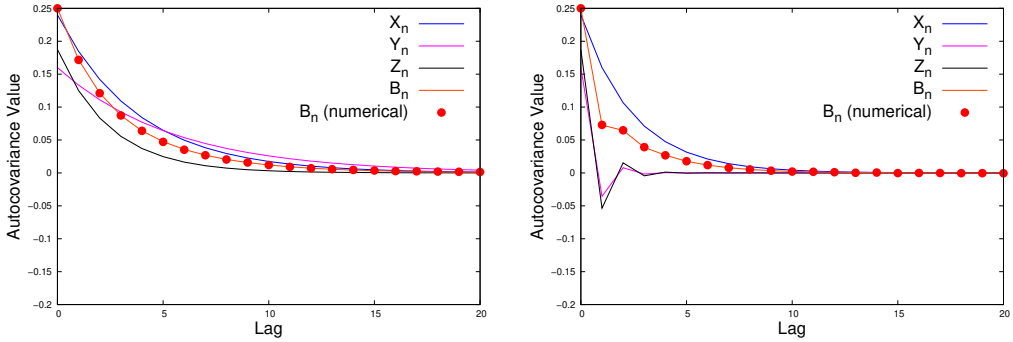
(**a**) Case 3-1a (**b**) Case 3-1b



(**c**) Case 3-2a (**d**) Case 3-2b

Figure 5. Autocovariance functions in Case 3.

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(**a**) Case 4a (**b**) Case 4b

Figure 6. Autocovariance functions in Case 4.

Figure 4 shows the results for Case 2, where $P_3(\ell) = 0$ for nonzero lags and $P_2(\ell)$ contributes in addition to $P_1(\ell)$. Since $\mu_X \neq \mu_Y$, the autocovariance of the control sequence Z_n contributes through the term

$$(\mu_X - \mu_Y)^2 C_Z(\ell). \tag{55}$$

The results show that this additional term modifies the autocovariance profile of B_n, while the numerical values remain consistent with the theoretical predictions. Figure 5 presents the results for Case 3. In this case, $P_2(\ell) = 0$ for nonzero lags, whereas the contribution

$$(1 - 2\mu_Z) C_Y(\ell) \tag{56}$$

remains because $\mu_Z \neq \frac{1}{2}$. The numerical examples confirm that the imbalance in the control sequence affects the autocovariance of the switched sequence through the autocovariance of Y_n, as predicted by (5). Finally, Figure 6 shows the results for Case 4, in which all three terms $P_1(\ell)$, $P_2(\ell)$, and $P_3(\ell)$ contribute. Despite the simultaneous presence of the three contributions, the numerical results closely follow the theoretical autocovariance functions obtained from (5).

For all the parameter settings considered, good agreement is observed between the theoretical and numerical autocovariance functions of B_n. These results are consistent with the theoretical expression derived in Theorem 1 and demonstrate that the switching operation can produce various autocovariance characteristics through appropriate combinations of the statistical properties of X_n, Y_n, and Z_n. A quantitative assessment of the numerical discrepancies is presented in the following subsection.

4.3. Quantitative Error Assessment

To quantitatively evaluate the agreement between the theoretical and numerical autocovariance functions, we calculate the root mean squared error (RMSE) over nonzero lags. For the r-th realization with sequence length N, the RMSE is defined as

$$e_r(N) = \sqrt{\frac{1}{L} \sum_{\ell=1}^{L} \left[\widehat{C}_B^{(r)}(\ell; N) - C_B(\ell) \right]^2}, \tag{57}$$

where $\widehat{C}_B^{(r)}(\ell; N)$ denotes the numerical autocovariance function obtained in the r-th realization, and $C_B(\ell)$ is the theoretical value calculated from (5). The maximum lag is set to $L = 20$. The zero lag is excluded because $\widehat{C}_B(0) = C_B(0) = 1/4$ holds identically for the binary sequences with $\mu_B = 1/2$ considered here.

For each parameter setting, 100 realizations with different sets of initial conditions were used. The mean RMSE and its standard deviation are calculated as

$$\bar{e}(N) = \frac{1}{M}\sum_{r=1}^{M} e_r(N), \tag{58}$$

and

$$s(N) = \sqrt{\frac{1}{M-1}\sum_{r=1}^{M}[e_r(N) - \bar{e}(N)]^2}, \tag{59}$$

respectively, where $M = 100$.

Table 2 summarizes the mean RMSE and standard deviation for all 18 parameter settings with $N = 10^6$. The mean RMSE ranges from 2.49×10^{-4} to 3.90×10^{-4}, indicating good quantitative agreement between the theoretical and numerical autocovariance functions in all cases.

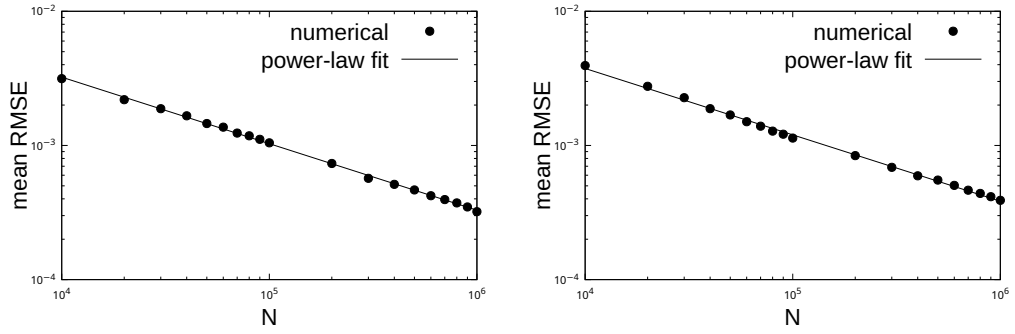
Table 2. Mean RMSE and standard deviation over 100 realizations for all 18 parameter settings ($N = 10^6$, $L = 20$).

Case	Mean RMSE	SD	Case	Mean RMSE	SD
(1-1a)	3.21×10^{-4}	9.10×10^{-5}	(1-1b)	2.59×10^{-4}	5.20×10^{-5}
(1-2a)	3.67×10^{-4}	1.38×10^{-4}	(1-2b)	2.92×10^{-4}	6.90×10^{-5}
(1-3a)	2.87×10^{-4}	6.70×10^{-5}	(1-3b)	2.92×10^{-4}	1.01×10^{-4}
(1-4a)	2.56×10^{-4}	4.90×10^{-5}	(1-4b)	2.52×10^{-4}	4.90×10^{-5}
(2-1a)	3.90×10^{-4}	1.59×10^{-4}	(2-1b)	3.33×10^{-4}	1.19×10^{-4}
(2-2a)	3.27×10^{-4}	1.16×10^{-4}	(2-2b)	2.49×10^{-4}	4.30×10^{-5}
(3-1a)	3.41×10^{-4}	1.07×10^{-4}	(3-1b)	2.87×10^{-4}	8.00×10^{-5}
(3-2a)	3.06×10^{-4}	1.10×10^{-4}	(3-2b)	3.05×10^{-4}	1.01×10^{-4}
(4a)	3.55×10^{-4}	1.33×10^{-4}	(4b)	2.93×10^{-4}	7.10×10^{-5}

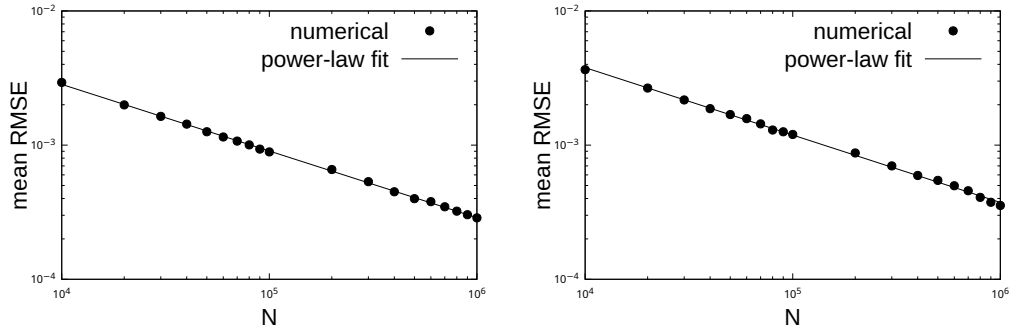
To examine the dependence of the numerical error on the sequence length, we further evaluated the mean RMSE for 19 values of N ranging from 10^4 to 10^6. Four representative parameter settings, Cases 1-1a, 2-1a, 3-1a, and 4a, were selected from the four classes considered in Section 4.1. The resulting mean RMSE values were fitted to the power-law function

$$\bar{e}(N) \approx cN^b. \tag{60}$$

Figure 7 shows the numerical results and the corresponding power-law fits. The fitted exponents b are -0.4959, -0.4947, -0.4968, and -0.5024 for Cases 1-1a, 2-1a, 3-1a, and 4a, respectively. Thus, the mean RMSE decreases approximately in proportion to $N^{-1/2}$ in all four cases, consistent with the expected scaling of finite-sample statistical errors. These quantitative results complement the autocovariance comparisons in Figures 3–6 and further support the agreement between the theoretical predictions and numerical realizations.



(**a**) Case 1-1a ($b = -0.496$, $c = 0.311$) (**b**) Case 2-1a ($b = -0.495$, $c = 0.357$)



(**c**) Case 3-1a ($b = -0.497$, $c = 0.275$) (**d**) Case 4a ($b = -0.502$, $c = 0.387$)

Figure 7. Mean RMSE versus sequence length N for four representative cases: (**a**) Case 1-1a, (**b**) Case 2-1a, (**c**) Case 3-1a, and (**d**) Case 4a. The symbols represent the numerical results averaged over 100 numerical runs with different initial conditions, and the solid lines represent the power-law fits cN^b.

5. Discussion

The main contribution of this study is an explicit characterization of the autocovariance function of a binary sequence generated by switching between two stationary binary sequences using an independent binary control sequence. Although Theorem 1 is derived from standard covariance identities under the mutual-independence assumption, the resulting expression is applicable to arbitrary stationary binary sequences satisfying this assumption, regardless of their generation mechanisms. In particular, the decomposition (14) clarifies how the statistical properties of the source and control sequences contribute to the autocovariance of the output sequence.

The present framework differs from those considered in previous studies [17,18]. Tsuneda and Fujikawa [17] employed a feedback-based switching mechanism in which the preceding output bit determines the selection between two chaotic binary sequences, and the source sequences are advanced according to the switching history. In contrast, the present study employs a separate, statistically independent control sequence and synchronous switching between the two source sequences. Tsuneda [18] investigated binary sequences generated by alternately applying two binary functions to a common chaotic orbit, deriving their autocorrelation functions using the autocorrelation and cross-correlation functions associated with these functions. Thus, these previous studies and the present work address different switching or alternating mechanisms. The present contribution is an explicit autocovariance characterization for the independent-control switching model, rather than a direct extension of the previous methods.

An important implication of Theorem 1 is that the output autocovariance can be influenced not only by the autocovariance functions of the component sequences but also by their mean values. For example, even when both source sequences have zero autocovariance at nonzero lags, temporal correlation in the control sequence can be reflected in the output autocovariance if the source mean values differ. Conversely, when the control sequence has zero autocovariance at nonzero lags, the output autocovariance becomes a weighted combination of the source autocovariance functions. These properties, discussed in Section 2.3, illustrate how different autocovariance characteristics can be obtained through appropriate selection of the component statistics.

The autocovariance function is also relevant to the sampling variability of statistics computed from finite-length binary sequences. For example, the variance of the sample mean of a stationary binary sequence $\{B_n\}$, defined by $\overline{B}_N = N^{-1}\sum_{n=0}^{N-1} B_n$, is given by

$$\mathrm{Var}(\overline{B}_N) = \frac{1}{N}\left[C_B(0) + 2\sum_{\ell=1}^{N-1}\left(1-\frac{\ell}{N}\right)C_B(\ell)\right]. \tag{61}$$

Thus, the temporal dependence of a binary sequence directly affects the variance of its finite-length sample mean. The theoretical expression derived in this study can therefore be useful for evaluating the statistical properties of binary sequences generated by switching operations, including those used in stochastic computing [14,15]. Further investigation is required to assess its usefulness in specific applications.

The numerical experiments in Section 4 illustrate the theoretical results using chaos-based binary sequences with analytically tractable mean values and autocovariance functions. The 18 parameter settings demonstrate a variety of output autocovariance characteristics, while the quantitative error assessment shows good agreement between the theoretical and numerical results. In particular, the mean RMSE decreases approximately in proportion to $N^{-1/2}$ for the four representative cases examined. Although chaotic maps provide convenient realizations for demonstrating these properties, the theoretical result itself is not restricted to chaos-based sequences.

The present analysis assumes mutual statistical independence among the three component sequences. If this assumption is violated, the output autocovariance generally depends on additional joint statistical properties of the component sequences and cannot necessarily be expressed solely in terms of their individual means and autocovariance functions. Extending the analysis to statistically dependent component sequences is therefore an important direction for future research. Another important problem is the inverse design of component sequences for realizing prescribed output autocovariance functions. Although Theorem 1 provides an explicit forward relationship, the feasibility constraints on the component statistics must also be considered in developing systematic inverse-design methods.

6. Conclusions

In this study, we analyzed the autocovariance function of a binary sequence generated by switching between two stationary binary source sequences using a third binary control sequence. Under the assumption that the three component sequences are mutually statistically independent, a general expression for the autocovariance function of the resulting sequence was derived in terms of their mean values and autocovariance functions. The derived expression clarifies how the statistical properties of the component sequences contribute to the autocovariance of the output sequence. Furthermore, several special cases were examined to illustrate the possibility of obtaining various autocovariance characteristics through appropriate combinations of the component statistics. The theoretical result is applicable to arbitrary stationary binary sequences satisfying the mutual-independence assumption, regardless of their generation mechanisms.

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As a concrete realization, chaos-based binary sequences generated by piecewise-linear maps and a skew tent map were employed. Numerical experiments were conducted for 18 parameter settings, demonstrating various autocovariance characteristics of the switched binary sequences. The quantitative error assessment showed good agreement between the theoretical and numerical autocovariance functions. In addition, the mean RMSE evaluated over 100 realizations decreased approximately in proportion to $N^{-1/2}$ for four representative cases as the sequence length N increased.

These results provide a general theoretical framework for analyzing the autocovariance characteristics of binary sequences generated by independent-control switching. Future work will include extending the analysis to statistically dependent component sequences and developing methods for designing component sequences to realize prescribed autocovariance characteristics.

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