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Modified Gronwall–Bellman-Type Retarded Integral Inequalities and Applications

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Abstract

We establish new nonlinear retarded integral inequalities of Gronwall–Bihari type involving time-dependent kernels, variable delays, nonconstant free terms, and iterated Volterra-type contributions. The first class of results provides explicit estimates for inequalities with kernels depending on both the observation time and the integration variable. A second class treats composite nonlinearities through successive comparison transformations and extends known retarded inequalities to the presence of an additional iterated kernel. The obtained estimates are then applied to nonlinear Volterra integral equations and neutral functional differential equations. For the Volterra problem, sufficient conditions for uniqueness, global existence, and boundedness are derived under Osgood-type assumptions. For neutral functional differential equations, the retarded estimates are combined with Driver’s continuation framework to obtain a global existence criterion. A delayed Liénard system is included as an illustrative application. These results extend several classical Gronwall–Bihari and retarded integral inequalities and provide a flexible tool for the qualitative analysis of nonlinear integral and functional differential equations.

Keywords: retarded; integral inequalities; Gronwall-type inequalities; nonlinear systems; time-delay differential equations; global existence; stability analysis

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1. Introduction

Integral inequalities constitute a fundamental tool in the qualitative analysis of differential, integral, and functional differential equations [1–5]. The classical inequality introduced by Gronwall [6] and its subsequent extension by Bellman [7] provide a basic comparison principle for obtaining explicit bounds on unknown functions. Bihari [8] later extended this approach to a nonlinear setting by replacing the linear dependence on the unknown function with a more general nonlinear comparison function. Since then, numerous variants of Gronwall–Bellman and Bihari-type inequalities have been developed and applied to nonlinear differential and integral equations; see, for instance, [9–14].

Such comparison estimates are particularly useful when explicit solutions are unavailable, since they permit qualitative properties such as uniqueness, boundedness, continuation, global existence, and stability to be studied through suitable a priori bounds. In problems with delay or memory, however, a direct reduction to a classical non-retarded inequality may suppress part of the information carried by the delayed integration interval. This provides one of the main motivations for retaining the retarded structure explicitly in the comparison process.

An important direction in this theory concerns retarded integral inequalities, in which one or more integration limits involve a retardation function α satisfying $\alpha(t) \leq t$. Such inequalities are naturally adapted to equations containing delay or memory effects. Lipovan [15–17] established several retarded Gronwall-type and nonlinear integral inequalities and applied them to delayed differential equations. Agarwal, Deng and Zhang [18] obtained a further generalization of a retarded Gronwall-like inequality, while Sun [19] developed power-type retarded inequalities extending the results of Lipovan. Further developments in this direction can be found in [20–22].

More general nonlinear retarded comparison frameworks were developed subsequently. Ferreira and Torres [23] considered retarded inequalities involving composite nonlinearities and time-dependent kernels $f(t, s)$ and $g(t, s)$. Their results include, as special cases, the power-type inequalities obtained by Sun [19], and for $m = 2$ and $n = 1$, the corresponding result of Lipovan [17]. Related retarded inequalities with time-dependent kernels and applications to delayed differential equations were also considered in [24].

Retarded inequalities with iterated integrals have also been studied in considerable generality. Ma and Pečarić [25] established nonlinear retarded inequalities involving several nested integral terms and obtained explicit upper bounds through suitable comparison functions. Wang, Huang and Li [26] further investigated generalized retarded nonlinear inequalities with time-dependent kernels, nonconstant free terms, composite nonlinearities, and multiple levels of iterated integration. Related results were also obtained by Abdeldaim and El-Deeb [27,28].

Recent contributions confirm that the theory remains active, particularly for delay inequalities with power-type nonlinearities and nested integral terms. Song and Meng [29] derived explicit bounds for delay Gronwall–Bellman inequalities with power and applied them to nonlinear integral equations. Boudeliou [30] subsequently considered nonlinear retarded Gronwall–Bellman inequalities with power, including nested Volterra-type contributions and applications to retarded Volterra and delay integro-differential equations. More recently, Chen and Abdul Rahman [31] developed Gronwall–Bellman–Bihari inequalities for composite nonlinear structures and fractional integral problems.

A complementary recent direction concerns weakly singular kernels. Dien [32], for example, established Henry–Gronwall–Bihari-type inequalities involving doubly singular kernels and applied them to fractional differential and integral equations. Such weakly singular frameworks require analytical techniques different from those used for the regular kernels considered in the present paper.

These developments also show that the usefulness of a Gronwall–Bihari-type estimate depends not only on the nonlinearity involved, but also on how faithfully the estimate preserves the original delay and kernel structure. This issue is particularly relevant when direct and iterated integral contributions occur simultaneously.

These contributions show that nonlinear retarded inequalities have already been developed in several directions, including variable retardation, time-dependent kernels, composite nonlinearities, and iterated integral terms. The purpose of the present paper is therefore not to introduce any of these features separately, but rather to derive explicit

comparison estimates for specific combinations of them that arise naturally in nonlinear integral and functional differential equations.

More precisely, the problems considered below are motivated by situations in which a time-dependent free term, two-variable kernels, a variable retardation, and an iterated Volterra contribution may appear in the same estimate. Treating these terms separately, or enlarging the retarded integration interval at an early stage, may lead to a coarser upper bound. Our aim is therefore to preserve this combined structure while obtaining explicit comparison estimates that remain directly applicable to nonlinear integral and functional differential equations.

Our first class of results concerns inequalities of the form

$$u(t) \leq k(t) + \int_{t_0}^t f(t, s) \omega(u(s)) ds + \int_{\alpha(t_0)}^{\alpha(t)} g(t, s) \omega(u(s)) ds,$$

where the free term k is time-dependent, the kernels $f(t, s)$ and $g(t, s)$ depend on both the observation time and the integration variable, and α is a variable retardation. Under suitable monotonicity assumptions, we obtain an explicit estimate involving

$$G(r) = \int_1^r \frac{ds}{\omega(s)}.$$

A feature of the resulting bound is that the retarded interval $[\alpha(t_0), \alpha(t)]$ is retained explicitly. This is consistent with the retarded comparison philosophy developed, for example, by Lipovan [17], where preserving the delayed integration interval may lead to sharper estimates than those obtained after replacing it by a non-retarded one.

This distinction is not merely formal. We show analytically that the retained-delay estimate is no larger than the corresponding non-retarded reduction and identify conditions under which the inequality is strict. A concrete specialization further demonstrates that preserving the retarded interval can produce a strictly smaller upper bound than the one obtained from the corresponding classical Bihari–Gronwall reduction.

Within the same variable-kernel framework, we next consider the iterated Volterra-type contribution

$$\int_{\alpha(t_0)}^{\alpha(t)} g(t, s) \int_{\alpha(t_0)}^s h(s, \tau) \omega(u(\tau)) d\tau ds.$$

Although iterated retarded integrals have already been treated in [25–28], the inequality considered here has a different additive structure: a direct variable-kernel term

$$\int_{t_0}^t f(t, s) \omega(u(s)) ds$$

is combined with a separate retarded Volterra contribution. The resulting estimate is obtained through a single nonlinear comparison function and yields a direct G^{-1} bound adapted to this structure.

A further feature of this result is that no monotonicity assumption is imposed on the inner kernel h . Its nonnegativity is the property used in the comparison step, while continuity is retained as a convenient regularity assumption in the classical formulation. The resulting upper bound depends on h through its accumulated integral contribution rather than through any pointwise monotonicity property.

We also consider a structured iterated-kernel version of the composite nonlinear comparison framework of Ferreira and Torres [23]. In their setting, ϕ is an outer increasing comparison function, while η and ω enter successive nonlinear comparison transformations. The corresponding retarded term has the form

$$\int_0^{\alpha(t)} g(t, s) \eta(u(s)) ds.$$

In the formulation considered here, this term is replaced by the nested Volterra contribution

$$\int_0^{\alpha(t)} g(t, s) \int_0^s h(s, \tau) \eta(u(\tau)) d\tau ds.$$

Under the assumptions imposed in the corresponding theorem, Fubini's theorem allows this nested contribution to be rewritten by means of the effective kernel

$$\tilde{g}(t, \tau) = \int_{\tau}^{\alpha(t)} g(t, s) h(s, \tau) ds.$$

Consequently, this formulation is not claimed to yield a uniformly stronger estimate than the Ferreira–Torres result. Its advantage is structural: the original nested kernels g and h remain explicit, which is useful when the two kernels represent distinct successive memory mechanisms. The resulting estimate retains the successive nonlinear transformations and involves the inverse functions G^{-1} and Ψ^{-1} . An analytical example illustrates this structured formulation and its relation to the corresponding effective-kernel representation.

As a consequence, we derive a power-type specialization by choosing

$$\phi(x) = x^m, \quad \eta(x) = \frac{m}{m-n} x^n, \quad m > n > 0.$$

This specialization is directly related to the power-type inequalities considered by Sun [19] and recovered in the more general framework of Ferreira and Torres [23]. In particular, the case $m = 2$ and $n = 1$ corresponds to the retarded inequality of Lipovan [17].

The resulting corollary should therefore be viewed as an iterated-kernel formulation of this power-type comparison structure rather than as a uniformly stronger power inequality. It keeps explicit the contribution

$$\int_0^{\alpha(t)} g(t, s) \int_0^s h(s, \tau) u^n(\tau) d\tau ds.$$

The theoretical estimates are then applied to two classes of problems. First, we consider nonlinear retarded Volterra integral equations and derive sufficient conditions for uniqueness, global existence, and boundedness under Osgood-type assumptions. The use of integral inequalities in the study of Volterra equations and related qualitative properties is well established; see, for example, [15,33–35].

The two Osgood-type conditions used in this application have distinct roles. The divergence condition near the origin is associated with the uniqueness argument, whereas the divergence condition at infinity prevents finite-time blow-up in the scalar comparison problem. These conditions are sufficient rather than necessary for every individual Volterra equation, and the uniqueness and growth hypotheses can be formulated with different comparison functions when greater flexibility is required.

Second, we study neutral functional differential equations. The local existence and continuation argument is based on Driver's framework [36], while the retarded estimates established in the present paper are used to obtain a sufficient condition preventing finite-

time blow-up. A delayed Liénard system is included as an illustrative application. Related delayed Liénard and Rayleigh systems were already considered by Lipovan [17] and Sun [19].

For this particular application, the change of variables used in the comparison argument leads to the same total coefficient integral as the corresponding non-retarded estimate. Thus, the Liénard example is not presented as producing a strictly sharper numerical bound. Rather, it illustrates how the delayed state structure can be handled directly within the same continuation framework. Genuine quantitative improvement from retaining the delayed interval occurs in the variable-kernel setting discussed above.

The present results are formulated for continuous kernels, together with differentiability with respect to the first variable whenever required by the comparison argument. Consequently, they do not directly cover weakly singular kernels of fractional type, such as $(t-s)^{\gamma-1}$ with $0 < \gamma < 1$, or logarithmically singular kernels. Recent weakly singular Gronwall-type results such as [32] address this complementary setting. Likewise, extending the present pointwise comparison arguments to ordered function spaces or to L^p settings would require additional order, measurability, and integral-operator arguments.

The remainder of the paper is organized as follows. Section 2 introduces the notation and recalls the preliminary results used throughout the paper, including Bihari's inequality, the generalized retarded inequality of Ferreira–Torres, and Driver's existence and continuation theorem. Section 3 contains the main results. We first derive retarded inequalities with variable kernels and then consider composite nonlinearities with iterated Volterra-type terms. Section 4 is devoted to applications to nonlinear Volterra integral equations and functional differential equations. Finally, Section 5 contains the concluding remarks.

2. Preliminaries

In this section, we introduce the notation and standing assumptions used throughout the paper and recall three results that will be needed later: Bihari's inequality, the nonlinear retarded inequality of Ferreira and Torres, and an extended existence theorem of Driver for neutral functional differential equations.

2.1. Notation and Standing Assumptions

We write $\mathbb{R}_+ = [0, \infty)$. For a vector $x \in \mathbb{R}^n$, $|x|$ denotes its Euclidean norm. The notation $AC([a, b], \mathbb{R}^n)$ stands for the space of absolutely continuous functions from $[a, b]$ into $\mathbb{R}^n$.

For two sets A and B , $C(A, B)$ denotes the space of continuous functions from A into B . When A is an interval, $C^1(A, B)$ denotes the class of continuously differentiable functions from A into B . We use $\mathbb{R}^n$ for the n -dimensional Euclidean space.

For the retarded inequalities in Subsection 3.1, let $I = [t_0, T) \subset \mathbb{R}_+$, where $0 \leq t_0 < T \leq \infty$. Unless otherwise stated, we assume that $u \in C(I, \mathbb{R}_+)$, $f, g, h \in C(I \times I, \mathbb{R}_+)$, and that $\alpha \in C^1(I, I)$ is nondecreasing and satisfies $\alpha(t) \leq t$, for all $t \in I$. In particular, $\alpha(t_0) = t_0$. We also assume that $\omega \in C(\mathbb{R}_+, \mathbb{R}_+)$ is nondecreasing and satisfies $\omega(r) > 0$ for $r > 0$.

Here u denotes the nonnegative function to be estimated, f, g , and h are nonnegative kernels, α is the retardation function, and ω is the nonlinear comparison function.

Whenever differentiability of a kernel with respect to its first variable is required, we write

$$\partial_t f(t, s) = \frac{\partial f}{\partial t}(t, s), \quad \partial_t g(t, s) = \frac{\partial g}{\partial t}(t, s).$$

For a strictly increasing function H , we denote by H^{-1} its inverse on its range and by $\text{Dom}(H^{-1})$ the domain of this inverse.

Accordingly, whenever the comparison functions G and Ψ are introduced below, G^{-1} and Ψ^{-1} denote their inverses on their respective ranges.

The assumptions needed for the nonlinear inequalities on $\mathbb{R}_+$ and for the functional differential equations considered in Section 4 will be stated explicitly when they are used.

2.2. Bihari's Inequality

We first recall the classical Bihari inequality, which will serve both as a basic comparison tool and as a reference for assessing the retarded estimates obtained below.

Lemma 1 (Bihari's inequality [8]). *Let $u \in C(I, \mathbb{R}_+)$ and $a \in C(I, \mathbb{R}_+)$. Let $\omega \in C(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing and satisfy $\omega(r) > 0$ for all $r > 0$. Suppose that, for some constant $k > 0$,*

$$u(t) \leq k + \int_{t_0}^t a(s)\omega(u(s)) ds, \quad t \in I. \quad (1)$$

Define

$$G(r) = \int_1^r \frac{ds}{\omega(s)}, \quad r > 0. \quad (2)$$

Then

$$u(t) \leq G^{-1}\left(G(k) + \int_{t_0}^t a(s) ds\right), \quad (3)$$

for every $t \in [t_0, t_1)$, where $t_1 \in I$ is chosen such that

$$G(k) + \int_{t_0}^t a(s) ds \in \text{Dom}(G^{-1})$$

for all $t \in [t_0, t_1)$.

2.3. A Nonlinear Retarded Inequality of Ferreira–Torres

The following nonlinear retarded inequality of Ferreira and Torres [23] will be used as a reference result when discussing the scope of the composite nonlinear inequalities established in Subsection 3.3. In this result, ϕ denotes the outer increasing comparison function, c the free term, and η and ω the nonlinear comparison functions entering the successive transformations.

Theorem 1 (Ferreira–Torres [23]). *Let $f, g \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with respect to their first variable. Let $\phi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be strictly increasing and satisfy $\lim_{x \rightarrow \infty} \phi(x) = \infty$. Assume that $c \in C(\mathbb{R}_+, (0, \infty))$ is nondecreasing and that $\eta, \omega \in C(\mathbb{R}_+, \mathbb{R}_+)$ are nondecreasing functions satisfying $\eta(x) > 0$ and $\omega(x) > 0$ for all $x > 0$.*

Let $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing and satisfy $\alpha(t) \leq t$, $t \geq 0$. Suppose that

$$\int_{x_0}^{\infty} \frac{ds}{\eta(\phi^{-1}(s))} = \infty, \quad (4)$$

where x_0 is chosen as specified below. If $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies

$$\phi(u(t)) \leq c(t) + \int_0^{\alpha(t)} \left[f(t, s)\eta(u(s))\omega(u(s)) + g(t, s)\eta(u(s)) \right] ds, \quad t \geq 0, \quad (5)$$

then there exists $\tau > 0$ such that, for every $t \in [0, \tau]$,

$$\Psi(p(t)) + \int_0^{\alpha(t)} f(t, s) ds \in \text{Dom}(\Psi^{-1}),$$

and

$$u(t) \leq \phi^{-1} \left(G^{-1} \left(\Psi^{-1} \left[\Psi(p(t)) + \int_0^{\alpha(t)} f(t, s) ds \right] \right) \right). \quad (6)$$

Here,

$$G(x) = \int_{x_0}^x \frac{ds}{\eta(\phi^{-1}(s))}. \quad (7)$$

The number x_0 is chosen so that $x \geq c(0) > x_0 > 0$ when

$$\int_0^x \frac{ds}{\eta(\phi^{-1}(s))} = \infty,$$

whereas $x \geq c(0) > x_0 \geq 0$ when

$$\int_0^x \frac{ds}{\eta(\phi^{-1}(s))} < \infty.$$

Furthermore,

$$p(t) = G(c(t)) + \int_0^{\alpha(t)} g(t, s) ds, \quad (8)$$

and, for an arbitrary $x_1 > 0$,

$$\Psi(x) = \int_{x_1}^x \frac{ds}{\omega(\phi^{-1}(G^{-1}(s)))}, \quad x > 0. \quad (9)$$

The functions G^{-1} and Ψ^{-1} denote the inverse functions of G and Ψ , respectively.

Theorem 1 will be compared with the iterated nonlinear inequality established in Theorem 6.

2.4. Driver's Extended Existence Theorem

For the application to neutral functional differential equations, we use the following specialization of Driver's extended existence theorem [36]. Here F denotes the vector field, α and q are delay functions, and φ denotes the prescribed initial history.

Theorem 2 (Driver [36]). Let $r > 0$ and consider the neutral functional differential equation

$$x'(t) = F(t, x(t), x(\alpha(t)), x'(q(t))), \quad t > 0, \quad (10)$$

with initial history

$$x(t) = \varphi(t), \quad -r \leq t \leq 0. \quad (11)$$

Assume that $F \in C(\mathbb{R}_+ \times \mathbb{R}^{3n}, \mathbb{R}^n)$, and that the functions $\alpha, q : \mathbb{R}_+ \rightarrow [-r, \infty)$ are continuous and satisfy, for all $t \geq 0$,

$$-r \leq \alpha(t) \leq t \quad \text{and} \quad -r \leq q(t) < t.$$

Moreover, suppose $q \in C^1(\mathbb{R}_+, \mathbb{R})$ with $q'(t) > 0$ for every $t \geq 0$. Let $\varphi \in AC([-r, 0], \mathbb{R}^n)$ be such that φ' is essentially bounded on $[-r, 0]$.

Then there exist $\beta > 0$ and an absolutely continuous function $x : [-r, \beta) \rightarrow \mathbb{R}^n$ extending φ (that is, $x = \varphi$ on $[-r, 0]$) such that the required differential relation holds on $[-r, \beta)$.

$$x'(t) = F(t, x(t), x(\alpha(t)), x'(q(t)))$$

for almost every $t \in (0, \beta)$.

Moreover, the solution can be continued as long as $(t, x(t), x(\alpha(t)))$ remains in a compact subset of the domain of F .

Remark 1. For the existence conclusion in Theorem 2, the condition $q'(t) > 0$ may be replaced by the assumption that q is increasing and that q^{-1} is absolutely continuous on the relevant intervals. We retain the stronger assumption

$$q \in C^1(\mathbb{R}_+, \mathbb{R}), \quad q'(t) > 0,$$

since it provides a convenient regularity framework for the applications considered below.

3. Main Results

In this section, we establish the main retarded integral inequalities of the paper. We first consider nonlinear inequalities involving time-dependent kernels and variable delays. We then extend the comparison procedure to composite nonlinearities and iterated Volterra-type terms.

3.1. Retarded Inequalities with Variable Kernels

Theorem 3. Assume that f and g are continuously differentiable with respect to their first variable and satisfy $\partial_t f(t, s) \geq 0$ and $\partial_t g(t, s) \geq 0$ for all $(t, s) \in I \times I$.

Let $k \in C^1(I, \mathbb{R}_+)$ be nondecreasing and satisfy $k(t_0) > 0$. Suppose that

$$u(t) \leq k(t) + \int_{t_0}^t f(t, s) \omega(u(s)) ds + \int_{\alpha(t_0)}^{\alpha(t)} g(t, s) \omega(u(s)) ds, \quad t \in I. \quad (12)$$

Define

$$G(r) = \int_1^r \frac{ds}{\omega(s)}, \quad r > 0. \quad (13)$$

Then

$$u(t) \leq G^{-1} \left(G(k(t)) + \int_{t_0}^t f(t, s) ds + \int_{\alpha(t_0)}^{\alpha(t)} g(t, s) ds \right), \quad (14)$$

for every $t \in [t_0, t_1)$, where $t_1 \in I$ is chosen such that

$$G(k(t)) + \int_{t_0}^t f(t, s) ds + \int_{\alpha(t_0)}^{\alpha(t)} g(t, s) ds \in \text{Dom}(G^{-1})$$

for all $t \in [t_0, t_1)$.

Remark 2. The monotonicity assumptions in Theorem 3 are directly related to the comparison argument. In particular, they ensure that the auxiliary function used in the proof is nondecreasing and therefore allow the estimate

$$\omega(u(s)) \leq \omega(U(s)) \leq \omega(U(t)), \quad s \leq t.$$

The monotonicity of k also yields

$$\frac{k'(t)}{\omega(U(t))} \leq \frac{k'(t)}{\omega(k(t))} = \frac{d}{dt} G(k(t)).$$

Thus, these assumptions are precisely those that lead to the explicit term $G(k(t))$ in (14). The differentiability assumptions could be weakened to suitable absolute continuity conditions, but the present formulation is adopted for clarity.

Remark 3. Theorem 3 extends the classical Gronwall–Bihari framework in two directions. First, the kernels $f(t, s)$ and $g(t, s)$ may depend simultaneously on the observation time t and the integration variable s . Second, the second contribution is evaluated over the retarded interval $[\alpha(t_0), \alpha(t)]$.

When $g \equiv 0$, the estimate reduces to a nonlinear inequality with a single time-dependent kernel. When $f \equiv 0$, it gives a purely retarded estimate. Moreover, unlike the classical Bihari inequality, Theorem 3 allows the free term k to depend on time, and the resulting estimate naturally involves $G(k(t))$.

It is useful to distinguish structural generality from sharpness. Theorem 3 is not claimed to provide a uniformly sharper estimate than every retarded inequality of Lipovan, Sun, or Ferreira–Torres, since these results are formulated for different nonlinear structures and under different hypotheses. The gain in the present estimate becomes explicit when it is compared with the corresponding non-retarded reduction of the same inequality.

Indeed, since $\alpha(t_0) = t_0$ and $\alpha(t) \leq t$, and since $g(t, s) \geq 0$, we have

$$\int_{\alpha(t_0)}^{\alpha(t)} g(t, s) ds \leq \int_{t_0}^t g(t, s) ds.$$

Consequently, whenever both arguments belong to $\text{Dom}(G^{-1})$, the estimate in (14) satisfies

$$\begin{aligned} G^{-1} \left(G(k(t)) + \int_{t_0}^t f(t, s) ds + \int_{\alpha(t_0)}^{\alpha(t)} g(t, s) ds \right) \\ \leq G^{-1} \left(G(k(t)) + \int_{t_0}^t f(t, s) ds + \int_{t_0}^t g(t, s) ds \right). \end{aligned}$$

Thus, preserving the retarded interval never enlarges the corresponding non-retarded upper bound. Moreover, the inequality is strict whenever $\alpha(t) < t$ and

$$\int_{\alpha(t)}^t g(t, s) ds > 0,$$

provided that both comparison arguments lie in the domain of G^{-1} .

Remark 4. The comparison above explains the relation with earlier retarded estimates more precisely. The advantage does not come from replacing the results of Lipovan [17], Sun [19], or Ferreira–Torres [23] in their full generality. Rather, it comes from retaining the delayed integration interval in situations where a classical reduction would extend that interval to $[t_0, t]$. The power-type settings considered by Lipovan and Sun, as well as the more general nonlinear framework of Ferreira–Torres, illustrate the same principle that preserving the retarded structure can lead to a more

informative comparison estimate. Theorem 3 complements these results by allowing simultaneously a time-dependent free term $k(t)$ and two kernels depending on both t and s .

Recent power-type developments provide a useful complementary comparison. Song and Meng [29] and Boudeliou [30] considered delay and retarded Gronwall–Bellman inequalities with power nonlinearities and nested integral terms. Their results are tailored to power-type structures, whereas Theorem 3 is formulated for a general nonlinear comparison function ω and permits kernels depending explicitly on both the observation time and the integration variable. Thus, the present result should be viewed as complementary to these recent power-type inequalities rather than as a uniform extension of them.

Proof. Define

$$U(t) = k(t) + \int_{t_0}^t f(t, s) \omega(u(s)) ds + \int_{\alpha(t_0)}^{\alpha(t)} g(t, s) \omega(u(s)) ds.$$

Then

$$u(t) \leq U(t), \quad U(t_0) = k(t_0) > 0.$$

Since k and α are nondecreasing, $\partial_t f \geq 0$, $\partial_t g \geq 0$, and all the functions appearing in the integral terms are nonnegative, U is nondecreasing on I .

Differentiating U , we obtain

$$U'(t) = k'(t) + f(t, t) \omega(u(t)) + \int_{t_0}^t \partial_t f(t, s) \omega(u(s)) ds + \alpha'(t) g(t, \alpha(t)) \omega(u(\alpha(t))) + \int_{\alpha(t_0)}^{\alpha(t)} \partial_t g(t, s) \omega(u(s)) ds.$$

Since $u(s) \leq U(s)$, ω is nondecreasing, and U is nondecreasing,

$$\omega(u(s)) \leq \omega(U(s)) \leq \omega(U(t)), \quad s \leq t.$$

Furthermore, $\alpha(t) \leq t$, and therefore

$$\omega(u(\alpha(t))) \leq \omega(U(\alpha(t))) \leq \omega(U(t)).$$

Consequently,

$$U'(t) \leq k'(t) + \omega(U(t)) \left[f(t, t) + \int_{t_0}^t \partial_t f(t, s) ds + \alpha'(t) g(t, \alpha(t)) + \int_{\alpha(t_0)}^{\alpha(t)} \partial_t g(t, s) ds \right].$$

Since $U(t) \geq k(t) > 0$, $k'(t) \geq 0$, and ω is nondecreasing,

$$\frac{k'(t)}{\omega(U(t))} \leq \frac{k'(t)}{\omega(k(t))} = \frac{d}{dt} G(k(t)).$$

Dividing the preceding estimate by $\omega(U(t))$ therefore gives

$$\begin{aligned} \frac{d}{dt}G(U(t)) &\leq \frac{d}{dt}G(k(t)) + f(t, t) + \int_{t_0}^t \partial_t f(t, s) ds \\ &\quad + \alpha'(t)g(t, \alpha(t)) + \int_{\alpha(t_0)}^{\alpha(t)} \partial_t g(t, s) ds. \end{aligned}$$

By Leibniz' rule,

$$\frac{d}{dt} \left(\int_{t_0}^t f(t, s) ds \right) = f(t, t) + \int_{t_0}^t \partial_t f(t, s) ds,$$

and

$$\frac{d}{dt} \left(\int_{\alpha(t_0)}^{\alpha(t)} g(t, s) ds \right) = \alpha'(t)g(t, \alpha(t)) + \int_{\alpha(t_0)}^{\alpha(t)} \partial_t g(t, s) ds.$$

Integrating from t_0 to t and using

$$U(t_0) = k(t_0)$$

yields

$$G(U(t)) \leq G(k(t)) + \int_{t_0}^t f(t, s) ds + \int_{\alpha(t_0)}^{\alpha(t)} g(t, s) ds.$$

Since G is strictly increasing, its inverse is increasing on its domain. Hence

$$U(t) \leq G^{-1} \left(G(k(t)) + \int_{t_0}^t f(t, s) ds + \int_{\alpha(t_0)}^{\alpha(t)} g(t, s) ds \right).$$

Finally, $u(t) \leq U(t)$, which proves (14). $\square$

Corollary 1. Let $a \in C(I, \mathbb{R}_+)$, and assume that g is continuously differentiable with respect to its first variable and satisfies $\partial_t g(t, s) \geq 0$ for $(t, s) \in I \times I$.

Suppose that, for some constant $k > 0$,

$$u(t) \leq k + \int_{t_0}^t a(s)\omega(u(s)) ds + \int_{\alpha(t_0)}^{\alpha(t)} g(t, s)\omega(u(s)) ds, \quad t \in I. \quad (15)$$

Then

$$u(t) \leq G^{-1} \left(G(k) + \int_{t_0}^t a(s) ds + \int_{\alpha(t_0)}^{\alpha(t)} g(t, s) ds \right), \quad (16)$$

for every $t \in [t_0, t_1)$, where $t_1 \in I$ is chosen such that

$$G(k) + \int_{t_0}^t a(s) ds + \int_{\alpha(t_0)}^{\alpha(t)} g(t, s) ds \in \text{Dom}(G^{-1})$$

for all $t \in [t_0, t_1)$, and G is defined by (13).

Remark 5. The estimate in Corollary 1 may be sharper than the one obtained by reducing the retarded inequality to the classical Bihari inequality.

Indeed, let $I = \mathbb{R}_+$, $t_0 = 0$, $a \equiv 0$, and consider

$$\omega(r) = r, \quad \alpha(t) = \ln(1 + t), \quad g(t, s) = t + \frac{1}{1 + s}.$$

Suppose that

$$u(t) \leq k + \int_0^{\ln(1+t)} \left(t + \frac{1}{1+s} \right) u(s) ds, \quad t \geq 0, \quad (17)$$

where $k > 0$.

Since $G(r) = \ln r$ and $G^{-1}(r) = e^r$, Corollary 1 gives

$$\begin{aligned} u(t) &\leq k \exp \left(\int_0^{\ln(1+t)} \left(t + \frac{1}{1+s} \right) ds \right) \\ &= k(1 + \ln(1+t)) \exp(t \ln(1+t)). \end{aligned}$$

On the other hand, for all fixed $t > 0$ and all $x \in [0, t]$,

$$\alpha(x) \leq x, \quad g(x, s) \leq t + \frac{1}{1+s}.$$

Hence (17) implies

$$u(x) \leq k + \int_0^x \left(t + \frac{1}{1+s} \right) u(s) ds.$$

Applying the classical Bihari inequality, or equivalently Gronwall's inequality in this linear case, yields

$$\begin{aligned} u(t) &\leq k \exp \left(\int_0^t \left(t + \frac{1}{1+s} \right) ds \right) \\ &= k(1+t)e^{t^2}. \end{aligned}$$

Since $\ln(1+t) \leq t$, we obtain $t \ln(1+t) \leq t^2$ and $1 + \ln(1+t) \leq 1+t$. Therefore,

$$k(1 + \ln(1+t))e^{t \ln(1+t)} \leq k(1+t)e^{t^2},$$

with strict inequality for every $t > 0$.

Thus, in this example, retaining the retarded structure yields a strictly sharper estimate than reducing the problem to the classical Bihari inequality.

Proof. Define the functions $\tilde{f}(t, s) = a(s)$ for $(t, s) \in I \times I$ and $\tilde{k}(t) = k$ for $t \in I$. Then $\tilde{f}$ is continuously differentiable with respect to its first variable and $\partial_t \tilde{f}(t, s) = 0$.

Moreover, $\tilde{k} \in C^1(I, \mathbb{R}_+)$, $\tilde{k}'(t) = 0$, and $\tilde{k}(t_0) = k > 0$. Hence, $\tilde{k}(t) = k$ for all $t \in I$, and in particular $\tilde{k}$ is nondecreasing. Therefore all the assumptions of Theorem 3 are satisfied. Applying Theorem 3 to (15) gives

$$u(t) \leq G^{-1} \left(G(k) + \int_{t_0}^t a(s) ds + \int_{\alpha(t_0)}^{\alpha(t)} g(t, s) ds \right),$$

for every $t \in [t_0, t_1)$ for which the argument of G^{-1} belongs to $\text{Dom}(G^{-1})$. This proves (16). $\square$

Theorem 4. Assume that f and g are continuously differentiable with respect to their first variable and satisfy $\partial_t f(t, s) \geq 0$ and $\partial_t g(t, s) \geq 0$ for all $(t, s) \in I \times I$.

Let $k \in C^1(I, \mathbb{R}_+)$ be nondecreasing and satisfy $k(t_0) > 0$.

Suppose that u satisfies

$$u(t) \leq k(t) + \int_{t_0}^t f(t, s) \omega(u(s)) ds + \int_{\alpha(t_0)}^{\alpha(t)} g(t, s) \int_{\alpha(t_0)}^s h(s, \tau) \omega(u(\tau)) d\tau ds, \quad t \in I. \quad (18)$$

Then

$$u(t) \leq G^{-1} \left(G(k(t)) + \int_{t_0}^t f(t, s) ds + \int_{\alpha(t_0)}^{\alpha(t)} g(t, s) \int_{\alpha(t_0)}^s h(s, \tau) d\tau ds \right), \quad (19)$$

for every $t \in [t_0, t_1)$, where $t_1 \in I$ is chosen such that

$$G(k(t)) + \int_{t_0}^t f(t, s) ds + \int_{\alpha(t_0)}^{\alpha(t)} g(t, s) \int_{\alpha(t_0)}^s h(s, \tau) d\tau ds \in \text{Dom}(G^{-1})$$

for all $t \in [t_0, t_1)$, and G is defined by (13).

Remark 6. The assumptions on k , f , g , and α play the same role as in Theorem 3. In particular, they guarantee that the auxiliary function introduced in the proof is nondecreasing and allow for the estimate

$$\omega(u(\tau)) \leq \omega(U(\tau)) \leq \omega(U(t)), \quad \tau \leq t.$$

No monotonicity assumption on the kernel h is required. Only its continuity and nonnegativity are used in the comparison argument.

More precisely, the proof never compares values of the form $h(s_1, \tau_1)$ and $h(s_2, \tau_2)$. The kernel h enters only multiplicatively through nonnegative integrals such as

$$\int_{\alpha(t_0)}^s h(s, \tau) \omega(u(\tau)) d\tau.$$

After using

$$\omega(u(\tau)) \leq \omega(U(t)), \quad \alpha(t_0) \leq \tau \leq s \leq \alpha(t) \leq t,$$

the factor $\omega(U(t))$ is taken outside the inner integral, giving

$$\int_{\alpha(t_0)}^s h(s, \tau) \omega(u(\tau)) d\tau \leq \omega(U(t)) \int_{\alpha(t_0)}^s h(s, \tau) d\tau.$$

Thus, no monotonicity property of h is involved at this step.

Remark 7. The continuity assumption on h is convenient rather than optimal. Inspection of the proof shows that h is not differentiated. Accordingly, the argument can in principle be extended to nonnegative locally integrable kernels, provided that the corresponding inner integral

$$s \mapsto \int_{\alpha(t_0)}^s h(s, \tau) d\tau$$

has sufficient regularity for the outer Volterra term to be absolutely continuous and for the Leibniz rule to hold almost everywhere. We retain continuity in Theorem 4 in order to keep the statement and proof within a classical framework.

Remark 8. The dependence of the estimate on the inner kernel h is explicit and monotone. Define

$$\mathcal{H}(t) = \int_{\alpha(t_0)}^{\alpha(t)} g(t, s) \int_{\alpha(t_0)}^s h(s, \tau) d\tau ds.$$

Then the bound in (19) can be written as

$$u(t) \leq G^{-1} \left(G(k(t)) + \int_{t_0}^t f(t, s) ds + \mathcal{H}(t) \right).$$

Consequently, if two nonnegative kernels h_1 and h_2 satisfy

$$h_1(s, \tau) \leq h_2(s, \tau)$$

on the relevant integration region, then

$$\mathcal{H}_1(t) \leq \mathcal{H}_2(t),$$

and, since G^{-1} is increasing, the upper bound corresponding to h_1 does not exceed the one corresponding to h_2 .

Thus, the influence of the iterated kernel is governed by its accumulated mass

$$\int_{\alpha(t_0)}^s h(s, \tau) d\tau,$$

rather than by any pointwise monotonicity property of h .

Remark 9. Theorem 4 develops the variable-kernel framework of Theorem 3 in a different direction by treating a separate iterated Volterra-type contribution.

If $g \equiv 0$, the iterated term disappears and Theorem 4 reduces to the single-kernel estimate

$$u(t) \leq k(t) + \int_{t_0}^t f(t, s) \omega(u(s)) ds.$$

When $\omega(r) = r$, the inequality becomes linear with respect to the unknown function and is closely related to classical retarded Gronwall-type inequalities involving iterated integral terms.

Within the particular additive structure considered in (18), the result retains a single explicit G^{-1} representation while accommodating both the direct variable-kernel contribution and the separate iterated retarded Volterra term.

Recent results of Song and Meng [29] and Boudeliou [30] also treat delay or retarded inequalities involving power nonlinearities and nested integral contributions. Their results are primarily adapted to power-type structures. In contrast, Theorem 4 is formulated for a general nonlinear comparison function ω , allows the outer kernels to depend on both variables, and requires no monotonicity assumption on the inner kernel h . These results are therefore complementary rather than ordered by a general sharpness relation.

Proof. Define

$$U(t) = k(t) + \int_{t_0}^t f(t, s) \omega(u(s)) ds + \int_{\alpha(t_0)}^{\alpha(t)} g(t, s) \int_{\alpha(t_0)}^s h(s, \tau) \omega(u(\tau)) d\tau ds.$$

Then $u(t) \leq U(t)$ and $U(t_0) = k(t_0) > 0$.

Since k and α are nondecreasing, $\partial_t f \geq 0$, $\partial_t g \geq 0$, and all kernels are nonnegative, the function U is nondecreasing on I . Differentiating U , we obtain

$$U'(t) = k'(t) + f(t, t) \omega(u(t)) + \int_{t_0}^t \partial_t f(t, s) \omega(u(s)) ds + \alpha'(t) g(t, \alpha(t)) \int_{\alpha(t_0)}^{\alpha(t)} h(\alpha(t), \tau) \omega(u(\tau)) d\tau + \int_{\alpha(t_0)}^{\alpha(t)} \partial_t g(t, s) \left(\int_{\alpha(t_0)}^s h(s, \tau) \omega(u(\tau)) d\tau \right) ds.$$

Since $u(s) \leq U(s)$, ω is nondecreasing, and U is nondecreasing,

$$\omega(u(s)) \leq \omega(U(s)) \leq \omega(U(t)), \quad s \leq t.$$

Similarly, whenever

$$\alpha(t_0) \leq \tau \leq s \leq \alpha(t) \leq t,$$

we have

$$\omega(u(\tau)) \leq \omega(U(\tau)) \leq \omega(U(t)).$$

Since $h(s, \tau) \geq 0$, multiplication preserves the inequality, and therefore

$$h(s, \tau) \omega(u(\tau)) \leq h(s, \tau) \omega(U(t)).$$

Integrating with respect to τ gives

$$\int_{\alpha(t_0)}^s h(s, \tau) \omega(u(\tau)) d\tau \leq \omega(U(t)) \int_{\alpha(t_0)}^s h(s, \tau) d\tau.$$

Notice that no comparison between different values of h is used; hence no monotonicity assumption on h is needed.

Therefore,

$$U'(t) \leq k'(t) + \omega(U(t)) \left[f(t, t) + \int_{t_0}^t \partial_t f(t, s) ds + \alpha'(t) g(t, \alpha(t)) \int_{\alpha(t_0)}^{\alpha(t)} h(\alpha(t), \tau) d\tau + \int_{\alpha(t_0)}^{\alpha(t)} \partial_t g(t, s) \int_{\alpha(t_0)}^s h(s, \tau) d\tau ds \right].$$

Since $U(t) \geq k(t) > 0$, $k'(t) \geq 0$, and ω is nondecreasing,

$$\frac{k'(t)}{\omega(U(t))} \leq \frac{k'(t)}{\omega(k(t))} = \frac{d}{dt} G(k(t)).$$

Dividing by $\omega(U(t))$ yields

$$\begin{aligned} \frac{d}{dt}G(U(t)) &\leq \frac{d}{dt}G(k(t)) + f(t, t) + \int_{t_0}^t \partial_t f(t, s) ds \\ &\quad + \alpha'(t)g(t, \alpha(t)) \int_{\alpha(t_0)}^{\alpha(t)} h(\alpha(t), \tau) d\tau \\ &\quad + \int_{\alpha(t_0)}^{\alpha(t)} \partial_t g(t, s) \int_{\alpha(t_0)}^s h(s, \tau) d\tau ds. \end{aligned}$$

By Leibniz' rule,

$$\frac{d}{dt} \left(\int_{t_0}^t f(t, s) ds \right) = f(t, t) + \int_{t_0}^t \partial_t f(t, s) ds,$$

and

$$\begin{aligned} \frac{d}{dt} \left(\int_{\alpha(t_0)}^{\alpha(t)} g(t, s) \int_{\alpha(t_0)}^s h(s, \tau) d\tau ds \right) &= \alpha'(t)g(t, \alpha(t)) \int_{\alpha(t_0)}^{\alpha(t)} h(\alpha(t), \tau) d\tau \\ &\quad + \int_{\alpha(t_0)}^{\alpha(t)} \partial_t g(t, s) \int_{\alpha(t_0)}^s h(s, \tau) d\tau ds. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{d}{dt}G(U(t)) &\leq \frac{d}{dt}G(k(t)) + \frac{d}{dt} \left(\int_{t_0}^t f(t, s) ds \right) \\ &\quad + \frac{d}{dt} \left(\int_{\alpha(t_0)}^{\alpha(t)} g(t, s) \int_{\alpha(t_0)}^s h(s, \tau) d\tau ds \right). \end{aligned}$$

Integrating from t_0 to t and using

$$U(t_0) = k(t_0)$$

gives

$$\begin{aligned} G(U(t)) &\leq G(k(t)) + \int_{t_0}^t f(t, s) ds \\ &\quad + \int_{\alpha(t_0)}^{\alpha(t)} g(t, s) \int_{\alpha(t_0)}^s h(s, \tau) d\tau ds. \end{aligned}$$

Since G^{-1} is increasing on its domain,

$$\begin{aligned} U(t) &\leq G^{-1} \left(G(k(t)) + \int_{t_0}^t f(t, s) ds \right. \\ &\quad \left. + \int_{\alpha(t_0)}^{\alpha(t)} g(t, s) \int_{\alpha(t_0)}^s h(s, \tau) d\tau ds \right). \end{aligned}$$

Finally, $u(t) \leq U(t)$, and therefore (19) follows. $\square$

Example 1. Let $t_0 = 0$,

$$\alpha(t) = \frac{t}{2}, \quad k(t) = k_0 > 0, \quad \omega(r) = r.$$

Assume that

$$u(t) \leq k_0 + \int_0^t \left(t + \frac{1}{1+s} \right) u(s) ds + \int_0^{t/2} (1+t+s) \int_0^s \frac{1}{1+\tau} u(\tau) d\tau ds. \quad (20)$$

The kernels

$$f(t, s) = t + \frac{1}{1+s}, \quad g(t, s) = 1 + t + s, \quad h(s, \tau) = \frac{1}{1+\tau}$$

satisfy the assumptions of Theorem 4.

Notice in particular that $h(s, \tau)$ is decreasing with respect to τ . Thus, this example also illustrates that the monotonicity of the inner kernel h is not required in Theorem 4.

Moreover,

$$G(r) = \ln r, \quad G^{-1}(x) = e^x.$$

Hence

$$u(t) \leq k_0 \exp \left[\int_0^t \left(t + \frac{1}{1+s} \right) ds + \int_0^{t/2} (1+t+s) \int_0^s \frac{d\tau}{1+\tau} ds \right].$$

Since

$$\int_0^t \left(t + \frac{1}{1+s} \right) ds = t^2 + \ln(1+t),$$

and

$$\int_0^s \frac{d\tau}{1+\tau} = \ln(1+s),$$

we obtain

$$u(t) \leq k_0 \exp \left[t^2 + \ln(1+t) + \int_0^{t/2} (1+t+s) \ln(1+s) ds \right], \quad t \geq 0. \quad (21)$$

Since the range of $G(r) = \ln r$ on $(0, \infty)$ is $\mathbb{R}$, no additional restriction arising from the domain of G^{-1} is required in this case. The estimate therefore shows explicitly how the iterated Volterra-type kernel contributes to the growth bound.

3.2. Retarded Inequalities with Composite Nonlinearities

We now consider a more general nonlinear structure involving two successive comparison transformations and an iterated Volterra-type retarded term.

Theorem 5. Let $f, g, h \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$, and assume that f and g are nondecreasing with respect to their first variable.

Let $\phi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be strictly increasing and satisfy

$$\lim_{r \rightarrow \infty} \phi(r) = \infty.$$

Assume that $c, \eta, \omega \in C(\mathbb{R}_+, \mathbb{R}_+)$ are nondecreasing functions satisfying $\eta(r) > 0$ and $\omega(r) > 0$ for all $r > 0$. Let $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing and satisfy $\alpha(t) \leq t$ for $t \geq 0$.

Choose x_0 such that

$$\phi(0) \leq x_0 < c(0)$$

and assume that

$$\int_{x_0}^x \frac{ds}{\eta(\phi^{-1}(s))} < \infty, \quad x > x_0, \quad (22)$$

while

$$\int_{x_0}^{\infty} \frac{ds}{\eta(\phi^{-1}(s))} = \infty. \quad (23)$$

Define

$$G(x) = \int_{x_0}^x \frac{ds}{\eta(\phi^{-1}(s))}, \quad x \geq x_0. \quad (24)$$

Suppose that $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies

$$\begin{aligned} \phi(u(t)) &\leq c(t) + \int_0^{\alpha(t)} f(t, s) \eta(u(s)) \omega(u(s)) ds \\ &+ \int_0^{\alpha(t)} g(t, s) \int_0^s h(s, \tau) \eta(u(\tau)) d\tau ds, \quad t \geq 0. \end{aligned} \quad (25)$$

Set

$$p(t) = G(c(t)) + \int_0^{\alpha(t)} g(t, s) \int_0^s h(s, \tau) d\tau ds, \quad (26)$$

and, for an arbitrary $x_1 > 0$, define

$$\Psi(x) = \int_{x_1}^x \frac{ds}{\omega(\phi^{-1}(G^{-1}(s)))}, \quad x > 0. \quad (27)$$

Then there exists $\tau_* > 0$ such that

$$\Psi(p(t)) + \int_0^{\alpha(t)} f(t, s) ds \in \text{Dom}(\Psi^{-1}), \quad t \in [0, \tau_*], \quad (28)$$

and

$$u(t) \leq \phi^{-1} \left(G^{-1} \left(\Psi^{-1} \left[\Psi(p(t)) + \int_0^{\alpha(t)} f(t, s) ds \right] \right) \right), \quad t \in [0, \tau_*]. \quad (29)$$

Remark 10. Condition (38) ensures that $G(x)$ is finite for every $x > x_0$, including the possible endpoint case $x_0 = \phi(0)$. Condition (39), on the other hand, guarantees that

$$\lim_{x \rightarrow \infty} G(x) = \infty.$$

Consequently, G is strictly increasing from $[x_0, \infty)$ onto $[0, \infty)$, and hence

$$G^{-1} : [0, \infty) \longrightarrow [x_0, \infty)$$

is well defined.

The local integrability condition is particularly useful when $x_0 = \phi(0)$ and $\eta(0) = 0$, since in that case the integrand in (40) may be singular at the lower endpoint.

Remark 11. The restriction to the interval $[0, \tau_*]$ arises only from the domain of Ψ^{-1} . If Ψ is unbounded from above, then its range is unbounded from above and estimate (45) extends to every $t \geq 0$ for which the quantities involved are defined.

If Ψ has a finite upper endpoint, the estimate remains valid as long as condition (44) is satisfied.

Remark 12. The relation between Theorem 6 and the Ferreira–Torres inequality in Theorem 1 can be described explicitly.

For fixed $t \geq 0$, Fubini’s theorem gives

$$\begin{aligned} \int_0^{\alpha(t)} g(t, s) \int_0^s h(s, \tau) \eta(u(\tau)) d\tau ds \\ = \int_0^{\alpha(t)} \tilde{g}(t, \tau) \eta(u(\tau)) d\tau, \end{aligned}$$

where

$$\tilde{g}(t, \tau) = \int_{\tau}^{\alpha(t)} g(t, s) h(s, \tau) ds, \quad 0 \leq \tau \leq \alpha(t). \quad (30)$$

For $\tau > \alpha(t)$, we set $\tilde{g}(t, \tau) = 0$. Since g , h , and α are continuous, the resulting effective kernel $\tilde{g}$ is continuous on $\mathbb{R}_+ \times \mathbb{R}_+$. Moreover, for each fixed τ , the mapping $t \mapsto \tilde{g}(t, \tau)$ is nondecreasing. Indeed, if $t_1 \leq t_2$, then $\alpha(t_1) \leq \alpha(t_2)$ and, using the monotonicity of g in its first variable and the nonnegativity of h ,

$$\begin{aligned} \tilde{g}(t_1, \tau) &= \int_{\tau}^{\alpha(t_1)} g(t_1, s) h(s, \tau) ds \\ &\leq \int_{\tau}^{\alpha(t_1)} g(t_2, s) h(s, \tau) ds \\ &\leq \int_{\tau}^{\alpha(t_2)} g(t_2, s) h(s, \tau) ds = \tilde{g}(t_2, \tau), \end{aligned}$$

whenever $\tau \leq \alpha(t_1)$; if $\alpha(t_1) < \tau \leq \alpha(t_2)$, then $\tilde{g}(t_1, \tau) = 0 \leq \tilde{g}(t_2, \tau)$, while the remaining case is immediate.

Thus, under the present assumptions, the iterated term can be viewed as a Ferreira–Torres term with the effective kernel $\tilde{g}$. Furthermore, by Fubini’s theorem,

$$\int_0^{\alpha(t)} \tilde{g}(t, \tau) d\tau = \int_0^{\alpha(t)} g(t, s) \int_0^s h(s, \tau) d\tau ds, \quad (31)$$

so that the function $p(t)$ in (42) coincides exactly with the corresponding quantity obtained from Theorem 1.

Accordingly, Theorem 6 is not claimed to yield a strictly stronger comparison estimate than the Ferreira–Torres theorem under the present assumptions. Its usefulness lies in giving the estimate directly in terms of the original nested kernels g and h , thereby preserving the explicit Volterra structure of the problem rather than absorbing it into the effective kernel $\tilde{g}$.

Recent work by Chen and Abdul Rahman [31] also develops Gronwall–Bellman–Bihari inequalities for composite nonlinear structures and fractional integral problems. Their setting is complementary to the one considered here: it emphasizes composite and fractional comparison mechanisms, whereas Theorem 6 focuses on a retarded nested Volterra structure with an explicit pair of kernels g and h . No general sharpness ordering between these frameworks is asserted.

Proof. Define the effective kernel $\tilde{g}$ by (30), with $\tilde{g}(t, \tau) = 0$ for $\tau > \alpha(t)$. As shown in Remark 16, $\tilde{g}$ is nonnegative, continuous, and nondecreasing with respect to its first variable.

By Fubini's theorem, inequality (41) becomes

$$\begin{aligned} \phi(u(t)) &\leq c(t) + \int_0^{\alpha(t)} f(t, s) \eta(u(s)) \omega(u(s)) ds \\ &\quad + \int_0^{\alpha(t)} \tilde{g}(t, s) \eta(u(s)) ds. \end{aligned}$$

All the hypotheses of Theorem 1 are therefore satisfied. Applying that theorem gives

$$u(t) \leq \phi^{-1} \left(G^{-1} \left(\Psi^{-1} \left[\Psi(\tilde{p}(t)) + \int_0^{\alpha(t)} f(t, s) ds \right] \right) \right),$$

where

$$\tilde{p}(t) = G(c(t)) + \int_0^{\alpha(t)} \tilde{g}(t, s) ds.$$

Using (31), we obtain

$$\begin{aligned} \tilde{p}(t) &= G(c(t)) + \int_0^{\alpha(t)} g(t, s) \int_0^s h(s, \tau) d\tau ds \\ &= p(t). \end{aligned}$$

Hence the Ferreira–Torres estimate reduces exactly to (45). The local restriction $t \in [0, \tau_*]$ follows from the domain condition for Ψ^{-1} , as in Theorem 1. $\square$

Example 2. Consider

$$\phi(r) = r^2, \quad \eta(r) = 2r, \quad \omega(r) = r, \quad \alpha(t) = \frac{t}{2},$$

and let

$$c(t) = 4, \quad f(t, s) = 1, \quad g(t, s) = 1 + t + s, \quad h(s, \tau) = e^{-\tau}.$$

Suppose that

$$\begin{aligned} u^2(t) &\leq 4 + 2 \int_0^{t/2} u^2(s) ds \\ &\quad + 2 \int_0^{t/2} (1 + t + s) \int_0^s e^{-\tau} u(\tau) d\tau ds. \end{aligned} \quad (32)$$

Choose $x_0 = 0$. Since

$$\eta(\phi^{-1}(s)) = 2\sqrt{s},$$

we obtain

$$G(x) = \int_0^x \frac{ds}{2\sqrt{s}} = \sqrt{x}, \quad G^{-1}(x) = x^2.$$

Moreover,

$$\phi^{-1}(G^{-1}(x)) = x,$$

and hence, choosing $x_1 = 1$,

$$\Psi(x) = \int_1^x \frac{ds}{s} = \ln x, \quad \Psi^{-1}(x) = e^x.$$

In the present case,

$$\begin{aligned} p(t) &= 2 + \int_0^{t/2} (1+t+s) \int_0^s e^{-\tau} d\tau ds \\ &= 2 + \int_0^{t/2} (1+t+s)(1-e^{-s}) ds. \end{aligned}$$

Therefore Theorem 6 yields

$$u(t) \leq p(t) \exp\left(\frac{t}{2}\right), \quad t \geq 0, \quad (33)$$

Since the range of $\Psi(x) = \ln x$ on $(0, \infty)$ is $\mathbb{R}$, no additional restriction arising from $\text{Dom}(\Psi^{-1})$ occurs in this example.

The integral entering $p(t)$ can also be evaluated explicitly:

$$p(t) = -\frac{t}{2} + \frac{5t^2}{8} + \left(2 + \frac{3t}{2}\right)e^{-t/2}.$$

For comparison, the corresponding effective kernel obtained from (30) is

$$\tilde{g}(t, \tau) = e^{-\tau} \left[(1+t) \left(\frac{t}{2} - \tau \right) + \frac{t^2}{8} - \frac{\tau^2}{2} \right], \quad 0 \leq \tau \leq \frac{t}{2}.$$

Thus, the same estimate may be recovered from the Ferreira–Torres theorem after introducing $\tilde{g}$. The advantage of Theorem 6 in this example is therefore structural rather than quantitative: the nested Volterra kernel is handled in its original form, while the reduction to a single kernel produces the more involved expression above.

Corollary 2. Let $m > n > 0$ and $c > 0$. Assume that $f, g, h \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$, with f and g nondecreasing with respect to their first variable.

Let $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing and satisfy $\alpha(t) \leq t$ for all $t \geq 0$, and let $\omega \in C(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing and satisfy $\omega(r) > 0$ for all $r > 0$.

Suppose that $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies

$$\begin{aligned} u^m(t) &\leq c^{\frac{m}{m-n}} + \frac{m}{m-n} \int_0^{\alpha(t)} f(t, s) u^n(s) \omega(u(s)) ds \\ &\quad + \frac{m}{m-n} \int_0^{\alpha(t)} g(t, s) \int_0^s h(s, \tau) u^n(\tau) d\tau ds, \quad t \geq 0. \end{aligned} \quad (34)$$

Define

$$\Psi(x) = \int_1^x \frac{ds}{\omega(s^{1/(m-n)})}, \quad x > 0. \quad (35)$$

Then there exists $\tau_* > 0$ such that

$$\Psi\left(c + \int_0^{\alpha(t)} g(t, s) \int_0^s h(s, \tau) d\tau ds\right) + \int_0^{\alpha(t)} f(t, s) ds \in \text{Dom}(\Psi^{-1}), \quad (36)$$

for all $t \in [0, \tau_*]$, and

$$u(t) \leq \left\{ \Psi^{-1} \left[\Psi \left(c + \int_0^{\alpha(t)} g(t, s) \int_0^s h(s, \tau) d\tau ds \right) + \int_0^{\alpha(t)} f(t, s) ds \right] \right\}^{1/(m-n)}, \quad t \in [0, \tau_*]. \quad (37)$$

Remark 13. The choice

$$\phi(x) = x^m, \quad \eta(x) = \frac{m}{m-n} x^n$$

is the same power-type specialization used by Ferreira and Torres [23] to recover the retarded inequality of Sun [19]. In particular, Ferreira and Torres showed that their corresponding result reduces to Sun's Theorem 2.1 and, when $m = 2$ and $n = 1$, to Lipovan's Theorem 1 [17].

Corollary 3 retains this power-type structure while writing the second retarded contribution directly in the iterated Volterra form

$$\int_0^{\alpha(t)} g(t, s) \int_0^s h(s, \tau) u^n(\tau) d\tau ds.$$

As explained in Remark 16, under the present hypotheses this term can be rewritten as a single retarded integral with an effective kernel. Thus, the corollary should be viewed as an iterated-kernel formulation of the corresponding Ferreira–Torres power-type comparison principle, rather than as a strictly stronger estimate. Its advantage is that the original Volterra structure is retained explicitly.

Proof. Applying Theorem 6 with

$$\phi(r) = r^m, \quad \eta(r) = \frac{m}{m-n} r^n, \quad c(t) = c^{m/(m-n)},$$

gives

$$\eta(\phi^{-1}(s)) = \frac{m}{m-n} s^{n/m}.$$

Let $x_0 = 0 = \phi(0)$. Because $m > n > 0$, we have $n/m < 1$, and therefore

$$\int_0^x s^{-n/m} ds < \infty, \quad x > 0.$$

Hence, the local integrability condition (38) is satisfied. Moreover,

$$\int_0^\infty s^{-n/m} ds = \infty,$$

so that (39) also holds.

The function G defined in Theorem 6 therefore becomes

$$G(x) = x^{(m-n)/m},$$

and hence

$$\phi^{-1}(G^{-1}(x)) = x^{1/(m-n)}.$$

Thus the function Ψ appearing in Theorem 6 reduces to

$$\Psi(x) = \int_1^x \frac{ds}{\omega(s^{1/(m-n)})}.$$

Also,

$$p(t) = c + \int_0^{\alpha(t)} g(t, s) \int_0^s h(s, \tau) d\tau ds.$$

Substituting these expressions into (45) yields

$$u(t) \leq \left\{ \Psi^{-1} \left[\Psi \left(c + \int_0^{\alpha(t)} g(t, s) \int_0^s h(s, \tau) d\tau ds \right) + \int_0^{\alpha(t)} f(t, s) ds \right] \right\}^{1/(m-n)},$$

for every $t \in [0, \tau_*]$ for which the argument of Ψ^{-1} belongs to $\text{Dom}(\Psi^{-1})$. This proves the result. $\square$

3.3. Retarded Inequalities with Composite Nonlinearities

We now consider a more general nonlinear structure involving two successive comparison transformations and an iterated Volterra-type retarded term.

Theorem 6. Let $f, g, \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$, and assume that f and g are nondecreasing with respect to their first variable. Let $\phi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be strictly increasing and satisfy $\lim_{r \rightarrow \infty} \phi(r) = \infty$.

Assume that $c, \eta, \omega \in C(\mathbb{R}_+, \mathbb{R}_+)$ are nondecreasing functions satisfying $\eta(r) > 0$ and $\omega(r) > 0$ for all $r > 0$. Let $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing and satisfy $\alpha(t) \leq t$ for $t \geq 0$.

Choose x_0 such that $\phi(0) \leq x_0 < c(0)$ and assume that

$$\int_{x_0}^x \frac{ds}{\eta(\phi^{-1}(s))} < \infty, \quad x > x_0, \quad (38)$$

while

$$\int_{x_0}^{\infty} \frac{ds}{\eta(\phi^{-1}(s))} = \infty. \quad (39)$$

Define

$$G(x) = \int_{x_0}^x \frac{ds}{\eta(\phi^{-1}(s))}, \quad x \geq x_0. \quad (40)$$

Suppose that $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies

$$\begin{aligned} \phi(u(t)) &\leq c(t) + \int_0^{\alpha(t)} f(t, s) \eta(u(s)) \omega(u(s)) ds \\ &\quad + \int_0^{\alpha(t)} g(t, s) \int_0^s h(s, \tau) \eta(u(\tau)) d\tau ds, \quad t \geq 0. \end{aligned} \quad (41)$$

Set

$$p(t) = G(c(t)) + \int_0^{\alpha(t)} g(t, s) \int_0^s h(s, \tau) d\tau ds, \quad (42)$$

and, for an arbitrary $x_1 > 0$, define

$$\Psi(x) = \int_{x_1}^x \frac{ds}{\omega(\phi^{-1}(G^{-1}(s)))}, \quad x > 0. \quad (43)$$

Then there exists $\tau_* > 0$ such that

$$\Psi(p(t)) + \int_0^{\alpha(t)} f(t, s) ds \in \text{Dom}(\Psi^{-1}), \quad t \in [0, \tau_*], \quad (44)$$

and

$$u(t) \leq \phi^{-1} \left(G^{-1} \left(\Psi^{-1} \left[\Psi(p(t)) + \int_0^{\alpha(t)} f(t, s) ds \right] \right) \right), \quad t \in [0, \tau_*]. \quad (45)$$

Remark 14. Condition (38) ensures that $G(x)$ is finite for every $x > x_0$, including the possible endpoint case $x_0 = \phi(0)$. Condition (39), on the other hand, guarantees that

$$\lim_{x \rightarrow \infty} G(x) = \infty.$$

Consequently, G is strictly increasing from $[x_0, \infty)$ onto $[0, \infty)$, and hence

$$G^{-1} : [0, \infty) \longrightarrow [x_0, \infty)$$

is well defined.

The local integrability condition is particularly useful when $x_0 = \phi(0)$ and $\eta(0) = 0$, since in that case the integrand in (40) may be singular at the lower endpoint.

Remark 15. The restriction to the interval $[0, \tau_*]$ arises only from the domain of Ψ^{-1} . If Ψ is unbounded from above, then its range is unbounded from above and estimate (45) extends to every $t \geq 0$ for which the quantities involved are defined.

If Ψ has a finite upper endpoint, the estimate remains valid as long as condition (44) is satisfied.

Remark 16. Theorem 6 is motivated by the nonlinear retarded inequality of Ferreira and Torres stated in Theorem 1. Both results involve the nonlinear structure generated by ϕ , η , and ω , together with the successive transformations G and Ψ .

The essential difference lies in the second retarded contribution. Theorem 1 contains the single-integral term

$$\int_0^{\alpha(t)} g(t, s) \eta(u(s)) ds,$$

whereas Theorem 6 treats the iterated term

$$\int_0^{\alpha(t)} g(t, s) \int_0^s h(s, \tau) \eta(u(\tau)) d\tau ds.$$

Thus, Theorem 6 extends the nonlinear comparison mechanism of Ferreira–Torres to a class of retarded inequalities in which the unknown function occurs inside an additional Volterra-type integral.

Proof. Since c is nondecreasing and $c(0) > x_0$, we have $c(t) > x_0$ for $t \geq 0$. Hence $G(c(t))$ is well defined. Moreover, conditions (38) and (39) imply that G is strictly increasing from $[x_0, \infty)$ onto $[0, \infty)$. Since $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and $\alpha(t) \leq t$, we have $\alpha(0) = 0$. Therefore,

$$p(0) = G(c(0)) > 0.$$

Moreover, p is nondecreasing because c and α are nondecreasing, g is nondecreasing with respect to its first variable, and all kernels are nonnegative.

Fix $\bar{t} > 0$. For $t \in [0, \bar{t}]$, define

$$\begin{aligned} z(t) &= c(\bar{t}) + \int_0^{\alpha(t)} f(\bar{t}, s) \eta(u(s)) \omega(u(s)) ds \\ &\quad + \int_0^{\alpha(t)} g(\bar{t}, s) \int_0^s h(s, \tau) \eta(u(\tau)) d\tau ds. \end{aligned}$$

Since c , f , and g are nondecreasing with respect to the appropriate variables, inequality (41) gives $\phi(u(t)) \leq z(t)$ for $0 \leq t \leq \bar{t}$. As ϕ is strictly increasing,

$$u(t) \leq \phi^{-1}(z(t)), \quad 0 \leq t \leq \bar{t}. \quad (46)$$

The function z is positive and nondecreasing. Differentiating gives

$$\begin{aligned} z'(t) &= \left[f(\bar{t}, \alpha(t)) \eta(u(\alpha(t))) \omega(u(\alpha(t))) \right. \\ &\quad \left. + g(\bar{t}, \alpha(t)) \int_0^{\alpha(t)} h(\alpha(t), \tau) \eta(u(\tau)) d\tau \right] \alpha'(t). \end{aligned}$$

By (46), the monotonicity of η and ω , and the fact that $\alpha(t) \leq t$,

$$\begin{aligned} \eta(u(\alpha(t))) &\leq \eta(\phi^{-1}(z(\alpha(t)))) \\ &\leq \eta(\phi^{-1}(z(t))), \end{aligned}$$

while

$$\omega(u(\alpha(t))) \leq \omega(\phi^{-1}(z(\alpha(t)))).$$

Similarly, for $0 \leq \tau \leq \alpha(t) \leq t$,

$$\eta(u(\tau)) \leq \eta(\phi^{-1}(z(t))).$$

Consequently,

$$\begin{aligned} \frac{z'(t)}{\eta(\phi^{-1}(z(t)))} &\leq \left[f(\bar{t}, \alpha(t)) \omega(\phi^{-1}(z(\alpha(t)))) \right. \\ &\quad \left. + g(\bar{t}, \alpha(t)) \int_0^{\alpha(t)} h(\alpha(t), \tau) d\tau \right] \alpha'(t). \end{aligned}$$

Since

$$G'(x) = \frac{1}{\eta(\phi^{-1}(x))},$$

integration over $[0, t]$ gives

$$G(z(t)) \leq G(c(\bar{t})) + \int_0^{\alpha(t)} f(\bar{t}, s) \omega(\phi^{-1}(z(s))) ds \\ + \int_0^{\alpha(t)} g(\bar{t}, s) \int_0^s h(s, \tau) d\tau ds.$$

Since $t \leq \bar{t}$,

$$\int_0^{\alpha(t)} g(\bar{t}, s) \int_0^s h(s, \tau) d\tau ds \leq \int_0^{\alpha(\bar{t})} g(\bar{t}, s) \int_0^s h(s, \tau) d\tau ds.$$

Hence

$$z(t) \leq G^{-1} \left[p(\bar{t}) + \int_0^{\alpha(t)} f(\bar{t}, s) \omega(\phi^{-1}(z(s))) ds \right]. \quad (47)$$

Define

$$v(t) = p(\bar{t}) + \int_0^{\alpha(t)} f(\bar{t}, s) \omega(\phi^{-1}(z(s))) ds.$$

Then $v(0) = p(\bar{t}) > 0$ and v is nondecreasing. Moreover,

$$v'(t) = f(\bar{t}, \alpha(t)) \omega(\phi^{-1}(z(\alpha(t)))) \alpha'(t).$$

By (47),

$$z(\alpha(t)) \leq G^{-1}(v(\alpha(t))).$$

Since v is nondecreasing and $\alpha(t) \leq t$, the monotonicity of G^{-1} , ϕ^{-1} , and ω therefore implies

$$\frac{v'(t)}{\omega(\phi^{-1}(G^{-1}(v(t))))} \leq f(\bar{t}, \alpha(t)) \alpha'(t).$$

By the definition of Ψ ,

$$\frac{d}{dt} \Psi(v(t)) \leq f(\bar{t}, \alpha(t)) \alpha'(t).$$

Integrating over $[0, t]$, we obtain

$$\Psi(v(t)) \leq \Psi(p(\bar{t})) + \int_0^{\alpha(t)} f(\bar{t}, s) ds.$$

Taking $t = \bar{t}$ gives

$$v(\bar{t}) \leq \Psi^{-1} \left[\Psi(p(\bar{t})) + \int_0^{\alpha(\bar{t})} f(\bar{t}, s) ds \right],$$

provided that the argument belongs to $\text{Dom}(\Psi^{-1})$.

Combining this estimate with (46) and (47) yields

$$u(\bar{t}) \leq \phi^{-1} \left(G^{-1} \left(\Psi^{-1} \left[\Psi(p(\bar{t})) + \int_0^{\alpha(\bar{t})} f(\bar{t}, s) ds \right] \right) \right).$$

Finally, define

$$H(t) = \Psi(p(t)) + \int_0^{\alpha(t)} f(t, s) ds.$$

The function H is continuous. Since $p(0) > 0$, $\Psi(p(0))$ belongs to the interior of the range of Ψ . Therefore, by continuity, there exists $\tau_* > 0$ such that

$$H(t) \in \text{Dom}(\Psi^{-1}), \quad t \in [0, \tau_*].$$

Since $\bar{t} \in (0, \tau_*)$ was arbitrary, (45) follows. $\square$

Corollary 3. Let $m > n > 0$ and $c > 0$. Assume that $f, g, h \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$, with f and g nondecreasing with respect to their first variable.

Let $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing and satisfy $\alpha(t) \leq t$ for all $t \geq 0$ and $\omega \in C(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing satisfying $\omega(r) > 0$ for all $r > 0$. Suppose that $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies

$$\begin{aligned} u^m(t) &\leq c^{\frac{m}{m-n}} + \frac{m}{m-n} \int_0^{\alpha(t)} f(t, s) u^n(s) \omega(u(s)) ds \\ &\quad + \frac{m}{m-n} \int_0^{\alpha(t)} g(t, s) \int_0^s h(s, \tau) u^n(\tau) d\tau ds, \quad t \geq 0. \end{aligned} \quad (48)$$

Define

$$\Psi(x) = \int_1^x \frac{ds}{\omega(s^{1/(m-n)})}, \quad x > 0. \quad (49)$$

Then there exists $\tau_* > 0$ such that

$$\Psi\left(c + \int_0^{\alpha(t)} g(t, s) \int_0^s h(s, \tau) d\tau ds\right) + \int_0^{\alpha(t)} f(t, s) ds \in \text{Dom}(\Psi^{-1}), \quad (50)$$

for all $t \in [0, \tau_*]$, and

$$\begin{aligned} u(t) &\leq \left\{ \Psi^{-1} \left[\Psi\left(c + \int_0^{\alpha(t)} g(t, s) \int_0^s h(s, \tau) d\tau ds\right) \right. \right. \\ &\quad \left. \left. + \int_0^{\alpha(t)} f(t, s) ds \right] \right\}^{1/(m-n)}, \quad t \in [0, \tau_*]. \end{aligned} \quad (51)$$

Remark 17. The choice

$$\phi(x) = x^m, \quad \eta(x) = \frac{m}{m-n} x^n$$

is the same power-type specialization used by Ferreira and Torres [23] to recover the retarded inequality of Sun [19]. In particular, Ferreira and Torres showed that their corresponding result reduces to Sun's Theorem 2.1 and, when $m = 2$ and $n = 1$, to Lipovan's Theorem 1 [17]. Corollary 3 preserves this power-type structure while incorporating the additional iterated retarded term

$$\int_0^{\alpha(t)} g(t, s) \int_0^s h(s, \tau) u^n(\tau) d\tau ds.$$

Thus, Corollary 3 extends the corresponding power-type retarded inequalities of Sun, Lipovan, and Ferreira–Torres to a class involving an additional iterated Volterra-type kernel.

Proof. Applying Theorem 6 with $\phi(r) = r^m$, $\eta(r) = \frac{m}{m-n}r^n$ and $c(t) = c^{m/(m-n)}$. gives $\eta(\phi^{-1}(s)) = \frac{m}{m-n}s^{n/m}$. Let $x_0 = 0 = \phi(0)$. Because $m > n > 0$, we have $n/m < 1$, and therefore

$$\int_0^x s^{-n/m} ds < \infty, \quad x > 0.$$

Hence, the local integrability condition (38) is satisfied. Moreover,

$$\int_0^\infty s^{-n/m} ds = \infty,$$

so that (39) also holds.

The function G defined in Theorem 6 therefore becomes $G(x) = x^{(m-n)/m}$, and then $\phi^{-1}(G^{-1}(x)) = x^{1/(m-n)}$. Thus the function Ψ appearing in Theorem 6 reduces to

$$\Psi(x) = \int_1^x \frac{ds}{\omega(s^{1/(m-n)})}.$$

Also,

$$p(t) = c + \int_0^{\alpha(t)} g(t, s) \int_0^s h(s, \tau) d\tau ds.$$

Substituting these expressions into (45) yields

$$u(t) \leq \left\{ \Psi^{-1} \left[\Psi \left(c + \int_0^{\alpha(t)} g(t, s) \int_0^s h(s, \tau) d\tau ds \right) + \int_0^{\alpha(t)} f(t, s) ds \right] \right\}^{1/(m-n)},$$

for every $t \in [0, \tau_*]$ for which the argument of Ψ^{-1} belongs to $\text{Dom}(\Psi^{-1})$. This proves the result. $\square$

4. Applications to Integral and Functional Differential Equations

We now apply the integral inequalities established in Section 3 to two classes of problems. We first consider nonlinear retarded Volterra integral equations and derive conditions for uniqueness, global existence, and boundedness. We then turn to neutral functional differential equations.

4.1. Nonlinear Volterra Integral Equations

Consider the nonlinear retarded Volterra integral equation

$$u(t) = k(t) + \int_0^t f(t, s) \omega(u(s)) ds + \int_0^{\alpha(t)} g(t, s) \omega(u(s)) ds, \quad t \geq 0, \quad (52)$$

where $k \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\omega \in C(\mathbb{R}_+, \mathbb{R}_+)$, and $\omega(0) = 0$.

Assume that $f, g \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ are continuously differentiable with respect to their first variable and satisfy $\partial_t f(t, s) \geq 0$, $\partial_t g(t, s) \geq 0$ for all $t, s \geq 0$. Let $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing and satisfy $\alpha(t) \leq t$ for $t \geq 0$.

Under the usual local existence assumptions for nonlinear Volterra integral equations, Equation (52) admits a solution on a maximal interval $[0, T)$; see, for instance, [33]. If $T < \infty$, the corresponding continuation principle implies that

$$\limsup_{t \rightarrow T^-} u(t) = \infty. \quad (53)$$

We now use Theorem 3 to derive sufficient conditions for uniqueness, global existence, and boundedness.

Proposition 1. *Assume that there exists a nondecreasing function $z \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that $z(r) > 0$ for all $r > 0$, and*

$$|\omega(x) - \omega(y)| \leq z(|x - y|), \quad x, y \in \mathbb{R}_+. \quad (54)$$

Suppose further that

$$\int_0^1 \frac{ds}{z(s)} = \infty, \quad \int_1^\infty \frac{ds}{z(s)} = \infty. \quad (55)$$

Then Equation (52) has a unique global solution.

Moreover, if k is bounded on $\mathbb{R}_+$ and

$$\sup_{t \geq 0} \left[\int_0^t f(t, s) ds + \int_0^{\alpha(t)} g(t, s) ds \right] < \infty, \quad (56)$$

then the global solution is bounded on $\mathbb{R}_+$.

Remark 18. *The two conditions in (55) play distinct roles.*

The condition

$$\int_0^1 \frac{ds}{z(s)} = \infty$$

is used in the uniqueness argument. Indeed, if

$$G(r) = \int_1^r \frac{ds}{z(s)}, \quad r > 0,$$

then $\lim_{r \rightarrow 0^+} G(r) = -\infty$. This allows the auxiliary parameter introduced in the uniqueness estimate to tend to zero. In contrast,

$$\int_1^\infty \frac{ds}{z(s)} = \infty$$

implies $\lim_{r \rightarrow \infty} G(r) = \infty$, which provides finite a priori bounds on every finite time interval and thereby rules out finite-time blow-up.

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These assumptions are sufficient conditions for the corresponding qualitative conclusions and are not claimed to be necessary for every equation of the form (52). Their significance is best understood at the level of the scalar comparison problem.

For uniqueness, the divergence condition near the origin is the standard threshold associated with the Osgood comparison mechanism. If it fails, uniqueness can in general be lost. For instance, for

$$z(r) = r^\gamma, \quad 0 < \gamma < 1,$$

we have

$$\int_0^1 \frac{ds}{z(s)} = \int_0^1 s^{-\gamma} ds < \infty,$$

and the scalar problem

$$y'(t) = y^\gamma(t), \quad y(0) = 0,$$

admits, besides the zero solution, delayed nontrivial solutions. Thus, without additional structure on the equation, the condition near zero cannot in general be removed from a uniqueness criterion based only on the modulus z .

Similarly, the divergence condition at infinity is close to the natural borderline for a growth criterion of this type. Indeed, for the scalar comparison equation

$$y'(t) = z(y(t)), \quad y(0) = y_0 > 0,$$

separation of variables gives

$$t = \int_{y_0}^{y(t)} \frac{ds}{z(s)}.$$

Hence, if

$$\int_{y_0}^\infty \frac{ds}{z(s)} < \infty,$$

finite-time blow-up may occur. Consequently, the divergence condition at infinity cannot be omitted from a general continuation result that uses only a scalar growth majorant.

The use of the same function z in both parts of (55) is, however, not essential. Inspection of the proof shows that the uniqueness argument requires a modulus z_0 satisfying

$$|\omega(x) - \omega(y)| \leq z_0(|x - y|), \quad \int_0^1 \frac{ds}{z_0(s)} = \infty,$$

whereas global continuation only requires a nondecreasing growth majorant z_∞ such that

$$\omega(r) \leq z_\infty(r), \quad \int_1^\infty \frac{ds}{z_\infty(s)} = \infty.$$

Thus, the hypotheses of Proposition 1 can be decoupled if a more flexible formulation is desired. We have retained a single function z in the statement in order to keep the criterion concise.

Remark 20. The uniqueness argument is based on the classical Osgood–Bihari mechanism. If the kernels depend only on the integration variable, the corresponding estimates may be obtained directly from Lemma 1.

Proposition 1, however, is adapted to the retarded Volterra structure in (52), where both $f(t, s)$ and $g(t, s)$ may depend simultaneously on the observation time t and the integration variable s , while the second integral has the variable upper limit $\alpha(t)$.

Condition (56) provides a uniform control of these two kernel contributions and, together with Theorem 3, yields a global bound for the solution.

In particular, the proposition should be interpreted as a sufficient comparison criterion adapted to the kernel structure of (52), rather than as a characterization of global existence or boundedness. Additional sign, dissipativity, or cancellation properties of a specific Volterra equation may yield global solutions under weaker growth assumptions, but such information is not used in the present comparison argument.

Proof. We first prove uniqueness.

Let u_1 and u_2 be two solutions of (52) on $[0, t_1]$, where $t_1 > 0$, and set

$$v(t) = |u_1(t) - u_2(t)|.$$

Using (54), we obtain

$$v(t) \leq \int_0^t f(t, s)z(v(s)) ds + \int_0^{\alpha(t)} g(t, s)z(v(s)) ds, \quad 0 \leq t \leq t_1. \quad (57)$$

Define

$$G(r) = \int_1^r \frac{ds}{z(s)}, \quad r > 0.$$

By (55) $\lim_{r \rightarrow 0^+} G(r) = -\infty$ and $\lim_{r \rightarrow \infty} G(r) = \infty$.

For every $\varepsilon > 0$, inequality (57) implies

$$v(t) \leq \varepsilon + \int_0^t f(t, s)z(v(s)) ds + \int_0^{\alpha(t)} g(t, s)z(v(s)) ds.$$

Applying Theorem 3 with the constant free term $k(t) \equiv \varepsilon$, we obtain

$$v(t) \leq G^{-1} \left(G(\varepsilon) + \int_0^t f(t, s) ds + \int_0^{\alpha(t)} g(t, s) ds \right). \quad (58)$$

For every fixed $t \in [0, t_1]$, the two integrals on the right-hand side are finite. Since $G(\varepsilon) \rightarrow -\infty$ as $\varepsilon \rightarrow 0^+$, and $G^{-1}(r) \rightarrow 0$ as $r \rightarrow -\infty$, letting $\varepsilon \rightarrow 0^+$ in (58) gives $v(t) = 0$. Hence $u_1(t) = u_2(t)$ for all $0 \leq t \leq t_1$ and uniqueness follows.

We next prove global existence.

Let $[0, T)$ be the maximal interval of existence of the unique solution and suppose, by contradiction, that $T < \infty$. Since k is continuous, there exists $K_T > 0$ such that $k(t) \leq K_T$ for $0 \leq t < T$. Because $\omega(0) = 0$, condition (54) gives

$$\omega(u(s)) = |\omega(u(s)) - \omega(0)| \leq z(u(s)).$$

Consequently,

$$u(t) \leq K_T + \int_0^t f(t, s)z(u(s)) ds + \int_0^{\alpha(t)} g(t, s)z(u(s)) ds, \quad 0 \leq t < T.$$

Applying Theorem 3 again yields

$$u(t) \leq G^{-1} \left(G(K_T) + \int_0^t f(t, s) ds + \int_0^{\alpha(t)} g(t, s) ds \right). \quad (59)$$

Since $T < \infty$, the continuity of f and g , together with $\alpha(t) \leq t$, implies

$$\sup_{0 \leq t < T} \left[\int_0^t f(t, s) ds + \int_0^{\alpha(t)} g(t, s) ds \right] < \infty.$$

The second condition in (55) implies that G is unbounded from above. Together with the first condition, its image is the whole real line. Hence G^{-1} is finite for every finite argument. It follows from (59) that

$$\sup_{0 \leq t < T} u(t) < \infty.$$

This contradicts the continuation condition (53). Therefore,

$$T = \infty.$$

Finally, suppose that k is bounded on $\mathbb{R}_+$ and that (56) holds. Choose $K > 0$ such that

$$k(t) \leq K, \quad t \geq 0.$$

Repeating the preceding estimate gives

$$u(t) \leq G^{-1} \left(G(K) + \int_0^t f(t, s) ds + \int_0^{\alpha(t)} g(t, s) ds \right), \quad t \geq 0.$$

By (56), the argument of G^{-1} is uniformly bounded on $\mathbb{R}_+$. Consequently,

$$\sup_{t \geq 0} u(t) < \infty.$$

This completes the proof. $\square$

4.2. Functional Differential Equations

We now consider the neutral functional differential equation

$$x'(t) = F(t, x(t), x(\alpha(t)), x'(q(t))), \quad t > 0, \quad (60)$$

supplemented with the initial history

$$x(t) = \varphi(t), \quad -r \leq t \leq 0, \quad (61)$$

where $r > 0$ and $F \in C(\mathbb{R}_+ \times \mathbb{R}^{3n}, \mathbb{R}^n)$.

Assume that the delay functions α and q , together with the initial history φ , satisfy the hypotheses of Theorem 2. Under these assumptions, Driver's theorem guarantees the existence of a maximal absolutely continuous solution of problem (60)–(61).

Our aim is to use the retarded integral inequalities developed above to obtain a sufficient condition ensuring that every such maximal solution is global.

Proposition 2. Consider problem (60)–(61) and assume that the hypotheses of Theorem 2 are satisfied.

Suppose, in addition, that $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$, $\alpha(0) = 0$, $\alpha'(t) > 0$, and $\alpha(t) \leq t$ for all $t \geq 0$.

Assume that there exist functions $a, b, c \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $\omega, d \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that ω is nondecreasing, $\omega(r) > 0$ for all $r > 0$, and

$$|F(t, x, y, z)| \leq a(t)\omega(|x|) + b(t)\omega(|y|) + c(t)d(|z|), \quad (62)$$

for all $t \geq 0$ and $(x, y, z) \in \mathbb{R}^{3n}$.

Suppose further that d is bounded on $\mathbb{R}_+$; that is, there exists $D > 0$ such that

$$d(r) \leq D, \quad r \geq 0, \quad (63)$$

and assume that

$$\int_1^\infty \frac{ds}{\omega(s)} = \infty. \quad (64)$$

Then every maximal solution of the problem (60)–(61) is defined on $[-r, \infty)$.

Remark 21. The assumptions on the two delay functions play different roles.

The condition $q'(t) > 0$ follows from Driver's existence framework and provides the regularity and monotonicity properties required there for the delayed derivative term $x'(q(t))$.

In contrast,

$$\alpha'(t) > 0$$

is used directly in the proof of Proposition 2 to perform the change of variables in the delayed state term

$$\int_0^t b(s)\omega(u(\alpha(s))) ds.$$

The boundedness of d allows the neutral contribution

$$c(t)d(|x'(q(t))|)$$

to be controlled without requiring an a priori bound on x' . This is compatible with Driver's framework, in which solutions are absolutely continuous and the differential equation is understood almost everywhere.

Remark 22. The condition $\alpha'(t) > 0$ is used only to justify the classical change of variables

$$\sigma = \alpha(s)$$

in the delayed state term. Accordingly, this assumption is stronger than what is conceptually required. The same argument remains valid whenever α is strictly increasing and α^{-1} is absolutely continuous on the relevant intervals, provided that the change-of-variables formula is valid almost everywhere.

We retain the C^1 assumption with $\alpha'(t) > 0$ in order to keep the proof within a classical framework.

Remark 23. The boundedness assumption on d is a sufficient and relatively restrictive hypothesis. Its role is to remove the delayed derivative from the a priori estimate:

$$\int_0^t c(s) d(|x'(q(s))|) ds \leq D \int_0^t c(s) ds.$$

Consequently, no separate estimate for x' is required.

More general growth conditions on d would require an additional comparison argument involving the delayed derivative itself and are not covered by Proposition 2. Thus, the proposition should be viewed as a sufficient continuation criterion rather than an optimal growth condition for genuinely neutral equations.

Remark 24. Driver also obtained global existence criteria for neutral functional differential equations under Osgood-type growth assumptions. In particular, for equations having a suitable linear dependence on the delayed derivative, his criterion involves a positive function L satisfying

$$\int^\infty \frac{dr}{L(r)} = \infty,$$

together with appropriate growth assumptions on the remaining terms.

Proposition 2 follows a different comparison strategy. The delayed derivative contribution is first controlled by the bounded function d , after which Corollary 1 is applied to the resulting retarded state inequality. The corresponding global continuation condition is

$$\int_1^\infty \frac{ds}{\omega(s)} = \infty.$$

Thus, Proposition 2 provides an alternative global continuation criterion adapted to the retarded Gronwall–Bihari framework developed in this paper. It should be noted that, after the change of variables used in the proof,

$$\int_0^{\alpha(t)} b(\alpha^{-1}(s)) (\alpha^{-1})'(s) ds = \int_0^t b(s) ds.$$

Hence, in this particular application, retaining the delayed interval does not lead to a strictly smaller numerical upper bound. The role of Corollary 1 is instead to provide a direct comparison framework consistent with the delayed state structure of the equation.

Proof. Let x be a maximal solution of problem (60)–(61), defined on $[-r, T)$, and suppose, by contradiction, that $T < \infty$. Set

$$u(t) = |x(t)|, \quad 0 \leq t < T.$$

Integrating (60) over $[0, t]$ and using (62), we obtain

$$\begin{aligned} u(t) &\leq |\varphi(0)| + \int_0^t a(s)\omega(u(s)) \, ds \\ &\quad + \int_0^t b(s)\omega(u(\alpha(s))) \, ds + \int_0^t c(s)d(|x'(q(s))|) \, ds. \end{aligned}$$

By (63),

$$\int_0^t c(s)d(|x'(q(s))|) \, ds \leq D \int_0^T c(s) \, ds.$$

Define

$$K_T = 1 + |\varphi(0)| + D \int_0^T c(s) \, ds.$$

Then

$$u(t) \leq K_T + \int_0^t a(s)\omega(u(s)) \, ds + \int_0^t b(s)\omega(u(\alpha(s))) \, ds. \quad (65)$$

Since $\alpha'(t) > 0$, the function α is strictly increasing. Therefore its inverse is continuously differentiable on its range.

Using the change of variables $\sigma = \alpha(s)$ in the last integral of (65), we obtain

$$\int_0^t b(s)\omega(u(\alpha(s))) \, ds = \int_0^{\alpha(t)} b(\alpha^{-1}(\sigma))(\alpha^{-1})'(\sigma)\omega(u(\sigma)) \, d\sigma.$$

Thus,

$$u(t) \leq K_T + \int_0^t a(s)\omega(u(s)) \, ds + \int_0^{\alpha(t)} b(\alpha^{-1}(s))(\alpha^{-1})'(s)\omega(u(s)) \, ds.$$

Applying Corollary 1, we obtain

$$u(t) \leq G^{-1} \left(G(K_T) + \int_0^t a(s) \, ds + \int_0^{\alpha(t)} b(\alpha^{-1}(s))(\alpha^{-1})'(s) \, ds \right),$$

where

$$G(r) = \int_1^r \frac{ds}{\omega(s)}.$$

Changing variables back in the last integral yields

$$u(t) \leq G^{-1} \left(G(K_T) + \int_0^t a(s) \, ds + \int_0^t b(s) \, ds \right). \quad (66)$$

Since $T < \infty$ and $a, b \in C(\mathbb{R}_+, \mathbb{R}_+)$, the argument of G^{-1} in (66) remains bounded for $0 \leq t < T$. Moreover, (64) implies that G is unbounded from above. Hence

$$\sup_{0 \leq t < T} |x(t)| < \infty.$$

Since $\alpha(t) \leq t$ and φ is continuous on $[-r, 0]$, it also follows that

$$\sup_{0 \leq t < T} |x(\alpha(t))| < \infty.$$

Therefore, $(t, x(t), x(\alpha(t)))$ remains in a compact subset of the domain of F as $t \rightarrow T^-$. By the continuation property in Theorem 2, the solution can be extended beyond T , contradicting the maximality of T .

Hence $T = \infty$, which proves the result. $\square$

Example 3 (A delayed Liénard system). Let $P \in C(\mathbb{R}, \mathbb{R})$ and $g \in C(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$, and consider the delayed Liénard system

$$\begin{aligned} x'(t) &= y(t) - P(x(t)), \\ y'(t) &= g(t, x(\alpha(t))), \quad t \geq 0, \end{aligned} \quad (67)$$

supplemented with an initial history satisfying the hypotheses required for local existence.

Assume that $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$, $\alpha(0) = 0$, $\alpha'(t) > 0$, and $\alpha(t) \leq t$ for all $t \geq 0$.

Suppose that there exist $a \in C(\mathbb{R}_+, \mathbb{R}_+)$ and a nondecreasing function $\omega \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that

$$|P(x)| \leq \omega(|x|), \quad (68)$$

and

$$|g(t, x)| \leq a(t)\omega(|x|), \quad (69)$$

for all $t \geq 0$ and $x \in \mathbb{R}$. Assume moreover that

$$\int_1^\infty \frac{ds}{s + \omega(s)} = \infty. \quad (70)$$

Then every maximal solution of (67) is global.

Remark 25. System (67) contains no delayed derivative. It is therefore a retarded special case of the functional differential framework considered in Proposition 2: the corresponding vector field is independent of its last argument, so the coefficient of the neutral term may be taken equal to zero.

Thus, Example 3 shows that the global continuation criterion of Proposition 2 is not restricted to genuinely neutral equations, but also applies to retarded differential systems with nonlinear delayed feedback.

The purpose of this example is illustrative rather than to claim a strict improvement over all classical Gronwall–Bihari estimates. Indeed, after the change of variables used in Proposition 2, the delayed contribution leads to the same total coefficient integral as the corresponding non-retarded comparison. The advantage here is therefore structural: the delayed state term is incorporated directly into the continuation argument without replacing the original functional differential equation by a non-delayed model.

A genuine quantitative improvement from retaining the delayed integration interval occurs in situations such as Remark 5, where the kernel depends explicitly on both the observation time and the integration variable.

Proof. Set

$$X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Then system (67) can be written in the form

$$X'(t) = \mathcal{F}(t, X(t), X(\alpha(t))),$$

where, for

$$X = (x, y), \quad Y = (\xi, \eta),$$

we define

$$\mathcal{F}(t, X, Y) = \begin{pmatrix} y - P(x) \\ g(t, \xi) \end{pmatrix}.$$

Introduce the function $\Omega(r) = r + \omega(r)$ for $r \geq 0$. Since ω is nondecreasing, Ω is also nondecreasing, and $\Omega(r) > 0$ for $r > 0$.

Using (68) and (69), we obtain

$$\begin{aligned} |\mathcal{F}(t, X, Y)| &\leq |y| + |P(x)| + |g(t, \xi)| \\ &\leq |X| + \omega(|X|) + a(t)\omega(|Y|) \\ &\leq \Omega(|X|) + a(t)\Omega(|Y|). \end{aligned}$$

Hence the growth condition (62) of Proposition 2 is satisfied with

$$a_1(t) = 1, \quad b_1(t) = a(t), \quad c_1(t) = 0,$$

and with Ω in place of ω .

Furthermore, condition (70) gives

$$\int_1^\infty \frac{ds}{\Omega(s)} = \int_1^\infty \frac{ds}{s + \omega(s)} = \infty.$$

Therefore all the growth assumptions of Proposition 2 are satisfied. It follows that every maximal solution of (67) is defined for all $t \geq 0$. $\square$

5. Conclusions

In this paper, we established nonlinear retarded integral inequalities with time-dependent kernels, variable delays, nonconstant free terms, and iterated Volterra-type contributions. The resulting estimates are expressed through suitable nonlinear comparison transformations and provide explicit upper bounds for the unknown function.

For the first class of inequalities, the delayed integration interval is retained explicitly. This may lead to a sharper estimate than the corresponding non-retarded reduction, as illustrated by a direct comparison example. For composite nonlinearities, we considered an iterated Volterra formulation related to the Ferreira–Torres comparison framework. Under

the present assumptions, this formulation is not claimed to yield a strictly stronger estimate; its main advantage is that the original nested kernel structure is kept explicit.

The obtained inequalities were applied to nonlinear retarded Volterra integral equations and neutral functional differential equations. For the Volterra problem, sufficient conditions for uniqueness, global existence, and boundedness were derived under Osgood-type assumptions. For neutral functional differential equations, the comparison estimates were combined with Driver's continuation framework to obtain a global existence criterion. A delayed Liénard system was included as an illustrative application.

Possible extensions include weaker regularity assumptions, weakly singular kernels, more general growth conditions for delayed derivative terms, and corresponding comparison principles in ordered or L^p -type function spaces.

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