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On SAC- $\mathcal{P}$ Spaces and Nowhere Densely Reflexive Properties

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Abstract

In this paper, we investigate how the properties of the nowhere dense subspaces reflect those of the entire space. We call a space X an SAC- $\mathcal{P}$ space if the set of accumulation points of any infinite family of mutually disjoint open subsets of X is a non-empty set having property $\mathcal{P}$. We prove that for a regular space X , X is SAC-compact precisely when it is compact. In general, for a regular space X , X is SAC- $\mathcal{P}$ (here, $\mathcal{P}$ is a property which closed subsets inherit) exactly when X is feebly compact and each closed nowhere dense subspace of X has property $\mathcal{P}$. Similar to discrete reflexivity, we introduce the notion of nowhere dense reflexivity. A property $\mathcal{P}$ is *nowhere densely reflexive in* $\mathcal{A}$ if each space $X \in \mathcal{A}$ has property $\mathcal{P}$ precisely when each closed nowhere dense subspace of X has property $\mathcal{P}$. We prove that compactness, linear Lindelöfness and initial κ -compactness are all nowhere densely reflexive both in the class of feebly compact spaces and in that of dense-in-itself spaces. For topological groups, we prove that if G is a non-compact connected feathered topological group, then G is metrizable precisely when each closed nowhere dense subgroup of G is metrizable.

Keywords: SAC- $\mathcal{P}$ spaces; nowhere dense subspaces; nowhere densely reflexive; topological groups

MSC: 54D20; 54A25; 54H11

1. Introduction

All spaces are assumed to be regular unless otherwise stated.

It is known that properties of small subspaces often reflect the properties of the whole space. In [1], Tkachuk proved the well-known theorem that if the closure of each discrete subspace of X is compact, then X must be compact. Then, he and other authors continued to study discrete subspaces and defined a series of notions such as *discrete reflexivity*, *discretely generated spaces* [2–5] and *weakly discretely generated spaces* [3,4], as well as covering properties defined by discrete or star-covering conditions [6–8]. Another kind of “small” subspace is the *nowhere dense* subspace. It is well-known that for a dense-in-itself space X , if all of its nowhere dense subspaces are countable, then X is hereditarily Lindelöf.

In this paper, we investigate how the properties of the nowhere dense subspaces reflect those of the whole space. We mainly study such properties through their interactions with the families of mutually disjoint open subsets.

Let $\mathcal{U}$ be a family of pairwise disjoint open subsets of a space X ; we call it a *cellular family* of X . A point p is called an *accumulation point* (or a *cluster point*) of $\mathcal{U}$ if every neighbourhood of p meets infinitely many members of $\mathcal{U}$. We abbreviate the set of all



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cluster points of $\mathcal{U}$ as its “cluster set”. Obviously, the cluster set of any cellular family of X is nowhere dense and closed in X .

In [9], Bella and Spadaro introduced the class of *cellular- $\mathcal{P}$* spaces where, for any cellular family $\mathcal{U}$, there exists a subspace P with property $\mathcal{P}$ such that every member of $\mathcal{U}$ intersects P . The class of *cellular-Lindelöf* spaces, obtained by taking $\mathcal{P}$ to be the Lindelöf property, has since been studied extensively [10–13]; for a study of cellular- $\mathcal{P}$ spaces with respect to other Lindelöf-type properties, see [14]. In this article, we wonder whether every infinite cellular family ‘accumulates’ at a subspace with property $\mathcal{P}$.

We call a space X an *SAC- $\mathcal{P}$* space if the set of accumulation points of any infinite cellular family of X is a non-empty set with property $\mathcal{P}$. Obviously, SAC- $\mathcal{P}$ spaces are feebly compact. Moreover, it turns out that whether a space is SAC- $\mathcal{P}$ has a close relationship with whether every closed nowhere dense subspace of it has property $\mathcal{P}$.

We first prove a special case: if a space X is regular, then it is SAC-compact precisely when it is compact. Then, we give a general result: a space X is SAC- $\mathcal{P}$, where $\mathcal{P}$ is a property inherited by closed subsets, precisely when X is feebly compact and all of its closed nowhere dense subspaces have property $\mathcal{P}$.

Recall that a property $\mathcal{P}$ is termed *discretely reflexive* in $\mathcal{A}$ if each space $X \in \mathcal{A}$ has property $\mathcal{P}$ precisely when the closure of every discrete subset of X has property $\mathcal{P}$. We can similarly define a property $\mathcal{P}$ to be *nowhere densely reflexive* in $\mathcal{A}$ if any space $X \in \mathcal{A}$ has property $\mathcal{P}$ exactly when all of its closed nowhere dense subspaces have property $\mathcal{P}$. It is easy to see that, in the class of dense-in-itself spaces, discretely reflexive properties, such as compactness, linear Lindelöfness, initial κ -compactness, etc., are nowhere densely reflexive. Note that countable tightness, for instance, is not discretely reflexive even in σ -compact spaces [15]. Whether the Lindelöf property is discretely reflexive is still unknown. We prove a weaker result: If every closed nowhere dense subset of a space X is Lindelöf and $c(X) \leq \omega$, then X is Lindelöf. For topological groups, we prove that if G is a non-compact connected feathered topological group, then G is metrizable if and only if every closed nowhere dense subgroup of G is metrizable. We also give several counterexamples.

2. Basic Facts and Terminologies

We denote the *boundary* of a subspace $P \subset X$ by $\text{bd}(P)$, the *interior* of P by P° and the *closure* of P by $\bar{P}$. $\text{bd}(P) = \bar{P} \setminus P^\circ$.

A point $p \in X$ is called an ω -*point* if there is an infinite cellular family $\mathcal{U}$ of X with $p \in \bar{U}$ for every $U \in \mathcal{U}$.

A subset $D \subset X$ is called *discrete* if for each point $x \in D$ there exists an open neighbourhood U_x of x satisfying $U_x \cap (D \setminus \{x\}) = \emptyset$. A subset D is called *strongly discrete* if each $x \in D$ admits a neighbourhood U_x with $\{U_x : x \in D\}$ cellular. We shall use the following theorem of Tkachuk several times [1] (Theorem 4.13): a space X is compact if and only if the closure of every discrete subspace of X is compact.

A space X is called a *feebly compact* space if every locally finite family consisting of open subsets of X is finite or, equivalently, every infinite cellular family of X has an accumulation point. A Tychonoff space with this property is called *pseudocompact*.

A space X is *dense-in-itself* if it has no isolated points.

Then, we state two lemmas used below.

Lemma 1. *Supposing I is an infinite index set, and supposing $\mathcal{V} = \{V_i : i \in I\}$ is a cellular family of a space X , $\mathcal{U} = \{U_i : i \in I\}$ is a family of disjoint subsets of X such that $U_i \subset V_i$ for every $i \in I$. Let E and F be the sets of accumulation points of $\mathcal{U}$ and $\mathcal{V}$ respectively. Then, every accumulation point of $\mathcal{U}$ is an accumulation point of $\mathcal{V}$, that is, $E \subset F$.*

Proof. Let p be an accumulation point of $\mathcal{U}$ and let W be a neighbourhood of p . First, we observe that $p \notin V_i$ for every $i \in I$. Indeed, if p belonged to V_j for some $j \in I$, then the open set V_j would be a neighbourhood of p meeting only the single member U_j of $\mathcal{U}$, which is impossible. Now, W meets infinitely many members of $\mathcal{U}$, and each U_i is contained in the corresponding V_i , so W meets infinitely many members of $\mathcal{V}$ as well. Hence, p is an accumulation point of $\mathcal{V}$. $\square$

Lemma 2. Suppose I is an infinite index set, and suppose $\mathcal{U} = \{U_i : i \in I\}$ and $\mathcal{V} = \{V_i : i \in I\}$ are two families of disjoint subsets of a space X such that $U_i \subset V_i$ and U_i is dense in V_i for every $i \in I$. Let E and F be the sets of accumulation points of $\mathcal{U}$ and $\mathcal{V}$ respectively. Then, every accumulation point of $\mathcal{V}$ is an accumulation point of $\mathcal{U}$, that is, $F \subset E$.

Proof. Let $p \in F$ and let W be an arbitrary neighbourhood of p . Since p is an accumulation point of $\mathcal{V}$, the set $I_W = \{i \in I : W \cap V_i \neq \emptyset\}$ is infinite. Fix $i \in I_W$. Since U_i is dense in V_i , we have $V_i \subset \overline{U_i}$; as $W \cap V_i \neq \emptyset$, and $W \cap \overline{U_i} \neq \emptyset$, and since W is open, we conclude that $W \cap U_i \neq \emptyset$. Hence, W meets infinitely many members of $\mathcal{U}$. Therefore, p is an accumulation point of $\mathcal{U}$, that is, $p \in E$. Thus, $F \subset E$. $\square$

Lemma 3. Let $\mathcal{U}$ be a cellular family in a space X and let E be its cluster set. Then,

$$E \cap \bigcup \mathcal{U} = \emptyset \quad \text{and} \quad E \subset \overline{\bigcup \mathcal{U}}.$$

Proof. If $p \in U_j$ for some j , then U_j is a neighbourhood of p which, by the disjointness of $\mathcal{U}$, meets no member of $\mathcal{U}$ different from U_j ; hence, $p \notin E$. This proves the first assertion. For the second, assume that $p \notin \overline{\bigcup \mathcal{U}}$. Then, there is an open neighbourhood W of p with $W \cap \bigcup \mathcal{U} = \emptyset$, so W meets no member of $\mathcal{U}$, and again $p \notin E$. $\square$

The rest of our notations are standard and follow the book [16].

3. Basic Properties of SAC-Compact Spaces

In this section, we define the notion of SAC-compactness and give some basic properties.

Definition 1. A space X is called SAC-compact if the set of accumulation points of any infinite cellular family $\mathcal{U}$ in X is a non-empty compact subspace.

Remark 1. The set F of accumulation points of $\mathcal{U}$ does not generally equal the set $\overline{\bigcup \mathcal{U}} \setminus \bigcup \mathcal{U}$ (we denote the set by $\text{bd}(\mathcal{U})$), as $\text{bd}(\mathcal{U})$ may contain such a point p that $p \in \overline{U}$ for some $U \in \mathcal{U}$. But an accumulation point of $\mathcal{U}$ should intersect infinitely many elements of $\mathcal{U}$. Hence, if $p \in F$, either there exists an infinite subfamily $\mathcal{V} \subset \mathcal{U}$ such that $p \notin \overline{V}$ for every $V \in \mathcal{V}$ and every neighbourhood of p intersects infinitely many members of $\mathcal{V}$, or there exists an infinite subfamily $\mathcal{W} \subset \mathcal{U}$ such that $p \in \overline{W}$ for every $W \in \mathcal{W}$ (such a point is called an ω -point).

Lemma 4. The set of accumulation points of a cellular family is closed and nowhere dense.

Proof. Let $\mathcal{U}$ be a cellular family and F be its set of accumulation points. First, we prove that F is closed. Suppose $p \in \overline{F}$; then, every neighbourhood W of p intersects F , so there exists a $q \in W \cap F$. Since $q \in F$ is an accumulation point of $\mathcal{U}$, every neighbourhood of q intersects infinitely many elements of $\mathcal{U}$. Since W is also a neighbourhood of q , it intersects infinitely many elements of $\mathcal{U}$. It follows that p is an accumulation point of $\mathcal{U}$, $p \in F$, i.e., F is closed.

Next, we prove that F is nowhere dense. Suppose F is dense in an open set; since F is closed, it has a non-empty interior U . If $p \in U \subset F$, then p has a neighbourhood U which intersects no element of $\mathcal{U}$, a contradiction. $\square$

From the definition, the following propositions immediately follow.

Proposition 1. *A compact space is SAC-compact.*

Proof. Let X be compact. By Lemma 4, the cluster set F of every infinite cellular family $\mathcal{U}$ is a non-empty closed subset of X . As a closed subspace of a compact space, F is compact. Hence, X is SAC-compact. $\square$

Proposition 2. *An SAC-compact space is feebly compact.*

Proof. A space is feebly compact precisely when every infinite cellular family possesses an accumulation point. Let X be SAC-compact; then, the set of accumulation points of each infinite cellular family is non-empty, so X is feebly compact. $\square$

Example 1. *The converse of Proposition 2 fails, and the failure is not caused by any lack of regularity. Let $X = \omega_1$ carry the order topology. Then, X is a linearly ordered space, hence normal and therefore regular [16] (Problem 1.7.4(d)), and it is countably compact, hence feebly compact [16] (Example 3.10.16 and Theorem 3.10.3): every countable subset of ω_1 is contained in some $[0, \alpha_0]$ with $\alpha_0 < \omega_1$, and $[0, \alpha_0]$ is compact. However, X is not SAC-compact. Indeed, $S = \{\alpha + 1 : \alpha < \omega_1\}$ is a cellular family of X , because every successor ordinal is isolated, and its cluster set is the set L of non-zero limit ordinals. The set L is closed and unbounded in ω_1 , and it is not compact, since $\{L \cap [0, \beta) : \beta \in L\}$ is an open cover of L admitting no finite subcover.*

Next, we will prove that every regular SAC-compact space is compact.

Lemma 5. *Let N be a nowhere dense subspace of a regular space X , then there exists a cellular family $\mathcal{V}$ of X such that N is included in the set of accumulation points E of $\mathcal{V}$.*

Proof. Since N is nowhere dense in X and X is regular, every point $p \notin \overline{N}$ has an open neighbourhood V with $\overline{V} \cap \overline{N} = \emptyset$. By Zorn's lemma we may choose a cellular family $\mathcal{V}$ of non-empty open sets which is maximal with respect to the requirement that $\overline{V} \cap \overline{N} = \emptyset$ for every $V \in \mathcal{V}$. We claim that $\overline{\bigcup \mathcal{V}} = X$. Otherwise, some non-empty open set O misses $\bigcup \mathcal{V}$. Since N is nowhere dense, $\overline{N}$ has an empty interior, so O is not contained in $\overline{N}$ and we may pick $p \in O \setminus \overline{N}$. Intersecting the neighbourhood supplied by regularity with O , we obtain a non-empty open set $W \subset O$ with $\overline{W} \cap \overline{N} = \emptyset$. Then, W is disjoint from every member of $\mathcal{V}$, so $\mathcal{V} \cup \{W\}$ is a larger cellular family of the same kind, contradicting maximality. Let E denote the set of accumulation points of $\mathcal{V}$ and take $q \in \overline{N}$. Then, $q \in \overline{\bigcup \mathcal{V}} = X$ and $q \notin \overline{V}$ for every $V \in \mathcal{V}$. To see that $q \in E$, let W be any neighbourhood of q . Since $q \in \overline{\bigcup \mathcal{V}}$, the set W meets some $V_1 \in \mathcal{V}$; since $q \notin \overline{V_1}$, the set $W \setminus \overline{V_1}$ is still a neighbourhood of q and therefore meets some $V_2 \in \mathcal{V}$, and $V_2 \neq V_1$ because V_1 misses $W \setminus \overline{V_1}$. Repeating this, $W \setminus (\overline{V_1} \cup \dots \cup \overline{V_n})$ remains a neighbourhood of q and hence meets a new member of $\mathcal{V}$ at every step, so W meets infinitely many members of $\mathcal{V}$. Thus, $q \in E$ and $\overline{N} \subset E$. The proof is done. $\square$

Theorem 1. *Suppose X is a regular space, then X is SAC-compact if and only if X is compact.*

Proof. By Theorem 4.13 in [1], a space is compact precisely when the closure of every discrete subset of it is compact. We only need to prove that the closure of every discrete

subset of X is compact. Let A be a discrete subset of X . We write $A = B \cup C$, where B is the set of all isolated points of X which are contained in A and $C = A \setminus B$. Then, C is a nowhere dense subset of X . By Lemma 5, there exists a cellular family $\mathcal{V}$ of X such that $\overline{C}$ is included in the set of accumulation points E of $\mathcal{V}$. Since X is SAC-compact, E is compact and so is $\overline{C}$. Then, we will prove that $\overline{B}$ is compact. If B is finite, the proof is done. Suppose B is infinite. Then,

$$\mathcal{B} = \{\{b\} \mid b \in B\}$$

is an infinite cellular family, and the cluster set F of $\mathcal{B}$ is compact by the SAC-compactness of X . Moreover, if U is an open neighbourhood of F , then $B \setminus U$ is finite, for otherwise the family

$$\{\{b\} \mid b \in B \setminus U\}$$

would be an infinite cellular family without cluster points, which contradicts the SAC-compactness of X . Then, it is easy to prove that $\overline{B} = B \cup F$ is compact. We have proved that $\overline{A} = \overline{B} \cup \overline{C}$ is compact which means that X is compact.

On the other hand, suppose X is compact, then the cluster set of every infinite cellular family is a non-empty closed subset of X and hence is compact. Therefore, X is SAC-compact. $\square$

4. SAC- $\mathcal{P}$ Spaces and Equivalent Conditions

Let $\mathcal{P}$ be a property. We generalize the class of SAC-compact spaces to SAC- $\mathcal{P}$ spaces.

Definition 2. A space X is called SAC- $\mathcal{P}$ if the set of accumulation points of every infinite cellular family $\mathcal{U}$ of X is a non-empty set with property $\mathcal{P}$.

It turns out that whether a space is SAC- $\mathcal{P}$ is closely related to the properties of its nowhere dense subsets. We will prove the following theorem:

Theorem 2. Let $\mathcal{P}$ be a property inherited by closed subsets and let X be a regular space. Then, X is SAC- $\mathcal{P}$ if and only if X is feebly compact and all closed nowhere dense subspaces of X have property $\mathcal{P}$.

We split the proof of the theorem into two lemmas.

Lemma 6. If X is feebly compact and $\mathcal{P}$ holds for all closed nowhere dense subspaces of X , then X is SAC- $\mathcal{P}$.

Proof. Take an infinite cellular family $\mathcal{V}$ of X . Since X is feebly compact, its cluster set E is non-empty, and by Lemma 4, E is closed and nowhere dense. The hypothesis indicates that E has property $\mathcal{P}$, so X is SAC- $\mathcal{P}$. $\square$

If $\mathcal{P}$ is a property inherited by closed subsets, we prove the reverse direction.

Lemma 7. Suppose $\mathcal{P}$ is a property inherited by closed subsets. If a regular space X is SAC- $\mathcal{P}$, then X is feebly compact and $\mathcal{P}$ holds for every closed nowhere dense subspace of X .

Proof. That X is feebly compact is obvious by Proposition 2. By Lemma 5, every closed nowhere dense subspace N of X is contained in the cluster set E of some cellular family. Since X is SAC- $\mathcal{P}$, E has property $\mathcal{P}$. Since $\mathcal{P}$ is a property inherited by closed subsets, N has property $\mathcal{P}$ as a closed subset of E . $\square$

By Lemmas 6 and 7, we have given the complete proof of Theorem 2.

Remark 2. Theorem 2 applies to every property $\mathcal{P}$ that is inherited by closed subsets. Familiar properties with this feature include compactness, Lindelöfness, σ -compactness, countable compactness, metrizability, normality, perfect normality, zero-dimensionality, and the property “ $|X| \leq \kappa$ ” for an infinite cardinal κ .

Then, we present a property of SAC- $\mathcal{P}$ spaces.

Theorem 3. Suppose $\mathcal{P}$ is a property inherited by closed subspaces. Then, every regular closed subspace of an SAC- $\mathcal{P}$ space is SAC- $\mathcal{P}$.

Proof. Let Y be an SAC- $\mathcal{P}$ space and let $X \subset Y$ satisfy $X = \text{cl}_Y(U)$ for some open subset U of Y . Take an infinite cellular family $\mathcal{V}$ of X and put $\mathcal{W} = \{V \cap U : V \in \mathcal{V}\}$. Both families are cellular in X , and by Lemmas 1 and 2 their cluster sets E and F in X coincide. Since X is closed in Y , the set F is also the cluster set of $\mathcal{W}$ computed in Y . As Y is SAC- $\mathcal{P}$, the set F has property $\mathcal{P}$, and hence so does E . Therefore, X is SAC- $\mathcal{P}$. $\square$

Remark 3. Lemmas 6 and 7 together yield the following: if X is a closed subspace of a regular SAC- $\mathcal{P}$ space Y , then X is SAC- $\mathcal{P}$ if and only if X is feebly compact (by Lemma 7, every closed nowhere dense subspace of Y has $\mathcal{P}$, and a closed nowhere dense subspace of the closed subspace X is also closed and nowhere dense in Y). The proof of Theorem 3 uses regular closedness only in order to guarantee that the open set U is dense in $X = \text{cl}_Y(U)$, and the word “regular” cannot be dropped in general: as the next example shows, closed subspaces of feebly compact spaces need not be feebly compact.

Example 2. Let $\mathcal{A}$ be an infinite maximal almost disjoint family of infinite subsets of ω , which exists by Zorn’s lemma. For each $A \in \mathcal{A}$, let $\hat{A}$ be a new point and put $\Psi = \omega \cup \{\hat{A} : A \in \mathcal{A}\}$; declare each $n \in \omega$ isolated and take the sets $\{\hat{A}\} \cup (A \setminus F)$, with F a finite subset of A , as basic neighbourhoods of $\hat{A}$. This is the classical Mrówka space, the space of [16] (Exercise 3.6.I(a)).

The space Ψ is SAC- $\mathcal{P}$ for $\mathcal{P}$ = “zero-dimensional”. Indeed, Ψ is zero-dimensional, since its standard base consists of clopen sets, and therefore every subspace of Ψ is zero-dimensional.

It remains to check that Ψ is feebly compact. By [16] (Exercise 3.10.G), the space Ψ is pseudocompact, and a Tychonoff pseudocompact space is feebly compact by [16] (Theorem 3.10.22).

The subspace $\mathcal{A}$ is closed in Ψ , since $\Psi \setminus \mathcal{A} = \omega$ is a union of isolated points, but it is not regular closed: no basic neighbourhood $\{\hat{A}\} \cup (A \setminus F)$ is contained in $\mathcal{A}$, so $\mathcal{A}$ has an empty interior and $\text{int } \overline{\mathcal{A}} = \emptyset \neq \mathcal{A}$. Finally, $\mathcal{A}$ carries the discrete topology, because $\{\hat{A}\} = \mathcal{A} \cap (\{\hat{A}\} \cup A)$, and it is infinite; an infinite discrete space is not feebly compact, since its singletons form an infinite locally finite family of non-empty open sets. As every SAC- $\mathcal{Q}$ space is feebly compact, the space $\mathcal{A}$ is therefore not SAC- $\mathcal{Q}$ for any property $\mathcal{Q}$.

Thus, a closed subspace of an SAC- $\mathcal{P}$ space need not be SAC- $\mathcal{P}$ itself, and “regular closed” cannot be weakened to “closed” in Theorem 3.

5. Nowhere Densely Reflexive Properties

As noted in Section 1, a topological property $\mathcal{P}$ is termed *discretely reflexive* in a class $\mathcal{A}$ if each space $X \in \mathcal{A}$ has $\mathcal{P}$ exactly when the closure of every discrete subspace of X has $\mathcal{P}$. Compactness and linear Lindelöfness are two familiar examples.

For a dense-in-itself space, the closure of a discrete subset is always nowhere dense, so we easily establish the following relationships.

Theorem 4. Suppose X is feebly compact and $\mathcal{P}$ is inherited by closed subsets. Then, the following statements satisfy (1) $\implies$ (2) $\implies$ (3):

- (1) X has property $\mathcal{P}$.

- (2) X is SAC- $\mathcal{P}$.
- (3) All closed nowhere dense subspaces of X have property $\mathcal{P}$.
- (4) The closure of every discrete subset of X has property $\mathcal{P}$.

Moreover, if X has no isolated points and $\mathcal{P}$ is discretely reflexive, then all four statements are equivalent.

Proof. Since X is feebly compact and $\mathcal{P}$ is inherited by closed subsets, $(1) \implies (2) \implies (3)$ is obvious. If X has no isolated points, then every discrete set is nowhere dense and $(3) \implies (4)$ holds. By [1] (Theorem 4.13), if $\mathcal{P}$ is discretely reflexive, then $(4) \iff (1)$ holds and all four statements are equivalent. The proof is done. $\square$

Corollary 1. If $\mathcal{P}$ is discretely reflexive and inherited by closed subsets, then a dense-in-itself feebly compact space X has property $\mathcal{P}$ if and only if X is SAC- $\mathcal{P}$.

Similar to the notion of discrete reflexivity, we can define the notion of nowhere dense reflexivity.

Definition 3. A property $\mathcal{P}$ is termed nowhere densely reflexive in a class $\mathcal{A}$ if each space $X \in \mathcal{A}$ has $\mathcal{P}$ exactly when every closed nowhere dense subspace (equivalently, the closure of every nowhere dense subset) of X has $\mathcal{P}$.

We then list some discretely reflexive properties and see if they are nowhere densely reflexive.

Recall that a space is *perfectly normal* if each closed subset of it is a G_δ -set.

From [17], we know that if X is a compact space, then it has property $\mathcal{P}$, where $\mathcal{P} \in \{\text{perfect normality, tightness} \leq \kappa, \text{character} \leq \kappa, \text{countability}\}$, if and only if $\overline{D}$ has $\mathcal{P}$ for any discrete set $D \subset X$. Then, we have the following similar corollary.

Corollary 2. Suppose $\mathcal{P} \in \{\text{perfect normality, tightness} \leq \kappa, \text{character} \leq \kappa, \text{countability}\}$ and let X be a compact dense-in-itself space. Then, X has $\mathcal{P}$ if and only if every closed nowhere dense subset $N \subset X$ has $\mathcal{P}$.

Proof. Let D be a discrete subset of X . We first claim that $\overline{D}$ is nowhere dense. Suppose, on the contrary, that $\overline{D}$ contains a non-empty open set U . Choose $d \in D \cap U$ and a neighbourhood W of d with $W \cap D = \{d\}$. Since X is dense-in-itself, the non-empty open set $W \cap U$ cannot be equal to $\{d\}$; hence, $(W \cap U) \setminus \{d\}$ is a non-empty open subset of $U \subset \overline{D}$ and therefore must meet D , contradicting $W \cap D = \{d\}$. Thus, $\overline{D}$ is nowhere dense and, being closed, is a closed nowhere dense subspace of X ; by the hypothesis it has $\mathcal{P}$. Since X is compact, ref. [17] indicates that X has $\mathcal{P}$. Conversely, if X has $\mathcal{P}$ then so does every closed nowhere dense subspace of X , since each of the four properties in question is inherited by closed subspaces. $\square$

Recall that a space X is *linearly Lindelöf* if every open cover of X , linearly ordered under inclusion, has a countable subcover. A space is *initially κ -compact* precisely when every open cover of size at most κ admits a finite subcover. We know from [17] (Proposition 2.4 and Theorem 2.6) that linear Lindelöfness and initial κ -compactness are discretely reflexive. We easily know that linear Lindelöfness and initial κ -compactness are nowhere densely reflexive in the class of dense-in-itself spaces. We can also prove that

Theorem 5. If X is SAC-linearly Lindelöf (resp. initially κ -compact), then X is linearly Lindelöf (resp. initially κ -compact).

Proof. By [17] (Proposition 2.4), linear Lindelöfness is discretely reflexive; it therefore suffices to show that the closure of every discrete subset of X is linearly Lindelöf. Let $\overline{D}$ be the closure of a discrete set D , so that $\overline{D} = S \cup N$, where S consists of the isolated points of X lying in D and $N = (\overline{D} \setminus S)$ is closed and nowhere dense. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an increasing open cover of $\overline{D}$. By hypothesis, N is linearly Lindelöf, so it is covered by countably many of the sets U_j ($j \in J \subset I$); put $U = \bigcup_{j \in J} U_j$. Arguing as in the proof of Theorem 1, only finitely many isolated points lie outside U . Hence, countably many members of $\mathcal{U}$ cover $\overline{D}$.

The proof for initially κ -compact spaces is similar. $\square$

By Theorem 2, the above theorem is equivalent to the following one.

Theorem 6. *If X is feebly compact and all closed nowhere dense subspaces of X are linearly Lindelöf (initially κ -compact), then X is linearly Lindelöf (initially κ -compact).*

Whether Lindelöfness is discretely reflexive is still an open question. However, we can prove the following partial results:

Theorem 7. *If a space X satisfies $c(X) \leq \omega$ and every closed nowhere dense subset of X is Lindelöf, then X is Lindelöf.*

Proof. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X , and well order $\mathcal{U} = \{U_i\}_{i \in I}$ so that its index set is an ordinal. Each $V_i = U_i \setminus \overline{\bigcup_{j < i} U_j}$ is open, being the difference of the open set U_i and the closed set $\overline{\bigcup_{j < i} U_j}$, and we prove that $\mathcal{V} = \{V_i\}_{i \in I}$ is a cellular family: if $j < i$, then V_i misses $\overline{\bigcup_{k < i} U_k}$ and hence misses V_j . Moreover, $\bigcup_{i \in I} V_i$ is dense in X . Indeed, given a non-empty open set O , let i be the least index with $O \cap U_i \neq \emptyset$ and pick $y \in O \cap U_i$; if $y \notin \overline{\bigcup_{j < i} U_j}$, then $y \in V_i$, whereas if $y \in \overline{\bigcup_{j < i} U_j}$, then the neighbourhood O of y meets $\bigcup_{j < i} U_j$, contradicting the minimality of i . Thus, $O \cap V_i \neq \emptyset$, and since O was an arbitrary non-empty open set, $\bigcup_{i \in I} V_i$ is dense in X . Discarding the empty members and using $c(X) \leq \omega$, we get $|\mathcal{V}| \leq \omega$. Let $V = \bigcup \mathcal{V}$ and $N = X \setminus V$. Then, N is a closed nowhere dense subspace of X and hence Lindelöf by hypothesis. Therefore, N can be covered by countably many members of $\mathcal{U}$. Since V can be covered by countably many members of $\mathcal{U}$ as well, X is Lindelöf. $\square$

Example 3. *The assumption $c(X) \leq \omega$ in Theorem 7 cannot be omitted. Let $X = \omega_1$ (the first uncountable ordinal) endowed with the discrete topology. Then, X is regular (in fact, discrete spaces are completely regular).*

- $c(X) = \omega_1 > \omega$, because the family $\{\{\alpha\} : \alpha < \omega_1\}$ is an uncountable cellular family.
- Every nowhere dense subset of a discrete space is empty, so every closed nowhere dense subset is trivially Lindelöf.
- X is not Lindelöf, since the open cover $\{\{\alpha\} : \alpha < \omega_1\}$ has no countable subcover.

Thus, X satisfies all conditions of Theorem 7 except $c(X) \leq \omega$, but fails to be Lindelöf.

Example 4. *The hypothesis on closed nowhere dense subspaces in Theorem 7 is essential as well. Let Ψ be the Mrówka space of Example 2 and let $\mathcal{A}$ be its closed discrete subspace. Then, $c(\Psi) = \omega$, because ω is dense in Ψ and a cellular family in a separable space is countable. However, $\mathcal{A}$ is a closed nowhere dense subspace of Ψ which is not Lindelöf, since it is infinite and discrete; consequently, Ψ itself is not Lindelöf, because a closed subspace of a Lindelöf space is Lindelöf [16] (Theorem 3.8.4). Thus, Ψ satisfies all the conditions of Theorem 7 except the one on closed nowhere dense subspaces, and yet fails to be Lindelöf. This is the counterpart of Example 3.*

Theorem 8. *If a space X is weakly Lindelöf and every closed nowhere dense subset of X is Lindelöf, then X is Lindelöf.*

Proof. Let $\mathcal{U}$ be an open cover of X . Since X is weakly Lindelöf, we can find a countable subfamily $\mathcal{V} \subset \mathcal{U}$ whose union $V = \bigcup \mathcal{V}$ is dense in X . Then, $N = X \setminus V$ is closed and nowhere dense in X , and the hypothesis makes it Lindelöf. So countably many members of $\mathcal{U}$ cover N . As $X = V \cup N$ and V is already covered by the countable subfamily $\mathcal{V}$, the space X is Lindelöf. $\square$

Weak Lindelöfness is one of several common weakenings of the Lindelöf property; for others, and for their relation to H-closedness, see [18,19].

In [17] (Proposition 2.1), the authors proved that the hereditary Lindelöf number is discretely reflexive.

Corollary 3. *If a space X is dense-in-itself and every closed nowhere dense subspace of X is hereditarily Lindelöf, then X is hereditarily Lindelöf.*

Proof. Since the hereditary Lindelöf number is discretely reflexive, it suffices to show that the closure of every discrete subspace of X is hereditarily Lindelöf. Let D be a discrete subspace of X . As X is dense-in-itself, D is nowhere dense, so $\overline{D}$ is a closed nowhere dense subspace of X and hence is hereditarily Lindelöf by hypothesis. Therefore, X is hereditarily Lindelöf. $\square$

Then, we see some results in topological groups. First, we give a Lemma.

Lemma 8. *For a topological group G with more than one point and an open neighbourhood U of the neutral element e , there exists a closed G_δ subgroup H of G with $H \subset U$ and $H \neq G$. Moreover, if G is connected, then H is nowhere dense.*

Proof. Fix a proper open neighbourhood $U_0 \subset U$ of e and choose a symmetric open neighbourhood W_1 of e with $W_1^2 \subset U_0$. Having chosen W_i , choose a symmetric open neighbourhood W_{i+1} of e with $W_{i+1}^2 \subset W_i$, and put $N = \bigcap_{i \in \omega} W_i$. Then, $N \subset W_1 \subset U_0 \subset U$, and $N \neq G$ because U_0 is a proper subset of G . The set N is a subgroup: if $x, y \in N$, then $x, y \in W_{i+1}$ for every i , from where $xy \in W_{i+1}^2 \subset W_i$ for every i and so $xy \in N$, while $x^{-1} \in W_i$ for every i by symmetry, so $x^{-1} \in N$. It is also closed: if $x \notin N$, say $x \notin W_n$, then xW_{n+1} is an open neighbourhood of x , and $y \in xW_{n+1} \cap N$ would give $x = yw$ with $y \in N \subset W_{n+1}$ and $w \in W_{n+1}$, so that $x \in W_{n+1}^2 \subset W_n$, a contradiction; hence, $xW_{n+1} \cap N = \emptyset$. Finally, N is a G_δ -set, being a countable intersection of open sets, and it is the required H .

For the second assertion, suppose that N contains a neighbourhood of e . Then, N is an open subgroup of G , and an open subgroup is also closed, being the complement of the union of its remaining cosets. As $N \neq G$, this would exhibit a non-empty proper clopen subset of G and contradict the connectedness of G . Hence, N has an empty interior, and being closed it is nowhere dense. $\square$

Recall that a topological group G is *feathered* if it has a non-empty compact subset K of countable character in G . B. A. Pasynkov proved that a topological group G is feathered precisely when it contains a compact subgroup H for which the left quotient space G/H is metrizable (see [20] (Theorem 4.3.20)).

Theorem 9. *If G is a non-compact connected feathered topological group, then G is metrizable if and only if every closed nowhere dense subgroup of G is metrizable.*

Proof. Since G is feathered, Pasynkov's theorem provides a compact subgroup H of G for which the left quotient space G/H is metrizable. As G is not compact, $H \neq G$. A subgroup with a non-empty interior is open, and an open subgroup is also closed; since H is a proper subgroup of G , a non-empty interior would give a proper non-empty clopen subset of G and thus contradict the connectedness of G . Hence, H has an empty interior and, being closed, is nowhere dense in G . By hypothesis, H is metrizable, and then [20] (Corollary 3.3.20) indicates that G is metrizable. For the converse, every subspace of a metrizable space is metrizable, so if G is metrizable then so is every closed nowhere dense subgroup of G . $\square$

Remark 4. The hypothesis that G is not compact is used only to ensure that the subgroup supplied by Pasynkov's theorem can be chosen differently from G ; if that subgroup equals G , then G is compact and the subgroup is not nowhere dense, so the argument above does not apply. The compact case is left open.

The class of feathered groups contains, among others, all locally compact groups, all Čech-complete groups and all discrete groups.

6. Conclusions and Future Works

We have introduced the class of $SAC-\mathcal{P}$ spaces, in which the cluster set of every infinite cellular family is non-empty and required to carry the property $\mathcal{P}$. For a regular space, SAC -compactness turned out to coincide with compactness, and for a property $\mathcal{P}$ inherited by closed subsets, the class $SAC-\mathcal{P}$ proved to be exactly the class of feebly compact spaces all of whose closed nowhere dense subspaces have $\mathcal{P}$. This reformulation was then applied to show that compactness, linear Lindelöfness and initial κ -compactness are nowhere densely reflexive in the class of feebly compact spaces, and that a non-compact connected feathered topological group is metrizable precisely when all of its closed nowhere dense subgroups are. Several questions suggested by these results are left open.

1. Is Lindelöfness discretely reflexive? Theorem 7 gives only a partial answer.
2. Is the class of $SAC-\mathcal{P}$ spaces finitely productive? Equivalently, if X and Y are SAC -compact regular spaces, must $X \times Y$ be SAC -compact?
3. Does there exist, in ZFC, a non-separable feebly compact space all of whose closed nowhere dense subspaces are countable?
4. Can regularity be omitted from Theorems 1 and 2?

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