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Improved Solution–Function Inequalities and Kneser-Type Oscillation Criteria for Noncanonical Neutral Differential Equations with Continuous Delay

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Abstract

This paper investigates the oscillatory behavior of second-order noncanonical neutral differential equations involving a neutral term with continuous delay and distributed deviating arguments. The main contribution is the establishment of a new and sharper relation between the solution and the corresponding neutral function, a problem that has not been addressed previously for noncanonical neutral equations with continuous delay. By combining iterative techniques with refined inequalities, we derive several new oscillation criteria, including Kneser-type conditions. The obtained results significantly extend the existing theory and improve a number of known oscillation tests in the literature. Illustrative examples and detailed comparisons with recent criteria demonstrate the effectiveness and superiority of the proposed approach in detecting oscillatory solutions.

Keywords: differential equations; distributed deviating arguments; oscillatory behavior; neutral delay equations; Kneser-type criteria; noncanonical equations

MSC: 34C10; 34K11

1. Introduction

Neutral differential equations (NDEs) are regarded as a principal category of functional differential equation (FDE), distinguished by the fact that the highest-order derivative of the solution depends on delayed as well as non-delayed terms. Their theoretical prominence, coupled with the notable analytical challenges encountered in analyzing their dynamical properties, has ensured that they remain a prominent subject of scholarly inquiry; see [1]. They likewise possess substantial practical significance, emerging naturally in the modeling of diverse systems, including problems associated with electric networks featuring lossless transmission lines—particularly within high-speed computing architectures where such lines interconnect switching circuits—as well as in the study of vibrating masses attached to elastic rods and in the formulation and resolution of variational problems that involve time-delay effects (see [2,3]).

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1.1. NDE with Distributed Deviating Arguments

This study applies an advanced analytical approach to investigate the oscillatory nature of solutions to the NDE:

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(r(u)\left[\frac{\mathrm{d}}{\mathrm{d}u}z(u)\right]^{\kappa}\right)+\int_{a}^{b}\phi(u,\rho)[x(\eta(u,\rho))]^{\kappa}\mathrm{d}\rho=0, \tag{1}$$

where $u \geq u_0$ and

$$z(u)=x(u)+\int_{l}^{k}p(u,\rho)x(\mu(u,\rho))\mathrm{d}\rho. \tag{2}$$

Throughout this study, we adopt the following assumptions:

(H1) $a, b, l, k \in [0,\infty)$, $a<b$, $l<k$, and $\kappa>0$ is a quotient of odd integers;
(H2) $r \in \mathbf{C}([u_0,\infty),(0,\infty))$ and $\delta(u_0)<\infty$ (noncanonical case), where

$$\delta(u):=\int_{u}^{\infty}[r(\rho)]^{-1/\kappa}\mathrm{d}\rho;$$

(H3) $\eta, \phi \in \mathbf{C}([u_0,\infty)\times[a,b],[0,\infty))$, $\eta(u,\rho)\leq u$, $\lim_{u\to\infty}\eta(u,\rho)=\infty$, $\eta(u,\rho)$ is increasing in u and ρ, and ϕ does not vanish identically on any half line $([u_*,\infty)\times[a,b])$ for $u_*\geq u_0$;
(H4) $\mu, p \in \mathbf{C}([u_0,\infty)\times[l,k],[0,\infty))$, $\mu(u,\rho)\leq u$, and $\lim_{u\to\infty}\mu(u,\rho)=\infty$.

We consider a function $x \in \mathbf{C}([u_x,\infty),\mathbb{R})$, with $u_x \geq u_0$, to be a solution of (1) provided that $z \in \mathbf{C}^1([u_x,\infty),\mathbb{R})$, $r[z']^{\kappa} \in \mathbf{C}^1([u_x,\infty),\mathbb{R})$, and that x satisfies (1) for every $u \geq u_x$. We restrict our attention to those solutions of (1) for which $\sup\{|x(u)| : u \geq u_*\}>0$ for all $u_* \geq u_x$. A solution (x) is termed oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is classified as nonoscillatory.

1.2. Review and Analysis of Previous Results

The qualitative theory of FDEs has remained a vibrant and evolving field, largely due to the broad spectrum of applications these equations model and the central role that oscillation theory plays in understanding their dynamic behavior. In recent years, considerable effort has been devoted to advancing and sharpening this theory. For second-order equations, the reader may consult [4–6] for delay-type problems and [7,8] for neutral equations. Furthermore, higher-order cases have been explored in works such as [9–11].

Another recent direction of research concerns the integration of analytical and computational approaches, where modern numerical methodologies can provide additional insight into the validation of theoretical results (see, for example, [12,13]).

One of the analytically challenging aspects in studying the oscillatory behavior of solutions to neutral differential equations lies in improving the relation between the solution and its associated function. Several recent works have focused on sharpening this relation; see [14–16] for second-order equations and [17,18] for higher-order cases. However, no previous results have addressed the situation in which takes the form given in (2). This gap is largely due to the analytical difficulties that arise when dealing with such a relatively intricate structure.

There has long been great interest in second-order functional differential equations with distributed deviating arguments. In the canonical case ($\delta(u_0)=\infty$), Wang [19], Gui and Xu [20], Ye and Xu [21], Zhang and Meng [22], Zhao and Meng [23], and Moaaz et al. [24] examined the oscillatory behavior of the solutions to the NDE:

$$\left(r(u)\psi(y(u))\left[y'(u)\right]^{\kappa}\right)'+\int_{a}^{b}\phi(u,\rho)[x(\eta(u,\rho))]^{\kappa}\mathrm{d}\rho=0,$$

where

$$y(u) = x(u) + p(u)x(\tau(u)).$$

Candan [25] considered Equation (1) in the canonical case and studied the oscillatory behavior of its solutions. In addition, the oscillatory behavior of neutral dynamic equations with distributed deviating arguments has been investigated in [26–28].

Conversely, the noncanonical case has garnered considerable interest among numerous researchers. In what follows, we provide a more detailed examination of the findings associated with this case.

In [29], Li et al. investigated the oscillatory behavior of the NDE:

$$\left(r(u)|y'(u)|^{\kappa-1}y'(u)\right)' + \int_a^b \phi(u,\rho)|x(\eta(u,\rho))|^{\kappa-1}x(\eta(u,\rho))\mathrm{d}\sigma(\rho) = 0. \tag{3}$$

They used an approach based on combining two different forms of Equation (3), instead of deriving a relation between x and y.

Theorem 1 ([29] Theorem 2.5)**.** *Assume that* $p(u) \leq p_0$, $\tau'(u) \geq \tau_0 > 0$, $\eta(\tau(u),\rho) = \tau(\eta(u,\rho))$, *and* $\eta(u,a) \leq \tau(u)$. *All solutions of (3) are oscillatory if there are* $\varrho \in \mathbf{C}^1([u_0,\infty),(0,\infty))$ *and* $\varphi \in \mathbf{C}^1([u_0,\infty),\mathbb{R})$ *such that* $\varphi(u) \geq u$, $\varphi'(u) \geq 0$,

$$\limsup_{u\to\infty}\int_{u_0}^{u}\left(\frac{\varrho(v)}{2^{\kappa-1}}\int_a^b L(v,\rho)\mathrm{d}\sigma(\rho) - \frac{1+\frac{p_0^\kappa}{\tau_0}}{(\kappa+1)^{1+\kappa}}\frac{r(\eta(v,a))[\varrho'(v)]^{\kappa+1}}{[\varrho(v)\eta'(v,a)]^\kappa}\right)\mathrm{d}v = \infty,$$

and

$$\limsup_{u\to\infty}\int_{u_0}^{u}\left(\frac{\delta^\kappa(\varphi(v))}{2^{\kappa-1}}\int_a^b L(v,\rho)\mathrm{d}\sigma(\rho) - \frac{\left(1+\frac{p_0^\kappa}{\tau_0}\right)\kappa^{1+\kappa}}{(\kappa+1)^{1+\kappa}}\frac{\varphi'(v)}{\delta(\varphi(v))r^{1/\kappa}(\varphi(v))}\right)\mathrm{d}v = \infty,$$

where $L_1(u,\rho) = \min\{\phi(u,\rho),\phi(\tau(u),\rho)\}$.

The results in [29] are notable in that they consider situations where p_0 may be either greater or less than 1, but they impose additional restrictions on the delay functions, such as $\eta(\tau(u),\rho) = \tau(\eta(u,\rho))$ and $\eta(u,a) \leq \tau(u)$.

Grace et al. [30] provided criteria for the oscillation of the damped NDE:

$$\left(r(u)\left(y'(u)\right)^\kappa\right)' + h(u)\left(y'(u)\right)^\kappa + \int_a^b \phi(u,\rho)f(u,x(\eta(u,\rho)))\mathrm{d}\rho = 0, \tag{4}$$

where $h \in \mathbf{C}([u_0,\infty),(0,\infty))$ and $f(u,y)/y^\kappa \geq l_0 > 0$.

Theorem 2 ([30] Theorem 1)**.** *Assume that*

$$\int_{u_0}^{\infty}\left(\frac{A(\mathfrak{z})}{r(\mathfrak{z})}\right)^{1/\kappa}\mathrm{d}\mathfrak{z} < \infty.$$

If there is a $\varrho \in \mathbf{C}^1([u_0,\infty),(0,\infty))$ *such that* $\varrho(u)h(u) \geq r(u)\varrho'(u)$,

$$\int_{u_0}^{u}\varrho(\mathfrak{z})\Phi_1(\mathfrak{z})\left(\frac{\eta(\mathfrak{z},b)}{\mathfrak{z}}\right)^\kappa\mathrm{d}\mathfrak{z} = \infty,$$

and

$$\int_{u_0}^{\infty}\left(\frac{A(\mathfrak{z})}{r(\mathfrak{z})}\int_{u_0}^{\mathfrak{z}}B^\kappa(\tau)\Phi_2(\tau)\mathrm{d}\tau\right)^{1/\kappa}\mathrm{d}\mathfrak{z} = \infty, \tag{5}$$

then all solutions of (4) are oscillatory, where

$$A(\mathfrak{z}) = \exp\left(-\int_{u_0}^{u} \frac{h(\mathfrak{z})}{r(\mathfrak{z})}\mathrm{d}\mathfrak{z}\right),\ B(u) = \int_{u}^{\infty}\left(\frac{A(\mathfrak{z})}{r(\mathfrak{z})}\right)^{1/\alpha}\mathrm{d}\mathfrak{z},$$

$$\Phi_1(u) = \int_a^b \phi(u,\rho)[1-p(\eta(u,\rho))]^{\kappa}\mathrm{d}\rho,$$

and

$$\Phi_2(u) = \int_a^b \phi(u,\rho)\left[1-p(\eta(u,\rho))\left(\frac{B(\tau(\eta(u,\rho)))}{B(\eta(u,\rho))}\right)\right]^{\kappa}\mathrm{d}\rho.$$

Although the presence of the f function allows the results, in Theorem 2, to be applied to a wider range of special cases, the condition of $\varrho(u)h(u) \geq r(u)\varrho'(u)$ increases the complexity of choosing the ϱ function.

In a recent study, Tunç [31] analyzed the oscillatory behavior of solutions of the NDE:

$$\left(r(u)\left(y'(u)\right)^{\kappa}\right)' + \int_a^b \phi(u,\rho)[x(\eta(u,\rho))]^{\gamma}\mathrm{d}\rho = 0, \tag{6}$$

where γ is a quotient of odd integers. These results are distinguished by the fact that they take into account the cases $\kappa = \gamma$ and $\kappa > \gamma$. Their results represent an extension of the approach adopted in Koplatadze et al. [32] and Baculikova [33].

Theorem 3 ([31] Theorem 2.5)**.** *All solutions of (6) are oscillatory if*

$$\limsup_{u\to\infty}\Psi(u) > \begin{cases} \kappa & \text{if } \kappa = \gamma, \\ 0 & \text{if } \kappa > \gamma, \end{cases}$$

where

$$\begin{aligned}\Psi(u) &= \delta(\eta(u,a))\int_{u_0}^{\eta(u,a)}[\delta(\mathfrak{z})]^{\kappa-1}W(\mathfrak{z})\mathrm{d}\mathfrak{z} + \int_{\eta(u,a)}^{u}[\delta(\mathfrak{z})]^{\kappa}W(\mathfrak{z})\mathrm{d}\mathfrak{z} \\ &\quad + [\delta(\eta(u,a))]^{-\gamma/\kappa}\int_u^{\infty}[\delta(\mathfrak{z})]^{\kappa}[\delta(\eta(\mathfrak{z},a))]^{\gamma/\kappa}W(\mathfrak{z})\mathrm{d}\mathfrak{z}\end{aligned}$$

and

$$W(u) = \int_a^b \phi(u,\rho)\left[1-p(\eta(u,\rho))\frac{\delta(\tau(\eta(u,\rho)))}{\delta(\eta(u,\rho))}\right]^{\kappa}\mathrm{d}\rho.$$

It follows from the preceding theorem that the results in [31] establish a single-condition oscillation criterion, which distinguishes them from the results in [29,30].

Considerable progress has been made in recent years in the oscillation theory of canonical differential equations, resulting in several refined oscillation criteria; see, for instance, [34–36]. The present paper contributes to this research direction by treating the more challenging noncanonical case with distributed deviating arguments.

The primary aim of our study is to provide new oscillation criteria for Equation (1) in the noncanonical case. The novelty of the results lies in two aspects: first, extending and broadening previous findings, and, second, improving upon the existing results in the literature. The main contributions of this study are outlined as follows:

- Consideration of the general case of the function z, which includes continuous delay, thereby complementing the results in [25];
- Improvement of the relation between x and z in the general case (2), which, to the best of our knowledge, has not been addressed in any previous studies;
- Attainment of Kneser-type criteria for Equation (1).

The effectiveness of the new criteria is demonstrated through comparison with the results in [29–31].

The proof of one of our main theorems relies on Theorem 2.1.1 from [37]. For convenience, below, we state the part of this theorem that will be used in the sequel.

Theorem 4 ([37]). *Let* $\tau, q \in \mathbf{C}([u_0, \infty), [0, \infty))$, $\tau(u) < u$, *and* $\lim_{u\to\infty} \tau(u) = \infty$. *If*

$$\liminf_{u\to\infty} \int_{\tau(u)}^{u} q(\mathfrak{z})\mathrm{d}\mathfrak{z} > \frac{1}{\mathbf{e}},$$

then the differential inequality expressed as

$$y'(u) + q(u)y(\tau(u)) \leq 0$$

has no eventually positive solution.

Remark 1. *Throughout the paper, all functional inequalities, sign properties, and asymptotic relations are understood to hold eventually, that is, for all sufficiently large values of u, unless explicitly stated otherwise.*

2. Main Results

In our results, we assume that $\mathbb{P}$ represents the set of solutions to Equation (1) that are eventually positive. The following notations and operators are introduced to facilitate the presentation of the results:

$$\begin{aligned} \Lambda_0[F(u)] &= F(u), \\ \Lambda_1[F(u)] &= \int_l^k F(\mu(u,\rho))\mathrm{d}\rho, \\ \Lambda_i[F(u)] &= \Lambda_{i-1}[\Lambda_1[F(u)]], \text{ for } i = 2, 3, \dots, \end{aligned}$$

and $L := 1/(k-l)$.

In order to derive refined relations, we impose the following condition:

(M) There exist constants $p_0 \in [0, L)$ and $p_1 \in (0, L)$ such that $p(u,\rho) \in [p_0, p_1]$ for $\rho \in [l, k]$. Moreover, there is a $p^* \in [0, 1)$ such that

$$p_1 \frac{\Lambda_1[\delta(u)]}{\delta(u)} \leq p^*. \tag{7}$$

We observe that in the case where $p_1 = 0$ and $p(u,\rho) \in [p_0, p_1]$, the p function must also be zero, whereupon the equation reduces to the non-neutral case. Moreover, Condition (M) is an additional assumption and does not follow from (H1)–(H4). Here, p^* denotes a positive constant independent of u satisfying (7). We emphasize that this requirement is stronger than the condition $p_1 < L$ and is imposed to control the contribution of the neutral term in the subsequent estimates.

Furthermore, we define

$$\eta_c(u) = \eta(u, c),\ \mu_c(u) = \mu(u, c), \text{ for any constant } c,$$

$$G(u) = \left(1 - \frac{p_1}{L}\right)^{\kappa} \int_a^b \phi(u,\rho)\mathrm{d}\rho,$$

$$\widetilde{\phi}_m(u) = \left[(1-p^*)\sum_{i=0}^{m}\left[p_0 p_1 (k-l)^2\right]^i\right]^{\kappa} \int_a^b \phi(u,\rho)\mathrm{d}\rho,$$

and

$$\widetilde{\phi}_{\infty}(u) = \left(\frac{(1-p^*)L^2}{L^2 - p_0 p_1}\right)^{\kappa} \int_a^b \phi(u,\rho)\mathrm{d}\rho.$$

Here, m denotes an arbitrary but fixed positive integer. The quantity $\widetilde{\phi}_m$ is defined with respect to this choice of m.

Since κ is assumed to be the ratio of two odd positive integers, the quantity s^{κ} is well defined for all $s \in \mathbb{R}$. In particular, the expression $(z'(u))^{\kappa}$ remains real-valued even when $z'(u) < 0$, and the sign of $(z'(u))^{\kappa}$ coincides with that of $z'(u)$.

2.1. Preliminary Results and Improved Inequalities

In this section, we establish structural properties of the positive solutions of the equation under consideration, followed by the derivation of relations and inequalities to be used in the remaining results.

Lemma 1. *Assume that $x \in \mathbb{P}$. Then, there exists a $u_1 \geq u_0$ such that $z(u) > 0$, z is monotone, and $r(u)[z'(u)]^{\kappa}$ is decreasing for all $u \geq u_1$.*

Proof. Let $x \in \mathbb{P}$. Then, x is eventually positive. Since $\mu(u,\rho) \to \infty$ and $\eta(u,\rho) \to \infty$ as $u \to \infty$, there exists a $u_1 \geq u_0$ such that $x(\mu(u,\rho))$ and $x(\eta(u,\rho))$ are positive for all $u \geq u_1$. Consequently, given the fact that p is positive on $[u_0,\infty) \times [l,k]$, we deduce from the definition of z that it is also positive for all $u \geq u_1$. Furthermore, based on the facts that $x(\eta(u,\rho)) > 0$ and $\phi(u,\rho) \geq 0$, we obtain from (1) that

$$\Big(r(u)\big[z'(u)\big]^{\kappa}\Big)' = -\int_a^b \phi(u,\rho)[x(\eta(u,\rho))]^{\kappa}\mathrm{d}\rho \leq 0, \quad \text{for } u \geq u_1.$$

It follows that $r(u)[z'(u)]^{\kappa}$ is monotone and therefore has a constant sign for all $u \geq u_1$. Since $r(u) > 0$ and the quantity $[z'(u)]^{\kappa}$ has the same sign as $z'(u)$, we conclude that $z'(u)$ also has a constant sign for all $u \geq u_1$. Hence, z is eventually monotone.

The proof is now complete. □

According to Lemma 1, z is eventually monotone. In the following lemma, we determine the direction of this monotonicity under an additional assumption.

Lemma 2. *Assume that $x \in \mathbb{P}$, and that condition (M) is satisfied. If*

$$\int_{u_0}^{\infty} \left(\frac{1}{r(\mathfrak{z})}\int_{u_0}^{\mathfrak{z}} G(\rho)\mathrm{d}\rho\right)^{1/\kappa} \mathrm{d}\mathfrak{z} = \infty, \tag{8}$$

then z exhibits a decreasing behavior and also confirms that z/δ is increasing.

Proof. Let $x \in \mathbb{P}$. According to Lemma 1, z is monotonic.

Suppose the contrary that $z'(u) > 0$ for $u \geq u_1$. Then,

$$x > z\Big(1 - \frac{p_1}{L}\Big),$$

and so Equation (1) becomes

$$\Big(r(u)\big[z'(u)\big]^{\kappa}\Big)' + G(u)[z(\eta_a(u))]^{\kappa} \leq 0, \tag{9}$$

and so

$$\frac{\big(r(u)[z'(u)]^{\kappa}\big)'}{[z(\eta_a(u))]^{\kappa}} \leq -G(u). \tag{10}$$

Next, we define

$$F(u) := \int_{u_0}^{u} \left(\frac{1}{r(\mathfrak{z})} \int_{u_0}^{\mathfrak{z}} G(\rho) \mathrm{d}\rho \right)^{1/\kappa} \mathrm{d}\mathfrak{z},$$

Hence, it follows from (8) that $\lim_{u\to\infty} F(u) = \infty$. Moreover, since $\mathfrak{z} \leq u$, we obtain

$$F(u) \leq \int_{u_0}^{u} \frac{1}{r^{1/\kappa}(\mathfrak{z})} \mathrm{d}\mathfrak{z} \left(\int_{u_0}^{u} G(\rho) \mathrm{d}\rho \right)^{1/\kappa},$$

which, as $u \to \infty$, becomes

$$\delta(u_0) \lim_{u\to\infty} \left(\int_{u_0}^{u} G(\rho) \mathrm{d}\rho \right)^{1/\kappa} = \infty.$$

From (H2), we have $\delta(u_0) < \infty$, and thus

$$\int_{u_1}^{u} G(\rho) \mathrm{d}\rho \to \infty \text{ as } u \to \infty. \tag{11}$$

Now, we define

$$w := \frac{r \left[z' \right]^{\kappa}}{\left[z(\eta_a) \right]^{\kappa}} > 0.$$

Hence, it follows from (10) that

$$\begin{aligned} w' &= \frac{\left(r \left[z' \right]^{\kappa} \right)'}{\left[z(\eta_a) \right]^{\kappa}} - \kappa \frac{r \left[z' \right]^{\kappa}}{\left[z(\eta_a) \right]^{2\kappa}} \left[z(\eta_a) \right]^{\kappa-1} z'(\eta_a) \, \eta_a' \\ &\leq -G - \kappa \frac{r \left[z' \right]^{\kappa}}{\left[z(\eta_a) \right]^{2\kappa}} \left[z(\eta_a) \right]^{\kappa-1} z'(\eta_a) \, \eta_a'. \end{aligned}$$

Since $z(u), z'(u) > 0$ and $\eta(u, \rho)$ is increasing in u, we obtain $w'(u) \leq -G(u)$. Integrating this relation, we get

$$\int_{u_1}^{u} G(\rho) \mathrm{d}\rho \leq w(u_1) - w(u). \tag{12}$$

Because w is positive and decreasing, Equation (12) stands in contradiction with (11). Now, we have $z'(u) < 0$. Thus, we obtain

$$z(u) \geq -\int_{u}^{\infty} \left[r(v) \right]^{1/\kappa} z'(v) \left[r(v) \right]^{-1/\kappa} \mathrm{d}v \geq -\delta(u) \left[r(u) \right]^{1/\kappa} z'(u), \tag{13}$$

and then z/δ is increasing.

The proof is now complete. □

Lemma 3. *Assume that $x \in \mathbb{P}$, (M) and (8) hold. Consequently,*

$$\Lambda_i[z(u)] \geq (k-l)^i z(u) \tag{14}$$

and

$$\Lambda_{i+1}[z(u)] \leq \frac{p^*}{p_1} \Lambda_i[z(u)], \tag{15}$$

for $i \in \mathbb{N}_0$.

Proof. Let $x \in \mathbb{P}$. According to Lemma 2, z exhibits a decreasing behavior and also indicates that z/δ is increasing. Thus, we obtain

$$z(u) \leq z(\mu(u,\rho)) \leq \frac{\delta(\mu(u,\rho))}{\delta(u)} z(u), \tag{16}$$

for $\rho \in [l,k]$. Moreover, we observe that

$$\Lambda_{m+1}[z(u)] = \Lambda_m[\Lambda_1[z(u)]] = \Lambda_1[\Lambda_m[z(u)]]. \tag{17}$$

We will prove relations (14) and (15) by applying the principle of induction.

First, for (14), one readily verifies that relation (14) is fulfilled when $i = 0$. Now, we assume that

$$\Lambda_m[z(u)] \geq (k-l)^m z(u), \tag{18}$$

for some $m > 0$. Using (16)–(18), we obtain

$$\begin{aligned} \Lambda_{m+1}[z(u)] &= \Lambda_1[\Lambda_m[z(u)]] \\ &\geq \Lambda_1\left[(k-l)^m z(u)\right] \\ &= (k-l)^m \int_l^k z(\mu(u,\rho))\mathrm{d}\rho \\ &\geq (k-l)^{m+1} z(u). \end{aligned} \tag{19}$$

Thus, (14) is fulfilled for every $i \in \mathbb{N}_0$.

Next, for (15), it follows from (16) that

$$\begin{aligned} \Lambda_1[z(u)] &= \int_l^k z(\mu(u,\rho))\mathrm{d}\rho \\ &\leq \int_l^k \frac{z(u)}{\delta(u)} \delta(\mu(u,\rho))\mathrm{d}\rho \\ &= \frac{z(u)}{\delta(u)} \Lambda_1[\delta(u)], \end{aligned}$$

which with (M) yields

$$\Lambda_1[z(u)] \leq \frac{p^*}{p_1} \Lambda_0[z(u)].$$

If we now assume that

$$\Lambda_m[z(u)] \leq \frac{p^*}{p_1} \Lambda_{m-1}[z(u)],$$

for some $m > 0$, then

$$\Lambda_{m+1}[z(u)] = \Lambda_1[\Lambda_m[z(u)]] \leq \frac{p^*}{p_1} \Lambda_1[\Lambda_{m-1}[z(u)]] = \frac{p^*}{p_1} \Lambda_m[z(u)],$$

and this implies that (15) is valid for every $i \in \mathbb{N}_0$.

The proof is now complete. □

Lemma 4. *Assume that $x \in \mathbb{P}$, (M) and (8) hold. Consequently,*

$$x(u) \geq (1-p^*) z(u) \sum_{i=0}^{m} \left[p_0 p_1 (k-l)^2\right]^i, \text{ for } m \geq 0, \tag{20}$$

and

$$x(u) \geq \frac{(1-p^*)L^2}{L^2 - p_0 p_1} z(u). \tag{21}$$

Furthermore, Equation (1) takes the form of

$$\left(r(u)\left[z'(u)\right]^{\kappa}\right)' + \widetilde{\phi}_{\infty}(u)[z(\eta_b(u))]^{\kappa} \leq 0. \tag{22}$$

Proof. Let $x \in \mathbb{P}$. From Lemma 2, z exhibits a decreasing behavior and also satisfies that z/δ is increasing.

By induction, we shall show that

$$x(u) \geq \sum_{i=0}^{m} (p_0 p_1)^i (\Lambda_{2i}[z(u)] - p_1 \Lambda_{2i+1}[z(u)]) + (p_0 p_1)^{m+1} \Lambda_{2m+2}[x(u)]. \tag{23}$$

First, it follows from (2) and (H4) that

$$z(u) - p_1 \Lambda_1[x(u)] \leq x(u) \leq z(u) - p_0 \Lambda_1[x(u)].$$

So, we arrive at

$$\begin{aligned} x(u) &\geq \Lambda_0[z(u)] - p_1 \Lambda_1[x(u)] \\ &\geq \Lambda_0[z(u)] - p_1 \Lambda_1[z(u) - p_0 \Lambda_1[x(u)]] \\ &= \Lambda_0[z(u)] - p_1 \Lambda_1[z(u)] + p_0 p_1 \Lambda_2[x(u)]. \end{aligned} \tag{24}$$

Hence, (23) is verified for $m = 0$. Subsequently, the assumption that (23) holds for $m = k$ leads to

$$x(u) \geq \sum_{i=0}^{k} (p_0 p_1)^i (\Lambda_{2i}[z(u)] - p_1 \Lambda_{2i+1}[z(u)]) + (p_0 p_1)^{k+1} \Lambda_{2k+2}[x(u)]. \tag{25}$$

Applying the operator Λ_{2k+2} to (24), we obtain

$$\Lambda_{2k+2}[x(u)] \geq \Lambda_{2k+2}[z(u)] - p_1 \Lambda_{2k+3}[z(u)] + p_0 p_1 \Lambda_{2k+4}[x(u)],$$

which with (25) gives

$$\begin{aligned} x(u) &\geq \sum_{i=0}^{k} (p_0 p_1)^i (\Lambda_{2i}[z(u)] - p_1 \Lambda_{2i+1}[z(u)]) \\ &\quad + (p_0 p_1)^{k+1} (\Lambda_{2k+2}[z(u)] - p_1 \Lambda_{2k+3}[z(u)]) + (p_0 p_1)^{k+2} \Lambda_{2k+4}[x(u)] \\ &= \sum_{i=0}^{k+1} (p_0 p_1)^i (\Lambda_{2i}[z(u)] - p_1 \Lambda_{2i+1}[z(u)]) + (p_0 p_1)^{k+2} \Lambda_{2k+4}[x(u)]. \end{aligned}$$

Consequently, (23) is true for any integer $m \geq 0$.

Since $x(u)$, the Λ_i operator preserves positivity, and given the fact that $p_0 p_1 \geq 0$, we have

$$(p_0 p_1)^{m+1} \Lambda_{2m+2}[x(u)] \geq 0.$$

Therefore, inequality (23) yields

$$x(u) \geq \sum_{i=0}^{m} (p_0 p_1)^i (\Lambda_{2i}[z(u)] - p_1 \Lambda_{2i+1}[z(u)]). \tag{26}$$

Based on Lemma 3, we find that (14) and (15) hold. Thus, (26) reduces to

$$\begin{aligned} x(u) &\geq (1 - p^*) \sum_{i=0}^{m} (p_0 p_1)^i \Lambda_{2i}[z(u)] \\ &\geq z(u)(1 - p^*) \sum_{i=0}^{m} \left[p_0 p_1 (k - l)^2\right]^i. \end{aligned}$$

Since $p_0 \le p_1 < L = 1/(k-l)$, we have $p_0 p_1 (k-l)^2 < 1$, so

$$\sum_{i=0}^{\infty} \left[p_0 p_1 (k-l)^2 \right]^i = \frac{1}{1 - p_0 p_1 (k-l)^2} = \frac{L^2}{L^2 - p_0 p_1}$$

Therefore,

$$x(u) \ge \frac{(1-p^*)L^2}{L^2 - p_0 p_1} z(u),$$

which, with (1), gives

$$\left(r(u) \left[z'(u) \right]^{\kappa} \right)' + \widetilde{\phi}_{\infty}(u) [z(\eta_b(u))]^{\kappa} \le 0.$$

The proof is now complete. □

Remark 2. *It can readily be observed that*

$$x(u) \ge (1-p^*) z(u) \left(1 + \sum_{i=1}^{m} \left[p_0 p_1 (k-l)^2 \right]^i \right) \ge (1-p^*) z(u).$$

Thus, relation (20) provides a genuine improvement over the classical estimate connecting the solution and its associated neutral function. Consequently, whenever this refined relation is incorporated into a given oscillation framework, the resulting oscillation criteria are expected to be sharper than those obtained using the traditional relation

2.2. Oscillation Criteria

Among the highly effective tools in oscillation theory is the comparison technique, which connects the oscillation of the given equation to that of a simpler or lower-order one. In the next theorem, we apply this method to derive oscillation criteria of the Kneser type.

Theorem 5. *Assume that $\kappa \ge 1$, (8) and (M) are satisfied. If all solutions of the equation expressed as*

$$\left(r^{1/\kappa}(u) \delta^2(u) w'(u) \right)' + \frac{1}{\kappa} \delta^{\kappa}(u) \widetilde{\phi}_{\infty}(u) \delta(\eta_b(u)) w(\eta_b(u)) = 0 \tag{27}$$

oscillate, then all solutions of (1) are likewise oscillatory.

Proof. Suppose the contrary that $x \in \mathbb{P}$. From Lemma 2, z exhibits a decreasing behavior and also satisfies that z/δ is increasing. Hence, we find that

$$-r^{1/\kappa} z' \le \frac{1}{\delta} z \le \frac{1}{\delta} (z \circ \eta_b),$$

or

$$\left[-r \left[z' \right]^{\kappa} \right]^{-1+1/\kappa} = \left[r \left[z' \right]^{\kappa} \right]^{-1+1/\kappa} \ge \frac{1}{\delta^{1-\kappa}} [z \circ \eta_b]^{1-\kappa}. \tag{28}$$

Thus, it follows from (22) and (28) that

$$\begin{aligned} \left(r^{1/\kappa} z' \right)' &= \frac{1}{\kappa} \left[r \left[z' \right]^{\kappa} \right]^{-1+1/\kappa} \left(r \left[z' \right]^{\kappa} \right)' \\ &\le -\frac{1}{\kappa} \widetilde{\phi}_{\infty} \left[r \left[z' \right]^{\kappa} \right]^{-1+1/\kappa} [z \circ \eta_b]^{\kappa} \\ &\le -\frac{1}{\kappa} \delta^{\kappa-1} \widetilde{\phi}_{\infty} (z \circ \eta_b), \end{aligned}$$

which implies that

$$\begin{aligned}\left(r^{1/\kappa}\,\delta^2\left[\frac{z}{\delta}\right]'\right)' &= \left(\delta\, r^{1/\kappa}\, z' + z\right)' \\ &= \delta\left(r^{1/\kappa}\, z'\right)' \\ &\leq -\frac{1}{\kappa}\,\delta^{\kappa}\,\widetilde{\phi}_{\infty}\,(z\circ\eta_b). \end{aligned} \tag{29}$$

Using the transformation that preserves oscillation $w = z/\delta > 0$, (29) becomes

$$\left(r^{1/\kappa}\,\delta^2\,w'\right)' + \frac{1}{\kappa}\,\delta^{\kappa}\,\widetilde{\phi}_{\infty}\,(\delta\circ\eta_b)\,(w\circ\eta_b) \leq 0.$$

According to Corollary 1 in [38], the corresponding Equation (27) must also have a positive solution, leading to a contradiction with the theorem's assumptions.

The proof is now complete. □

In view of the fact that $\delta(u_0) < \infty$, we deduce that

$$\varphi(u) := \int_{u_0}^{u} \frac{1}{r^{1/\kappa}(v)\delta^2(v)}\mathrm{d}v = \frac{1}{\delta(u)} - \frac{1}{\delta(u_0)} \to \infty \text{ as } u \to \infty,$$

which implies that Equation (27) is in the canonical form. Using the results established in [39], we can now obtain Kneser-type oscillation criteria, as formulated in the next corollary.

Corollary 1. *Assume that $\kappa \geq 1$, (8) and (M) are satisfied. If*

$$\lambda := \lim_{u\to\infty} \frac{\varphi(u)}{\varphi(\eta_b(u))} < \infty$$

and

$$\liminf_{u\to\infty}\left[r^{1/\kappa}(u)\delta^{\kappa+2}(u)\delta(\eta_b(u))\varphi(u)\varphi(\eta_b(u))\widetilde{\phi}_{\infty}(u)\right] > \kappa\gamma, \tag{30}$$

then all solutions of Equation (1) are oscillatory, where

$$\gamma := \max\left\{\frac{\epsilon(1-\epsilon)}{\lambda^{\epsilon}} : \epsilon \in (0,1)\right\}.$$

Proof. Suppose the contrary, i.e., that $x \in \mathbb{P}$. According to Lemma 2, z exhibits a decreasing behavior and also indicates z/δ is increasing. By proceeding as in the proof of Theorem 5, one concludes that Equation (27) has a positive solution. Yet Theorem 2 in [39] asserts that, under condition (30), all solutions of Equation (27) must be oscillatory. This contradiction completes the argument. □

In the following theorems, we extend and improve the approach used in Theorem 2.2 in [40] to obtain the appropriate oscillation criterion.

Theorem 6. *Assume that $\kappa \geq 1$, (8) and (M) are satisfied. If*

$$\limsup_{u\to\infty}\left[\widetilde{\delta}(u)\right]^{\kappa}\int_{u_1}^{u}\widetilde{\phi}_{\infty}(\mathfrak{z})\mathrm{d}\mathfrak{z} > 1, \tag{31}$$

then all solutions of Equation (1) are oscillatory, where

$$\widetilde{\delta}(u) = \delta(u) + \frac{1}{\kappa}\frac{\delta(u)}{\delta(\eta_b(u))}\int_{u}^{\infty}\delta^{\kappa}(\mathfrak{z})\delta(\eta_b(\mathfrak{z}))\widetilde{\phi}_{\infty}(\mathfrak{z})\mathrm{d}\mathfrak{z}. \tag{32}$$

Proof. Suppose the contrary, i.e., that $x \in \mathbb{P}$. According to Lemma 2, z exhibits a decreasing behavior and also indicates that z/δ is increasing.

By proceeding in the same manner as in the proof of Theorem 5, we arrive at (29). If we define $H = \delta r^{1/\kappa} z' + z$, then it follows from (13) and (29) that $H > 0$ and $H' < 0$. Integrating (29), we obtain

$$z(u) \geq -\delta(u)[r(u)]^{1/\kappa} z'(u) + \frac{1}{\kappa} \int_u^\infty \delta^\kappa(\mathfrak{z}) \widetilde{\phi}_\infty(\mathfrak{z}) z(\eta_b(\mathfrak{z})) \mathrm{d}\mathfrak{z}. \tag{33}$$

Since $z' < 0$, $(z/\delta)' \geq 0$, $\eta_b(u) \leq u$, and $\left(r^{1/\kappa} z'\right)' < 0$, we find that

$$z(\eta_b(\mathfrak{z})) \geq \frac{\delta(\eta_b(\mathfrak{z}))}{\delta(\eta_b(u))} z(\eta_b(u)) \geq \frac{\delta(\eta_b(\mathfrak{z}))}{\delta(\eta_b(u))} z(u) \geq -\frac{\delta(\eta_b(\mathfrak{z}))}{\delta(\eta_b(u))} r^{1/\kappa}(u) z'(u) \delta(u)$$

and substituting it into relation (33) leads to

$$\begin{aligned} z(u) &\geq -[r(u)]^{1/\kappa} z'(u) \delta(u) \left[1 + \frac{1}{\kappa} \frac{1}{\delta(\eta_b(u))} \int_u^\infty \delta^\kappa(\mathfrak{z}) \delta(\eta_b(\mathfrak{z})) \widetilde{\phi}_\infty(\mathfrak{z}) \mathrm{d}\mathfrak{z}\right] \\ &= -\widetilde{\delta}(u)[r(u)]^{1/\kappa} z'(u). \end{aligned} \tag{34}$$

Now, integrating (22), we get

$$\begin{aligned} -r(u)\left[z'(u)\right]^\kappa &\geq -r(u_1)\left[z'(u_1)\right]^\kappa + \int_{u_1}^u \widetilde{\phi}_\infty(\mathfrak{z})[z(\eta_b(\mathfrak{z}))]^\kappa \mathrm{d}\mathfrak{z} \\ &\geq [z(\eta_b(u))]^\kappa \int_{u_1}^u \widetilde{\phi}_\infty(\mathfrak{z}) \mathrm{d}\mathfrak{z}, \end{aligned} \tag{35}$$

which, with (34), gives

$$-r(u)\left[z'(u)\right]^\kappa \geq -\left[\widetilde{\delta}(u)\right]^\kappa r(u)\left[z'(u)\right]^\kappa \int_{u_1}^u \widetilde{\phi}_\infty(\mathfrak{z}) \mathrm{d}\mathfrak{z},$$

or

$$1 \geq \left[\widetilde{\delta}(u)\right]^\kappa \int_{u_1}^u \widetilde{\phi}_\infty(\mathfrak{z}) \mathrm{d}\mathfrak{z},$$

which contradicts (31).

The proof is now complete. □

Theorem 7. *Assume that $\eta_b(u) < u$, $\kappa \geq 1$, (8) and (M) are satisfied. If*

$$\liminf \int_{\eta_b(u)}^u \left(\frac{1}{r(\tau)} \int_{u_1}^\tau \widetilde{\phi}_\infty(\mathfrak{z}) \mathrm{d}\mathfrak{z}\right)^{1/\kappa} \mathrm{d}\tau > \frac{1}{\mathbf{e}}, \tag{36}$$

then all solutions of Equation (1) are oscillatory.

Proof. Suppose the contrary, i.e., that $x \in \mathbb{P}$. According to Lemma 2, z exhibits a decreasing behavior and also indicates that z/δ is increasing.

By proceeding in the same manner as in the proof of Theorem 6, we arrive at (35). Thus,

$$z'(u) + z(\eta_b(u)) \left(\frac{1}{r(u)} \int_{u_1}^u \widetilde{\phi}_\infty(\mathfrak{z}) \mathrm{d}\mathfrak{z}\right)^{1/\kappa} \leq 0. \tag{37}$$

So, z is a positive solution of (37). It is readily seen that $\eta_b \in \mathbf{C}([u_0, \infty), [0, \infty))$, $\eta_b(u) < u$, $\eta_b(u) \to \infty$ as $u \to \infty$, and

$$\left(\frac{1}{r(u)} \int_{u_1}^u \widetilde{\phi}_\infty(\mathfrak{z}) \mathrm{d}\mathfrak{z}\right)^{1/\kappa} \in \mathbf{C}([u_0, \infty), [0, \infty)).$$

Therefore, all assumptions of Theorem 4 are satisfied. Hence, condition (36) implies that inequality (37) admits no eventually positive solution, which is a contradiction.

The proof is now complete. □

Finally, we present a criterion that serves as an extension of the results in [40,41], which is based on the generalized Riccati substitution. The next lemma is an immediate extension of Lemma 2.5 in [40]; hence, its proof is omitted.

Lemma 5. *Assume that $x \in \mathbb{P}$ and there is a $\sigma > 0$ such that*

$$\liminf \int_{\eta_b(u)}^{u} \left(\frac{1}{r(\tau)} \int_{u_1}^{\tau} \widetilde{\phi}_\infty(\mathfrak{z}) \mathrm{d}\mathfrak{z} \right)^{1/\kappa} \mathrm{d}\tau \geq \sigma. \tag{38}$$

If z is decreasing on $[u_1, \infty)$, then there is $u_2 \geq \eta_b^{-(2+n)}(u_1)$ such that

$$\frac{z(\eta_b(u))}{z(u)} \geq H_n(\sigma),$$

for $u \geq u_2$ and for some $n \in \mathbb{N}_0$, where $H_0(\sigma) = 1$ and $H_{n+1}(\sigma) = \exp(\sigma H_n(\sigma))$.

Theorem 8. *Assume that (8), (M), and (38) are satisfied. If there exists a $\theta \in \mathbf{C}^1([u_0, \infty), (0, \infty))$ such that*

$$\limsup_{u \to \infty} \left[\frac{\delta^\kappa(u)}{\theta(u)} \int_{u_1}^{u} \left(\theta(\mathfrak{z}) \widetilde{\phi}_\infty(\mathfrak{z}) \overline{H}_n^\kappa(\sigma) - \frac{r(\mathfrak{z}) [\theta'(\mathfrak{z})]^{\kappa+1}}{(1+\kappa)^{1+\kappa} [\theta(\mathfrak{z})]^\kappa} \right) \mathrm{d}\mathfrak{z} \right] > 1, \tag{39}$$

then all solutions of Equation (1) are oscillatory, where

$$\overline{H}_n(\sigma) = \begin{cases} 1, & \text{if } \eta_b(u) = u, \\ H_n(\sigma), & \text{if } \eta_b(u) < u, \end{cases}$$

where $H_n(\sigma)$ is defined as in Lemma 5.

Proof. Suppose the contrary, i.e., that $x \in \mathbb{P}$. According to Lemma 2, z exhibits a decreasing behavior and also indicates that z/δ is increasing.

Now, we define

$$Q = \theta \left(r \left[\frac{z'}{z} \right]^\kappa + \frac{1}{\delta^\kappa} \right) \geq 0. \tag{40}$$

Then, it follows from (22) and (40) that

$$\begin{aligned} Q' &= \frac{\theta'}{\theta} Q + \theta \left(\frac{\left(r \left[z' \right]^\kappa \right)'}{z^\kappa} - \kappa r \left[\frac{z'}{z} \right]^{\kappa+1} - \frac{\kappa \delta'}{\delta^{\kappa+1}} \right) \\ &\leq \frac{\theta'}{\theta} Q - \theta \widetilde{\phi}_\infty \frac{[z(\eta_b)]^\kappa}{z^\kappa} - \frac{\kappa}{\theta^{1/\kappa} r^{1/\kappa}} \left(Q - \frac{\theta}{\delta^\kappa} \right)^{1+1/\kappa} - \kappa \frac{\theta \delta'}{\delta^{\kappa+1}}. \end{aligned}$$

Using Lemma 5, we find that

$$Q' \leq -\theta \widetilde{\phi}_\infty \overline{H}_n^\kappa(\sigma) + \frac{\theta'}{\theta} Q - \frac{\kappa}{\theta^{1/\kappa} r^{1/\kappa}} \left(Q - \frac{\theta}{\delta^\kappa} \right)^{1+1/\kappa} - \kappa \frac{\theta \delta'}{\delta^{\kappa+1}}. \tag{41}$$

Using the fact that (see ([40] Lemma 2.6)

$$A_1 y - A_2 (y - A_3)^{1+1/\kappa} \leq A_1 A_3 + \kappa^\kappa (1+\kappa)^{-1-\kappa} A_1^{1+\kappa} A_2^{-\kappa},$$

with

$$A_1 = \frac{\rho'}{\rho}, A_2 = \frac{\kappa}{\theta^{1/\kappa} r^{1/\kappa}}, \text{ and } A_3 = \frac{\theta}{\delta^\kappa},$$

we arrive at

$$\frac{\theta'}{\theta} Q - \frac{\kappa}{\theta^{1/\kappa} r^{1/\kappa}} \left(Q - \frac{\theta}{\delta^\kappa} \right)^{1+1/\kappa} \leq \frac{\theta'}{\delta^\kappa} + (1+\kappa)^{-1-\kappa} \frac{r\,[\theta']^{\kappa+1}}{\theta^\kappa},$$

which turns (41) into the form

$$Q' \leq -\theta\, \widetilde{\phi}_\infty\, \overline{H}_n^\kappa(\sigma) + (1+\kappa)^{-1-\kappa} \frac{r\,[\theta']^{\kappa+1}}{\theta^\kappa} + \frac{\theta'}{\delta^\kappa} - \kappa \frac{\theta\,\delta'}{\delta^{\kappa+1}},$$

or

$$\theta\, \widetilde{\phi}_\infty\, \overline{H}_n^\kappa(\sigma) - (1+\kappa)^{-1-\kappa} \frac{r\,[\theta']^{\kappa+1}}{\theta^\kappa} \leq -\left(Q - \frac{\theta}{\delta^\kappa} \right)'.$$

Integrating this relation, we get

$$\begin{aligned} \int_{u_1}^{u} \left(\theta(\mathfrak{z}) \widetilde{\phi}_\infty(\mathfrak{z}) \overline{H}_n^\kappa(\sigma) - (1+\kappa)^{-1-\kappa} \frac{r(\mathfrak{z})[\theta'(\mathfrak{z})]^{\kappa+1}}{[\theta(\mathfrak{z})]^\kappa} \right) \mathrm{d}\mathfrak{z} &\leq \left(\frac{\theta(\mathfrak{z})}{\delta^\kappa(\mathfrak{z})} - Q(\mathfrak{z}) \right) \Big]_{u_1}^{u} \\ &\leq -\theta(u) r(u) \left[\frac{z'(u)}{z(u)} \right]^\kappa \\ &\leq \frac{\theta(u)}{\delta^\kappa(u)}. \end{aligned}$$

This implies that

$$\frac{\delta^\kappa(u)}{\theta(u)} \int_{u_1}^{u} \left(\theta(\mathfrak{z}) \widetilde{\phi}_\infty(\mathfrak{z}) \overline{H}_n^\kappa(\sigma) - (1+\kappa)^{-1-\kappa} \frac{r(\mathfrak{z})[\theta'(\mathfrak{z})]^{\kappa+1}}{[\theta(\mathfrak{z})]^\kappa} \right) \mathrm{d}\mathfrak{z} \leq 1,$$

which contradicts (39).

The proof is now complete. □

Corollary 2. *Assume that (8) and (M) are satisfied. If, for any integer ($m \geq 0$),*

$$\limsup_{u \to \infty} \left[\frac{\delta^\kappa(u)}{\theta(u)} \int_{u_1}^{u} \left(\theta(\mathfrak{z}) \widetilde{\phi}_\infty(\mathfrak{z}) - \frac{r(\mathfrak{z})[\theta'(\mathfrak{z})]^{\kappa+1}}{(1+\kappa)^{1+\kappa}[\theta(\mathfrak{z})]^\kappa} \right) \mathrm{d}\mathfrak{z} \right] > 1,$$

then all solutions of Equation (1) are oscillatory.

Proof. In the case where $z'(u) < 0$, we have $z(\eta_b(u)) \geq z(u)$. Accordingly, replacing $\overline{H}_n(\sigma)$ by 1 yields the same conclusions as those of Theorem 8, while eliminating the necessity of condition (38). □

3. Application and Discussion

This section applies our results to specific special cases of the equation under study. Furthermore, a comparison with prior literature is provided to illustrate the efficiency and practical effectiveness of our criteria.

Example 1. *Consider the equation*

$$\frac{\mathrm{d}}{\mathrm{d}u} \left(u^{\kappa+1} \left[\frac{\mathrm{d}}{\mathrm{d}u} z(u) \right]^\kappa \right) + \phi_0 \int_a^b [x(\rho\, u)]^\kappa \mathrm{d}\rho = 0, \tag{42}$$

where $u \geq u_0$, $\kappa \geq 1$,

$$z(u) = x(u) + p_0 x(\ell u),$$

$a < b \in (0,1]$, $p_0 \in \left[0, \sqrt[\kappa]{\ell}\right)$, $\ell \in (0,1]$, *and* $\phi_0 > 0$. *We see that* $l = 0$, $k = 1$, $p(u,\rho) = p_0 = p_1$, $\mu(u,\rho) = \ell u$, $r(u) = u^{\kappa+1}$, $\phi(u,\rho) = \phi_0$, *and* $\eta(u,\rho) = \rho u$. *Then, it is not difficult to conclude that* $L = 1$,

$$\delta(u) = \frac{\kappa}{\sqrt[\kappa]{u}},$$

and

$$p^* = p_1 \frac{\Lambda_1[\delta(u)]}{\delta(u)} = \frac{p_0}{\sqrt[\kappa]{\ell}}.$$

So, (M) is satisfied. Furthermore,

$$G(u) = (b-a)\phi_0(1-p_0)^\kappa,$$

$$\widetilde{\phi}_m(u) = \phi_0(b-a)\left[\left(1 - \frac{p_0}{\sqrt[\kappa]{\ell}}\right)\sum_{i=0}^{m} p_0^{2i}\right]^\kappa,$$

and

$$\widetilde{\phi}_\infty(u) = \phi_0(b-a)\left[\left(1 - \frac{p_0}{\sqrt[\kappa]{\ell}}\right)\frac{1}{1-p_0^2}\right]^\kappa = \phi_0^*.$$

Now, condition (8) takes the form

$$(b-a)\phi_0(1-p_0)^\kappa \int_{u_0}^{\infty}\left(\frac{1}{\mathfrak{z}^{\kappa+1}}\int_{u_0}^{\mathfrak{z}} \mathrm{d}\rho\right)^{1/\kappa}\mathrm{d}\mathfrak{z} \to \infty.$$

Based on our findings, all solutions of Equation (42) exhibit oscillatory behavior in one of the following cases:

(C1) Corollary 1: if $\kappa^\kappa \phi_0^* > \gamma$, *where*

$$\gamma := \max\left\{\epsilon(1-\epsilon)b^{\epsilon/\kappa} : \epsilon \in (0,1)\right\};$$

(C2) Theorem 6: if $\phi_0^*\left[\phi_1^*\right]^\kappa > 1$, *where*

$$\widetilde{\delta}(u) = \kappa[1 + \kappa^\kappa \phi_0^*]\frac{1}{\sqrt[\kappa]{u}} = \phi_1^*\frac{1}{\sqrt[\kappa]{u}};$$

(C3) Theorem 7: if $[\phi_0^*]^{1/\kappa}\ln(1/b) > 1/\mathbf{e}$;

(C4) Theorem 8: if $\sigma := [\phi_0^*]^{1/\kappa}\ln(1/b) \leq 1/\mathbf{e}$ *and*

$$\overline{H}_n^\kappa(\sigma)\phi_0^* > \frac{1}{(1+\kappa)^{1+\kappa}}.$$

Remark 3. *Consider Equation (42) when* $\kappa = 1$, *where it reduces to the form of*

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(u^2\frac{\mathrm{d}}{\mathrm{d}u}[x(u) + p_0 x(\ell u)]\right) + \phi_0\int_a^b x(\rho\, u)\mathrm{d}\rho = 0. \tag{43}$$

We observe that Theorem 1 guarantees the oscillation of this equation when $a \leq \ell$ *and*

$$(\mathrm{C5}) : \phi_0 > \frac{1}{4(b-a)}\left(1 + \frac{p_0}{\ell}\right).$$

In contrast, Theorem 3 states the oscillation condition in the following form

$$(\mathrm{C6}) : \phi_0 > \frac{1}{(b-a)(1-p_0/\ell)\left(2+\ln\frac{1}{a}\right)}.$$

Furthermore, Theorem 2 cannot be applied because condition (5) is not met.

Now, let us take into account the more particular case

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(u^2\frac{\mathrm{d}}{\mathrm{d}u}\left[x(u)+\frac{1}{10}x(\ell u)\right]\right)+\phi_0\int_{0.1}^{0.5}x(\rho\, u)\mathrm{d}\rho=0.$$

Figure 1 illustrates the sharpness of criterion (C1) compared with (C5) and (C6) for all $\ell \in (0.189, 1)$.

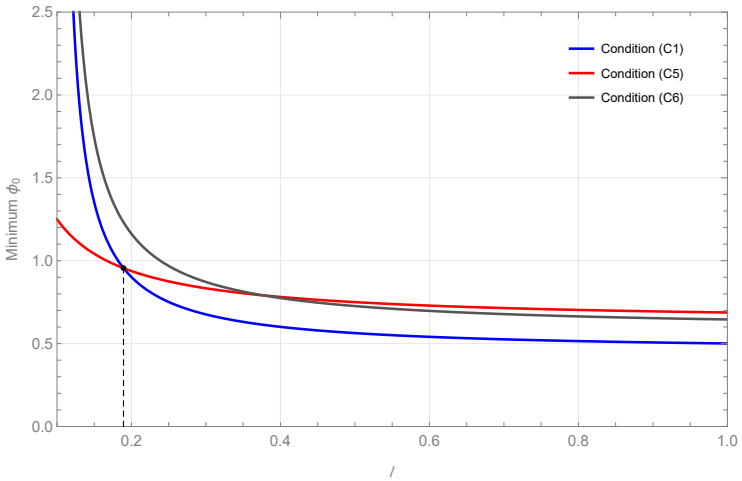


Figure 1. Minimal values of ϕ_0 obtained under criteria (C1), (C5), and (C6) for $\ell \in (0, 1)$.

Remark 4. *In Example 1, with a few straightforward computations, it can be seen that criteria (C2)–(C4) yield results that are less sharp than those of criterion (C1). However, Theorems 6–8 constitute a direct enhancement of their counterparts (see [40,41]) that employed the classical, non-improved inequalities. For instance, by considering the equation*

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(u^2\frac{\mathrm{d}}{\mathrm{d}u}\left[x(u)+\frac{4}{5}x\left(\frac{9u}{10}\right)\right]\right)+\phi_0 x(\alpha u)=0,\ \alpha\in(0,1), \tag{44}$$

we observe that Theorem 2.2 in [40] establishes an oscillation criterion for this equation in the form $\phi_0 > 9$, *whereas Theorem 6 guarantees the oscillation of Equation (44) under the condition of* $\phi_0 > 2.0173$. *Therefore, Theorem 6 constitutes not only an extension of Theorem 2.2 in [40] but also an enhancement of the methodology adopted in it.*

Example 2. *Consider the equation expressed as*

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(u^2\frac{\mathrm{d}}{\mathrm{d}u}z(u)\right)+\phi_0\int_0^1\left(\frac{3+\mathbf{e}^{-u}\sin(2\pi\rho)}{2}\right)x(\rho u)\mathrm{d}\rho=0, \tag{45}$$

with

$$z(u)=x(u)+\int_1^2\frac{3+\cos(\rho u)}{7}x\left(\frac{u}{\rho}\right)\mathrm{d}\rho,$$

where $u \geq u_0$ *and* $\phi_0 > 0$. *One can easily check that* $L = 1$, $\delta(u) = 1/u$,

$$\Lambda_1[\delta(u)] = \int_1^2 \delta(\mu(u,\rho))\mathrm{d}\rho = \frac{3}{2u},$$

$$p(u,\rho) = \frac{3+\cos(\rho u)}{5} \in [p_0, p_1],\ p_0 = \frac{2}{7},\ p_1 = \frac{4}{7},$$

and

$$p^* = p_1 \frac{\Lambda_1[\delta(u)]}{\delta(u)} = \frac{6}{7} < 1.$$

Moreover, we obtain

$$G(u) = \left(1 - \frac{4}{7}\right) \int_0^1 \left(\frac{3 + \mathbf{e}^{-u}\sin(2\pi\rho)}{2}\right)\mathrm{d}\rho = \frac{9}{14}\phi_0,$$

$$\widetilde{\phi}_m(u) = \phi_0\left(1 - \frac{6}{7}\right)\sum_{i=0}^{m}\left(\frac{8}{49}\right)^i \int_0^1 \left(\frac{3 + \mathbf{e}^{-u}\sin(2\pi\rho)}{2}\right)\mathrm{d}\rho,$$

and

$$\widetilde{\phi}_\infty(u) = \frac{7}{41}\phi_0 \int_0^1 \left(\frac{3 + \mathbf{e}^{-u}\sin(2\pi\rho)}{2}\right)\mathrm{d}\rho = \frac{21}{82}\phi_0.$$

Thus, condition (8) is satisfied. Now, to apply Theorem 5, we find that $\varphi(u) := u - u_0$, $\lambda = 1$, *and* $\gamma = 1/4$. *Consequently, all solutions of Equation (45) oscillate whenever* $\phi_0 > 0.97618$.

Example 3. *Consider the equation expressed as*

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(u^{4/3}\left[\frac{\mathrm{d}}{\mathrm{d}u}z(u)\right]^{1/3}\right) + \phi_0 \int_0^b (\rho+1)[x(\rho u)]^{1/3}\mathrm{d}\rho = 0, \tag{46}$$

where $u \geq u_0$,

$$z(u) = x(u) + p_0 \int_1^2 x\left(\frac{u}{\rho}\right)\mathrm{d}\rho,$$

$b \in (0,1]$, $\phi_0 > 0$, *and* $p_0 \in [0, 4/15)$. *One can easily check that* $L = 1$, $p_1 = p_0$,

$$\delta(u) = \frac{1}{3u^3},$$

$$\Lambda_1[\delta(u)] = \int_1^2 \frac{1}{3(u/\rho)^3}\mathrm{d}\rho = \frac{5}{4u^3},$$

and

$$p^* = p_1 \frac{\Lambda_1[\delta(u)]}{\delta(u)} = \frac{15}{4}p_0.$$

Furthermore, we obtain

$$G(u) = \frac{1}{2}b(b+2)\phi_0\sqrt[3]{1-p_0},$$

$$\widetilde{\phi}_m(u) = \frac{1}{2\sqrt[3]{4}}b(b+2)\phi_0\sqrt[3]{(4-15p_0)\sum_{i=0}^{m}p_0^2},$$

and

$$\widetilde{\phi}_\infty(u) = \frac{1}{2\sqrt[3]{4}}b(b+2)\phi_0\sqrt[3]{\frac{(4-15p_0)}{1-p_0^2}} := \phi_0^*.$$

Thus, condition (8) is satisfied. Using Corollary 2 with $\theta(u) = \delta^{\kappa}(u)$*, all solutions of Equation (46) oscillate if*

$$\phi_0^* > \sqrt[3]{\left(\frac{3}{4}\right)^4},$$

or

$$\frac{1}{2\sqrt[3]{4}} b(b+2)\phi_0 \sqrt[3]{\frac{4-15p_0}{1-p_0^2}} > \sqrt[3]{\left(\frac{3}{4}\right)^4}.$$

Remark 5. *It is evident that the results in [29–31] are not applicable to Examples 2 and 3. Moreover, it should be noted that Theorems 5–7 do not succeed in establishing the oscillation of (46), as in the case of* $\kappa = 1/3 < 1$.

4. Conclusions

The oscillation theory of neutral differential equations remains an active area of research due to its theoretical significance and the analytical challenges arising from the presence of deviating arguments. In this work, we addressed several aspects of oscillation theory for noncanonical second-order neutral differential equations. The main contributions of this study can be summarized as follows:

- We established a sharper relation between an eventually positive solution (x) and its associated neutral function (z) in the presence of a continuous delay.
- We extended the analysis to the general case involving a continuous deviating argument, thereby complementing and broadening existing results in the literature.
- We developed new oscillation criteria for Equation (1) in the noncanonical case by employing different analytical approaches.
- We derived a Kneser-type oscillation criterion for Equation (1).
- The effectiveness of the obtained criteria was demonstrated through illustrative examples and comparisons with related results from the literature.

Based on the examples and remarks presented in Section 3, we arrived at the following conclusions:

- The Kneser-type oscillation criterion (Corollary 1) is more effective in detecting oscillation than the other obtained criteria.
- The proposed results generally provide stronger oscillation tests than those reported in [29–31].
- Theorem 8 remains applicable for the case of $\kappa < 1$, whereas Theorems 5–7 require $\kappa \geq 1$.
- Theorem 6 provides an improved framework for deriving oscillation criteria in the form of (31).

The methodology developed herein is closely linked to the structural form of the equation rather than to its order. Therefore, extending the present approach to third- and higher-order neutral differential equations, particularly those involving distributed deviating arguments, represents a natural and promising direction for future research. It is also of interest to refine condition (M) in order to better capture the influence of the delay function (μ) within the oscillation criteria. Furthermore, the development of numerical procedures for validating the oscillation thresholds obtained herein, following recent advances in computational methods for differential equations [12,13], constitutes another worthwhile direction for future investigation.

One possible extension of the theoretical framework developed in this work is to examine the fractional-order counterpart of Equation (1)—more precisely, consider the neutral differential equation involving Caputo fractional derivatives of order $\alpha \in (0, 1]$:

$$ {}^{C}D^{\alpha}_{u_0}\left(r(u)\left[{}^{C}D^{\alpha}_{u_0}z(u)\right]^{\kappa}\right) + \int_a^b \phi(u,\rho)[x(\eta(u,\rho))]^{\kappa}\, d\rho = 0, \tag{47} $$

where $z(u)$ preserves the continuous delay structure specified in (2) and ${}^{C}D^{\alpha}_{u_0}$ denotes the Caputo fractional derivative operator.

Strictly speaking, the structural reduction technique established in Lemma 4 depends primarily on the integral representation of $z(u)$ and on the positivity properties of the solution components. Consequently, an analogue of the inequality connecting $x(u)$ and $z(u)$ can be established for (47). Nevertheless, extending the oscillation criteria derived in Theorems 5–8 to the fractional setting introduces several mathematical difficulties that prevent a direct application of the existing arguments.

1. Non-locality of monotonicity: Unlike the classical case, where $z'(u) < 0$ implies pointwise monotonic decrease, the condition of ${}^{C}D^{\alpha}_{u_0}z(u) \leq 0$ generally requires fractional mean-value arguments and introduces memory effects over the interval of $[u_0, u]$.
2. Failure of standard differential rules: The comparison techniques and Riccati-type transformations employed in Theorems 5 and 8 rely on classical product and quotient rules. In the fractional setting,

$$ {}^{C}D^{\alpha}_{u_0}(fg) \neq f\,{}^{C}D^{\alpha}_{u_0}g + g\,{}^{C}D^{\alpha}_{u_0}f, $$

   which generates additional terms requiring separate estimates.
3. Interaction between fractional kernels and delays: The combination of continuous deviating arguments ($\eta(u,\rho)$) with fractional integration operators introduces singular kernels in the form of $(u-t)^{-\alpha}$, leading to additional analytical difficulties and potentially more restrictive assumptions on the delay structure.

Thus, although the methodology developed in this paper provides a solid structural foundation, the establishment of rigorous oscillation criteria for (47) requires these operator-specific challenges be addressed rather than relying on a straightforward generalization of the present results.

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