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Matrix Transformations and Scale Separation in Non-Archimedean λ-Almost Convergent Sequence Spaces

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Abstract

Let F be a complete, non-trivially valued, non-Archimedean field of characteristic zero. This paper studies the space $\widehat{V}_\lambda$ of λ-almost convergent sequences over F and its paranormed generalization $\widehat{V}_\lambda(p)$, establishing it as a complete non-Archimedean topological vector space for every admissible λ. The classical inclusion $c \subset \widehat{V}_\lambda$ holds if and only if F has residue characteristic zero and $m - \lambda_m$ is unbounded. The matrix class $(c, \widehat{V}_\lambda)$ is characterized by means of an ultrametric gliding hump argument. The strong almost space $[\widehat{V}, \lambda]$ and the almost λ-statistical space $\widehat{s}_\lambda$ coincide on ℓ_∞, yet over $\mathbb{Q}_p$ this common space is not contained in $\widehat{V}_\lambda$, a separation that reflects the arithmetic of the base field.

Keywords: non-Archimedean field; ultrametric; λ-almost convergence; matrix transformation; strong almost convergence; almost λ-statistical convergence; gliding hump

MSC: 40C05; 46S10; 40H05; 40A35; 47B37

1. Introduction

The notion of almost convergence was introduced by Lorentz [1] as a limit process for bounded sequences that is invariant under translation and that returns the ordinary limit whenever it exists. Its defining feature is that the moving averages of a sequence converge uniformly with respect to their starting point, a condition met by many oscillating sequences that lie beyond the reach of ordinary convergence. Lorentz's original description rests on Banach limits [2], the positive translation invariant functionals on the space of bounded sequences. All such functionals agree on a given sequence exactly when its averages converge uniformly, and it is this averaged formulation that has guided most later work.

The summability side of the theory developed soon afterward. King [3] introduced almost conservative and almost regular matrices and determined when an infinite matrix carries every convergent sequence to an almost convergent one, and when it also preserves the limit. Maddox [4,5] studied strongly summable sequences and the stronger notion of strong almost convergence, in which it is the averaged modulus of the deviations from the limit, rather than the deviations themselves, that is required to vanish. The interplay between infinite matrices and almost convergence was carried further by Nanda [6,7], who characterized matrix classes mapping various sequence spaces into the almost convergent sequences, and by Mursaleen [8], who placed almost convergence within the broader framework of invariant (σ-) means, recovering Lorentz's notion as the case of the translation operator.

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A parallel line of work replaces the uniform average by one taken over windows of variable length. To a non-decreasing sequence $\lambda = (\lambda_m)$ tending to infinity one associates the generalized de la Vallée–Poussin mean, and with it the method of (V, λ)-summability and a family of associated sequence spaces. Malkowsky and Savaş [9] examined λ-sequence spaces defined through such means together with a modulus function, and Savaş [10] introduced almost λ-statistical convergence and related it to strong almost (V, λ)-summability, showing in particular that the two notions agree on bounded sequences. Over the real or complex numbers these theories all rest on one Archimedean feature: the averaging factor tends to zero, so that finite perturbations are diluted and the remaining oscillation is controlled by density estimates.

In a non-Archimedean valued field this mechanism is unavailable. The model case is the field $\mathbb{Q}_p$ of p-adic numbers, where the ultrametric inequality replaces the ordinary triangle inequality and where no order is compatible with the topology. The absence of such an order already removes the Banach-limit description of almost convergence and makes the averaged formulation the natural definition. More fundamentally, in a non-Archimedean field every integer has absolute value at most one, so the reciprocal of a large integer need not be small. Averaging therefore cannot dilute finite perturbations as it does over $\mathbb{R}$, and inclusions that are automatic in the classical theory may fail. Summability over non-Archimedean fields has a substantial history of its own. Monna [11] showed that an infinite matrix is convergence-preserving over such a field precisely under conditions closely parallel to those of classical summability. Rangachari and Srinivasan [12] proved that the convergent sequences of a non-Archimedean Banach space again form such a space and developed the corresponding matrix theory. Nanda [13] extended matrix-transformation theory to non-Archimedean spaces and studied paranormed sequence spaces over valued fields [14]. Natarajan [15,16] established Steinhaus- and Schur-type theorems for ultrametric summability. The topological side of the subject is treated in the monographs of Schikhof [17] and of Perez-Garcia and Schikhof [18], and Natarajan [19] collects the theory of sequence spaces over valued fields. On the classical side, the books of Başar [20] and of Mursaleen and Başar [21] survey the present state of sequence space theory and matrix transformations, and matrix transformations continue to be studied in new settings, for instance over n-normed spaces [22]. In this setting, λ-almost convergence has not been studied.

The present paper develops that theory. It studies the space $\widehat{V}_\lambda$ of λ-almost convergent sequences over F, together with a paranormed generalization $\widehat{V}_\lambda(p)$, and shows that the latter is a complete non-Archimedean topological vector space under no restriction on λ beyond admissibility. The inclusion $c \subset \widehat{V}_\lambda$ is shown to hold exactly when F has residue characteristic zero and the window slides past every fixed prefix. The class of infinite matrices that map every convergent sequence into $\widehat{V}_\lambda$ is characterized completely, the characterization resting on a uniform boundedness estimate for the averaged functionals, obtained here by an ultrametric gliding hump argument, in the manner of [11,16], over any complete non-trivially valued field. Finally, the scales attached to λ are compared with $\widehat{V}_\lambda$. Strong almost convergence and almost λ-statistical convergence are shown to coincide on the bounded sequences for every λ and every base field, yet over $\mathbb{Q}_p$ this common space is not contained in $\widehat{V}_\lambda$. This separation is genuinely non-Archimedean, with no counterpart over $\mathbb{R}$ or $\mathbb{C}$.

The passage from $\mathbb{R}$ to F is therefore not a matter of replacing the absolute value by the valuation. Over $\mathbb{R}$ the inclusion $c \subset \widehat{V}_\lambda$ is immediate, while over F it holds for some λ if and only if the residue characteristic of F is zero (Proposition 1). The boundedness estimate behind Theorem 3 is obtained by an explicit gliding hump construction, valid with no

restriction on the value group of F, and the separation in Theorem 5 has no Archimedean analog. These three results are the contribution of the paper.

The paper is organized as follows: Section 2 introduces the base field, the admissible window sequences, the field and real means, and the sequence spaces under study. Section 3 characterizes the fields and the windows for which the classical inclusion $c \subset \widehat{V}_\lambda$ holds. Section 4 establishes the topological vector space structure and completeness of $\widehat{V}_\lambda(p)$. Section 5 obtains the boundedness estimate for the averaged functionals by an ultrametric gliding hump and uses it to characterize the matrix class $(c, \widehat{V}_\lambda)$. Section 6 compares the scales with $\widehat{V}_\lambda$ and establishes the separation over $\mathbb{Q}_p$. Section 7 closes with a summary of the results.

2. Preliminaries

Throughout, F denotes a complete, non-trivially valued, non-Archimedean field with valuation $|\cdot|$ and $\operatorname{char} F = 0$. Non-triviality means that some $a \in F$ has $0 < |a| < 1$. The valuation obeys the ultrametric inequality $|a+b| \le \max(|a|, |b|)$, and every integer n, regarded as an element of F, satisfies $|n| \le 1$. Write $V = \{a \in F : |a| \le 1\}$ for the valuation ring. Since $\operatorname{char} F = 0$, the prime field $\mathbb{Q}$ embeds in F and every nonzero integer is invertible in F. The residue characteristic of F is the characteristic of the residue field $\{a : |a| \le 1\}/\{a : |a| < 1\}$. It is either 0, in which case $|n| = 1$ for every nonzero integer n, or a prime p, in which case $|p| < 1$.

Let ω be the F-vector space of all sequences $x = (x_k)_{k \ge 1}$ in F. The relevant subspaces are

$$\ell_\infty = \Big\{ x \in \omega : \sup_k |x_k| < \infty \Big\},$$
$$c = \Big\{ x \in \ell_\infty : \lim_k x_k \text{ exists in } F \Big\},$$
$$c_0 = \Big\{ x \in c : \lim_k x_k = 0 \Big\},$$

each normed by $\|x\|_\infty = \sup_k |x_k|$ and each a non-Archimedean Banach space. Write $e = (1, 1, 1, \dots)$ and $e^{(k)}$ for the k-th unit sequence. A series $\sum_k a_k$ in F converges if and only if $\lim_k a_k = 0$, in which case $\big|\sum_k a_k\big| \le \sup_k |a_k|$. A sum or a supremum over k with no range indicated runs over all $k \in \mathbb{N}$.

The window structure is governed by a class of admissible sequences. Let Δ be the class of non-decreasing $\lambda = (\lambda_m) \subset \mathbb{N}$ with $\lambda_1 = 1$, $\lambda_{m+1} \le \lambda_m + 1$, and $\lim_m \lambda_m = \infty$. For $m \in \mathbb{N}$, set $I_m = [\, m - \lambda_m + 1,\ m \,] \cap \mathbb{N}$, a block of $|I_m| = \lambda_m$ consecutive integers ending at m. The constraint $\lambda_{m+1} \le \lambda_m + 1$ keeps the window from growing faster than its right endpoint, so that $I_m \subset \{1, \dots, m\}$. The same constraint gives $(m+1) - \lambda_{m+1} \ge m - \lambda_m$, so $m - \lambda_m$ is non-decreasing. Apart from real intervals, square brackets are used only here, for integer intervals, and in the name $[\widehat{V}, \lambda]$ of the strong space below. The prototype is $\lambda_m = m$, for which $I_m = \{1, \dots, m\}$. More generally, $\lambda_m = \lfloor m^\alpha \rfloor$ with $0 < \alpha \le 1$ lies in Δ after adjustment of finitely many initial terms, since $\lfloor (m+1)^\alpha \rfloor - \lfloor m^\alpha \rfloor \in \{0, 1\}$ for large m.

A single arithmetic fact separates this theory from its classical counterpart. Because every integer n, regarded as an element of F, satisfies $|n| \le 1$, the field reciprocal obeys $|\lambda_m^{-1}| = |\lambda_m|^{-1} \ge 1$, and this quantity may be unbounded. Over $\mathbb{Q}_p$ with $\lambda_m = m$ one has $|\lambda_m^{-1}|_p = p^{v_p(m)}$, unbounded along $m = p^j$. Set

$$C_\lambda := \sup_m |\lambda_m^{-1}| \in [1, \infty].$$

Since λ increases in unit steps to infinity, it takes every sufficiently large integer value. Hence, $C_\lambda = 1$ when the residue characteristic is 0, while when it is a prime p the values $\lambda_m = p^j$ occur for every large j and $C_\lambda = \infty$. Thus, C_λ depends only on F, not on λ, and this fact underlies every divergence from the Archimedean theory below.

With the field and basic spaces in place, the averaging operators underlying the spaces of interest can now be introduced. The generalized de la Vallée–Poussin field mean is

$$\varphi_{m,n}(x) = \frac{1}{\lambda_m} \sum_{k \in I_m} x_{k+n} \in F, \qquad m \in \mathbb{N},\ n \in \mathbb{N}_0,$$

with $\lambda_m^{-1} \in F$. Here $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and every statement made uniformly in n refers to $n \in \mathbb{N}_0$. At $m = 1$ one has $I_m = \{1\}$ and $\varphi_{1,n}(x) = x_{1+n}$.

A sequence $x \in \ell_\infty$ is *almost convergent* to $\ell \in F$ if

$$\lim_{r \to \infty} \frac{1}{r} \sum_{i=0}^{r-1} x_{k+i} = \ell \qquad \text{uniformly in } k,$$

the mean and the limit taken in F. The set of such sequences is denoted as $\widehat{c} \subset \ell_\infty$ and $\ell = \widehat{c}\text{-}\lim x$. Over $\mathbb{R}$ or $\mathbb{C}$, Lorentz's theorem identifies almost convergence with the agreement of all Banach limits. A Banach limit is a positive shift-invariant functional, and positivity is an order-theoretic notion, whereas a non-Archimedean field such as $\mathbb{Q}_p$ carries no order compatible with its topology, so the Banach-limit formulation is unavailable over F. The mean condition above uses only field arithmetic and the valuation, and all properties of $\widehat{c}$ needed here follow from it, so it serves as the definition, as is standard in the non-Archimedean literature [12,15].

The central objects of the paper are now defined. A sequence $x \in \ell_\infty$ is *λ-almost convergent* to $\ell \in F$ if

$$\lim_{m \to \infty} |\varphi_{m,n}(x) - \ell| = 0 \qquad \text{uniformly in } n,$$

the set of such sequences being $\widehat{V}_\lambda$, with $\ell = \widehat{V}_\lambda\text{-}\lim x$. Given a bounded sequence $p = (p_m)$ of strictly positive reals, put $M = \max(1, \sup_m p_m)$. The paranormed space $\widehat{V}_\lambda(p)$ consists of those $x \in \ell_\infty$ for which $\lim_{m \to \infty} |\varphi_{m,n}(x) - \ell|^{p_m} = 0$ uniformly in n for some $\ell \in F$, and carries the functional

$$g(x) = \sup_{m,n} |\varphi_{m,n}(x)|^{p_m/M}.$$

The exponent sequence (p_m) always carries a subscript and is not to be confused with the prime p of $\mathbb{Q}_p$.

Alongside the field-valued mean, two real-valued scales are associated with each λ. A sequence $x \in \omega$ is *strongly almost (V, λ)-summable* to $L \in F$ if

$$\lim_{m \to \infty} \frac{1}{\lambda_m} \sum_{k \in I_m} |x_{k+n} - L| = 0 \qquad \text{uniformly in } n,$$

where $|x_{k+n} - L| \in \mathbb{R}_{\geq 0}$ is the valuation, $1/\lambda_m \in \mathbb{R}$ is the real reciprocal, and the limit is taken in $\mathbb{R}$. The space is denoted $[\widehat{V}, \lambda]$. The sequence x is *almost λ-statistically convergent* to L if, for every $\varepsilon > 0$,

$$\lim_{m \to \infty} \frac{1}{\lambda_m} \left| \{k \in I_m : |x_{k+n} - L| \geq \varepsilon\} \right| = 0 \qquad \text{uniformly in } n,$$

the cardinality and the quotient being real. The space is denoted $\widehat{s}_\lambda$. Both $[\widehat{V}, \lambda]$ and $\widehat{s}_\lambda$ are defined through real quantities and the real reciprocal $1/\lambda_m$. The Cesàro dilution therefore behaves exactly as over $\mathbb{R}$, and F enters only as the source of values and through its valuation. By contrast, $\widehat{V}_\lambda$ uses the field mean with $\lambda_m^{-1} \in F$ and $|\lambda_m^{-1}| \geq 1$, and admits no dilution. This distinction is what makes the inclusion theorems of Section 6 hold for the real spaces and fail for $\widehat{V}_\lambda$.

3. The Inclusion $c \subset \widehat{V}_\lambda$

The inclusion $c \subset \widehat{V}_\lambda$, automatic over $\mathbb{R}$ or $\mathbb{C}$, can fail over F. Over $\mathbb{R}$ or $\mathbb{C}$ it holds because $\frac{1}{\lambda_m} \sum_{k \in I_m} (x_{k+n} - \ell)$ is a genuine average of small terms. Over F this breaks down. Taking $F = \mathbb{Q}_p$, $\lambda_m = m$, and $x = (1, 0, 0, \dots) \in c$ with $\ell = 0$ gives

$$\varphi_{m,0}(x) = \frac{1}{m} \sum_{k=1}^{m} x_k = \frac{1}{m} \cdot 1 = m^{-1}, \qquad |\varphi_{m,0}(x)|_p = p^{\,v_p(m)}, \tag{1}$$

which is unbounded along $m = p^j$, so that $x \notin \widehat{V}_\lambda$. The root cause is the absence of dilution, since the ultrametric bounds a sum by its maximum and the prefactor $|\lambda_m^{-1}| = p^{v_p(m)} \geq 1$ amplifies rather than averages.

The inclusion is restored exactly when the residue characteristic is zero and the window slides past every fixed prefix.

Proposition 1. *Let $\lambda \in \Delta$. Then $c \subset \widehat{V}_\lambda$ if and only if*

(i) F has residue characteristic zero, so that $|\lambda_m| = 1$ for every m and the prefactor does not amplify;

(ii) $m - \lambda_m$ grows without bound, so that the window slides past every fixed prefix.

In that case, $\widehat{V}_\lambda\text{-}\lim x = \lim_k x_k$ for every $x \in c$.

Proof. Suppose $x \in c$ with $\lim_k x_k = \ell$, and let $\varepsilon > 0$. Choose K so that $|x_k - \ell| < \varepsilon$ whenever $k > K$. Condition (ii) provides an M_0 with $m - \lambda_m > K$ for all $m \geq M_0$, and then every $k \in I_m$ and every $n \geq 0$ satisfy $k + n \geq m - \lambda_m + 1 > K$, so that $|x_{k+n} - \ell| < \varepsilon$. By (i), $|\lambda_m^{-1}| = 1$ for every m. For $m \geq M_0$ the ultrametric inequality now bounds the deviation of the mean by the largest deviation in the window,

$$|\varphi_{m,n}(x) - \ell| = \left|\lambda_m^{-1} \sum_{k \in I_m} (x_{k+n} - \ell)\right| \leq |\lambda_m^{-1}| \max_{k \in I_m} |x_{k+n} - \ell| < \varepsilon,$$

uniformly in n, so that $x \in \widehat{V}_\lambda$ with limit ℓ.

For necessity, suppose first that (ii) fails. Then, $m - \lambda_m$, which is non-decreasing by Section 2, is eventually constant, say $m - \lambda_m = c_0$ for $m \geq M_0$, so every window I_m with $m \geq M_0$ has the same left endpoint $c_0 + 1$. The unit spike $x = e^{(c_0+1)} \in c_0 \subset c$, with limit 0, then meets each such window at the single coordinate $c_0 + 1$, so

$$|\varphi_{m,0}(x)| = |\lambda_m^{-1} x_{c_0+1}| = |\lambda_m^{-1}| \geq 1$$

by the valuation estimate of Section 2. For the same $m \geq M_0$ the shift $n = m$ moves the spike out of the window, giving $\varphi_{m,m}(x) = 0$, so

$$|\varphi_{m,0}(x) - \varphi_{m,m}(x)| = |\lambda_m^{-1}| \geq 1.$$

If x were in $\widehat{V}_\lambda$, the means $\varphi_{m,n}(x)$ would converge, as m increases, uniformly in n, forcing this difference to tend to 0. Since it does not, $x \in c \setminus \widehat{V}_\lambda$, whatever the residue characteristic.

Suppose next that (ii) holds and (i) fails, so that the residue characteristic is a prime p and $|p| < 1$. Since λ takes every large integer value and $m - \lambda_m$ grows without bound, indices $m_1 < m_2 < \cdots$ and exponents $e_1 < e_2 < \cdots$ can be chosen inductively with $\lambda_{m_j} = p^{e_j}$ and $m_{j+1} - \lambda_{m_{j+1}} > 2m_j$. The windows I_{m_j} are then pairwise disjoint, and $I_{m_{j+1}}$ lies beyond $2m_j$. Choose any $k_j \in I_{m_j}$ and set $x_{k_j} = \lambda_{m_j} = p^{e_j}$, with all other entries 0. Then, $|x_{k_j}| = |p|^{e_j}$, which tends to 0, so $x \in c_0 \subset c$. By disjointness I_{m_j} meets the support of x only at k_j, so

$$\varphi_{m_j,0}(x) = \lambda_{m_j}^{-1} x_{k_j} = \lambda_{m_j}^{-1} \lambda_{m_j} = 1$$

for every j. The shift $n = m_j$ moves the window to $\{m_j + 1, \ldots, 2m_j\}$, which contains no point of the support, since $k_i \le m_i \le m_j$ for $i \le j$ and $k_i > 2m_j$ for $i > j$. Hence $\varphi_{m_j,m_j}(x) = 0$, and $|\varphi_{m_j,0}(x) - \varphi_{m_j,m_j}(x)| = 1$ for every j. If x were in $\widehat{V}_\lambda$, this difference would tend to 0. Since it does not, $x \in c \setminus \widehat{V}_\lambda$. In every case, the amplifying prefactor, not the size of the entries, destroys convergence. □

Remark 1. *The inclusion $c \subset \widehat{V}_\lambda$ depends on the arithmetic of the field and not on the rate at which λ grows. Over $\mathbb{Q}_p$ it fails for every $\lambda \in \Delta$, for the fast-growing prototype $\lambda_m = m$ by the computation (1), and equally for the slowly growing $\lambda_m = \lfloor m^\alpha \rfloor$ with $0 < \alpha < 1$, where $m - \lambda_m \ge m(1 - m^{\alpha-1})$ grows without bound and the obstruction comes from the residue characteristic alone. Over a field of residue characteristic zero, for instance the completion of $\mathbb{Q}(t)$ under the t-adic valuation, the inclusion holds for every $\lambda \in \Delta$ with $m - \lambda_m$ unbounded.*

4. Topological Vector Space Structure and Completeness

Throughout this section $\lambda \in \Delta$, and $p = (p_m)$ is a sequence of strictly positive reals that is bounded and bounded away from 0. Put $M = \max(1, \sup_m p_m)$ and $q_m := p_m / M \in (0, 1]$, and write $\delta := \inf_m q_m$, which is positive. The argument uses repeatedly that $t \mapsto t^q$ is non-decreasing on $[0, \infty)$ for $q > 0$ and that $(\max(a, b))^q = \max(a^q, b^q)$ for $a, b \ge 0$.

These conventions in hand, the space $\widehat{V}_\lambda(p)$ carries a natural paranorm under which it becomes a complete non-Archimedean topological vector space. The paranorm and the topological vector space structure come first.

Theorem 1. *The functional $g(x) = \sup_{m,n} |\varphi_{m,n}(x)|^{q_m}$ is finite on $\widehat{V}_\lambda(p)$ and is a total paranorm satisfying the strong subadditivity*

$$g(x + y) \le \max\big(g(x), g(y)\big) \qquad \text{for all } x, y \in \widehat{V}_\lambda(p).$$

The induced metric $d(x, y) = g(x - y)$ is a translation invariant ultrametric, and $\widehat{V}_\lambda(p)$, with the topology it generates, is a non-Archimedean topological vector space.

Proof. *Finiteness.* Let $x \in \widehat{V}_\lambda(p)$. Then, $x \in \ell_\infty$ and there is $\ell \in F$ for which $|\varphi_{m,n}(x) - \ell|$ tends to 0 uniformly in n. Pick M_0 with $|\varphi_{m,n}(x) - \ell| < 1$ for all $m \ge M_0$ and all n. Then by the ultrametric inequality $|\varphi_{m,n}(x)| \le \max(1, |\ell|)$ for $m \ge M_0$. For each of the finitely many $m < M_0$, $|\varphi_{m,n}(x)| \le |\lambda_m^{-1}| \, \|x\|_\infty$. Hence, the single quantity

$$A := \max\Big(\max(1, |\ell|), \max_{m < M_0} |\lambda_m^{-1}| \, \|x\|_\infty\Big)$$

is a finite uniform bound for $|\varphi_{m,n}(x)|$ over all m, n, and therefore

$$g(x) \le \sup_m A^{q_m} \le \max(A, A^\delta) < \infty.$$

Only the finiteness of each $|\lambda_m^{-1}|$ is used, which holds since λ_m is a nonzero integer and $\operatorname{char} F = 0$, so no hypothesis on λ enters.

Paranorm axioms. Clearly $g(0) = 0$. If $g(x) = 0$ then $\varphi_{m,n}(x) = 0$ for all m, n. At $m = 1$, $\varphi_{1,n}(x) = x_{1+n} = 0$ for all $n \geq 0$, so $x = 0$. Symmetry $g(-x) = g(x)$ is immediate. For subadditivity, linearity of the mean and the ultrametric inequality give

$$\begin{aligned} |\varphi_{m,n}(x+y)|^{q_m} &\leq \max\big(|\varphi_{m,n}(x)|, |\varphi_{m,n}(y)|\big)^{q_m} = \max\big(|\varphi_{m,n}(x)|^{q_m}, |\varphi_{m,n}(y)|^{q_m}\big) \\ &\leq \max(g(x), g(y)), \end{aligned}$$

and taking the supremum over (m, n) yields $g(x + y) \leq \max(g(x), g(y))$.

Continuity of scalar multiplication. For $\mu \in F$, $|\varphi_{m,n}(\mu x)|^{q_m} = |\mu|^{q_m} |\varphi_{m,n}(x)|^{q_m}$. If $|\mu| \leq 1$ then $|\mu|^{q_m} \leq |\mu|^{\delta}$, since $q_m \geq \delta$ and the base is at most 1, so $g(\mu x) \leq |\mu|^{\delta} g(x)$, and the right side tends to 0 as μ tends to 0. This is where $\delta > 0$ is used. For any μ and any m, $|\mu|^{q_m} \leq \max(1, |\mu|)$, so $g(\mu x) \leq \max(1, |\mu|) g(x)$, which vanishes as $g(x)$ does. Joint continuity of the map $(\mu, x) \mapsto \mu x$ follows from these two estimates together with subadditivity, and continuity of addition is immediate from subadditivity. Hence, g is a total, ultrametric paranorm.

Topological vector space. Set $d(x, y) = g(x - y)$. It is symmetric, vanishes exactly on the diagonal, is translation invariant, and satisfies the strong triangle inequality $d(x, z) \leq \max(d(x, y), d(y, z))$, so it is a translation invariant ultrametric. The balls $S_r = \{x : g(x) < 1/r\}$, $r \in \mathbb{N}$, are nested and form a neighborhood base of 0. Each S_r is balanced, since for $\mu \in V$ and $x \in S_r$ one has $g(\mu x) \leq \max(1, |\mu|) g(x) = g(x) < 1/r$, and each is absolutely convex, since for $\mu, \eta \in V$ and $x, y \in S_r$, $g(\mu x + \eta y) \leq \max(g(\mu x), g(\eta y)) \leq \max(g(x), g(y)) < 1/r$. The existence of this base of balanced, absolutely convex neighborhoods of 0, together with the continuity of the operations, shows that $\widehat{V}_\lambda(p)$ is a non-Archimedean topological vector space. □

The construction uses no property of λ beyond $\lambda \in \Delta$. The unweighted case is worth singling out, since it also locates $\widehat{V}_\lambda$ relative to ℓ_∞.

Corollary 1. *When $p_m \equiv 1$, so that $M = 1$ and $\delta = 1$, the functional g reduces to*

$$\|x\|_{\widehat{V}_\lambda} := \sup_{m,n} |\varphi_{m,n}(x)|,$$

a non-Archimedean norm on $\widehat{V}_\lambda$. Evaluating at $m = 1$ gives $\sup_n |x_n| \leq \|x\|_{\widehat{V}_\lambda}$, so $\widehat{V}_\lambda \subset \ell_\infty$ unconditionally. If the residue characteristic is zero, then $C_\lambda = 1$ and $\|x\|_{\widehat{V}_\lambda} = \|x\|_\infty$, so the two norms coincide on $\widehat{V}_\lambda$.

Proof. The two inequalities are the cases $p_m \equiv 1$ of the bounds established in the proof of Theorem 1. The identity $|\varphi_{1,n}(x)| = |x_{1+n}|$ gives the lower bound, while $|\varphi_{m,n}(x)| \leq |\lambda_m^{-1}| \max_k |x_{k+n}| \leq C_\lambda \|x\|_\infty$ gives the upper bound, with $C_\lambda = 1$ in residue characteristic zero. □

For the completeness theorem it is convenient to isolate a bound that converts the q_m-weighted paranorm into a statement about the means $\varphi_{m,n}$ themselves.

Lemma 1. *If $y \in \omega$ satisfies $g(y) \leq 1$, then $\sup_{m,n} |\varphi_{m,n}(y)| \leq g(y)$.*

Proof. Fix m, n. Since $|\varphi_{m,n}(y)|^{q_m} \leq g(y) \leq 1$ and $1/q_m \geq 1$, raising both sides to the power $1/q_m$ gives $|\varphi_{m,n}(y)| \leq g(y)^{1/q_m} \leq g(y)$. The supremum over m, n gives the claim. □

Completeness now follows, in both the scalar and the vector-valued settings.

Theorem 2. *If F is complete, then $(\widehat{V}_\lambda(p), g)$ is complete. In particular, $(\widehat{V}_\lambda, \|\cdot\|_{\widehat{V}_\lambda})$ is a non-Archimedean Banach space. If E is a non-Archimedean Banach space over F, then the vector-valued space $\widehat{V}_\lambda(p; E)$, with $\|\cdot\|_E$ replacing $|\cdot|$ throughout, is complete.*

Proof. *Scalar case.* Let $(x^{(i)})_{i\geq 1}$ be a Cauchy sequence for the metric d of Theorem 1, that is, for every $\varepsilon > 0$ there is $I = I(\varepsilon)$ with $g(x^{(i)} - x^{(j)}) < \varepsilon$ for $i, j \geq I$. The argument first extracts a coordinatewise limit, then shows convergence to it in g, and finally identifies the limit of the means.

Fix $k \geq 1$. At $m = 1$ and $n = k - 1$ the mean is $\varphi_{1,k-1}(y) = y_k$, so for $y = x^{(i)} - x^{(j)}$,

$$|x_k^{(i)} - x_k^{(j)}|^{q_1} = |\varphi_{1,k-1}(x^{(i)} - x^{(j)})|^{q_1} \leq g(x^{(i)} - x^{(j)}).$$

The right side tends to 0 as i, j increase, and $q_1 > 0$, so $(x_k^{(i)})_i$ is Cauchy in F and, by completeness, converges to some $x_k \in F$. Put $x = (x_k)$.

This candidate limit is approached in g. Fix $\varepsilon > 0$, $I = I(\varepsilon)$, indices m, n, and $i \geq I$. For every $j \geq I$, $|\varphi_{m,n}(x^{(i)} - x^{(j)})|^{q_m} \leq g(x^{(i)} - x^{(j)}) < \varepsilon$. Since $\varphi_{m,n}(x^{(j)})$ is a finite F-linear combination of coordinates of $x^{(j)}$, the index set I_m being finite, the coordinatewise convergence just established gives $\lim_{j\to\infty} \varphi_{m,n}(x^{(j)}) = \varphi_{m,n}(x)$. By continuity of $|\cdot|$ and of $t \mapsto t^{q_m}$,

$$|\varphi_{m,n}(x^{(i)} - x)|^{q_m} = \lim_{j\to\infty} |\varphi_{m,n}(x^{(i)} - x^{(j)})|^{q_m} \leq \varepsilon.$$

Taking the supremum over (m, n), $g(x^{(i)} - x) \leq \varepsilon$ for $i \geq I$, so $x^{(i)}$ converges to x in g. By subadditivity $g(x) \leq \max(g(x^{(I)}), g(x^{(I)} - x)) < \infty$, and the $m = 1$ slice gives $\sup_k |x_k|^{q_1} \leq g(x) < \infty$, so $x \in \ell_\infty$.

It remains to place x in $\widehat{V}_\lambda(p)$. Each $x^{(i)} \in \widehat{V}_\lambda(p)$ has a unique $\ell^{(i)} \in F$ for which $|\varphi_{m,n}(x^{(i)}) - \ell^{(i)}|$ tends to 0 uniformly in n as m increases, and the limits $\ell^{(i)}$ themselves form a Cauchy sequence. Indeed, for $\varepsilon \in (0, 1)$ and $I = I(\varepsilon)$, whenever $i, j \geq I$ one has $g(x^{(i)} - x^{(j)}) < \varepsilon < 1$, so Lemma 1 gives $|\varphi_{m,n}(x^{(i)}) - \varphi_{m,n}(x^{(j)})| \leq g(x^{(i)} - x^{(j)}) < \varepsilon$ for all m, n. Fix such i, j and choose m and n with $|\varphi_{m,n}(x^{(i)}) - \ell^{(i)}| < \varepsilon$ and $|\varphi_{m,n}(x^{(j)}) - \ell^{(j)}| < \varepsilon$. Writing $\ell^{(i)} - \ell^{(j)}$ as the sum of $\ell^{(i)} - \varphi_{m,n}(x^{(i)})$, $\varphi_{m,n}(x^{(i)}) - \varphi_{m,n}(x^{(j)})$, and $\varphi_{m,n}(x^{(j)}) - \ell^{(j)}$, the ultrametric inequality gives $|\ell^{(i)} - \ell^{(j)}| < \varepsilon$. By completeness, $(\ell^{(i)})$ converges to some $\ell \in F$.

Finally, $\varphi_{m,n}(x)$ converges to this ℓ uniformly. Given $\varepsilon \in (0, 1)$, choose i with $|\ell^{(i)} - \ell| < \varepsilon$ and, from the convergence in g already shown, $g(x - x^{(i)}) < \varepsilon < 1$, so Lemma 1 gives $|\varphi_{m,n}(x) - \varphi_{m,n}(x^{(i)})| < \varepsilon$ for all m, n. For this i, pick m_0 with $|\varphi_{m,n}(x^{(i)}) - \ell^{(i)}| < \varepsilon$ for $m \geq m_0$ and all n. Then for $m \geq m_0$ and all n,

$$\varphi_{m,n}(x) - \ell = \big(\varphi_{m,n}(x) - \varphi_{m,n}(x^{(i)})\big) + \big(\varphi_{m,n}(x^{(i)}) - \ell^{(i)}\big) + \big(\ell^{(i)} - \ell\big),$$

and the ultrametric inequality gives $|\varphi_{m,n}(x) - \ell| < \varepsilon$. Thus, $|\varphi_{m,n}(x) - \ell|$ tends to 0 uniformly in n, and since (p_m) is bounded, so does $|\varphi_{m,n}(x) - \ell|^{p_m}$. Together with $x \in \ell_\infty$, this gives $x \in \widehat{V}_\lambda(p)$ and $x^{(i)}$ converges to x in $\widehat{V}_\lambda(p)$, so the space is complete. The case $p_m \equiv 1$ shows that $\widehat{V}_\lambda$ is a non-Archimedean Banach space.

Vector case. Replace the values by E and $|\cdot|$ by $\|\cdot\|_E$, keeping the scalars $\lambda_m^{-1} \in F$. The mean $\varphi_{m,n}(x) = \lambda_m^{-1} \sum_{k\in I_m} x_{k+n} \in E$ satisfies the ultrametric inequality and $\|\lambda_m^{-1} v\|_E = |\lambda_m^{-1}|\, \|v\|_E$. Lemma 1 holds verbatim, the coordinatewise limits exist in E by completeness of E, and the remaining two parts of the argument are unchanged. Hence, $\widehat{V}_\lambda(p; E)$ is complete. □

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Remark 2. *As a set, $\widehat{V}_\lambda(p)$ does not depend on p. Since $\inf_m p_m =: \delta_0 > 0$ and (p_m) is bounded, write $s := \sup_m p_m < \infty$. For non-negative reals (a_m) the condition that $a_m^{p_m}$ tends to 0 holds if and only if a_m tends to 0. Indeed, if a_m tends to 0 then eventually $a_m < 1$ and $a_m^{p_m} \le a_m^{\delta_0}$, which tends to 0. Conversely, if $a_m^{p_m}$ tends to 0 then eventually $a_m < 1$, and since $1/s \le 1/p_m$ with base $a_m^{p_m} \in (0,1)$, $a_m = (a_m^{p_m})^{1/p_m} \le (a_m^{p_m})^{1/s}$, which tends to 0. The membership conditions therefore coincide, so $\widehat{V}_\lambda(p) = \widehat{V}_\lambda$ as sets and p affects only the paranorm and the topology. Neither Theorem 1 nor Theorem 2 uses the residue characteristic. It enters only through the comparison with ℓ_∞ in Corollary 1.*

5. Matrix Transformations of $(c, \widehat{V}_\lambda)$

Let $A = (a_{rk})_{r,k\ge1}$ be an infinite matrix over F. The A-transform of $x \in \omega$ is $Ax = ((Ax)_r)_r$ with $(Ax)_r = \sum_k a_{rk}x_k$, defined whenever every row series converges. Write

$$\varphi_{m,n}(Ax) = \sum_k t(n,k,m)\, x_k, \qquad t(n,k,m) = \frac{1}{\lambda_m} \sum_{i\in I_m} a_{n+i,k},$$

so that $\varphi_{m,n}(Ax)$ is the field mean of the rows of Ax. For sequence spaces X, Y, write $A \in (X,Y)$ if $Ax \in Y$ for every $x \in X$, and $A \in (X,Y;P)$ if in addition A preserves limits.

The characterization rests on a boundedness estimate for the averaged functionals, established first by an ultrametric gliding hump.

Lemma 2. *Let F be complete and non-trivially valued, let N be an arbitrary index set, and let $(b_{\nu,k})_{\nu\in N,k\ge1}$ be a family in F such that, for each ν, $b_{\nu,k}$ tends to 0 as k increases, so that $f_\nu(x) = \sum_k b_{\nu,k}x_k$ converges for all $x \in \ell_\infty$. If $\sup_\nu |f_\nu(x)| < \infty$ for every $x \in c_0$, then $\sup_{\nu,k} |b_{\nu,k}| < \infty$.*

Proof. The argument is by contraposition: assuming $\sup_{\nu,k} |b_{\nu,k}| = \infty$, a sequence $x \in c_0$ is built with $\sup_\nu |f_\nu(x)| = \infty$. Fix $\pi \in F$ with $0 < \rho := |\pi| < 1$. Every scalar below is an integer power of π, which always exists, so no assumption on the value group of F is made.

Suppose first that some column is unbounded, say $\sup_\nu |b_{\nu,k_0}| = \infty$. Then $e^{(k_0)} \in c_0$ and $f_\nu(e^{(k_0)}) = b_{\nu,k_0}$ give $\sup_\nu |f_\nu(e^{(k_0)})| = \infty$, as required.

Suppose instead that every column is bounded, so that $\beta_k := \sup_\nu |b_{\nu,k}| < \infty$ for all k. For each finite K, $\sup_{\nu,k\le K} |b_{\nu,k}| = \max_{k\le K} \beta_k < \infty$, so $\sup_{\nu,k} |b_{\nu,k}| = \infty$ forces

$$\sup_{\nu,k>K} |b_{\nu,k}| = \infty \qquad \text{for every } K. \tag{2}$$

Construct inductively columns $0 = \kappa_0 < \kappa_1 < \cdots$ and indices $\nu_j \in N$ such that, writing $d_j := |b_{\nu_j,\kappa_j}|\,\rho^j$,

(I) $d_j > \rho^{-j}$;

(II) $d_j > \max_{i<j} \beta_{\kappa_i}\rho^i$, so the new hump exceeds every earlier column;

(III) $|b_{\nu_i,\kappa_j}|\,\rho^j < d_i$ for $i < j$, so the new column is negligible in every earlier row.

For the inductive step, suppose $\kappa_0, \ldots, \kappa_{j-1}$ and $\nu_1, \ldots, \nu_{j-1}$, hence $d_1, \ldots, d_{j-1}$, are already chosen. For each $i < j$, the convergence of $b_{\nu_i,k}$ to 0 in k provides K_i with $|b_{\nu_i,k}| < \rho^{-j}d_i$ for $k > K_i$, so any $\kappa_j > \max_{i<j} K_i$ secures (III). With $T := \max_{i<j}(\rho^{-j}, \beta_{\kappa_i}\rho^i) < \infty$, apply

(2) with $K = \max(\kappa_{j-1}, \max_{i<j} K_i)$ to choose ν_j and $\kappa_j > K$ with $|b_{\nu_j,\kappa_j}| > \rho^{-j}T$. Then $d_j > T \geq \max(\rho^{-j}, \max_{i<j} \beta_{\kappa_i}\rho^i)$, giving (I) and (II). This completes the induction.

Now put $x_{\kappa_j} = \pi^j$ and $x_k = 0$ otherwise. Then $|x_{\kappa_j}| = \rho^j$, which tends to 0, so $x \in c_0$. Fix j and split

$$f_{\nu_j}(x) = \underbrace{b_{\nu_j,\kappa_j}\pi^j}_{\text{hump}} + \underbrace{\sum_{i<j} b_{\nu_j,\kappa_i}\pi^i}_{\text{head}} + \underbrace{\sum_{i>j} b_{\nu_j,\kappa_i}\pi^i}_{\text{tail}}.$$

The hump has modulus $d_j > \rho^{-j}$ by (I). Each head term, with $i < j$, satisfies $|b_{\nu_j,\kappa_i}\pi^i| \leq \beta_{\kappa_i}\rho^i < d_j$, the first inequality because β_{κ_i} is the supremum of $|b_{\nu,\kappa_i}|$ over ν, the second by (II). Each tail term, with $i > j$, satisfies $|b_{\nu_j,\kappa_i}\pi^i| = |b_{\nu_j,\kappa_i}|\rho^i < d_j$ by (III), applied at stage i to the earlier row ν_j. Every term other than the hump is therefore strictly smaller than d_j in absolute value, and the strong triangle inequality, which gives $|a+b| = |a|$ whenever $|b| < |a|$, yields $|f_{\nu_j}(x)| = d_j > \rho^{-j}$. Hence, $\sup_\nu |f_\nu(x)| \geq \sup_j \rho^{-j} = \infty$, contradicting the hypothesis. Therefore, $\sup_{\nu,k} |b_{\nu,k}| < \infty$. □

With the boundedness estimate in hand, the matrix class $(c, \widehat{V}_\lambda)$ can be characterized.

Theorem 3. *Let $\lambda \in \Delta$ and F complete. An infinite matrix $A = (a_{rk})$ belongs to $(c, \widehat{V}_\lambda)$ if and only if*

(i) $\lim_{k\to\infty} a_{rk} = 0$ *for each row r;*

(ii) $\sup_{n,k,m} |t(n,k,m)| < \infty$;

(iii) $\lim_{m\to\infty} t(n,k,m) = \alpha_k$ *uniformly in n, for each k;*

(iv) $\lim_{m\to\infty} \sum_{k=1}^{\infty} t(n,k,m) = \alpha$ *uniformly in n.*

The first condition guarantees that the transform Ax is defined. The remaining three characterize membership. In that case, for every $x \in c$ with $\lim_k x_k = L$,

$$\widehat{V}_\lambda\text{-}\lim Ax = \alpha L + \sum_{k=1}^{\infty} \alpha_k (x_k - L),$$

the series converging because $x_k - L$ tends to 0 and $\sup_k |\alpha_k| \leq \sup_{n,k,m} |t(n,k,m)| < \infty$.

Proof. For necessity, assume $A \in (c, \widehat{V}_\lambda)$. For $x = e \in c$ the row series $\sum_k a_{rk}$ must converge, which forces a_{rk} to tend to 0 in k, giving (i). Testing the units $e^{(k)} \in c_0 \subset c$ gives $\varphi_{m,n}(Ae^{(k)}) = t(n,k,m)$, and since $Ae^{(k)} \in \widehat{V}_\lambda$ the mean $t(n,k,m)$ converges to some α_k uniformly in n, which is (iii). Testing e gives $\varphi_{m,n}(Ae) = \sum_k t(n,k,m)$, converging to some α uniformly, which is (iv). For (ii), every $x \in c_0$ has $Ax \in \widehat{V}_\lambda$, and $\|Ax\|_{\widehat{V}_\lambda} = \sup_{m,n} |\varphi_{m,n}(Ax)|$ is finite by the finiteness part of Theorem 1. Applying Lemma 2 with $\nu = (n,m)$ and $b_{\nu,k} = t(n,k,m)$, the column hypothesis holding by (i), yields $\sup_{n,k,m} |t(n,k,m)| < \infty$.

For sufficiency, assume (i)–(iv) and let $x \in c$ with $\lim_k x_k = L$. By (i) and the boundedness of x, $a_{rk}x_k$ tends to 0 in k, so Ax and the means $\varphi_{m,n}(Ax) = \sum_k t(n,k,m)x_k$ are

defined. Put $C = \sup_{n,k,m} |t(n,k,m)|$. Letting m increase in (iii) gives $|\alpha_k| \le C$, so $\sum_k \alpha_k(x_k - L)$ converges. Write

$$\varphi_{m,n}(Ax) = \sum_k t(n,k,m)(x_k - L) + L\sum_k t(n,k,m).$$

By (iv) the second term converges to $L\alpha$ uniformly in n. For the first, set

$$S_{m,n} = \sum_k (t(n,k,m) - \alpha_k)(x_k - L),$$

where $|t(n,k,m) - \alpha_k| \le C$ by the ultrametric inequality. Given $\varepsilon \in (0,1)$, choose K with $|x_k - L| < \varepsilon/C$ for $k > K$, the case $C = 0$ being trivial. The tail, over the indices $k > K$, satisfies for all n, m

$$\Big|\sum_{k>K} (t(n,k,m) - \alpha_k)(x_k - L)\Big| \le \sup_{k>K} |t(n,k,m) - \alpha_k|\,|x_k - L| < C \cdot \frac{\varepsilon}{C} = \varepsilon.$$

The remaining part, over the finitely many indices $k \le K$, is controlled by (iii), since $\max_{k\le K} \sup_n |t(n,k,m) - \alpha_k|$ tends to 0 as m increases, so

$$\sup_n \Big|\sum_{k\le K} (t(n,k,m) - \alpha_k)(x_k - L)\Big| \le \Big(\max_{k\le K} \sup_n |t(n,k,m) - \alpha_k|\Big)\,\|x - L\|_\infty,$$

which tends to 0. Choose m_0 making it less than ε for $m \ge m_0$ and all n. By the ultrametric inequality $|S_{m,n}| < \varepsilon$ for $m \ge m_0$ and all n, so $S_{m,n}$ tends to 0 uniformly, that is, $\sum_k t(n,k,m)(x_k - L)$ converges to $\sum_k \alpha_k(x_k - L)$ uniformly. Adding the two parts, $\varphi_{m,n}(Ax)$ converges to $\alpha L + \sum_k \alpha_k(x_k - L)$ uniformly in n. Finally, $Ax \in \ell_\infty$, since at $m = 1$ the mean is $\varphi_{1,n}(Ax) = (Ax)_{1+n}$, so $\sup_n |(Ax)_{1+n}| \le \sup_{m,n} |\varphi_{m,n}(Ax)| < \infty$. Hence, $Ax \in \widehat{V}_\lambda$ with the stated limit. □

Corollary 2. *$A \in (c, \widehat{V}_\lambda; P)$ if and only if* (i) *and* (ii) *hold together with*

$$\text{(iii}') \quad \lim_{m\to\infty} t(n,k,m) = 0 \ \textit{uniformly in } n, \textit{ for each } k,$$

$$\text{(iv}') \quad \lim_{m\to\infty} \sum_{k=1}^{\infty} t(n,k,m) = 1 \ \textit{uniformly in } n.$$

Proof. By Theorem 3 the image limit is $\alpha L + \sum_k \alpha_k(x_k - L)$. Preservation means this equals L for all $x \in c$. Taking $x = e$, with $L = 1$, forces $\alpha = 1$, and taking $x = e^{(k)}$, with $L = 0$, forces $\alpha_k = 0$. Conversely these reduce the formula to L. Substituting into (iii) and (iv) gives (iii′) and (iv′). □

Remark 3. *The characterization extends to the paranormed domain. Let $p = (p_k)$ be a bounded sequence of strictly positive reals with $\inf_k p_k > 0$. Then, $c(p) = c$ and $\widehat{V}_\lambda(p) = \widehat{V}_\lambda$ as sets, the latter by Remark 2, so a matrix maps $c(p)$ into $\widehat{V}_\lambda(p)$ exactly when it maps c into $\widehat{V}_\lambda$. The four conditions of Theorem 3 therefore characterize the class $(c(p), \widehat{V}_\lambda(p))$ as well. In the weighted form found in the literature, condition (ii) is written with a fixed integer $B > 1$ as $\sup_{n,k,m} |t(n,k,m)|\, B^{-1/p_k} < \infty$, which is equivalent to condition (ii) as stated, since the factor B^{-1/p_k} stays within the fixed interval $[B^{-1/\inf_k p_k}, 1)$. The field element λ_m^{-1} enters only through the definition of $t(n,k,m)$, so the characterization is valid for every $\lambda \in \Delta$, with no further restriction on λ.*

6. The Statistical Scale and Its Separation from $\widehat{V}_\lambda$

The real-arithmetic spaces $[\widehat{V},\lambda]$ and $\widehat{s}_\lambda$ are compared first with each other and then with the field-arithmetic space $\widehat{V}_\lambda$. On bounded sequences the two real scales coincide.

Theorem 4. *Let $\lambda \in \Delta$ and let F be complete and non-trivially valued. Then*

(a) $[\widehat{V},\lambda] \subseteq \widehat{s}_\lambda$, *with the same limit;*
(b) *If $x \in \ell_\infty \cap \widehat{s}_\lambda$ with limit L, then $x \in [\widehat{V},\lambda]$ with limit L;*
(c) *Consequently $\widehat{s}_\lambda \cap \ell_\infty = [\widehat{V},\lambda]$.*

Proof. Write $A_{m,n}(\varepsilon) = \{k \in I_m : |x_{k+n} - L| \geq \varepsilon\}$, and note that all quantities here are non-negative reals and $1/\lambda_m \in \mathbb{R}$.

For part (a), let $x \in [\widehat{V},\lambda]$ with limit L and let $\varepsilon > 0$. Then,

$$\frac{1}{\lambda_m}\sum_{k\in I_m}|x_{k+n}-L| \geq \frac{1}{\lambda_m}\sum_{k\in A_{m,n}(\varepsilon)}|x_{k+n}-L| \geq \frac{\varepsilon}{\lambda_m}\,|A_{m,n}(\varepsilon)|,$$

and the left side tends to 0 uniformly in n, so $\frac{1}{\lambda_m}|A_{m,n}(\varepsilon)|$ tends to 0 uniformly. Hence, $x \in \widehat{s}_\lambda$ with limit L. This inclusion can be strict in general, the gap $[\widehat{V},\lambda] \subsetneq \widehat{s}_\lambda$ being a feature of unbounded sequences, as in the classical real setting of Savaş [10]. Parts (b) and (c) show that on ℓ_∞ the two spaces coincide.

For part (b), let $x \in \ell_\infty \cap \widehat{s}_\lambda$ with limit L and set $H := \sup_k |x_k - L| < \infty$. Fix $\varepsilon > 0$ and split

$$\frac{1}{\lambda_m}\sum_{k\in I_m}|x_{k+n}-L| = \frac{1}{\lambda_m}\sum_{k\in A_{m,n}(\varepsilon)}|x_{k+n}-L| + \frac{1}{\lambda_m}\sum_{k\in I_m\setminus A_{m,n}(\varepsilon)}|x_{k+n}-L|.$$

The first sum has at most $|A_{m,n}(\varepsilon)|$ terms, each at most H, so it is at most $H\,\frac{|A_{m,n}(\varepsilon)|}{\lambda_m}$. The second sum has at most λ_m terms, each below ε, so it is below $\frac{1}{\lambda_m}\cdot\lambda_m\cdot\varepsilon = \varepsilon$. This is a genuine real Cesàro estimate, valid because $1/\lambda_m \in \mathbb{R}$, as noted for the real scales in Section 2. Thus,

$$\frac{1}{\lambda_m}\sum_{k\in I_m}|x_{k+n}-L| \leq H\,\frac{|A_{m,n}(\varepsilon)|}{\lambda_m} + \varepsilon,$$

and the first term tends to 0 uniformly in n because $x \in \widehat{s}_\lambda$. Hence,

$$\limsup_{m\to\infty}\sup_n \frac{1}{\lambda_m}\sum_{k\in I_m}|x_{k+n}-L| \leq \varepsilon.$$

As ε was arbitrary, the limit is 0 uniformly, that is, $x \in [\widehat{V},\lambda]$.

Part (c) is immediate from (a) and (b). □

Over $\mathbb{Q}_p$ this common real scale is not contained in the field space, so the two are genuinely distinct.

Theorem 5. *Let $F = \mathbb{Q}_p$ and $\lambda_m = m$, so $I_m = \{1,\dots,m\}$ and $|\lambda_m^{-1}|_p = p^{v_p(m)}$. The bounded sequence $x = \mathbf{1}_{\{p,p^2,p^3,\dots\}}$ (the indicator of the powers of p) satisfies*

$$x \in [\widehat{V},\lambda]\cap\ell_\infty = \widehat{s}_\lambda\cap\ell_\infty \qquad \textit{but} \qquad x \notin \widehat{V}_\lambda.$$

Hence $[\widehat{V},\lambda] \not\subseteq \widehat{V}_\lambda$ and $\widehat{s}_\lambda \cap \ell_\infty \not\subseteq \widehat{V}_\lambda$: the real-arithmetic statistical scale is not contained in the field-arithmetic space $\widehat{V}_\lambda$.

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Proof. First, we show that $x \in \widehat{s}_\lambda$ with limit 0. Fix $\varepsilon \in (0,1]$. The indices $k \in I_m$ with $|x_{k+n}|_p \geq \varepsilon$ are those for which $k+n$ is a power of p. Let p^a be the smallest and p^b the largest power of p in the window $\{n+1,\ldots,n+m\}$, if the window contains any. Then $p^{b-a} = p^b/p^a \leq (n+m)/(n+1) \leq m+1$, so $b-a \leq \log_p(m+1)$, and the window contains exactly $b-a+1 \leq \log_p(m+1)+1$ powers of p. Hence,

$$\sup_n \frac{1}{\lambda_m}\big|\{k \in I_m : |x_{k+n}|_p \geq \varepsilon\}\big| \leq \frac{\log_p(m+1)+1}{m},$$

and the right side tends to 0 as m increases, so $x \in \widehat{s}_\lambda$ with limit 0, and $x \in [\widehat{V}, \lambda]$ by Theorem 4(b).

Next, we show that $x \notin \widehat{V}_\lambda$. At $n = 0$ the mean is

$$\varphi_{m,0}(x) = m^{-1}\sum_{k=1}^{m} x_k = m^{-1}N(m), \qquad N(m) = \big|\{i \geq 1 : p^i \leq m\}\big| = \lfloor \log_p m \rfloor,$$

an integer multiple of m^{-1} in $\mathbb{Q}_p$. Along $m = p^j$, where $N(p^j) = j$,

$$|\varphi_{p^j,0}(x)|_p = |p^{-j}j|_p = p^j|j|_p \geq p^j \cdot p^{-\log_p j} = \frac{p^j}{j}.$$

Here, $p^{v_p(j)}$ divides j, so $v_p(j) \leq \log_p j$ and $|j|_p = p^{-v_p(j)} \geq p^{-\log_p j} = 1/j$. The right side grows without bound as j increases. Thus, $\varphi_{m,0}(x)$ is unbounded in F and cannot converge, so $x \notin \widehat{V}_\lambda$. Combining the two parts proves all assertions. □

Remark 4. *Theorems 4 and 5 together describe the relationship between the scales. The real strong and statistical spaces coincide on ℓ_∞, yet over $\mathbb{Q}_p$ this common real space is not contained in $\widehat{V}_\lambda$, because field averaging by λ_m^{-1} lacks the Cesàro dilution that real averaging by $1/\lambda_m$ enjoys. The separation is thus a feature of the arithmetic of $\mathbb{Q}_p$, where $|\lambda_m^{-1}|$ is unbounded for every $\lambda \in \Delta$.*

7. Conclusions

Over a complete non-Archimedean field the space $\widehat{V}_\lambda$ of λ-almost convergent sequences, and its paranormed form $\widehat{V}_\lambda(p)$, is a complete non-Archimedean topological vector space for every admissible λ, with no condition on the field beyond completeness. The residue characteristic enters elsewhere. It decides whether $c \subset \widehat{V}_\lambda$, jointly with the growth of $m - \lambda_m$, and it separates the field space from the real statistical scale, which is not contained in $\widehat{V}_\lambda$ over $\mathbb{Q}_p$. The matrix class $(c, \widehat{V}_\lambda)$ is characterized by four conditions of classical form, the boundedness condition resting on an ultrametric gliding hump that requires nothing of the value group of F.

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