

Article

Global Solution, Decay, and Blow-Up of a Viscoelastic Kirchhoff System with Logarithmic Nonlinearity and Acoustic Fractional Delay Boundary Conditions

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Abstract

We examine a nonlinear viscoelastic Kirchhoff model endowed with a logarithmic source. Assuming suitable hypotheses and under the inclusion of nonlinear distributed delay feedback terms, in addition to the acoustic and fractional boundary feedback terms, we obtain global existence, general decay estimates, and finite-time blow-up for negative initial energy, all for a wide range of relaxation kernels. The Kirchhoff term $\mathcal{K}(\|\nabla \mathcal{W}\|_2^2)$ significantly changes the structure of the Nehari manifold and requires a new energy functional based on the primitive $\hat{\mathcal{K}}$. Our analysis identifies the precise blow-up mechanism and establishes sharp dimension-dependent restrictions on the exponent p . Our outcomes broaden and enhance recent contributions on classical wave equations to the physically meaningful Kirchhoff framework.

Keywords: Kirchhoff equation; general decay; logarithmic source; acoustic boundary; partial differential equations; fractional derivative; distributed delay

MSC: 35B40; 35L70; 74D99; 93D20

1. Introduction

The viscoelastic Kirchhoff models have been of particular interest in mathematical physics and engineering due to their ability to describe correctly the vibrations of elastic strings and beams and to include the effect of tension changes caused by their stretching. The nonlocal coefficient $\mathcal{K}(\|\nabla \mathcal{W}\|_2^2)$ was originally introduced by Kirchhoff in 1876 to describe the transverse oscillations of a stretched string that is subject to a non-uniform tension. The inclusion of memory phenomena leads to a viscoelastic Kirchhoff system that can effectively describe materials whose mechanical behaviour is affected by previous deformations.

The presence of a logarithmic nonlinearity significantly complicates the analysis, since solutions may blow up in finite time depending on the parameter regime, which makes the investigation of stability and long-term dynamics considerably more delicate. In this work, we establish the global-in-time existence and asymptotic decay of solutions for a viscoelastic Kirchhoff equation endowed with a logarithmic source, subject to acoustic boundary dissipation, fractional derivatives, and nonlinear time delays. Elucidating how these interacting mechanisms collectively shape solution behavior is of paramount importance



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for the design of effective damping strategies and for predicting the long-term response of viscoelastic materials.

In [1], the authors carried out an in-depth analysis of finite-time blow-up for viscoelastic wave equations endowed with logarithmic nonlinearity, carefully accounting for the influence of time delays and acoustic boundary conditions while extracting sharp blow-up thresholds. Nevertheless, their framework is strictly semi-linear ($\mathcal{K} \equiv 1$), so the quasilinear tension effects that characterize Kirchhoff systems are absent. Building upon this framework, in [2], the same collaborators turned their attention to global existence and asymptotic decay properties of nonlinear wave equations equipped with acoustic and fractional boundary conditions, logarithmic source terms, and delay feedback. Yet, the Kirchhoff nonlocal coefficient was again omitted, which means the energy functional is quadratic and the potential-well geometry is substantially simpler.

Other related works including [3–17] have treated fixed, time-varying, or distributed delays, as well as acoustic and fractional dissipation, but always in the semi-linear setting. The recent contribution [2] analysed a special case of (1) with $\mathcal{K} \equiv 1$, establishing general decay estimates for a broad family of relaxation kernels; however, the authors explicitly noted that the Kirchhoff case remained open. Consequently, a rigorous treatment of the combined effects of nonlocal Kirchhoff tension, logarithmic focusing, acoustic boundary dynamics, fractional dissipation, and distributed delay has been missing from the literature.

Compared with the aforementioned semi-linear contributions, the inclusion of the nonlocal Kirchhoff modulus $\mathcal{K}(\|\nabla \mathcal{W}\|_2^2)$ fundamentally alters the principal operator from linear to quasilinear. This necessitates a genuinely non-quadratic energy functional built upon the primitive $\hat{\mathcal{K}}$, a modified Nehari geometry, and new multiplier estimates that exploit the lower bound $\mathcal{K}(s) \geq 1$. Moreover, the simultaneous presence of acoustic boundary conditions, fractional Caputo dissipation, and distributed delay feedback on the same active boundary Γ_0 creates a fully coupled dissipative structure that has not been previously analysed in the Kirchhoff framework. Our results therefore constitute the first complete dynamic theory global existence, general decay, and finite-time blow-up for this physically motivated configuration.

This manuscript goes beyond the works of similar studies by including the Kirchhoff nonlocal coefficient $\mathcal{K}(\|\nabla \mathcal{W}\|_2^2)$. This change transforms the linear wave operator into a quasilinear nonlocal operator, which requires a new energy functional based on the primitive $\hat{\mathcal{K}}$ of $\mathcal{K}$, a modified Nehari functional, and careful treatment of the nonlocal term in the estimates of the multiplier. As far as we are aware, no previous study has addressed the asymptotic behaviour of a viscoelastic Kirchhoff-type equation that simultaneously includes a logarithmic source, acoustic boundary conditions, a fractional derivative, and distributed delay.

The specific viscoelastic Kirchhoff system under scrutiny reads

$$\begin{cases} \mathcal{W}_{tt} - \mathcal{K}(\|\nabla \mathcal{W}\|_2^2) \Delta \mathcal{W}(t) + \int_0^t \mathcal{H}(t-\theta) \Delta \mathcal{W}(\theta) d\theta = |\mathcal{W}|^{p-2} \mathcal{W} \ln |\mathcal{W}|, & \text{in } \Omega \times \mathbb{R}_+, \\ \mathcal{K}(\|\nabla \mathcal{W}\|_2^2) \frac{\partial \mathcal{W}}{\partial n} - \int_0^t \mathcal{H}(t-\theta) \frac{\partial \mathcal{W}}{\partial n}(\theta) d\theta + \mathcal{F}(\mathcal{W}_t) + c \partial_t^{\gamma, \delta} \mathcal{W} = \mathcal{S}_t, & x \in \Gamma_0, t > 0, \\ \mathcal{W}_t + \alpha(x) \mathcal{S} + \beta(x) \mathcal{S}_t = 0, & x \in \Gamma_0, t > 0, \\ \mathcal{W}(x, t) = 0, & \text{on } \Gamma_1 \times \mathbb{R}_+, \\ \mathcal{W}(x, 0) = \mathcal{W}_0(x), \quad \mathcal{W}_t(x, 0) = \mathcal{W}_1(x), & x \in \Omega, \\ \mathcal{W}_t(x, -t) = w_0(x, t), & x \in \Gamma_0, t \in (0, \delta_2), \\ \mathcal{S}(x, 0) = \mathcal{S}_0(x), & x \in \Gamma_0, \end{cases} \quad (1)$$

where

$$\mathcal{F}(\mathcal{W}_t) := a_1 \mathcal{F}_1(\mathcal{W}_t) + \int_{\delta_1}^{\delta_2} a_2(z) \mathcal{F}_2(\mathcal{W}_t(t-z)) dz. \quad (2)$$

Throughout the manuscript, Ω stands for a bounded open subset of $\mathbb{R}^d$ with $d \geq 1$, whose boundary $\partial\Omega$ decomposes into two non-overlapping closed pieces Γ_0 and Γ_1 so that $\partial\Omega = \Gamma_0 \cup \Gamma_1$. By n , we mean the unit normal vector field pointing outward from Ω . The parameters a_1 and c are strictly positive reals, the exponent p is taken larger than 2, and the delay bounds satisfy $0 \leq \delta_1 < \delta_2$. The map $a_2 : [\delta_1, \delta_2] \rightarrow \mathbb{R}$ governs the distributed time lag, $\mathcal{H}$ plays the role of a positive relaxation kernel, the pair $\mathcal{F}_1, \mathcal{F}_2$ are prescribed nonlinearities, and $\mathcal{K} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ signifies the nonlocal Kirchhoff modulus.

The generalized Caputo fractional derivative is expressed, for $0 < \gamma < 1$, as [18,19]

$$\partial_t^{\gamma,\delta} \mathcal{W}(t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-z)^{-\gamma} e^{-\delta(t-z)} \mathcal{W}_z(z) dz, \quad \delta \geq 0, \quad (3)$$

where

$$\partial_t^{\gamma,\delta} \mathcal{W}(t) = I^{1-\gamma,\delta} \mathcal{W}_t(t), \quad (4)$$

and

$$I^{\gamma,\delta} \mathcal{W}(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-z)^{\gamma-1} e^{-\delta(t-z)} \mathcal{W}(z) dz, \quad \delta \geq 0. \quad (5)$$

Here, Γ and $I^{\gamma,\delta}$ denote the Euler gamma function and the exponential fractional integro-differential operator, respectively.

The model (1) describes a viscoelastic string (or membrane) whose tension at each instant is determined by the global deformation $\|\nabla \mathcal{W}\|_2^2$ through the Kirchhoff law $\mathcal{K}$. The boundary Γ_0 is an acoustically active surface (e.g., a piezoelectric actuator) coupled to a fluid pressure potential $\mathcal{S}$. The fractional derivative $\partial_t^{\gamma,\delta}$ models frequency-dependent viscous losses at the boundary, while the distributed delay term captures temporally nonlocal feedback that arises in digital or networked control loops. The logarithmic source $|\mathcal{W}|^{p-2} \mathcal{W} \ln |\mathcal{W}|$ represents a weak superlinear focusing effect that can destabilise the system when the initial energy is negative.

The nonlinearity $|\mathcal{W}|^{p-2} \mathcal{W} \ln |\mathcal{W}|$ has been extensively studied: several authors have studied global existence, general decay, and blow-up using the logarithmic Sobolev inequality and the potential well method, among other techniques. Choucha et al. [20,21] and Gongwei et al. [22] have carried out representative studies. In many physical situations, the reaction to an applied force is not immediate, leading to a natural occurrence of time delay. This makes the transport of material and information critically dependent on the temporal variable. In recent years, the study of delay phenomena has been a flourishing research area, and much research have focused on this kind of damping due to its strong effect on the existence of solutions in a system and the stability of the system.

When the logarithmic forcing is dropped, the literature on acoustic and fractional boundary conditions specialized to the linear damping case where $(\mathcal{F}_1(Y) = \mathcal{F}_2(Y) = Y)$ branches into three canonical delay structures: fixed pointwise retardation $(\mathcal{W}_t(x, t - \delta))$ [3–5], temporally shifting lag $(\mathcal{W}_t(x, t - \delta(t)))$ [6–8,23], and spatially averaged distributed memory $(\int_{\delta_1}^{\delta_2} a(z) \mathcal{W}_t(t - z) dz)$ [9–15]. Beyond these linear prototypes, a substantial body of work has also treated fully nonlinear damping pairs $(\mathcal{F}_1, \mathcal{F}_2)$, with retardation entering either through the interior equation or via the boundary dissipation law. A representative treatment is found in [16], where the authors examined

$$\mathcal{W}_{tt} - \Delta \mathcal{W} + \int_0^t h(t-z) \Delta \mathcal{W}(z) dz + \varsigma_1 g_1(\mathcal{W}_t) + \varsigma_2 g_2(\mathcal{W}_t(t - \delta)) = 0,$$

proving a general stability result and the existence of global solutions. In [17], the system

$$\begin{cases} \mathcal{W}_{tt} - \Delta \mathcal{W} = 0, \\ \mathcal{W}(x, t) = 0, \quad \text{on } \Gamma_0 \times \mathbb{R}_+, \\ \frac{\partial \mathcal{W}}{\partial n} + \varsigma_1 \mathcal{W}_t + \varsigma_2 \mathcal{W}_t(t - \delta) = 0, \quad \text{on } \Gamma_1 \times \mathbb{R}_+, \end{cases}$$

was studied, and exponential stability of the general energy was established under the condition $\varsigma_2 \leq \varsigma_1$. The problem considered in [8] yielded results on asymptotic behavior and global existence. The viscoelastic literature is extensive, we refer the reader to [24–30] and the references therein. Over the past few decades, Caputo fractional derivatives have witnessed substantial development, supported by numerous definitions and results, see, for example, [31–33].

Fractional dissipation acting through the boundary was investigated by Akil and Wehbe [34], who stabilized the multidimensional wave equation by means of a fractional dissipation law localized on a subset of the boundary under sharp geometric conditions; in contrast, the Caputo fractional derivative in (1) acts on the whole active boundary Γ_0 and is intrinsically coupled with the acoustic dynamics and the distributed delay. In the same spirit, Akil, Badawi, Nicaise, and Wehbe [35] established stability results for the Kirchhoff plate equation with delay terms entering the dynamical boundary controls; compared with that model, our delay feedback is a distributed memory of the boundary velocity appearing inside the damping law, and the equation is a Kirchhoff string/membrane with a logarithmic interior source rather than a plate equation.

Acoustic boundary conditions trace their origins to the seminal work of Ingard and Morse [36], and have subsequently been incorporated into a broad spectrum of problems while undergoing continual refinement by various research groups. Readers seeking additional background are directed to [37], whereas [38] treats questions centered on boundary dissipation. In a related recent development, stripping away the acoustic boundary entirely, the authors of [22] analysed a special case of (1) with $\mathcal{K} \equiv 1$, establishing general decay estimates for a broad family of relaxation kernels.

Inspired by the above discussion, we consider problem (1) having the logarithmic source term and a nonlocal Kirchhoff coefficient. The nonlinear distributed delay in the boundary feedback is considered in conjunction with the acoustic and fractional boundary conditions. This combination makes our study unique from previous work. After making appropriate assumptions on the different functions used, we make inferences about the asymptotic behavior of solutions.

The rest of this paper is organized as follows. Assumptions, notation and preliminary lemmas are given in Section 2. The result on global existence of weak solutions is given in Section 3. A brief overview of the general decay of solutions is given in Section 4 and blow-up in Section 5, and concluding remarks are given in Section 6.

2. Preliminaries

This section collects the standing assumptions on $a_2, p, \beta, \mathcal{H}, \mathcal{F}_1, \mathcal{F}_2$, and $\mathcal{K}$.

Assumption 1. $\mathcal{H} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a non-increasing $\mathcal{L}^1$ function satisfying

$$\mathcal{H}(0) > 0, \quad h_0 = \int_0^\infty \mathcal{H}(\vartheta) d\vartheta < \infty, \quad 1 - h_0 = \ell > 0. \quad (6)$$

Assumption 2. There exists a non-increasing function $\rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ of class $\mathcal{C}^1$ such that

$$\rho(t)\mathcal{H}(t) + \mathcal{H}'(t) \leq 0, \quad \forall t \geq 0. \quad (7)$$

Assumption 3. $\mathcal{F}_1 : \mathbb{R} \rightarrow \mathbb{R}$ is a non-decreasing $\mathcal{C}^1$ function, and $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a convex, increasing function belonging to $\mathcal{C}^1(\mathbb{R}_+) \cap \mathcal{C}^1([0, \infty[)$, with

$$\Phi(0) = 0,$$

and either Φ is linear on $[0, \eta]$, or $\Phi'(0) = 0$ and $\Phi''(t) > 0$ on $]0, \eta]$. Moreover,

$$\begin{aligned} c_0|z| &\leq \mathcal{F}_1(z) \leq c_1|z| & \text{if } |z| \geq \eta, \\ z^2 + \mathcal{F}_1^2(z) &\leq \Phi^{-1}(z\mathcal{F}_1(z)) & \text{if } |z| \leq \eta, \end{aligned} \quad (8)$$

where Φ^{-1} denotes the inverse of Φ and η, c_0, c_1 are positive constants.

Assumption 4. $\mathcal{F}_2 : \mathbb{R} \rightarrow \mathbb{R}$ is an odd non-decreasing C^1 function for which there exist $\tau_1 > \frac{1}{2}$ and $c_2, \tau_2 > 0$ such that

$$|\mathcal{F}_2'(z)| \leq c_2, \quad (9)$$

$$\tau_1 z \mathcal{F}_2(z) \leq \mathcal{G}(z) \leq \tau_2 z \mathcal{F}_1(z), \quad (10)$$

where $\mathcal{G}(z) = \int_0^z \mathcal{F}_2(\omega) d\omega$.

Assumption 5. $a_2 : [\delta_1, \delta_2] \rightarrow \mathbb{R}$ is a bounded function satisfying

$$2\tau_2 \int_{\delta_1}^{\delta_2} |a_2(z)| dz < a_1. \quad (11)$$

Assumption 6. There exist constants $\alpha_j, \beta_j > 0$ ($j = 0, 1$) such that $\alpha_0 \leq \alpha(x) \leq \alpha_1$ and $\beta_0 \leq \beta(x) \leq \beta_1$ for all $x \in \Gamma_0$.

Assumption 7. $\mathcal{K} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a C^1 nondecreasing function. Let $\widehat{\mathcal{K}}(s) = \int_0^s \mathcal{K}(\tau) d\tau$ be its primitive. We assume

$$\mathcal{K}(s) \geq 1, \quad \widehat{\mathcal{K}}(s) \geq s, \quad \forall s \geq 0. \quad (12)$$

Assumption 8 (Dimension-dependent restrictions). In addition to $p > 2$, we require the exponent p to satisfy the Sobolev embedding restrictions

$$\begin{cases} 2 < p < \infty, & d = 1, 2, \\ 2 < p \leq \frac{d+2}{d-2}, & d \geq 3, \end{cases} \quad (13)$$

and, for the blow-up estimates of Section 5, the additional bound

$$p < \frac{2d-2}{d-2} \quad \text{whenever } d = 3, 4. \quad (14)$$

For $d \geq 5$, condition (13) already implies (14), so no extra restriction is needed.

Remark 1. The assumption $\mathcal{K}(s) \geq 1$ guarantees that the Kirchhoff tension is at least as large as the classical wave tension (corresponding to $\mathcal{K} \equiv 1$). A canonical example is $\mathcal{K}(s) = 1 + \sigma s^\rho$ with $\sigma \geq 0, \rho \geq 0$, for which $\widehat{\mathcal{K}}(s) = s + \frac{\sigma}{\rho+1} s^{\rho+1} \geq s$. It is instructive to compare the present results with the semilinear case $\mathcal{K} \equiv 1$

- (i) *Energy functional.* For $\mathcal{K} \equiv 1$, the term $\frac{1}{2}\widehat{\mathcal{K}}(\|\nabla \mathcal{W}\|_2^2)$ reduces to $\frac{1}{2}\|\nabla \mathcal{W}\|_2^2$, and the energy becomes quadratic. For non-constant $\mathcal{K}$, the energy is genuinely non-quadratic, which changes the depth and geometry of the potential well.
- (ii) *Global existence.* The lower bound $\widehat{\mathcal{K}}(s) - (\int_0^t \mathcal{H})s \geq \kappa s$ used in (51) holds with $\kappa = \ell$ even when $\mathcal{K} \equiv 1$, so the global existence proof survives; however, the uniform bound (52) is weaker because the coercivity is linear rather than superlinear.
- (iii) *Decay estimates.* The multiplier estimate (68) uses $\mathcal{K}(\|\nabla \mathcal{W}\|_2^2)\|\nabla \mathcal{W}\|_2^2 \geq \|\nabla \mathcal{W}\|_2^2$. This is trivial for $\mathcal{K} \equiv 1$, but for non-constant $\mathcal{K}$, it requires the hypothesis $\mathcal{K}(s) \geq 1$.

- (iv) *Blow-up.* The deficit energy $\mathcal{E}_-(t)$ contains the term $\frac{1}{2}[\widehat{\mathcal{K}}(\|\nabla \mathcal{W}\|_2^2) - (\int \mathcal{H})\|\nabla \mathcal{W}\|_2^2] \geq \frac{\ell}{2}\|\nabla \mathcal{W}\|_2^2$. This coercivity is essential for closing the differential inequality (170); without it (or with $\ell = 0$), the concavity argument would fail.

The following lemma will be used frequently.

Lemma 1 (Jensen's Inequality). *If Φ is a convex function on $[a, b]$, $\mathcal{A} : \Sigma \rightarrow [a, b]$ and $\mathcal{W}$ are integrable on Σ , $\mathcal{W}(x) \geq 0$, and $\int_{\Sigma} \mathcal{W}(x) dx = \mathcal{N} > 0$, then*

$$\Phi\left(\frac{1}{\mathcal{N}} \int_{\Sigma} \mathcal{A}(x) \mathcal{W}(x) dx\right) \leq \frac{1}{\mathcal{N}} \int_{\Sigma} \Phi(\mathcal{A}(x)) \mathcal{W}(x) dx. \quad (15)$$

Throughout the rest of the paper, we also define

$$(\mathcal{H} \circ \mathcal{W})(t) := \int_{\Omega} \int_0^t \mathcal{H}(t - \vartheta) |\mathcal{W}(t) - \mathcal{W}(\vartheta)|^2 d\vartheta dx, \quad (16)$$

where

$$H_{\Gamma_1}^1(\Omega) = \left\{ \mathcal{W} \in H^1(\Omega), \mathcal{W} = 0 \text{ on } \Gamma_1 \right\}. \quad (17)$$

For every $q \in [1, \infty]$, the symbol $\|\cdot\|_q$ stands for the norm in the Lebesgue space $L^q(\Omega)$, whereas $\|\cdot\|_{q,\Gamma}$ denotes the norm in $L^q(\Gamma)$ for any measurable subset $\Gamma \subseteq \partial\Omega$. The subscript is omitted when $q = 2$, i.e., $\|\cdot\| = \|\cdot\|_2$.

Lemma 2 (Sobolev–Poincaré Inequality [39]). *Assume $1 \leq s \leq +\infty$ for $d = 1, 2$, or $1 \leq s \leq \frac{d+2}{d-2}$ for $d \geq 3$. Then, there exists $C_E > 0$ such that*

$$\|\mathcal{W}\|_{s+1} \leq C_E \|\nabla \mathcal{W}\|_2, \quad \forall \mathcal{W} \in H_0^1(\Omega). \quad (18)$$

Lemma 3 (Trace-Sobolev embedding [40,41]). *1. If either*

$$1 \leq p \leq \frac{d+2}{d-2}, \quad (d \geq 3) \quad \text{or} \quad 1 \leq p \leq \infty, \quad (d = 1, 2), \quad (19)$$

then $H_{\Gamma_1}^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$, i.e.,

$$\|\mathcal{W}\|_{p+1} \leq C_{p,\Omega} \|\nabla \mathcal{W}\|_2, \quad \forall \mathcal{W} \in H_{\Gamma_1}^1(\Omega). \quad (20)$$

2. If either

$$1 \leq s \leq \frac{d}{d-2}, \quad (d \geq 3) \quad \text{or} \quad 1 \leq s \leq \infty, \quad (d = 1, 2), \quad (21)$$

then $H_{\Gamma_1}^1(\Omega) \hookrightarrow L^{s+1}(\Gamma_0)$, i.e.,

$$\|\mathcal{W}\|_{s+1,\Gamma_0} \leq C_{s,\Gamma_0} \|\nabla \mathcal{W}\|_2, \quad \forall \mathcal{W} \in H_{\Gamma_1}^1(\Omega), \quad (22)$$

where $C_{p,\Omega}$ and C_{s,Γ_0} are the embedding constants.

Lemma 4 ([28]). *For any $\rho > 0$, we have the elementary inequalities*

$$\ln \mathcal{W} \leq \frac{\mathcal{W}^\rho}{e\rho}, \quad \forall \mathcal{W} \geq 1, \quad (23)$$

and

$$|\ln \mathcal{W}| \leq \frac{\mathcal{W}^{-\rho}}{e\rho}, \quad \forall 0 < \mathcal{W} < 1. \quad (24)$$

Next, we recall a result that allows us to rewrite the fractional boundary condition in an equivalent first-order form.

Theorem 1 ([42]). *Let*

$$\xi(\mu) = |\mu|^{(2\gamma-1)/2}, \quad 0 < \gamma < 1, \mu \in \mathbb{R}. \quad (25)$$

Set

$$\mathcal{P} = I^{1-\gamma, \delta} \mathcal{U}. \quad (26)$$

Then, ϕ satisfies

$$\partial_t \phi(\mu, t) + (\mu^2 + \delta) \phi(\mu, t) - \mathcal{U}(\mathcal{Q}, t) \xi(\mu) = 0, \quad t > 0, \delta \geq 0, \mu \in \mathbb{R}, \quad (27)$$

$$\phi(\mu, 0) = 0, \quad (28)$$

$$\mathcal{P}(t) = \frac{\sin(\gamma\pi)}{\pi} \int_{-\infty}^{+\infty} \phi(\mu, t) \xi(\mu) d\mu, \quad \mu \in \mathbb{R}, t > 0. \quad (29)$$

Following [15], we introduce the auxiliary variables

$$\psi(x, \varrho, z, t) = \mathcal{W}_t(x, t - z\varrho), \quad (x, \varrho, z, t) \in \mathcal{D} = \Gamma_0 \times (0, 1) \times (\delta_1, \delta_2) \times \mathbb{R}_+, \quad (30)$$

which satisfy

$$\begin{cases} z\psi_t(x, \varrho, z, t) + \psi_\varrho(x, \varrho, z, t) = 0, \\ \psi(x, 0, z, t) = \mathcal{W}_t(x, t). \end{cases} \quad (31)$$

Remark 2. *The initial data for the transformed system (33) must satisfy the natural compatibility conditions*

$$\mathcal{W}_0|_{\Gamma_1} = 0, \quad \mathcal{W}_1|_{\Gamma_1} = 0, \quad w_0(x, 0, z) = \mathcal{W}_1(x) \text{ for } x \in \Gamma_0, z \in (\delta_1, \delta_2), \quad \phi_0(\mu) = 0. \quad (32)$$

These guarantee that the boundary and initial values match continuously at $t = 0$.

Consequently, by Theorem 1 and (4), problem (1) can be rewritten as

$$\begin{cases} \mathcal{W}_t - \mathcal{K}(\|\nabla \mathcal{W}\|_2^2) \Delta \mathcal{W}(t) + \int_0^t \mathcal{H}(t - \vartheta) \Delta \mathcal{W}(\vartheta) d\vartheta = |\mathcal{W}|^{p-2} \mathcal{W} \ln |\mathcal{W}|, & \text{in } \Omega \times \mathbb{R}_+, \\ \mathcal{K}(\|\nabla \mathcal{W}\|_2^2) \frac{\partial \mathcal{W}}{\partial n} - \int_0^t \mathcal{H}(t - \vartheta) \frac{\partial \mathcal{W}}{\partial n}(\vartheta) d\vartheta + a_1 \mathcal{F}_1(\mathcal{W}_t) + c_1 \int_{-\infty}^{+\infty} \phi(\mu, t) \xi(\mu) d\mu \\ \quad + \int_{\delta_1}^{\delta_2} a_2(z) \mathcal{F}_2(\psi(x, 1, z, t)) dz = \mathcal{S}_t, & x \in \Gamma_0, t > 0, \\ \mathcal{W}_t + \alpha(x) \mathcal{S} + \beta(x) \mathcal{S}_t = 0, & x \in \Gamma_0, t > 0, \\ z\psi_t(x, \varrho, z, t) + \psi_\varrho(x, \varrho, z, t) = 0, & x \in \Gamma_0, \varrho \in (0, 1), z \in (\delta_1, \delta_2), t > 0, \\ \partial_t \phi(\mu, t) + (\mu^2 + \delta) \phi(\mu, t) - \mathcal{W}_t(x, t) \xi(\mu) = 0, & \mu \in \mathbb{R}, x \in \Gamma_0, t > 0, \\ \mathcal{W}(x, t) = 0, & \text{on } \Gamma_1 \times \mathbb{R}_+, \\ \mathcal{W}(x, 0) = \mathcal{W}_0(x), \quad \mathcal{W}_t(x, 0) = \mathcal{W}_1(x), & x \in \Omega, \\ \psi(x, \varrho, z, 0) = w_0(x, \varrho, z), & x \in \Gamma_0, \varrho \in (0, 1), z \in (\delta_1, \delta_2), \\ \phi(\mu, 0) = \phi_0(\mu), & \mu \in \mathbb{R}, \\ \mathcal{S}(x, 0) = \mathcal{S}_0(x), & x \in \Gamma_0. \end{cases} \quad (33)$$

Remark 3. *In the boundary condition (33)₂, the coefficient c_1 is related to the original fractional damping coefficient c appearing in (1) through the identity $c_1 = c \sin(\gamma\pi)/\pi$, which follows from Theorem 1 and Formula (29). This guarantees that the reformulated system (33) is exactly equivalent to the original model (1).*

For the reader's convenience, the main notation and function spaces used throughout the paper are summarized in Table 1.

Table 1. Summary of notation and function spaces.

Symbol	Meaning
$\Omega \subset \mathbb{R}^d$	Bounded spatial domain
Γ_0, Γ_1	Disjoint boundary components, $\partial\Omega = \Gamma_0 \cup \Gamma_1$
$H_{\Gamma_1}^1(\Omega)$	H^1 functions vanishing on Γ_1 (see (17))
$\mathcal{K}(s)$	Kirchhoff modulus, $\mathcal{K} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$
$\hat{\mathcal{K}}(s)$	Primitive $\int_0^s \mathcal{K}(\tau) d\tau$
$(\mathcal{H} \circ \mathcal{W})(t)$	Memory convolution (16)
$\psi(x, \varrho, z, t)$	Delay history variable on $\Gamma_0 \times (0, 1) \times (\delta_1, \delta_2)$
$\phi(\mu, t)$	Fractional-diffusion auxiliary variable, $\mu \in \mathbb{R}$
$\mathcal{S}(x, t)$	Acoustic boundary potential on Γ_0
$\mathcal{I}(t)$	Total energy functional (34)
$\mathcal{D}(t), \mathcal{B}(t)$	Potential-well auxiliary functionals (46) and (47)
$\mathcal{F}_1, \mathcal{F}_2$	Nonlinear damping and delay feedback functions

The following existence result for problem (33) rests on a synthesis of the Faedo–Galerkin approximation scheme and the technical machinery developed in [43].

Theorem 2. *Assumptions (1)–(8). Then, for any initial data $\mathcal{W}_0, \mathcal{W}_1 \in H_{\Gamma_1}^1(\Omega) \cap L^2(\Omega)$, $w_0 \in L^2(\Gamma_0 \times (0, 1) \times (\delta_1, \delta_2))$, $\phi_0 \in L^2(\mathbb{R})$, and $\mathcal{S}_0 \in L^2(\Gamma_0)$ satisfying the compatibility conditions (32), problem (33) admits a weak solution $(\mathcal{W}, \phi, \psi, \mathcal{S})$ with $\mathcal{W}, \mathcal{W}_t \in L^\infty([0, T^*], H_{\Gamma_1}^1(\Omega)) \cap W^{1,\infty}([0, T^*], L^2(\Omega))$, $\mathcal{W}_{tt} \in L^\infty([0, T^*], L^2(\Omega))$, $\psi \in L^\infty([0, T^*], L^2(\Gamma_0 \times (0, 1) \times (\delta_1, \delta_2)))$, $\phi \in L^\infty([0, T^*]; L^2(\mathbb{R}))$, $\mathcal{S}, \mathcal{S}_t \in L^2(\mathbb{R}_+, L^2(\Gamma_0))$.*

Proof. Let $\{e_j\}_{j \geq 1}$ be a basis of $H_{\Gamma_1}^1(\Omega)$ and let

$$\mathcal{W}_m(t) = \sum_{j=1}^m d_{jm}(t) e_j$$

be the Faedo–Galerkin approximations. Since $\|\nabla \mathcal{W}_m(t)\|_2^2$ is a scalar function of time, the Kirchhoff coefficient $\mathcal{K}(\|\nabla \mathcal{W}_m\|_2^2)$ enters each approximate problem as a bounded, positive, uniformly Lipschitz multiplier; using $\mathcal{K} \geq 1$, the approximate energy identity (the analogue of (35)) provides bounds that are uniform in m , namely

$$\|\mathcal{W}_m\|_{L^\infty(0,T;H_{\Gamma_1}^1)} + \|\mathcal{W}_{mt}\|_{L^\infty(0,T;H_{\Gamma_1}^1)} + \|\mathcal{W}_{mtt}\|_{L^\infty(0,T;L^2)} \leq C,$$

with analogous bounds for ψ_m, ϕ_m , and $\mathcal{S}_m$. The Aubin–Lions compactness lemma then yields, along a subsequence, $\nabla \mathcal{W}_m \rightarrow \nabla \mathcal{W}$ strongly in $L^2(\Omega \times (0, T))$; since $\mathcal{K} \in \mathcal{C}^1$, this implies

$$\mathcal{K}(\|\nabla \mathcal{W}_m\|_2^2) \longrightarrow \mathcal{K}(\|\nabla \mathcal{W}\|_2^2) \quad \text{uniformly in } t,$$

which is exactly the convergence needed to pass to the limit in the nonlocal term $\mathcal{K}(\|\nabla \mathcal{W}\|_2^2) \Delta \mathcal{W}$. The logarithmic source is continuous on $H_{\Gamma_1}^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$ under (Assumption 8), and the fractional and delay terms are treated as in [43]. Uniqueness follows from the energy inequality (35), and the continuation principle stated in Remark 6 extends the local solution to its maximal time. $\square$

Lemma 5. *Conservation and dissipation of total energy*

$$\begin{aligned} \mathcal{I}(t) = & \frac{1}{2} \|\mathcal{W}_t\|_2^2 + \frac{1}{2} \widehat{\mathcal{K}}(\|\nabla \mathcal{W}(t)\|_2^2) - \frac{1}{2} \left(\int_0^t \mathcal{H}(\vartheta) d\vartheta \right) \|\nabla \mathcal{W}(t)\|_2^2 + \frac{1}{p^2} \|\mathcal{W}\|_p^p - \frac{1}{p} \int_{\Omega} \ln |\mathcal{W}| |\mathcal{W}|^p dx \\ & + \frac{1}{2} \int_{\Gamma_0} \alpha(x) \mathcal{S}^2 d\Gamma + \frac{c_1}{2} \int_{\Gamma_0} \int_{-\infty}^{+\infty} |\phi(\mu, t)|^2 d\mu d\Gamma \\ & + \frac{1}{2} (\mathcal{H} \circ \nabla \mathcal{W})(t) + \int_{\Gamma_0} \int_0^1 \int_{\delta_1}^{\delta_2} z |a_2(z)| \mathcal{G}(\psi(x, q, z, t)) dz dq d\Gamma \end{aligned} \quad (34)$$

satisfies

$$\begin{aligned} \mathcal{I}'(t) \leq & -\sigma_1 \int_{\Gamma_0} \mathcal{W}_t \mathcal{F}_1(\mathcal{W}_t) d\Gamma + \frac{1}{2} (\mathcal{H}' \circ \nabla \mathcal{W})(t) - \int_{\Gamma_0} \beta(x) \mathcal{S}_t^2 d\Gamma \\ & - c_1 \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\mu^2 + \delta) |\phi(\mu, t)|^2 d\mu d\Gamma - \frac{1}{2} \mathcal{H}(t) \|\nabla \mathcal{W}(t)\|_2^2 \\ & - \sigma_2 \int_{\Gamma_0} \int_{\delta_1}^{\delta_2} |a_2(z)| \psi(x, 1, z, t) \mathcal{F}_2(\psi(x, 1, z, t)) dz d\Gamma \leq 0, \end{aligned} \quad (35)$$

where $\sigma_1 = a_1 - 2\tau_2 \int_{\delta_1}^{\delta_2} |a_2(z)| dz > 0$ and $\sigma_2 = 2\tau_1 - 1 > 0$.

Proof. Multiplying (33)₁ by $\mathcal{W}_t$ and (33)₃ by $\mathcal{S}_t$, then integrating over Ω and Γ_0 , respectively, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left\{ \|\mathcal{W}_t(t)\|_2^2 + \widehat{\mathcal{K}}(\|\nabla \mathcal{W}(t)\|_2^2) - \left(\int_0^t \mathcal{H}(\vartheta) d\vartheta \right) \|\nabla \mathcal{W}(t)\|_2^2 + (\mathcal{H} \circ \nabla \mathcal{W})(t) + \int_{\Gamma_0} \alpha(x) \mathcal{S}^2 d\Gamma \right\} \\ & + \frac{d}{dt} \left\{ \frac{1}{p^2} \|\mathcal{W}\|_p^p - \frac{1}{p} \int_{\Omega} \ln |\mathcal{W}| |\mathcal{W}|^p dx \right\} + a_1 \int_{\Gamma_0} \mathcal{W}_t \mathcal{F}_1(\mathcal{W}_t) d\Gamma - \frac{1}{2} (\mathcal{H}' \circ \nabla \mathcal{W})(t) \\ & + c_1 \int_{\Gamma_0} \mathcal{W}_t(x, t) \int_{-\infty}^{+\infty} \xi(\mu) \phi(\mu, t) d\mu d\Gamma + \frac{1}{2} \mathcal{H}(t) \|\nabla \mathcal{W}(t)\|_2^2 \\ & + \int_{\Gamma_0} \beta(x) \mathcal{S}_t^2 d\Gamma + \int_{\Gamma_0} \int_{\delta_1}^{\delta_2} |a_2(z)| \mathcal{F}_2(\psi(x, 1, z, t)) \mathcal{W}_t dz d\Gamma = 0. \end{aligned} \quad (36)$$

Next, multiplying (33)₄ by $|a_2(z)| \mathcal{F}_2(\psi(x, q, z, t))$ and integrating over $\Gamma_0 \times (0, 1) \times (\delta_1, \delta_2)$, and using (31)₂, we obtain

$$\begin{aligned} & \frac{d}{dt} \int_{\Gamma_0} \int_0^1 \int_{\delta_1}^{\delta_2} z |a_2(z)| \mathcal{G}(\psi(x, q, z, t)) dz dq d\Gamma \\ & = - \int_{\Gamma_0} \int_0^1 \int_{\delta_1}^{\delta_2} |a_2(z)| \mathcal{F}_2(\psi) \psi_q dz dq d\Gamma \\ & = - \int_{\Gamma_0} \int_0^1 \int_{\delta_1}^{\delta_2} |a_2(z)| \frac{d}{dq} \mathcal{G}(\psi(x, q, z, t)) dz dq d\Gamma \\ & = \int_{\Gamma_0} \int_{\delta_1}^{\delta_2} |a_2(z)| (\mathcal{G}(\psi(x, 0, z, t)) - \mathcal{G}(\psi(x, 1, z, t))) dz d\Gamma \\ & = \left(\int_{\delta_1}^{\delta_2} |a_2(z)| dz \right) \int_{\Gamma_0} \mathcal{G}(\mathcal{W}_t) d\Gamma - \int_{\Gamma_0} \int_{\delta_1}^{\delta_2} |a_2(z)| \mathcal{G}(\psi(x, 1, z, t)) dz d\Gamma. \end{aligned} \quad (37)$$

Furthermore, multiplying (33)₅ by $c_1 \phi$ and integrating over $\Gamma_0 \times (-\infty, +\infty)$ gives

$$\begin{aligned} & \frac{c_1}{2} \frac{d}{dt} \int_{\Gamma_0} \int_{-\infty}^{+\infty} |\phi(\mu, t)|^2 d\mu d\Gamma + c_1 \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\mu^2 + \delta) |\phi(\mu, t)|^2 d\mu d\Gamma \\ & - c_1 \int_{\Gamma_0} \mathcal{W}_t(x, t) \int_{-\infty}^{+\infty} \xi(\mu) \phi(\mu, t) d\mu d\Gamma = 0. \end{aligned} \quad (38)$$

Combining the above identities yields (35) and

$$\begin{aligned} \mathcal{I}'(t) = & -a_1 \int_{\Gamma_0} \mathcal{W}_t \mathcal{F}_1(\mathcal{W}_t) d\Gamma + \frac{1}{2} (\mathcal{H}' \circ \nabla \mathcal{W})(t) - \frac{1}{2} \mathcal{H}(t) \|\nabla \mathcal{W}(t)\|_2^2 \\ & - \int_{\Gamma_0} \beta(x) \mathcal{S}_t^2 d\Gamma - \int_{\Gamma_0} \int_{\delta_1}^{\delta_2} |a_2(z)| \mathcal{F}_2(\psi(x, 1, z, t)) \mathcal{W}_t dz d\Gamma \\ & + \left(\int_{\delta_1}^{\delta_2} |a_2(z)| dz \right) \int_{\Gamma_0} \mathcal{G}(\mathcal{W}_t) d\Gamma - \int_{\Gamma_0} \int_{\delta_1}^{\delta_2} |a_2(z)| \mathcal{G}(\psi(x, 1, z, t)) dz d\Gamma \\ & - c_1 \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\mu^2 + \delta) |\phi(\mu, t)|^2 d\mu d\Gamma. \end{aligned} \quad (39)$$

Let $\mathcal{G}^*$ denote the convex conjugate of $\mathcal{G}$:

$$\mathcal{G}^*(q) = \sup_{t \geq 0} (qt - \mathcal{G}(t)). \quad (40)$$

Furthermore, $\mathcal{G}^*$ identifies with the Legendre transform of $\mathcal{G}$ [44], admitting the explicit characterization

$$\mathcal{G}^*(q) = q(\mathcal{G}')^{-1}(q) - \mathcal{G}[(\mathcal{G}')^{-1}(q)], \quad \forall q \geq 0. \quad (41)$$

In addition, this conjugate functional obeys the generalized Young inequality

$$qt \leq \mathcal{G}^*(q) + \mathcal{G}(t), \quad \forall q, t \geq 0. \quad (42)$$

By definition of $\mathcal{G}$, we have

$$\mathcal{G}^*(q) = q\mathcal{F}_2^{-1}(q) - \mathcal{G}(\mathcal{F}_2^{-1}(q)), \quad \forall q \geq 0. \quad (43)$$

Hence, by (41) and (10), we obtain

$$\mathcal{G}^*(\mathcal{F}_2(\psi(x, 1, z, t))) = \psi(x, 1, z, t) \mathcal{F}_2(\psi(x, 1, z, t)) - \mathcal{G}(\psi(x, 1, z, t)) \leq (1 - \tau_1) \psi(x, 1, z, t) \mathcal{F}_2(\psi(x, 1, z, t)). \quad (44)$$

Using (42), (43), and (10), we find

$$\begin{aligned} \mathcal{I}'(t) \leq & -a_1 \int_{\Gamma_0} \mathcal{W}_t \mathcal{F}_1(\mathcal{W}_t) d\Gamma + \frac{1}{2} (\mathcal{H}' \circ \nabla \mathcal{W})(t) - \frac{1}{2} \mathcal{H}(t) \|\nabla \mathcal{W}(t)\|_2^2 \\ & - \int_{\Gamma_0} \beta(x) \mathcal{S}_t^2 d\Gamma - \int_{\Gamma_0} \int_{\delta_1}^{\delta_2} |a_2(z)| \mathcal{G}(\psi(x, 1, z, t)) dz d\Gamma - \int_{\Gamma_0} \alpha(x) \mathcal{S}_t^2 d\Gamma + 2 \left(\int_{\delta_1}^{\delta_2} |a_2(z)| dz \right) \int_{\Gamma_0} \mathcal{G}(\mathcal{W}_t) d\Gamma \\ & - \int_{\Gamma_0} \int_{\delta_1}^{\delta_2} |a_2(z)| \mathcal{G}(\psi(x, 1, z, t)) dz d\Gamma - c_1 \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\mu^2 + \delta) |\phi(\mu, t)|^2 d\mu d\Gamma \\ \leq & - \left(a_1 - 2\tau_2 \int_{\delta_1}^{\delta_2} |a_2(z)| dz \right) \int_{\Gamma_0} \mathcal{W}_t \mathcal{F}_1(\mathcal{W}_t) d\Gamma + \frac{1}{2} (\mathcal{H}' \circ \nabla \mathcal{W})(t) \\ & - \frac{1}{2} \mathcal{H}(t) \|\nabla \mathcal{W}(t)\|_2^2 - \int_{\Gamma_0} \beta(x) \mathcal{S}_t^2 d\Gamma - (2\tau_1 - 1) \int_{\Gamma_0} \int_{\delta_1}^{\delta_2} |a_2(z)| \psi(x, 1, z, t) \mathcal{F}_2(\psi(x, 1, z, t)) dz d\Gamma \\ & - c_1 \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\mu^2 + \delta) |\phi(\mu, t)|^2 d\mu d\Gamma \leq 0. \end{aligned} \quad (45)$$

Setting $\sigma_1 = a_1 - 2\tau_2 \int_{\delta_1}^{\delta_2} |a_2(z)| dz$ and $\sigma_2 = 2\tau_1 - 1$, we obtain (35). Consequently, Assumptions (1)–(7) imply that $\mathcal{I}$ is non-increasing. $\square$

3. Global Existence

The present section addresses the global existence of weak solutions to (33).

To secure uniform estimates that preclude finite-time degeneracy and guarantee indefinite continuation, we shall construct a pair of auxiliary functionals

$$\mathcal{B}(t) = \frac{1}{2} \widehat{\mathcal{K}}(\|\nabla \mathcal{W}\|_2^2) - \frac{1}{2} \left(\int_0^t \mathcal{H}(\vartheta) d\vartheta \right) \|\nabla \mathcal{W}\|_2^2 - \frac{1}{p} \int_{\Omega} \ln |\mathcal{W}| |\mathcal{W}|^p dx + \frac{1}{p^2} \|\mathcal{W}\|_p^p, \quad (46)$$

and

$$\mathcal{D}(t) = \widehat{\mathcal{K}}(\|\nabla \mathcal{W}\|_2^2) - \left(\int_0^t \mathcal{H}(\vartheta) d\vartheta \right) \|\nabla \mathcal{W}\|_2^2 - \int_{\Omega} \ln |\mathcal{W}| |\mathcal{W}|^p dx. \quad (47)$$

From the definitions of $\mathcal{D}(t)$, $\mathcal{B}(t)$, and $\mathcal{I}(t)$, we have

$$\mathcal{B}(t) = \frac{1}{p} \mathcal{D}(t) + \left(\frac{p-2}{2p} \right) \left[\widehat{\mathcal{K}}(\|\nabla \mathcal{W}\|_2^2) - \left(\int_0^t \mathcal{H}(\vartheta) d\vartheta \right) \|\nabla \mathcal{W}\|_2^2 \right] + \frac{1}{p^2} \|\mathcal{W}\|_p^p, \quad (48)$$

and

$$\begin{aligned} \mathcal{I}(t) &= \frac{1}{2} \|\mathcal{W}_t\|_2^2 + \mathcal{B}(t) + \frac{1}{2} \int_{\Gamma_0} \alpha(x) \mathcal{S}^2 d\Gamma + \frac{c_1}{2} \int_{\Gamma_0} \int_{-\infty}^{+\infty} |\phi(\mu, t)|^2 d\mu d\Gamma \\ &\quad + \frac{1}{2} (\mathcal{H} \circ \nabla \mathcal{W})(t) + \int_{\Gamma_0} \int_0^1 \int_{\delta_1}^{\delta_2} z |a_2(z)| \mathcal{G}(\psi(x, \varrho, z, t)) dz d\varrho d\Gamma. \end{aligned} \quad (49)$$

Lemma 6. Let the initial data satisfy $(\mathcal{W}_0, \mathcal{W}_1, \phi_0, w_0, \mathcal{S}_0) \in (H_{\Gamma_1}^1(\Omega) \times L^2(\Omega) \times L^2(\mathbb{R}) \times L^2(\Gamma_0 \times (0, 1) \times (\delta_1, \delta_2)) \times L^2(\Gamma_0))$, and assume $\mathcal{D}(0) > 0$ together with

$$\theta_0 := \frac{C_{p,\Omega}^{p+1}}{\kappa} \left(\frac{2p}{\kappa(p-2)} \mathcal{I}(0) \right)^{\frac{p-1}{2}} < 1. \quad (50)$$

Then $\mathcal{D}(t) > 0$ for every $t \in [0, T^*]$.

Proof. Because $\mathcal{D}(0)$ is strictly positive and the mapping $t \mapsto \mathcal{D}(t)$ is continuous, one can select a horizon $T_* \in (0, T^*]$ throughout which $\mathcal{D}(t) \geq 0$. Therefore,

$$\begin{aligned} \mathcal{B}(t) &\geq \frac{1}{p} \mathcal{D}(t) + \left(\frac{p-2}{2p} \right) \left[\widehat{\mathcal{K}}(\|\nabla \mathcal{W}\|_2^2) - \left(\int_0^t \mathcal{H}(\vartheta) d\vartheta \right) \|\nabla \mathcal{W}\|_2^2 \right] + \frac{1}{p^2} \|\mathcal{W}\|_p^p \\ &\geq \left(\frac{p-2}{2p} \right) \left[\widehat{\mathcal{K}}(\|\nabla \mathcal{W}\|_2^2) - \left(\int_0^t \mathcal{H}(\vartheta) d\vartheta \right) \|\nabla \mathcal{W}\|_2^2 \right] \\ &\geq \frac{(p-2)\kappa}{2p} \|\nabla \mathcal{W}\|_2^2, \quad \forall t \in [0, T_*], \end{aligned} \quad (51)$$

where we have used (Assumption 7) to infer $\widehat{\mathcal{K}}(s) - \left(\int_0^t \mathcal{H} \right) s \geq s - (1 - \kappa)s = \kappa s$. Therefore, by (35) and (51),

$$\kappa \|\nabla \mathcal{W}\|_2^2 \leq \frac{2p}{p-2} \mathcal{B}(t) \leq \frac{2p}{p-2} \mathcal{I}(t) \leq \frac{2p}{p-2} \mathcal{I}(0), \quad \forall t \in [0, T_*]. \quad (52)$$

The embedding $H_{\Gamma_1}^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$ yields

$$\int_{\Omega} \ln |\mathcal{W}| |\mathcal{W}|^p dx \leq \|\mathcal{W}\|_{p+1}^{p+1} \leq C_{p,\Omega}^{p+1} \|\nabla \mathcal{W}\|_2^{p+1}. \quad (53)$$

By (52), we obtain

$$\begin{aligned} \int_{\Omega} \ln |\mathcal{W}| |\mathcal{W}|^p dx &< \frac{C_{p,\Omega}^{p+1}}{\kappa} \left(\frac{2p \mathcal{I}(0)}{\kappa(p-2)} \right)^{\frac{p-1}{2}} \kappa \|\nabla \mathcal{W}(t)\|_2^2 \\ &< \theta_0 \kappa \|\nabla \mathcal{W}(t)\|_2^2, \end{aligned} \quad (54)$$

where

$$\theta_0 := \frac{C_{p,\Omega}^{p+1}}{\kappa} \left(\frac{2p\mathcal{I}(0)}{\kappa(p-2)} \right)^{\frac{p-1}{2}}. \quad (55)$$

According to (47), (52), and (54), we deduce

$$\mathcal{D}(t) > (1 - \theta_0) \left[\widehat{\mathcal{K}}(\|\nabla \mathcal{W}(t)\|_2^2) - \left(\int_0^t \mathcal{H}(\vartheta) d\vartheta \right) \|\nabla \mathcal{W}(t)\|_2^2 \right] > 0, \quad \forall t \in [0, T_*]. \quad (56)$$

Because the bound (56) depends only on the structural constants and on $\mathcal{I}(0)$, the argument can be iterated: the uniform estimate prevents the solution norm from blowing up at T_* , hence by the standard continuation principle for hyperbolic systems (see, e.g., [45], Chapter 1) the local solution extends beyond T_* . Repeating this argument shows that $T_* = T^*$ and, since T^* was arbitrary, global existence follows. $\square$

Remark 4. Under the hypotheses of Lemma 6, we have $\mathcal{B}(t) \geq 0$ and therefore $\mathcal{I}(t) \geq 0$ for all $t \in [0, T^*]$. Moreover, by (51) and (52),

$$\begin{aligned} \|\mathcal{W}_t(t)\|_2^2 &\leq 2\mathcal{I}(0), \\ \|\mathcal{W}(t)\|_p^p &\leq p^2\mathcal{I}(0), \\ \int_{\Gamma_0} \int_{-\infty}^{+\infty} |\phi(\mu, t)|^2 d\mu d\Gamma &\leq \frac{2}{c_1}\mathcal{I}(0), \\ (\mathcal{H} \circ \nabla \mathcal{W})(t) &\leq 2\mathcal{I}(0), \\ \int_{\Gamma_0} \alpha(x) \mathcal{S}^2 d\Gamma &\leq 2\mathcal{I}(0), \\ \int_{\Gamma_0} \int_0^1 \int_{\delta_1}^{\delta_2} z |a_2(z)| \mathcal{G}(\psi(x, \varrho, z, t)) dz d\varrho d\Gamma &\leq \mathcal{I}(0). \end{aligned} \quad (57)$$

Theorem 3. Under the standing assumptions of Lemma 6 and (Assumption 8), the weak solution generated by problem (33) persists for all time and remains uniformly bounded.

Proof. The argument reduces to establishing that

$$\begin{aligned} \|(\mathcal{W}, \phi, \psi, \mathcal{S})\|_{\mathcal{F}} &:= \|\mathcal{W}_t\|_2^2 + \|\nabla \mathcal{W}\|_2^2 + \|\mathcal{W}\|_p^p + \|\mathcal{S}\|_{\Gamma_0}^2 + \int_{\Gamma_0} \int_{-\infty}^{+\infty} |\phi(\mu, t)|^2 d\mu d\Gamma \\ &\quad + (\mathcal{H} \circ \nabla \mathcal{W}) + \int_{\Gamma_0} \int_0^1 \int_{\delta_1}^{\delta_2} z |a_2(z)| \mathcal{G}(\psi(x, \varrho, z, t)) dz d\varrho d\Gamma \end{aligned} \quad (58)$$

remains uniformly controlled for all $t \geq 0$. Toward this goal, we appeal to (49)

$$\begin{aligned} \mathcal{I}(0) > \mathcal{I}(t) &= \mathcal{B}(t) + \frac{1}{2} \|\mathcal{W}_t\|_2^2 + \frac{c_1}{2} \int_{\Gamma_0} \int_{-\infty}^{+\infty} |\phi(\mu, t)|^2 d\mu d\Gamma + \frac{1}{2} \int_{\Gamma_0} \alpha(x) \mathcal{S}^2 d\Gamma \\ &\quad + \frac{1}{2} (\mathcal{H} \circ \nabla \mathcal{W})(t) + \int_{\Gamma_0} \int_0^1 \int_{\delta_1}^{\delta_2} z |a_2(z)| \mathcal{G}(\psi(x, \varrho, z, t)) dz d\varrho d\Gamma. \end{aligned} \quad (59)$$

Consequently,

$$\begin{aligned} \mathcal{I}(0) &\geq \frac{1}{2} \|\mathcal{W}_t\|_2^2 + \frac{c_1}{2} \int_{\Gamma_0} \int_{-\infty}^{+\infty} |\phi(\mu, t)|^2 d\mu d\Gamma + \frac{1}{2} (\mathcal{H} \circ \nabla \mathcal{W})(t) \\ &\quad + \int_{\Gamma_0} \int_0^1 \int_{\delta_1}^{\delta_2} z |a_2(z)| \mathcal{G}(\psi(x, \varrho, z, t)) dz d\varrho d\Gamma + \frac{\alpha_0}{2} \|\mathcal{S}\|_{\Gamma_0}^2 \\ &\quad + \frac{(p-2)\kappa}{2p} \|\nabla \mathcal{W}\|_2^2 + \frac{1}{p^2} \|\mathcal{W}\|_p^p. \end{aligned} \quad (60)$$

Hence,

$$\|(\mathcal{W}, \phi, \psi, \mathcal{S})\|_{\mathcal{F}} \leq C\mathcal{I}(0), \quad (61)$$

for a suitable structural constant $C > 0$ determined exclusively by p, κ , and β_0 . $\square$

4. General Decay

The objective of the present section is to derive quantitative decay estimates for the dynamical system governed by (33). As a preparatory step, we introduce the following Lyapunov-type functionals:

$$\begin{aligned} J(t) &:= \int_{\Omega} \mathcal{W}(t) \mathcal{W}_t(t) dx + \int_{\Gamma_0} \mathcal{W} \mathcal{S} d\Gamma + \frac{1}{2} \int_{\Gamma_0} \beta(x) \mathcal{S}^2 d\Gamma, \\ K(t) &:= - \int_{\Omega} \mathcal{W}_t \int_0^t \mathcal{H}(t - \vartheta) (\mathcal{W}(t) - \mathcal{W}(\vartheta)) d\vartheta dx, \\ L(t) &:= \frac{c_1}{2} \int_{\Gamma_0} \int_{-\infty}^{+\infty} \frac{1}{\mu^2 + \delta} \Sigma^2(x, \mu, t) d\mu d\Gamma, \\ N(t) &:= \int_{\Gamma_0} \int_0^1 \int_{\delta_1}^{\delta_2} z e^{-z\varrho} |a_2(z)| \mathcal{G}(\psi(x, \varrho, z, t)) dz d\varrho d\Gamma, \end{aligned} \quad (62)$$

where

$$\Sigma(x, \mu, t) := (\mu^2 + \delta) \int_0^t \phi(\mu, z) dz + \mathcal{W}_0(x) \xi(\mu). \quad (63)$$

Lemma 7. Let $(\mathcal{W}, \phi, \psi, \mathcal{S})$ be the solution of (33). Then,

$$\begin{aligned} \int_{\Gamma_0} \int_{-\infty}^{+\infty} \phi(\mu, t) \Sigma(x, \mu, t) d\mu d\Gamma &= \int_{\Gamma_0} \mathcal{W}(x, t) \int_{-\infty}^{+\infty} \phi(\mu, t) \xi(\mu) d\mu d\Gamma \\ &\quad - \int_{\Gamma_0} \int_{-\infty}^{+\infty} |\phi(\mu, t)|^2 d\mu d\Gamma. \end{aligned} \quad (64)$$

Proof. From (33)₅, we have

$$(\mu^2 + \delta) \phi(\mu, t) = \mathcal{W}_t(x, t) \xi(\mu) - \partial_t \phi(\mu, t), \quad \forall x \in \Gamma_0. \quad (65)$$

Integrating (65) over $(0, t)$ and using the initial data, we obtain

$$\int_0^t (\mu^2 + \delta) \phi(\mu, z) dz = \mathcal{W}(x, t) \xi(\mu) - \mathcal{W}_0(x) \xi(\mu) - \phi(\mu, t), \quad \forall x \in \Gamma_0, \quad (66)$$

whence

$$\Sigma(x, \mu, t) = \mathcal{W}(x, t) \xi(\mu) - \phi(\mu, t), \quad \forall x \in \Gamma_0. \quad (67)$$

Multiplying by ϕ and integrating over $\Gamma_0 \times (-\infty, +\infty)$ yields (64). $\square$

Lemma 8. For any $\epsilon_1, \epsilon_2, \epsilon_3 > 0$, the functionals $J(t) + L(t)$ satisfy

$$\begin{aligned} J'(t) + L'(t) &\leq -(1 - 2\epsilon_1 - \epsilon_2 c_p - \epsilon_3 c_p^*) \|\nabla \mathcal{W}\|_2^2 + \|\mathcal{W}_t\|_2^2 - \int_{\Gamma_0} \alpha(x) \mathcal{S}^2 d\Gamma \\ &\quad + c(\epsilon_1) \int_{\Gamma_0} \mathcal{F}_1^2(\mathcal{W}_t) d\Gamma + c(\epsilon_1) \int_{\Gamma_0} \mathcal{S}_t^2 d\Gamma + c(\epsilon_2) (\mathcal{H} \circ \nabla \mathcal{W})(t) \\ &\quad + c(\epsilon_3) \int_{\Gamma_0} \int_{\delta_1}^{\delta_2} |a_2(z)| \mathcal{F}_2^2(\psi(x, 1, z, t)) dz d\Gamma \\ &\quad - c_1 \int_{\Gamma_0} \int_{-\infty}^{+\infty} |\phi(\mu, t)|^2 d\mu d\Gamma + \int_{\Omega} |\mathcal{W}|^p \ln |\mathcal{W}| dx. \end{aligned} \quad (68)$$

Proof. Differentiating (62)₁ and using (33)₁ and (33)₃, we obtain

$$\begin{aligned}
 J'(t) + L'(t) &= \|\mathcal{W}_t\|_2^2 + \int_{\Omega} \mathcal{W} \mathcal{W}_{tt} dx + \int_{\Gamma_0} \mathcal{W} \mathcal{S}_t d\Gamma + \int_{\Gamma_0} \mathcal{W}_t \mathcal{S} d\Gamma \\
 &\quad + \int_{\Gamma_0} \beta(x) \mathcal{S}_t \mathcal{S} d\Gamma + c_1 \int_{\Gamma_0} \int_{-\infty}^{+\infty} \phi(\mu, t) \Sigma(x, \mu, t) d\mu d\Gamma \\
 &= \|\mathcal{W}_t\|_2^2 - \mathcal{K}(\|\nabla \mathcal{W}\|_2^2) \|\nabla \mathcal{W}\|_2^2 + \int_{\Omega} \nabla \mathcal{W}(t) \int_0^t \mathcal{H}(t - \vartheta) (\nabla \mathcal{W}(t) - \nabla \mathcal{W}(\vartheta)) d\vartheta dx \\
 &\quad - \int_{\Gamma_0} \beta(x) \mathcal{S}^2 d\Gamma + \int_{\Omega} |\mathcal{W}|^p \ln |\mathcal{W}| dx \\
 &\quad + \underbrace{\int_{\Gamma_0} \mathcal{W} \mathcal{S}_t d\Gamma}_{I_0} + \underbrace{\int_{\Omega} \nabla \mathcal{W}(t) \int_0^t \mathcal{H}(t - \vartheta) (\nabla \mathcal{W}(t) - \nabla \mathcal{W}(\vartheta)) d\vartheta dx}_{I_1} \\
 &\quad - \underbrace{a_1 \int_{\Gamma_0} \mathcal{W} \mathcal{F}_1(\mathcal{W}_t) d\Gamma}_{I_2} + \underbrace{\int_{\Gamma_0} \mathcal{W} \int_{\delta_1}^{\delta_2} a_2(z) \mathcal{F}_2(\psi(x, 1, z, t)) dz d\Gamma}_{I_3}.
 \end{aligned} \tag{69}$$

Invoking Lemma 7 gives

$$\begin{aligned}
 J'(t) + L'(t) &= \|\mathcal{W}_t\|_2^2 - \mathcal{K}(\|\nabla \mathcal{W}\|_2^2) \|\nabla \mathcal{W}\|_2^2 - \int_{\Gamma_0} \alpha(x) \mathcal{S}^2 d\Gamma \\
 &\quad - c_1 \int_{\Gamma_0} \int_{-\infty}^{+\infty} |\phi(\mu, t)|^2 d\mu d\Gamma + \int_{\Omega} |\mathcal{W}|^p \ln |\mathcal{W}| dx + \underbrace{\int_{\Gamma_0} \mathcal{W} \mathcal{S}_t d\Gamma}_{I_0} \\
 &\quad + \underbrace{\int_{\Omega} \nabla \mathcal{W}(t) \int_0^t \mathcal{H}(t - \vartheta) (\nabla \mathcal{W}(t) - \nabla \mathcal{W}(\vartheta)) d\vartheta dx}_{I_1} - \underbrace{a_1 \int_{\Gamma_0} \mathcal{W} \mathcal{F}_1(\mathcal{W}_t) d\Gamma}_{I_2} \\
 &\quad + \underbrace{\int_{\Gamma_0} \mathcal{W} \int_{\delta_1}^{\delta_2} a_2(z) \mathcal{F}_2(\psi(x, 1, z, t)) dz d\Gamma}_{I_3}.
 \end{aligned} \tag{70}$$

We now estimate the last four terms on the right-hand side of (70). By Poincaré's, Young's, and Hölder's inequalities together with (Assumption 1), for arbitrary $\epsilon_j > 0$ ($j = 1, 2, 3$) we obtain

$$I_0 \leq \epsilon_1 \|\nabla \mathcal{W}\|_2^2 + c(\epsilon_1) \int_{\Gamma_0} \mathcal{S}_t^2 d\Gamma, \tag{71}$$

$$I_1 \leq \epsilon_1 \|\nabla \mathcal{W}\|_2^2 + c(\epsilon_1) (\mathcal{H} \circ \nabla \mathcal{W})(t), \tag{72}$$

$$I_2 \leq \epsilon_2 c_p \|\nabla \mathcal{W}\|_2^2 + c(\epsilon_2) \int_{\Gamma_0} \mathcal{F}_1^2(\mathcal{W}_t) d\Gamma, \tag{73}$$

and

$$I_3 \leq \epsilon_3 c_p^* \|\nabla \mathcal{W}\|_2^2 + c(\epsilon_3) \int_{\Gamma_0} \int_{\delta_1}^{\delta_2} |a_2(z)| \mathcal{F}_2^2(\psi(x, 1, z, t)) dz d\Gamma. \tag{74}$$

Combining (71)–(74) with (70), and using (Assumption 7) to bound $\mathcal{K}(\|\nabla \mathcal{W}\|_2^2) \|\nabla \mathcal{W}\|_2^2 \geq \|\nabla \mathcal{W}\|_2^2$, we arrive at (68). $\square$

Lemma 9. For any $\epsilon_1, \epsilon_2, \epsilon_3 > 0$, the functional $K(t)$ defined in (62)₂ satisfies

$$\begin{aligned}
 K'(t) &\leq - \left(\int_0^t \mathcal{H}(\vartheta) d\vartheta - \epsilon_2 \right) \|\mathcal{W}_t\|_2^2 + \epsilon_1 (2 - \kappa + \mathcal{H}^*) \|\nabla \mathcal{W}\|_2^2 - c(\epsilon_2) (\mathcal{H}' \circ \nabla \mathcal{W})(t) \\
 &\quad + \epsilon_3 \int_{\Gamma_0} \int_{-\infty}^{+\infty} |\phi(\mu, t)|^2 d\mu d\Gamma + c(\epsilon_1, \epsilon_3) (\mathcal{H} \circ \nabla \mathcal{W})(t) + c \int_{\Gamma_0} \mathcal{S}_t^2 d\Gamma \\
 &\quad + c \int_{\Gamma_0} \int_{\delta_1}^{\delta_2} |a_2(z)| \mathcal{F}_2^2(\psi(x, 1, z, t)) dz d\Gamma + c \int_{\Gamma_0} \mathcal{F}_1^2(\mathcal{W}_t) d\Gamma.
 \end{aligned} \tag{75}$$

Proof. Differentiating (62)₂ and using (33)₁, we obtain

$$\begin{aligned}
 K'(t) &= - \int_{\Omega} \mathcal{W}_{tt} \int_0^t \mathcal{H}(t-\vartheta)(\mathcal{W}(t) - \mathcal{W}(\vartheta)) d\vartheta dx \\
 &\quad - \int_{\Omega} \mathcal{W}_t \int_0^t \mathcal{H}'(t-\vartheta)(\mathcal{W}(t) - \mathcal{W}(\vartheta)) d\vartheta dx - \left(\int_0^t \mathcal{H}(\vartheta) d\vartheta \right) \|\mathcal{W}_t\|_2^2 \\
 &= - \int_{\Omega} \left[\mathcal{K}(\|\nabla \mathcal{W}\|_2^2) \Delta \mathcal{W} - \int_0^t \mathcal{H}(t-\vartheta) \Delta \mathcal{W}(\vartheta) d\vartheta \right] \left[\int_0^t \mathcal{H}(t-\vartheta)(\mathcal{W}(t) - \mathcal{W}(\vartheta)) d\vartheta \right] dx \\
 &\quad - \int_{\Omega} \left[|\mathcal{W}|^{p-2} \mathcal{W} \ln |\mathcal{W}| \right] \left[\int_0^t \mathcal{H}(t-\vartheta)(\mathcal{W}(t) - \mathcal{W}(\vartheta)) d\vartheta \right] dx \\
 &\quad - \int_{\Omega} \mathcal{W}_t \int_0^t \mathcal{H}'(t-\vartheta)(\mathcal{W}(t) - \mathcal{W}(\vartheta)) d\vartheta dx - \left(\int_0^t \mathcal{H}(\vartheta) d\vartheta \right) \|\mathcal{W}_t\|_2^2.
 \end{aligned} \tag{76}$$

We now proceed to bound each contribution appearing on the right-hand side of (76). As a first step, we partition the domain Ω into

$$\Omega_1 = \{x \in \Omega : |\mathcal{W}| \leq 1\} \quad \text{and} \quad \Omega_2 = \{x \in \Omega : |\mathcal{W}| > 1\}. \tag{77}$$

By Young's inequality,

$$\mathcal{L}_0 \leq \varepsilon_1 \int_{\Omega} \left(|\mathcal{W}|^{p-2} \mathcal{W} \ln |\mathcal{W}| \right)^2 dx + c(\varepsilon_1)(\mathcal{H} \circ \nabla \mathcal{W})(t). \tag{78}$$

Next, using Lemma 4 and the condition $2 < 2(p-1) < \frac{2d}{d-2}$ (which is guaranteed by (Assumption 8)), we can choose $\varrho_1, \varrho_2 > 0$ such that

$$2 < 2(p-1-\varrho_1) < \frac{2d}{d-2} \quad \text{and} \quad 2 < 2(p-1+\varrho_2) < \frac{2d}{d-2}. \tag{79}$$

By the embedding $H_0^1(\Omega) \hookrightarrow L^{2(p-1+\varrho)}(\Omega)$, we deduce from (78) that

$$\begin{aligned}
 \int_{\Omega} \left(|\mathcal{W}|^{p-2} \mathcal{W} \ln |\mathcal{W}| \right)^2 dx &\leq (e\varrho_1)^{-2} \int_{\Omega_1} |\mathcal{W}|^{2(p-1-\varrho_1)} dx + (e\varrho_2)^{-2} \int_{\Omega_2} |\mathcal{W}|^{2(p-1+\varrho_2)} dx \\
 &\leq \left(\frac{C_{2(p-1-\varrho_1)}^{p-1-\varrho_1}}{e\varrho_1} \right)^2 \left(\frac{2p\mathcal{I}(0)}{\kappa(p-2)} \right)^{p-2-\varrho_1} \|\nabla \mathcal{W}\|_2^2 \\
 &\quad + \left(\frac{C_{2(p-1+\varrho_2)}^{p-1-\varrho_2}}{e\varrho_2} \right)^2 \left(\frac{2p\mathcal{I}(0)}{\kappa(p-2)} \right)^{p-2-\varrho_2} \|\nabla \mathcal{W}\|_2^2.
 \end{aligned} \tag{80}$$

Substituting (80) into (78), we obtain

$$\mathcal{L}_0 \leq \varepsilon_1 \mathcal{H}^* \|\nabla \mathcal{W}\|_2^2 + c(\varepsilon_1)(\mathcal{H} \circ \nabla \mathcal{W})(t), \tag{81}$$

where

$$\mathcal{H}^* = \left(\frac{C_{2(p-1-\varrho_1)}^{p-1-\varrho_1}}{e\varrho_1} \right)^2 \left(\frac{2p\mathcal{I}(0)}{\kappa(p-2)} \right)^{p-2-\varrho_1} + \left(\frac{C_{2(p-1+\varrho_2)}^{p-1-\varrho_2}}{e\varrho_2} \right)^2 \left(\frac{2p\mathcal{I}(0)}{\kappa(p-2)} \right)^{p-2-\varrho_2}. \tag{82}$$

The remaining terms are handled via Poincaré's, Young's, and Hölder's inequalities together with (Assumption 1):

$$\mathcal{L}_1 \leq \varepsilon_1 \|\nabla \mathcal{W}\|_2^2 + c(\varepsilon_1)(\mathcal{H} \circ \nabla \mathcal{W})(t), \tag{83}$$

$$\begin{aligned}\mathcal{L}_2 &\leq \left(\int_0^t \mathcal{H}(\vartheta) d\vartheta \right) \int_{\Omega} \left(\nabla \mathcal{W}(t) \int_0^t \mathcal{H}(t-\vartheta) (\nabla \mathcal{W}(t) - \nabla \mathcal{W}(\vartheta)) d\vartheta \right) dx \\ &\quad - \int_{\Omega} \left(\int_0^t \mathcal{H}(t-\vartheta) (\nabla \mathcal{W}(t) - \nabla \mathcal{W}(\vartheta)) d\vartheta \right)^2 dx \\ &\leq \varepsilon_1 (1 - \kappa) \|\nabla \mathcal{W}\|_2^2 + c(\varepsilon_1) (\mathcal{H} \circ \nabla \mathcal{W})(t).\end{aligned}\quad (84)$$

Similarly,

$$\begin{aligned}\mathcal{L}_3 &\leq c \int_{\Gamma_0} \mathcal{F}_1^2(\mathcal{W}_t) d\Gamma + c(\mathcal{H} \circ \nabla \mathcal{W})(t), \\ \mathcal{L}_4 &\leq c \int_{\Gamma_0} \int_{\delta_1}^{\delta_2} |a_2(z)| \mathcal{F}_2^2(\psi(x, 1, z, t)) dz d\Gamma + c(\mathcal{H} \circ \nabla \mathcal{W})(t), \\ \mathcal{L}_5 &\leq \varepsilon_2 \|\mathcal{W}_t\|_2^2 - c(\varepsilon_2) (\mathcal{H}' \circ \nabla \mathcal{W})(t), \\ \mathcal{L}_6 &\leq c \int_{\Gamma_0} \mathcal{S}_t^2 d\Gamma + c(\mathcal{H} \circ \nabla \mathcal{W})(t).\end{aligned}\quad (85)$$

Finally, Young's inequality yields, for any $\varepsilon_3 > 0$

$$\mathcal{L}_7 \leq \varepsilon_3 \int_{\Gamma_0} \int_{-\infty}^{+\infty} |\phi(\mu, t)|^2 d\mu d\Gamma + c(\varepsilon_3) (\mathcal{H} \circ \nabla \mathcal{W})(t) \int_{-\infty}^{+\infty} \xi^2(\mu) d\mu. \quad (86)$$

By Lemma 2.1 in [12]

$$\int_{-\infty}^{+\infty} \xi^2(\mu) d\mu \leq B_0. \quad (87)$$

Inserting the above estimates into (76) yields (75). $\square$

Lemma 10. The functional $N(t)$ defined in (62)₄ satisfies

$$\begin{aligned}N'(t) &\leq -\psi_1 \int_{\Gamma_0} \int_0^1 \int_{\delta_1}^{\delta_2} z |a_2(z)| \mathcal{G}(\psi(x, \varrho, z, t)) dz d\varrho d\Gamma + a_1 \int_{\Gamma_0} \mathcal{W}_t \mathcal{F}_1(\mathcal{W}_t) d\Gamma \\ &\quad - \psi_1 \tau_1 \int_{\Gamma_0} \int_{\delta_1}^{\delta_2} |a_2(z)| \psi(x, 1, z, t) \mathcal{F}_2(\psi(x, 1, z, t)) dz d\Gamma.\end{aligned}\quad (88)$$

Proof. Differentiating $N(t)$ and using (33)₄, we obtain

$$\begin{aligned}N'(t) &= - \int_{\Gamma_0} \int_0^1 \int_{\delta_1}^{\delta_2} e^{-z\varrho} |a_2(z)| \psi_{\varrho} \mathcal{F}_2(\psi(x, \varrho, z, t)) dz d\varrho d\Gamma \\ &= - \int_{\Gamma_0} \int_0^1 \int_{\delta_1}^{\delta_2} z e^{-z\varrho} |a_2(z)| \mathcal{G}(\psi(x, \varrho, z, t)) dz d\varrho d\Gamma \\ &\quad - \int_{\Gamma_0} \int_{\delta_1}^{\delta_2} |a_2(z)| [e^{-z} \mathcal{G}(\psi(x, 1, z, t)) - \mathcal{G}(\psi(x, 0, z, t))] dz d\Gamma.\end{aligned}\quad (89)$$

Since $\psi(x, 0, z, t) = \mathcal{W}_t(x, t)$ and $e^{-z} \leq e^{-z\varrho} \leq 1$ for $0 < \varrho < 1$, setting $\psi_1 = e^{-\delta_2}$ gives

$$\begin{aligned}N'(t) &\leq -\psi_1 \int_{\Gamma_0} \int_0^1 \int_{\delta_1}^{\delta_2} z |a_2(z)| \mathcal{G}(\psi(x, \varrho, z, t)) dz d\varrho d\Gamma \\ &\quad - \psi_1 \int_{\Gamma_0} \int_{\delta_1}^{\delta_2} |a_2(z)| \mathcal{G}(\psi(x, 1, z, t)) dz d\Gamma + \left(\int_{\delta_1}^{\delta_2} |a_2(z)| dz \right) \int_{\Gamma_0} \mathcal{G}(\mathcal{W}_t) d\Gamma.\end{aligned}\quad (90)$$

By Assumptions (4) and (5), we obtain (88). $\square$

Now, introduce the perturbed functional

$$\mathcal{Z}(t) := \mathcal{N}\mathcal{I}(t) + J(t) + L(t) + \mathcal{K}\mathcal{K}(t) + N(t), \quad (91)$$

where $\mathcal{N}, \mathcal{K} > 0$ are constants to be chosen later.

Lemma 11. *There exist constants $C_j, t_1 > 0$ ($j = 1, \dots, 5$) such that*

$$\mathcal{Z}'(t) \leq -\mathfrak{C}_1 \mathcal{I}(t) + \mathfrak{C}_2 \int_{\Gamma_0} \mathcal{F}_1^2(\mathcal{W}_t) d\Gamma + \mathfrak{C}_3 (\mathcal{H} \circ \nabla \mathcal{W})(t), \quad \forall t \geq t_1, \quad (92)$$

and

$$\mathfrak{C}_4 \mathcal{I}(t) \leq \mathcal{Z}(t) \leq \mathfrak{C}_5 \mathcal{I}(t). \quad (93)$$

Proof. Since $\mathcal{H}$ is continuous and positive, for any $t_1 > 0$, we have

$$\int_0^t \mathcal{H}(\vartheta) d\vartheta \geq \int_0^{t_1} \mathcal{H}(\vartheta) d\vartheta := \mathcal{H}_0, \quad \forall t \geq t_1. \quad (94)$$

Differentiating (91), using (35), Lemmas 8–10, and adding and subtracting $\lambda \mathcal{I}(t)$ with $\lambda > 0$, we obtain after lengthy computations

$$\begin{aligned} \mathcal{Z}'(t) &:= \mathcal{N}'\mathcal{I}'(t) + J'(t) + L'(t) + \mathcal{K}K'(t) + N'(t) \\ &\leq -\left(\mathcal{K}(\mathcal{H}_0 - \varepsilon_2) - 1 - \frac{\lambda}{2}\right) \|\mathcal{W}_t\|_2^2 - \left(1 - \frac{\lambda}{2}\right) \int_{\Gamma_0} \alpha(x) \mathcal{S}^2 d\Gamma \\ &\quad + \left(1 - \frac{\lambda}{p}\right) \int_{\Omega} |\mathcal{W}|^p \ln |\mathcal{W}| dx + \frac{\lambda}{p^2} \|\mathcal{W}\|_p^p - (\mathcal{N}\beta_0 - c\mathcal{K} - c(\varepsilon_1)) \int_{\Gamma_0} \mathcal{S}_t^2 d\Gamma \\ &\quad - \mathcal{N}c_1 \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\mu^2 + \delta) |\phi(\mu, t)|^2 d\mu d\Gamma - \lambda \mathcal{I}(t) \\ &\quad - \left(c_1 - \varepsilon_3 \mathcal{K} - \frac{c_1 \lambda}{2}\right) \int_{\Gamma_0} \int_{-\infty}^{+\infty} |\phi(\mu, t)|^2 d\mu d\Gamma \\ &\quad - \left\{ \kappa - 2\varepsilon_1 - c_p \varepsilon_2 - c_p^* \varepsilon_3 - \mathcal{K} \varepsilon_1 (2 - \kappa + \mathcal{H}^*) - \frac{\lambda}{2} \kappa \right\} \|\nabla \mathcal{W}\|_2^2 \\ &\quad + \left(c(\varepsilon_2) + \mathcal{K}c(\varepsilon_1) + \frac{\mathcal{N}}{2}\right) (\mathcal{H} \circ \nabla \mathcal{W})(t) + \left(\frac{\mathcal{N}}{2} - \mathcal{K}c(\varepsilon_2)\right) (\mathcal{H}' \circ \nabla \mathcal{W})(t) \\ &\quad + (c(\varepsilon_1) + c\mathcal{K}) \int_{\Gamma_0} \mathcal{F}_1^2(\mathcal{W}_t) d\Gamma - (\sigma_1 \mathcal{N} - a_1) \int_{\Gamma_0} \mathcal{W}_t \mathcal{F}_1(\mathcal{W}_t) d\Gamma \\ &\quad + (c(\varepsilon_3) + \mathcal{K}c) \int_{\Gamma_0} \int_{\delta_1}^{\delta_2} |a_2(z)| \mathcal{F}_2^2(\psi(x, 1, z, t)) dz d\Gamma \\ &\quad - (\psi_1 - \lambda) \int_{\Gamma_0} \int_0^1 \int_{\delta_1}^{\delta_2} z |a_2(z)| \mathcal{G}(\psi(x, q, z, t)) dz dq d\Gamma \\ &\quad - (\sigma_2 \mathcal{N} + \psi_1 \tau_1) \int_{\Gamma_0} \int_{\delta_1}^{\delta_2} |a_2(z)| \psi(x, 1, z, t) \mathcal{F}_2(\psi(x, 1, z, t)) dz d\Gamma. \end{aligned} \quad (95)$$

We estimate $\|\mathcal{W}\|_p^p$ as follows

$$\begin{aligned} \|\mathcal{W}\|_p^p &\leq C_{p-1, \Omega}^p \|\nabla \mathcal{W}\|_2^p \leq C_{p-1, \Omega}^p \|\nabla \mathcal{W}\|_2^{\frac{p-2}{2}} \|\nabla \mathcal{W}\|_2^2 \\ &\leq C_{p-1, \Omega}^p \left(\frac{2p\mathcal{I}(0)}{\kappa(p-2)}\right)^{\frac{p-2}{2}} \|\nabla \mathcal{W}\|_2^2 = C_B \theta_0 \kappa \|\nabla \mathcal{W}\|_2^2, \end{aligned} \quad (96)$$

where $C_B = \frac{C_{p-1, \Omega}^p}{C_{p, \Omega}^{p+1}}$, and by (54), we have

$$\int_{\Omega} \ln |\mathcal{W}| |\mathcal{W}|^p dx \leq \theta_0 \kappa \|\nabla \mathcal{W}\|_2^2. \quad (97)$$

Substituting (96) and (97) into the expression for $\mathcal{Z}'(t)$ yields

$$\begin{aligned} \mathcal{Z}'(t) \leq & -\left(\mathcal{K}(\mathcal{H}_0 - \varepsilon_2) - 1 - \frac{\lambda}{2}\right) \|\mathcal{W}_t\|_2^2 - \left(1 - \frac{\lambda}{2}\right) \int_{\Gamma_0} \alpha(x) \mathcal{S}^2 d\Gamma \\ & + \left(1 - \frac{\lambda}{p}\right) \theta_0 \kappa \|\nabla \mathcal{W}\|_2^2 + \frac{\lambda}{p^2} C_B \theta_0 \kappa \|\nabla \mathcal{W}\|_2^2 - (\mathcal{N} \beta_0 - c\mathcal{K} - c(\varepsilon_1)) \int_{\Gamma_0} \mathcal{S}_t^2 d\Gamma \\ & - \mathcal{N} c_1 \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\mu^2 + \delta) |\phi(\mu, t)|^2 d\mu d\Gamma - \lambda \mathcal{I}(t) \\ & - \left(c_1 - \varepsilon_3 \mathcal{K} - \frac{c_1 \lambda}{2}\right) \int_{\Gamma_0} \int_{-\infty}^{+\infty} |\phi(\mu, t)|^2 d\mu d\Gamma \\ & - \left\{ \kappa - 2\varepsilon_1 - c_p \varepsilon_2 - c_p^* \varepsilon_3 - \mathcal{K} \varepsilon_1 (2 - \kappa + \mathcal{H}^*) - \frac{\lambda}{2} \kappa \right\} \|\nabla \mathcal{W}\|_2^2 \\ & + \left(c(\varepsilon_2) + \mathcal{K} c(\varepsilon_1) + \frac{\mathcal{N}}{2}\right) (\mathcal{H} \circ \nabla \mathcal{W})(t) + \left(\frac{\mathcal{N}}{2} - \mathcal{K} c(\varepsilon_2)\right) (\mathcal{H}' \circ \nabla \mathcal{W})(t) \\ & + (c(\varepsilon_1) + c\mathcal{K}) \int_{\Gamma_0} \mathcal{F}_1^2(\mathcal{W}_t) d\Gamma - (\sigma_1 \mathcal{N} - a_1) \int_{\Gamma_0} \mathcal{W}_t \mathcal{F}_1(\mathcal{W}_t) d\Gamma \\ & + (c(\varepsilon_3) + \mathcal{K} c) \int_{\Gamma_0} \int_{\delta_1}^{\delta_2} |a_2(z)| \mathcal{F}_2^2(\psi(x, 1, z, t)) dz d\Gamma \\ & - (\psi_1 - \lambda) \int_{\Gamma_0} \int_0^1 \int_{\delta_1}^{\delta_2} z |a_2(z)| \mathcal{G}(\psi(x, q, z, t)) dz dq d\Gamma \\ & - (\sigma_2 \mathcal{N} + \psi_1 \tau_1) \int_{\Gamma_0} \int_{\delta_1}^{\delta_2} |a_2(z)| \psi(x, 1, z, t) \mathcal{F}_2(\psi(x, 1, z, t)) dz d\Gamma. \end{aligned} \quad (98)$$

Now, choose ε_j ($j = 1, 2, 3$) sufficiently small so that

$$\kappa_1 := \kappa(1 - \theta_0) - 2\varepsilon_1 - c_p \varepsilon_2 - c_p^* \varepsilon_3 > 0. \quad (99)$$

Set

$$\varepsilon_2 = \frac{\mathcal{H}_0}{2}, \quad \varepsilon_3 = \frac{c_1}{2\mathcal{K}}, \quad (100)$$

and pick $\mathcal{K}$ large enough that

$$\frac{\mathcal{H}_0}{2} \mathcal{K} - 1 > 0, \quad (101)$$

which guarantees

$$\kappa_2 = \kappa_1 - \varepsilon_1 \mathcal{K} (2 - \kappa + \mathcal{H}^*) > 0. \quad (102)$$

Next, choose $\lambda > 0$ small enough so that

$$\begin{aligned} \beta_1 &= \kappa_2 - \lambda \left(\frac{\kappa}{2} + \frac{\theta_0 \kappa}{p^2} C_B - \frac{\theta_0 \kappa}{p} \right) > 0, \\ \beta_2 &= \frac{1}{2} - \frac{\lambda}{2} > 0, \quad \beta_3 = \psi_1 - \lambda > 0. \end{aligned} \quad (103)$$

Accordingly, once the positive parameters d_j ($j = 1, \dots, 5$) are fixed at suitable values, estimate (98) takes the form

$$\begin{aligned}
\mathcal{Z}'(t) \leq & -d_1 \|\mathcal{W}_t\|_2^2 - d_2 \|\nabla \mathcal{W}\|_2^2 + d_3 (\mathcal{H} \circ \nabla \mathcal{W})(t) + \left(\frac{\mathcal{N}}{2} - c\right) (\mathcal{H}' \circ \nabla \mathcal{W})(t) \\
& - \lambda \mathcal{I}(t) - \int_{\Gamma_0} \alpha(x) \mathcal{S}^2 d\Gamma - (\mathcal{N}\beta_0 - c) \int_{\Gamma_0} \mathcal{S}_t^2 d\Gamma + d_4 \int_{\Gamma_0} \mathcal{F}_1^2(\mathcal{W}_t) d\Gamma \\
& - (\sigma_1 \mathcal{N} - a_1) \int_{\Gamma_0} \mathcal{W}_t \mathcal{F}_1(\mathcal{W}_t) d\Gamma - \mathcal{N}c_1 \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\mu^2 + \delta) |\phi(\mu, t)|^2 d\mu d\Gamma \\
& - \frac{c_1}{2} \int_{\Gamma_0} \int_{-\infty}^{+\infty} |\phi(\mu, t)|^2 d\mu d\Gamma - \beta_3 \int_{\Gamma_0} \int_0^1 \int_{\delta_1}^{\delta_2} z |a_2(z)| \mathcal{G}(\psi(x, \varrho, z, t)) dz d\varrho d\Gamma \\
& - (\sigma_2 \mathcal{N} + \psi_1 \tau_1 - c_2 d_5) \int_{\Gamma_0} \int_{\delta_1}^{\delta_2} |a_2(z)| \psi(x, 1, z, t) \mathcal{F}_2(\psi(x, 1, z, t)) dz d\Gamma.
\end{aligned} \tag{104}$$

Using (Assumption 4), we obtain

$$\begin{aligned}
\mathcal{Z}'(t) \leq & -d_1 \|\mathcal{W}_t\|_2^2 - d_2 \|\nabla \mathcal{W}\|_2^2 + d_3 (\mathcal{H} \circ \nabla \mathcal{W})(t) + \left(\frac{\mathcal{N}}{2} - c\right) (\mathcal{H}' \circ \nabla \mathcal{W})(t) \\
& - \lambda \mathcal{I}(t) - \int_{\Gamma_0} \alpha(x) \mathcal{S}^2 d\Gamma - (\mathcal{N}\beta_0 - c) \int_{\Gamma_0} \mathcal{S}_t^2 d\Gamma + d_4 \int_{\Gamma_0} \mathcal{F}_1^2(\mathcal{W}_t) d\Gamma \\
& - (\sigma_1 \mathcal{N} - a_1) \int_{\Gamma_0} \mathcal{W}_t \mathcal{F}_1(\mathcal{W}_t) d\Gamma - \mathcal{N}c_1 \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\mu^2 + \delta) |\phi(\mu, t)|^2 d\mu d\Gamma \\
& - \frac{c_1}{2} \int_{\Gamma_0} \int_{-\infty}^{+\infty} |\phi(\mu, t)|^2 d\mu d\Gamma - \beta_3 \int_{\Gamma_0} \int_0^1 \int_{\delta_1}^{\delta_2} z |a_2(z)| \mathcal{G}(\psi(x, \varrho, z, t)) dz d\varrho d\Gamma \\
& - (\sigma_2 \mathcal{N} + \psi_1 \tau_1 - c_2 d_5) \int_{\Gamma_0} \int_{\delta_1}^{\delta_2} |a_2(z)| \psi(x, 1, z, t) \mathcal{F}_2(\psi(x, 1, z, t)) dz d\Gamma.
\end{aligned} \tag{105}$$

Conversely, from (63), we have

$$\Sigma^2(x, \mu, t) = |\phi(\mu, t)|^2 + |\mathcal{W}(x, t)|^2 \xi^2(\mu) - 2\phi(\mu, t) \mathcal{W}(x, t) \xi(\mu). \tag{106}$$

Dividing by $(\mu^2 + \delta)$ and integrating over $\Gamma_0 \times (-\infty, +\infty)$, we obtain

$$\begin{aligned}
|L(t)| \leq & \frac{c_1}{2} \int_{\Gamma_0} \int_{-\infty}^{+\infty} \frac{|\phi(\mu, t)|^2}{(\mu^2 + \delta)} d\mu d\Gamma + \frac{c_1}{2} \int_{\Gamma_0} \int_{-\infty}^{+\infty} \frac{|\mathcal{W}(x, t)|^2 \xi^2(\mu)}{(\mu^2 + \delta)} d\mu d\Gamma \\
& + c_1 \int_{\Gamma_0} \int_{-\infty}^{+\infty} \frac{|\phi(\mu, t) \mathcal{W}(x, t) \xi(\mu)|}{(\mu^2 + \delta)} d\mu d\Gamma.
\end{aligned} \tag{107}$$

By Young's inequality, for any $\alpha > 0$,

$$\begin{aligned}
\int_{\Gamma_0} \int_{-\infty}^{+\infty} \frac{|\phi(\mu, t) \mathcal{W}(x, t) \xi(\mu)|}{(\mu^2 + \delta)} d\mu d\Gamma \leq & \frac{1}{4\alpha} \int_{\Gamma_0} \int_{-\infty}^{+\infty} \frac{|\phi(\mu, t)|^2}{(\mu^2 + \delta)} d\mu d\Gamma \\
& + \alpha \int_{\Gamma_0} \int_{-\infty}^{+\infty} \frac{|\mathcal{W}(x, t)|^2 \xi^2(\mu)}{(\mu^2 + \delta)} d\mu d\Gamma.
\end{aligned} \tag{108}$$

Combining (107) and (108), and using $\frac{1}{\mu^2 + \delta} < \frac{1}{\delta}$, we obtain

$$\begin{aligned}
|L(t)| \leq & \left(\frac{c_1(2\alpha + 1)}{4\alpha\delta}\right) \int_{\Gamma_0} \int_{-\infty}^{+\infty} |\phi(\mu, t)|^2 d\mu d\Gamma \\
& + \frac{c_1(2\alpha + 1)}{2\delta} \int_{\Gamma_0} |\mathcal{W}(x, t)|^2 \int_{-\infty}^{+\infty} \xi^2(\mu) d\mu d\Gamma.
\end{aligned} \tag{109}$$

By Lemma 2.1 in [12],

$$|L(t)| \leq \frac{c_1(2\alpha + 1)}{4\alpha\delta} \int_{\Gamma_0} \int_{-\infty}^{+\infty} |\phi(\mu, t)|^2 d\mu d\Gamma + B_0 C_{1,\Gamma_0}^2 \frac{c_1(2\alpha + 1)}{2\delta} \|\nabla \mathcal{W}\|_2^2, \tag{110}$$

where C_{s,Γ_0} are the trace-Sobolev embedding constants.

By combining the structural identities in (62) with the classical estimates of Young, Poincaré, and Hölder, and further incorporating the bound (110), we infer

$$\begin{aligned} |\mathcal{Z}(t) - \mathcal{NI}(t)| &\leq \frac{1}{2} \|\mathcal{W}_t(t)\|_2^2 + \left(\frac{\beta_1}{2\alpha_0} + \frac{1}{2\alpha_0} \right) \int_{\Gamma_0} \alpha(x) \mathcal{S}^2 d\Gamma \\ &\quad + \left(c_p + B_0 C_{1,\Gamma_0}^2 \frac{c_1(2\alpha+1)}{2\delta} \right) \|\nabla \mathcal{W}(t)\|_2^2 \\ &\quad + \frac{\mathcal{K}}{2} \left(\|\mathcal{W}_t(t)\|_2^2 + c_p(\mathcal{H} \circ \nabla \mathcal{W})(t) \right) \\ &\quad + \frac{c_1(2\alpha+1)}{4\alpha\delta} \int_{\Gamma_0} \int_{-\infty}^{+\infty} |\phi(\mu, t)|^2 d\mu d\Gamma \\ &\quad + \int_{\Gamma_0} \int_0^1 \int_{\delta_1}^{\delta_2} z e^{-z\varrho} |a_2(z)| \mathcal{G}(\psi(x, \varrho, z, t)) dz d\varrho d\Gamma. \end{aligned} \quad (111)$$

Since $e^{-z\varrho} < 1$, choosing $\alpha > 0$ appropriately yields

$$|\mathcal{Z}(t) - \mathcal{NI}(t)| \leq C\mathcal{I}(t). \quad (112)$$

Hence,

$$(\mathcal{N} - C)\mathcal{I}(t) \leq \mathcal{Z}(t) \leq (\mathcal{N} + C)\mathcal{I}(t). \quad (113)$$

Finally, select $\mathcal{N}$ large enough that

$$\mathcal{N} - C > 0, \quad \sigma_1 \mathcal{N} - a_1 > 0, \quad \frac{\mathcal{N}}{2} - c > 0, \quad \sigma_2 \mathcal{N} + \psi_1 \tau_1 - c_2 d_5 > 0, \quad \mathcal{N} \beta_0 - c > 0. \quad (114)$$

Exploiting (35), (104), and (112), we obtain (92) and (93). $\square$

Theorem 4. Assumptions (1)–(8). Then, there exist positive constants μ_1, μ_2, t_1 , and $\epsilon_0 \in (0, \eta]$ such that the energy of (33) satisfies

$$\mathcal{I}(t) \leq \mu_1 \Phi^{-1} \left\{ \mu_2 \left(1 + \int_{t_1}^t \rho(\omega) d\omega \right) \right\}, \quad \forall t \geq t_1, \quad (115)$$

where

$$\Phi(t) := \int_t^1 \frac{1}{\mathfrak{h}(\varrho)} d\varrho, \quad (116)$$

with

$$\mathfrak{h}(t) = \begin{cases} t, & \text{if } \Phi \text{ is linear on } [0, \eta], \\ t\Phi'(\epsilon_0 t), & \text{if } \Phi'(0) = 0 \text{ and } \Phi'' > 0 \text{ on } (0, \eta]. \end{cases} \quad (117)$$

Proof. Although the methodology is classical, we record every step so the presentation remains self-contained. Multiplying inequality (92) by the non-increasing weight $\rho(t)$ and exploiting hypothesis (Assumption 1) together with the energy identity (35), we obtain

$$\begin{aligned} \rho(t) \mathcal{Z}'(t) &\leq -\mathfrak{C}_1 \rho(t) \mathcal{I}(t) + \mathfrak{C}_2 \rho(t) \int_{\Gamma_0} \mathcal{F}_1^2(\mathcal{W}_t) d\Gamma + \mathfrak{C}_3 \rho(t) (\mathcal{H} \circ \nabla \mathcal{W})(t) \\ &\leq -\mathfrak{C}_1 \rho(t) \mathcal{I}(t) + \mathfrak{C}_2 \rho(t) \int_{\Gamma_0} \mathcal{F}_1^2(\mathcal{W}_t) d\Gamma - \mathfrak{C}_3 (\mathcal{H}' \circ \nabla \mathcal{W})(t) \\ &\leq -\mathfrak{C}_1 \rho(t) \mathcal{I}(t) + \mathfrak{C}_2 \rho(t) \int_{\Gamma_0} \mathcal{F}_1^2(\mathcal{W}_t) d\Gamma - 2\mathfrak{C}_3 \mathcal{I}'(t). \end{aligned} \quad (118)$$

Since $\rho(t)$ is non-increasing,

$$\frac{d}{dt}(\rho(t)\mathcal{Z}(t) + 2\mathfrak{C}_3\mathcal{I}(t)) \leq -\mathfrak{C}_1\rho(t)\mathcal{I}(t) + \mathfrak{C}_2\rho(t) \int_{\Gamma_0} \mathcal{F}_1^2(\mathcal{W}_t) d\Gamma. \quad (119)$$

Set

$$\mathcal{Q}(t) := \rho(t)\mathcal{Z}(t) + 2\mathfrak{C}_3\mathcal{I}(t) \sim \mathcal{I}(t). \quad (120)$$

Thus,

$$\mathcal{Q}'(t) \leq -\mathfrak{C}_1\rho(t)\mathcal{I}(t) + \mathfrak{C}_2\rho(t) \int_{\Gamma_0} \mathcal{F}_1^2(\mathcal{W}_t) d\Gamma, \quad \forall t \geq t_1. \quad (121)$$

To complete the proof, we must estimate the last term in (121). Define

$$\Gamma_0^1 := \{x \in \Gamma_0 : |\mathcal{W}_t| > \eta\} \quad \text{and} \quad \Gamma_0^2 := \{x \in \Gamma_0 : |\mathcal{W}_t| \leq \eta\}. \quad (122)$$

By (Assumption 3) and (35),

$$\mathfrak{C}_2\rho(t) \int_{\Gamma_0^1} \mathcal{F}_1^2(\mathcal{W}_t) d\Gamma \leq \mathfrak{C}_2\rho(0) \int_{\Gamma_0^1} \mathcal{F}_1^2(\mathcal{W}_t) d\Gamma \leq -\mu_3\mathcal{I}'(t), \quad (123)$$

where $\mu_3 = \frac{\mathfrak{C}_2\mathfrak{C}_1\rho(0)}{\sigma_1}$. We distinguish two cases.

Case 1: Φ is linear on $[0, \eta]$. By (Assumption 3) and (35),

$$\mathfrak{C}_2\rho(t) \int_{\Gamma_0^2} \mathcal{F}_1^2(\mathcal{W}_t) d\Gamma \leq \mathfrak{C}_2\rho(0) \int_{\Gamma_0^2} \mathcal{F}_1^2(\mathcal{W}_t) d\Gamma \leq -\mu_4\mathcal{I}'(t), \quad (124)$$

where $\mu_4 = \frac{\mathfrak{C}_2\mathfrak{C}_1\rho(0)}{\sigma_1}$. Substituting (123) and (124) into (121) yields

$$\begin{aligned} \mathcal{Q}'_1(t) &\leq -\mathfrak{C}_1\rho(t)\mathcal{I}(t) \\ &= -\mu_5\rho(t)\mathfrak{h}\left(\frac{\mathcal{I}(t)}{\mathcal{I}(0)}\right), \quad \forall t \geq t_1, \end{aligned} \quad (125)$$

where

$$\mathcal{Q}_1(t) = (\mathcal{Q}(t) + \mu\mathcal{I}(t)) \sim \mathcal{I}(t), \quad (126)$$

with $\mu = \mu_3 + \mu_4$ and $\mu_5 = \mathfrak{C}_1\mathcal{I}(0)$. Integrating (125) over (t_1, t) and using (126), we obtain (115).

Case 2: Φ is nonlinear. From (Assumption 3), (35), and Jensen's inequality Lemma 1 applied with $\Sigma = \Gamma_0^2$, $\mathcal{W}(x) = 1$, and $\mathcal{A}(x) = \Phi^{-1}(\mathcal{W}_t(x)\mathcal{F}_1(\mathcal{W}_t(x)))$, we obtain

$$\begin{aligned} \mu_2\rho(t) \int_{\Gamma_0^2} \mathcal{F}_1^2(\mathcal{W}_t) d\Gamma &\leq \mu_2\rho(t) \int_{\Gamma_0^2} \Phi^{-1}(\mathcal{W}_t\mathcal{F}_1(\mathcal{W}_t)) d\Gamma \\ &\leq \mu_2\rho(t)|\Gamma_0^2|\Phi^{-1}\left(\frac{1}{|\Gamma_0^2|} \int_{\Gamma_0^2} \mathcal{W}_t\mathcal{F}_1(\mathcal{W}_t) d\Gamma\right) \\ &\leq \mu_2\rho(t)|\Gamma_0^2|\Phi^{-1}\left(-\frac{\mathcal{I}'(t)}{\sigma_1|\Gamma_0^2|}\right). \end{aligned} \quad (127)$$

Substituting (123) and (127) into (121), we obtain

$$\mathcal{Q}'_2(t) \leq -\mathfrak{C}_1\rho(t)\mathcal{I}(t) + \mu_6\rho(t)\Phi^{-1}\left(-\frac{\mathcal{I}'(t)}{\sigma_1|\Gamma_0^2|}\right), \quad \forall t \geq t_1, \quad (128)$$

where

$$\mathcal{Q}_2(t) = (\mathcal{Q}(t) + \mu_3 \mathcal{I}(t)) \sim \mathcal{I}(t), \quad (129)$$

and $\mu_6 = \mu_2 |\Gamma_0^2|$.

Now, for $0 < \epsilon_0 < \eta$ and $\delta_0 > 0$, we compute

$$\begin{aligned} & \left\{ \Phi' \left(\epsilon_0 \frac{\mathcal{I}(t)}{\mathcal{I}(0)} \right) \mathcal{Q}_2(t) + \delta_0 \mathcal{I}(t) \right\}' \\ &= \epsilon_0 \frac{\mathcal{I}'(t)}{\mathcal{I}(0)} \Phi'' \left(\epsilon_0 \frac{\mathcal{I}(t)}{\mathcal{I}(0)} \right) \mathcal{Q}_2(t) + \Phi' \left(\epsilon_0 \frac{\mathcal{I}(t)}{\mathcal{I}(0)} \right) \mathcal{Q}_2'(t) + \delta_0 \mathcal{I}'(t) \\ &\leq -\mathfrak{C}_1 \rho(t) \Phi' \left(\epsilon_0 \frac{\mathcal{I}(t)}{\mathcal{I}(0)} \right) \mathcal{I}(t) + \mu_6 \rho(t) \Phi' \left(\epsilon_0 \frac{\mathcal{I}(t)}{\mathcal{I}(0)} \right) \Phi^{-1} \left(-\frac{\mathcal{I}'(t)}{\sigma_1 |\Gamma_0^2|} \right) + \delta_0 \mathcal{I}'(t). \end{aligned} \quad (130)$$

Choose ϵ_0 small enough that

$$\mu_7 := \mathfrak{C}_1 \mathcal{I}(0) - \mu_6 \epsilon_0 > 0, \quad (131)$$

and δ_0 so large that

$$\delta_0 - \mu_6 \rho(0) \sigma_1 |\Gamma_0^2| > 0. \quad (132)$$

Then,

$$\begin{aligned} \left\{ \Phi' \left(\epsilon_0 \frac{\mathcal{I}(t)}{\mathcal{I}(0)} \right) \mathcal{Q}_2(t) + \delta_0 \mathcal{I}(t) \right\}' &\leq -\mu_7 \rho(t) \Phi' \left(\epsilon_0 \frac{\mathcal{I}(t)}{\mathcal{I}(0)} \right) \frac{\mathcal{I}(t)}{\mathcal{I}(0)} \\ &= -\mu_7 \rho(t) \mathfrak{h} \left(\epsilon_0 \frac{\mathcal{I}(t)}{\mathcal{I}(0)} \right), \quad \forall t \geq t_1. \end{aligned} \quad (133)$$

Define

$$\mathcal{Q}_3(t) = \begin{cases} \mathcal{Q}_1(t), & \text{if } \Phi \text{ is linear on } [0, \eta], \\ \Phi' \left(\epsilon_0 \frac{\mathcal{I}(t)}{\mathcal{I}(0)} \right) \mathcal{Q}_2(t) + \delta_0 \mathcal{I}(t), & \text{if } \Phi'(0) = 0 \text{ and } \Phi'' > 0 \text{ on } (0, \eta], \end{cases} \quad (134)$$

Then, from (125) and (133),

$$\mathcal{Q}_3'(t) \leq -\mu_8 \rho(t) \mathfrak{h} \left(\epsilon_0 \frac{\mathcal{I}(t)}{\mathcal{I}(0)} \right), \quad \forall t \geq t_1. \quad (135)$$

Since $\mathcal{Q}_3(t) \sim \mathcal{I}(t)$, there exist $\mathcal{A}_1, \mathcal{A}_2 > 0$ such that

$$\mathcal{A}_1 \mathcal{Q}_3(t) \leq \mathcal{I}(t) \leq \mathcal{A}_2 \mathcal{Q}_3(t). \quad (136)$$

Introduce

$$\mathcal{Q}(t) := \mathcal{A}_1 \frac{\mathcal{Q}_3(t)}{\mathcal{I}(0)}. \quad (137)$$

By (136),

$$\mathcal{Q}(t) \leq \frac{\mathcal{I}(t)}{\mathcal{I}(0)} < 1. \quad (138)$$

Using (135), (136), (137), and the monotonicity of Φ , we obtain

$$\begin{aligned} \mathcal{Q}'(t) &\leq -\frac{\mathcal{A}_1 \mu_8}{\mathcal{I}(0)} \rho(t) \mathfrak{h} \left(\frac{\mathcal{I}(t)}{\mathcal{I}(0)} \right) \\ &\leq -\mu_9 \rho(t) \mathfrak{h} \left(\frac{\mathcal{I}(t)}{\mathcal{I}(0)} \right). \end{aligned} \quad (139)$$

Integrating (139) over (t_1, t) and using $\Phi'(t) = -\frac{1}{h(t)}$, we arrive at

$$\Phi(\mathcal{Q}(t)) - \Phi(\mathcal{Q}(0)) \geq \mu_9 \int_{t_1}^t \rho(\omega) d\omega. \quad (140)$$

Since Φ^{-1} is decreasing,

$$\mathcal{Q}(t) \leq \Phi^{-1} \left(\Phi(\mathcal{Q}(0)) + \mu_9 \int_{t_1}^t \rho(\omega) d\omega \right), \quad \forall t \geq t_1. \quad (141)$$

By the equivalence $\mathcal{Q}(t) \sim \mathcal{I}(t)$, we conclude (115). This completes the proof. $\square$

5. Finite-Time Blow-Up

This section demonstrates that solutions may suffer finite-time blow-up when the initial configuration carries negative total energy. The mechanism is a subtle resonance between the logarithmic source, the nonlocal Kirchhoff tension, and the boundary delay feedback acting on the acoustic boundary.

We first assemble the requisite analytical apparatus: a rigorous control of the logarithmic nonlinearity and interpolation inequalities that control the logarithmic nonlinearities in the natural energy space.

Lemma 12 (Control of the negative logarithmic part). *Let $\mathcal{W} \in H_{\Gamma_1}^1(\Omega)$ and let $\Omega_- = \{x \in \Omega : |\mathcal{W}(x)| < 1\}$. Then, for any $\rho \in (0, p-2)$, there exists $C_\rho = C(\rho, p, \Omega) > 0$ such that*

$$\int_{\Omega_-} |\mathcal{W}|^p \ln |\mathcal{W}| dx \leq C_\rho \|\nabla \mathcal{W}\|_2^2. \quad (142)$$

Consequently,

$$\int_{\Omega} |\mathcal{W}|^p \ln |\mathcal{W}| dx \geq \int_{\Omega_+} |\mathcal{W}|^p \ln |\mathcal{W}| dx - C_\rho \|\nabla \mathcal{W}\|_2^2, \quad (143)$$

where $\Omega_+ = \{x \in \Omega : |\mathcal{W}(x)| \geq 1\}$.

Proof. On Ω_- we have $0 < |\mathcal{W}| < 1$. By Lemma 4 with exponent $\rho \in (0, p-2)$,

$$|\ln |\mathcal{W}|| \leq \frac{|\mathcal{W}|^{-\rho}}{e\rho}.$$

Hence,

$$|\mathcal{W}|^p \ln |\mathcal{W}| \leq \frac{1}{e\rho} |\mathcal{W}|^{p-\rho}.$$

Since $p-\rho > 2$, the Sobolev–Poincaré embedding gives $|\mathcal{W}|^{p-\rho} \in L^1(\Omega)$ with

$$\int_{\Omega_-} |\mathcal{W}|^{p-\rho} dx \leq \int_{\Omega} |\mathcal{W}|^{p-\rho} dx \leq C_\rho \|\nabla \mathcal{W}\|_2^{p-\rho}.$$

Because $\|\nabla \mathcal{W}\|_2$ is bounded on bounded sets, the right-hand side is controlled by $C'_\rho \|\nabla \mathcal{W}\|_2^2$ (using $\|\nabla \mathcal{W}\|_2^{p-\rho} \leq C \|\nabla \mathcal{W}\|_2^2$ when $\|\nabla \mathcal{W}\|_2 \leq M$). In the blow-up regime, we shall only apply this to trajectories whose energy norm is bounded initially, so (142) follows. $\square$

Remark 5. For a solution with negative initial energy $\mathcal{E}(0) < 0$, the deficit energy

$$\mathcal{E}_-(t) = -\mathcal{E}(t) = \frac{1}{p} \int_{\Omega} |\mathcal{W}|^p \ln |\mathcal{W}| dx - \frac{1}{p^2} \|\mathcal{W}\|_p^p - \frac{1}{2} \|\mathcal{W}_t\|_2^2 - \dots$$

satisfies $\mathcal{E}_-(t) \geq \mathcal{E}_-(0) > 0$. Since all subtracted terms are non-negative, we immediately obtain the dynamical lower bound

$$\int_{\Omega} |\mathcal{W}|^p \ln |\mathcal{W}| dx \geq p \mathcal{E}_-(t) \geq p \mathcal{E}_-(0) > 0, \quad \forall t \in [0, T^*]. \quad (144)$$

Thus, along trajectories with $\mathcal{E}(0) < 0$, the logarithmic integral is strictly positive, even though it is not positive for arbitrary H^1 functions.

Lemma 13. *There exists $\mathfrak{C}_{**} = \mathfrak{C}_{**}(p, \Omega) > 0$ such that for every $\mathcal{W} \in H_{\Gamma_1}^1(\Omega)$ with $\int_{\Omega} |\mathcal{W}|^p \ln |\mathcal{W}| dx \geq 0$,*

$$\|\mathcal{W}\|_p^p \leq \mathfrak{C}_{**} \left(\int_{\Omega} |\mathcal{W}|^p \ln |\mathcal{W}| dx + \|\nabla \mathcal{W}\|_2^2 \right). \quad (145)$$

Lemma 14. *Assume $p \geq 2$. Then, there exists $\mathfrak{C}_{\diamond} > 0$ such that for all $2 \leq s \leq p$,*

$$\|\mathcal{W}\|_s^s \leq \mathfrak{C}_{\diamond} \left(\|\mathcal{W}\|_p^p + \|\mathcal{W}\|_2^2 \right), \quad (146)$$

and the same for the boundary delay distribution ψ .

Corollary 1. *There exists $\mathfrak{C}_{\bullet} = \mathfrak{C}_{\bullet}(p, \Omega) > 0$ such that for any $\mathcal{W}$ with $\int_{\Omega} |\mathcal{W}|^p \ln |\mathcal{W}| dx \geq 0$,*

$$\|\mathcal{W}\|_2^2 \leq \mathfrak{C}_{\bullet} \left\{ \left(\int_{\Omega} |\mathcal{W}|^p \ln |\mathcal{W}| dx \right)^{\frac{2}{p}} + \|\nabla \mathcal{W}\|_2^{\frac{4}{p}} \right\}, \quad (147)$$

and the same for the acoustic potential $\mathcal{S}$ on Γ_0 .

Define the deficit energy functional the mirror of the total energy that captures the excess of nonlinear focusing over all dissipative and restorative mechanisms:

$$\begin{aligned} \mathcal{E}_-(t) = -\mathcal{E}(t) &= \frac{1}{p} \int_{\Omega} |\mathcal{W}|^p \ln |\mathcal{W}| dx - \frac{1}{p^2} \|\mathcal{W}\|_p^p - \frac{1}{2} \|\mathcal{W}_t\|_2^2 - \frac{1}{2} \widehat{\mathcal{K}}(\|\nabla \mathcal{W}\|_2^2) + \frac{1}{2} \left(\int_0^t \mathcal{H}(\vartheta) d\vartheta \right) \|\nabla \mathcal{W}\|_2^2 \\ &\quad - \frac{1}{2} (\mathcal{H} \circ \nabla \mathcal{W})(t) - \frac{1}{2} \int_{\Gamma_0} \alpha(x) \mathcal{S}^2 d\Gamma - \frac{c_1}{2} \int_{\Gamma_0} \int_{-\infty}^{+\infty} |\phi(\mu, t)|^2 d\mu d\Gamma \\ &\quad - \int_{\Gamma_0} \int_0^1 \int_{\delta_1}^{\delta_2} z |a_2(z)| \mathcal{G}(\psi) dz d\varrho d\Gamma. \end{aligned} \quad (148)$$

Remark 6. *The local existence result (Theorem 2) provides a maximal existence time $T_{\max} \in (0, +\infty]$. By the standard continuation principle for hyperbolic systems, if $T_{\max} < +\infty$ then necessarily*

$$\lim_{t \rightarrow T_{\max}^-} \|(\mathcal{W}, \mathcal{W}_t, \mathcal{S}, \psi, \phi)\|_{\mathcal{F}} = +\infty,$$

where $\|\cdot\|_{\mathcal{F}}$ is defined in (58). Consequently, to prove finite-time blow-up, it suffices to exhibit an auxiliary functional that blows up in finite time and that dominates the norm $\|\cdot\|_{\mathcal{F}}$.

Theorem 5. *Assumptions (1)–(8). Let the initial data satisfy the compatibility conditions (32). Define the total initial energy $\mathcal{E}(0)$ by (34) evaluated at $t = 0$. If $\mathcal{E}(0) < 0$, then the solution $(\mathcal{W}, \mathcal{S}, \psi, \phi)$ of (33) blows up in finite time in the sense that there exists $T^* < +\infty$ such that*

$$\lim_{t \rightarrow T^*-} \left(\|\nabla \mathcal{W}(t)\|_2^2 + \|\mathcal{W}_t(t)\|_2^2 + \|\mathcal{W}(t)\|_p^p \right) = +\infty. \quad (149)$$

Moreover, the blow-up time T^* satisfies the explicit estimate

$$T^* \leq t_1 + \frac{2(1-\xi)}{\mathfrak{C}_4 \xi \mathcal{A}_1^{\frac{\xi}{1-\xi}}}, \quad (150)$$

where $\mathcal{A}(t)$ is the collapse accelerator functional defined in (154), $\mathcal{A}_1$ and t_1 are as in (172), $\xi \in (0, \frac{p-2}{2p})$, and $\mathfrak{C}_4 > 0$ is a structural constant.

Proof. The argument unfolds in three movements: establishing the energy deficit as a Lyapunov functional, constructing an auxiliary quantity that captures the acceleration of blow-up, and demonstrating that this quantity satisfies a superlinear differential inequality whose finite-time singularity is unavoidable.

From the dissipation inequality (35) and the hypothesis $\mathcal{E}(0) < 0$, we inherit the monotonicity

$$\mathcal{E}(t) \leq \mathcal{E}(0) < 0 \implies \mathcal{E}_-(t) = -\mathcal{E}(t) \geq \mathcal{E}_-(0) > 0. \quad (151)$$

The time derivative of the deficit is controlled by the dissipation itself

$$\begin{aligned} \mathcal{E}'_-(t) = -\mathcal{E}'(t) &\geq \sigma_1 \int_{\Gamma_0} \mathcal{W}_t \mathcal{F}_1(\mathcal{W}_t) d\Gamma + \int_{\Gamma_0} \beta(x) \mathcal{S}_t^2 d\Gamma + c_1 \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\mu^2 + \delta) |\phi(\mu, t)|^2 d\mu d\Gamma \\ &+ \sigma_2 \int_{\Gamma_0} \int_{\delta_1}^{\delta_2} |a_2(z)| \psi(x, 1, z, t) \mathcal{F}_2(\psi(x, 1, z, t)) dz d\Gamma \geq 0, \end{aligned} \quad (152)$$

for the structural constants $\sigma_1, \sigma_2 > 0$ appearing in Lemma 5. Consequently, the deficit energy is not merely positive but actively growing, bounded below by the nonlinear sources

$$0 < \mathcal{E}_-(0) \leq \mathcal{E}_-(t) \leq \frac{1}{p} \int_{\Omega} |\mathcal{W}|^p \ln |\mathcal{W}| dx. \quad (153)$$

Here, the last inequality follows from the definition (148) because all terms subtracted from $\frac{1}{p} \int |\mathcal{W}|^p \ln |\mathcal{W}| dx$ are non-negative; crucially, Remark 5 guarantees that the integral itself is positive along this trajectory.

Introduce the finite-time blow-up accelerator a weighted combination of the deficit energy and the mechanical correlation

$$\mathcal{A}(t) = \mathcal{E}_-^{1-\xi}(t) + \epsilon \int_{\Omega} \mathcal{W} \mathcal{W}_t dx + \frac{\epsilon \eta_0}{2} \int_{\Omega} \mathcal{W}^2 dx + \epsilon \int_{\Gamma_0} \mathcal{W} \mathcal{S} d\Gamma + \frac{\epsilon}{2} \int_{\Gamma_0} \beta(x) \mathcal{S}^2 d\Gamma + \mathcal{M} \mathcal{E}_-(t), \quad (154)$$

where $\epsilon > 0$ is small, $\eta_0 > 0$ is a fixed weight parameter, $\mathcal{M} > 0$ is a sufficiently large constant to be fixed in (161), and ξ obeys

$$0 < \xi < \frac{p-2}{2p}. \quad (155)$$

Differentiating (154) and invoking the governing Equation (33) yields the raw acceleration:

$$\begin{aligned} \mathcal{A}'(t) &= (1-\xi) \mathcal{E}_-^{-\xi}(t) \mathcal{E}'_-(t) + \epsilon \|\mathcal{W}_t\|_2^2 + \epsilon \int_{\Omega} \mathcal{W} \mathcal{W}_{tt} dx + \epsilon \eta_0 \int_{\Omega} \mathcal{W} \mathcal{W}_t dx \\ &+ \epsilon \int_{\Gamma_0} \mathcal{W}_t \mathcal{S} d\Gamma + \epsilon \int_{\Gamma_0} \mathcal{W} \mathcal{S}_t d\Gamma + \epsilon \int_{\Gamma_0} \beta(x) \mathcal{S} \mathcal{S}_t d\Gamma + \mathcal{M} \mathcal{E}'_-(t). \end{aligned} \quad (156)$$

Using the interior Equation (33)₁ together with integration by parts and the boundary condition (33)₂, the mechanical term expands as

$$\begin{aligned} \int_{\Omega} \mathcal{W} \mathcal{W}_{tt} dx &= -\mathcal{K}(\|\nabla \mathcal{W}\|_2^2) \|\nabla \mathcal{W}\|_2^2 + \int_{\Omega} |\mathcal{W}|^p \ln |\mathcal{W}| dx + \int_{\Omega} \nabla \mathcal{W}(t) \cdot \int_0^t \mathcal{H}(t-\vartheta) \nabla \mathcal{W}(\vartheta) d\vartheta dx \\ &+ \int_{\Gamma_0} \mathcal{W} \left[\mathcal{S}_t - a_1 \mathcal{F}_1(\mathcal{W}_t) - c_1 \int_{-\infty}^{+\infty} \phi(\mu, t) \xi(\mu) d\mu \right] d\Gamma \\ &- \int_{\Gamma_0} \mathcal{W} \int_{\delta_1}^{\delta_2} a_2(z) \mathcal{F}_2(\psi(x, 1, z, t)) dz d\Gamma. \end{aligned} \quad (157)$$

The memory integral is absorbed via Young's inequality and the hypothesis (Assumption 1):

$$\int_{\Omega} \nabla \mathcal{W}(t) \cdot \int_0^t \mathcal{H}(t-\vartheta) \nabla \mathcal{W}(\vartheta) d\vartheta dx \leq \frac{1-\ell}{2} \|\nabla \mathcal{W}(t)\|_2^2 + \frac{1}{2} (\mathcal{H} \circ \nabla \mathcal{W})(t). \quad (158)$$

Here, the hypothesis $\widehat{\mathcal{K}}(s) \geq s$ from (Assumption 7) is essential: combined with (Assumption 1) it yields the coercivity

$$\widehat{\mathcal{K}}(s) - \left(\int_0^t \mathcal{H}(\vartheta) d\vartheta \right) s \geq s - (1-\ell)s = \ell s, \quad \forall s \geq 0,$$

which is precisely what controls the negative memory contribution in (157); the condition $\mathcal{K}(s) \geq 1$ is used through $\mathcal{K}(\|\nabla \mathcal{W}\|_2^2) \|\nabla \mathcal{W}\|_2^2 \geq \|\nabla \mathcal{W}\|_2^2$ in (162) below.

The boundary terms are controlled explicitly as follows. For arbitrary $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$, by the trace embedding and Young's inequality,

$$\begin{aligned} \left| \int_{\Gamma_0} \mathcal{W} \mathcal{S}_t d\Gamma \right| &\leq \varepsilon_1 C_{tr}^2 \|\nabla \mathcal{W}\|_2^2 + C(\varepsilon_1) \|\mathcal{S}_t\|_{2,\Gamma_0}^2, \\ \left| \int_{\Gamma_0} \mathcal{W} \mathcal{F}_1(\mathcal{W}_t) d\Gamma \right| &\leq \varepsilon_2 C_{tr}^2 \|\nabla \mathcal{W}\|_2^2 + C(\varepsilon_2) \int_{\Gamma_0} \mathcal{F}_1^2(\mathcal{W}_t) d\Gamma, \\ \left| \int_{\Gamma_0} \mathcal{W} \int_{-\infty}^{+\infty} \phi \xi d\mu d\Gamma \right| &\leq \varepsilon_3 C_{tr}^2 \|\nabla \mathcal{W}\|_2^2 + \frac{C(\varepsilon_3)}{4} \int_{\Gamma_0} \int_{-\infty}^{+\infty} |\phi|^2 d\mu d\Gamma, \end{aligned} \quad (159)$$

where C_{tr} denotes the trace embedding constant. The acoustic identity $\mathcal{W}_t + \alpha \mathcal{S} + \beta \mathcal{S}_t = 0$ further yields

$$\begin{aligned} \int_{\Gamma_0} \mathcal{W}_t \mathcal{S} d\Gamma + \int_{\Gamma_0} \mathcal{W} \mathcal{S}_t d\Gamma + \int_{\Gamma_0} \beta(x) \mathcal{S} \mathcal{S}_t d\Gamma \\ = \int_{\Gamma_0} (\mathcal{W}_t + \beta(x) \mathcal{S}) \mathcal{S} d\Gamma + \int_{\Gamma_0} \mathcal{W} \mathcal{S}_t d\Gamma = - \int_{\Gamma_0} \alpha(x) \mathcal{S}^2 d\Gamma + \int_{\Gamma_0} \mathcal{W} \mathcal{S}_t d\Gamma. \end{aligned} \quad (160)$$

Because the dissipation inequality (152) dominates $\|\mathcal{S}_t\|_{2,\Gamma_0}^2$, $\int_{\Gamma_0} \mathcal{W}_t \mathcal{F}_1(\mathcal{W}_t) d\Gamma$, and the fractional flux ϕ , each boundary contribution is bounded by a fixed multiple of $\mathcal{E}'_-(t)$; more precisely, the last three terms in (160) together with the estimates (159) contribute at most $\varepsilon C(\varepsilon_1, \varepsilon_2, \varepsilon_3) \mathcal{E}'_-(t)$. We now fix the constant $\mathcal{M} > 0$ in (154) large enough that

$$(1-\xi) \mathcal{E}_-^{-\xi}(t) \mathcal{E}'_-(t) + \left(\mathcal{M} - \varepsilon C(\varepsilon_1, \varepsilon_2, \varepsilon_3) \right) \mathcal{E}'_-(t) \geq \frac{1-\xi}{2} \mathcal{E}_-^{-\xi}(t) \mathcal{E}'_-(t) + \frac{\mathcal{M}}{2} \mathcal{E}'_-(t), \quad (161)$$

which is legitimate because $\mathcal{E}'_-(t) \geq 0$ and no lower bound on $\mathcal{E}_-^{-\xi}(t)$ is required: the dangerous boundary terms are absorbed by $\mathcal{M} \mathcal{E}'_-(t)$ itself, while the term $(1-\xi) \mathcal{E}_-^{-\xi}(t) \mathcal{E}'_-(t)$

is left intact. Substituting (157)–(161) into (156) and using the Kirchhoff hypothesis $\mathcal{K}(s) \geq 1$ together with $\widehat{\mathcal{K}}(s) \geq s$, we obtain the intermediate acceleration

$$\begin{aligned} \mathcal{A}'(t) &\geq \mathfrak{c}_0 \mathcal{E}_-^{-\xi}(t) \mathcal{E}'_-(t) + \epsilon \left(1 + \frac{p}{2}\right) \|\mathcal{W}_t\|_2^2 + \epsilon \left(\frac{p\ell}{2} - 1\right) \|\nabla \mathcal{W}\|_2^2 \\ &\quad + \epsilon p \mathcal{E}_-(t) + \frac{\epsilon}{p} \|\mathcal{W}\|_p^p + \frac{\epsilon}{2} (\mathcal{H} \circ \nabla \mathcal{W})(t) \\ &\quad + \epsilon \eta_0 \int_{\Omega} \mathcal{W} \mathcal{W}_t dx - \epsilon \int_{\Gamma_0} \alpha(x) \mathcal{S}^2 d\Gamma + \frac{\mathcal{M}}{2} \mathcal{E}'_-(t). \end{aligned} \quad (162)$$

The logarithmic sources dominate the deficit energy through the exact decomposition

$$\epsilon \int_{\Omega} |\mathcal{W}|^p \ln |\mathcal{W}| dx = \epsilon p (1 - \alpha_1) \mathcal{E}_-(t) + \epsilon \alpha_1 \int_{\Omega} |\mathcal{W}|^p \ln |\mathcal{W}| dx + \frac{\epsilon p (1 - \alpha_1)}{2} \|\mathcal{W}_t\|_2^2 + \frac{\epsilon p (1 - \alpha_1) \ell}{2} \|\nabla \mathcal{W}\|_2^2, \quad (163)$$

for any $\alpha_1 \in (0, 1)$. An analogous identity holds for the boundary delay and fractional flux terms when extracted from $\mathcal{E}_-(t)$. Choosing α_1 so that the coefficients of $\|\nabla \mathcal{W}\|_2^2$ and $\|\mathcal{W}_t\|_2^2$ are positive, (162) simplifies to

$$\mathcal{A}'(t) \geq \mathfrak{C}^* \left\{ \mathcal{E}_-(t) + \|\mathcal{W}_t\|_2^2 + \|\mathcal{W}\|_p^p + \|\nabla \mathcal{W}\|_2^2 + (\mathcal{H} \circ \nabla \mathcal{W})(t) \right\} + \frac{\mathcal{M}}{2} \mathcal{E}'_-(t), \quad (164)$$

so that in particular $\mathcal{A}'(t) \geq \mathfrak{C}^* \mathcal{E}_-(0) > 0$, $\mathcal{A}$ is non-decreasing, and

$$\mathcal{A}(t) \geq \mathcal{A}(0) \geq \mathcal{E}_-(0)^{1-\xi} + \mathcal{M} \mathcal{E}_-(0) > 0, \quad \forall t > 0.$$

By Hölder's inequality and the Sobolev–Poincaré embedding,

$$\int_{\Omega} \mathcal{W} \mathcal{W}_t dx \leq \|\mathcal{W}\|_2 \|\mathcal{W}_t\|_2 \leq \mathfrak{C} (\|\mathcal{W}\|_p + \|\mathcal{W}_t\|_2). \quad (165)$$

Let $\mathfrak{C} > 0$ be the optimal constant. Young's inequality gives

$$\left| \int_{\Omega} \mathcal{W} \mathcal{W}_t dx \right|^{\frac{1}{1-\xi}} \leq \mathfrak{C}_1 \left(\|\mathcal{W}_t\|_2^{\frac{\psi'}{1-\xi}} + \|\mathcal{W}\|_p^{\frac{\psi'}{1-\xi}} \right), \quad (166)$$

with $\frac{1}{r} + \frac{1}{s} = 1$. Using Lemma 12 with $\psi' = 2(1 - \xi)$,

$$\frac{\psi'}{1 - \xi} = \frac{2}{1 - 2\xi} \leq p. \quad (167)$$

Thus, for $r = \frac{2}{1-2\xi}$:

$$\left| \int_{\Omega} \mathcal{W} \mathcal{W}_t dx \right|^{\frac{1}{1-\xi}} \leq \mathfrak{C}_2 \left(\|\mathcal{W}_t\|_2^2 + \|\mathcal{W}\|_p^p + \|\nabla \mathcal{W}\|_2^2 \right), \quad (168)$$

where $\mathfrak{C}_2 > 0$. The boundary correlation $\int_{\Gamma_0} \mathcal{W} \mathcal{S} d\Gamma$ obeys the analogous bound via the trace embedding, and the acoustic term is controlled by

$$\int_{\Gamma_0} \beta(x) \mathcal{S}^2 d\Gamma \leq \frac{\beta_1}{\alpha_0} \int_{\Gamma_0} \alpha(x) \mathcal{S}^2 d\Gamma \leq C (\mathcal{E}_-(t) + \mathcal{E}_-(t)^r) \leq \eta \mathcal{E}_-(t) + C_{\eta},$$

for some $r > 1$ and arbitrarily small $\eta > 0$, where we used $\|\nabla \mathcal{W}\|_2^2 \leq \frac{2}{\xi} \mathcal{E}_-(t)$ and Young's inequality. Consequently,

$$\begin{aligned} \mathcal{A}^{\frac{1}{1-\xi}}(t) &\leq \left(\mathcal{E}_-^{1-\xi}(t) + \int_{\Omega} \mathcal{W} \mathcal{W}_t dx + \int_{\Gamma_0} \mathcal{W} \mathcal{S} d\Gamma \right)^{\frac{1}{1-\xi}} \\ &\leq \mathfrak{C}_3 \left(\mathcal{E}_-^{1-\xi}(t) + \left| \int_{\Omega} \mathcal{W} \mathcal{W}_t dx \right|^{\frac{1}{1-\xi}} + \left| \int_{\Gamma_0} \mathcal{W} \mathcal{S} d\Gamma \right|^{\frac{1}{1-\xi}} \right) \\ &\leq \mathfrak{C}_3 \left(\mathcal{E}_-(t) + \|\mathcal{W}_t\|_2^2 + \|\mathcal{W}\|_p^p + \|\nabla \mathcal{W}\|_2^2 \right) + \mathfrak{C}_6 \end{aligned} \quad (169)$$

for all $t \geq 0$, with $\mathfrak{C}_3, \mathfrak{C}_6 > 0$. Combining (164) and (169):

$$\mathcal{A}'(t) \geq \mathfrak{C}_4 \mathcal{A}^{\frac{1}{1-\xi}}(t) - \mathfrak{C}_5, \quad t \geq 0, \quad (170)$$

where $\mathfrak{C}_4, \mathfrak{C}_5 > 0$. Set

$$\mathcal{A}_1 := \left(\frac{2\mathfrak{C}_5}{\mathfrak{C}_4} \right)^{1-\xi}, \quad (171)$$

and let $t_1 \geq 0$ be the first time at which $\mathcal{A}(t_1) \geq \mathcal{A}_1$; such a time exists and satisfies $t_1 \leq \max\{0, \mathcal{A}_1 - \mathcal{A}(0)\} / (\mathfrak{C}^* \mathcal{E}_-(0))$, because $\mathcal{A}'(t) \geq \mathfrak{C}^* \mathcal{E}_-(0) > 0$. For every $t \geq t_1$, we then have $\mathfrak{C}_4 \mathcal{A}^{1/(1-\xi)}(t) - \mathfrak{C}_5 \geq \frac{\mathfrak{C}_4}{2} \mathcal{A}^{1/(1-\xi)}(t)$, and integrating (170) over (t_1, t) yields

$$\mathcal{A}^{\frac{\xi}{1-\xi}}(t) \geq \frac{1}{\mathcal{A}^{\frac{-\xi}{1-\xi}}(t_1) - \frac{\mathfrak{C}_4 \xi (t-t_1)}{2(1-\xi)}}. \quad (172)$$

Therefore, the solution must cease to exist at a finite time T^* bounded by

$$T^* \leq t_1 + \frac{2(1-\xi)}{\mathfrak{C}_4 \xi \mathcal{A}_1^{\frac{\xi}{1-\xi}}}. \quad (173)$$

In view of Remark 6, the divergence of $\mathcal{A}(t)$ forces the energy norm $\|(\mathcal{W}, \mathcal{W}_t, \mathcal{S}, \psi, \phi)\|_{\mathcal{F}}$ to blow up; hence (149) holds. This completes the proof that the solution of system (33) cannot persist indefinitely when $\mathcal{E}(0) < 0$; the logarithmic focusing, acoustic dispersion, and boundary delay conspire to force finite-time blow-up. $\square$

6. Conclusions

In this paper, we examined the global existence of weak solutions, general decay, and finite-time blow-up for a viscoelastic Kirchhoff equation with a logarithmic source term, nonlinear distributed delay boundary feedback, and acoustic and fractional boundary conditions. The nonlocal Kirchhoff coefficient $\mathcal{K}(\|\nabla \mathcal{W}\|_2^2)$ has completely altered the principal operator, which requires a new energy functional expressed in terms of the primitive $\hat{\mathcal{K}}$ and forces the potential-well and multiplier arguments to be modified accordingly. We have rigorously identified the dimension-dependent restrictions on the exponent p , corrected the blow-up argument by controlling the negative part of the logarithmic nonlinearity, and established a precise continuation criterion.

From a physical viewpoint, our results indicate that the interplay between the Kirchhoff tension, viscoelastic memory, and boundary damping with fractional/delay terms determines the long-term fate of the system: for small initial energy, the dissipation prevails and solutions decay to equilibrium; for negative initial energy the logarithmic focusing dominates and drives finite-time blow-up. These findings suggest that in engineering applications involving viscoelastic actuators or acoustic boundary control, careful tuning

of the delay kernel $a_2(\cdot)$ and the fractional order γ is necessary to guarantee stability. Our results complement several recent contributions on evolution equations with logarithmic nonlinearities, memory, and delay. Local existence and blow-up for logarithmic wave equations with delay were obtained in [46], while global existence versus blow-up for hyperbolic equations with combined logarithmic and power-type sources was analysed in [47]. Uniform decay under mixed viscoelastic and frictional dissipation was established in [48], and the Cauchy problem for doubly dispersive nonlinear wave equations was treated in [49]. Kirchhoff-type problems with exponential growth were studied in [50], and the long-time behaviour of systems with logarithmic damping or logarithmic sources has been investigated for swelling porous models [51], infinite-memory wave equations [52], viscoelastic pseudo-parabolic Kirchhoff equations [53], and degenerate parabolic equations [54]. Finally, the interplay between delay, fractional damping, and logarithmic sources has recently been examined for wave equations and piezoelectric beams in [55–57]. In contrast with these works, the present paper combines a nonlocal Kirchhoff operator with acoustic, fractional, and distributed-delay boundary mechanisms in a single model.

Our results generalize the theory for the classical wave Equation (with $\mathcal{K} \equiv 1$); they hold for a very general class of relaxation kernels. Numerical verification of the abstract decay rates and explicit blow-up profiles, as well as the treatment of additional nonlinear damping terms, will be studied in future work.

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