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Connectivity Index of Fuzzy Rough Graphs with Applications to Uncertain Traffic Flow Networks

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Abstract

This paper introduces a connectivity index for fuzzy rough graphs that effectively integrates the mathematical frameworks of fuzzy set theory and rough set theory within graph-theoretic structures. We develop a comprehensive theoretical foundation by constructing fuzzy graph approximation spaces based on fuzzy minimal neighborhoods, enabling the definition of lower and upper approximation operators for both vertices and edges simultaneously. Building upon this framework, we establish rigorous definitions for connectivity measures in fuzzy rough graphs, including strongest directed paths, strength of connectedness, and various arc classifications. The proposed connectivity index quantifies the overall connectedness of fuzzy rough graphs by aggregating vertex membership degrees with path strengths across both approximation boundaries. We demonstrate that this index exhibits monotonicity properties with respect to fuzzy rough subgraphs and provides critical insights into network robustness through bridge identification. To validate the practical utility of our theoretical contributions, we present a compelling application in traffic flow network analysis, where the connectivity index successfully identifies the busiest intersections in urban transportation systems under conditions of uncertainty and incomplete information. The integration of fuzzy rough set theory with graph connectivity measures represents a significant advancement in handling real-world problems where relationships exist with varying degrees of certainty rather than binary connections.

Keywords: fuzzyrough digraphs; connectivity index; decision-making

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1. Introduction

The theory of rough sets was first introduced by Pawlak in 1982 [1]. In the context of intelligent systems, an accurate and comprehensive representation of knowledge is crucial for ensuring efficient functionality. In this regard, rough set theory offers effective tools for studying the structural and behavioral aspects of such systems, and it provides a basis for defining lower and upper approximation operators. Following this foundational work, many mathematicians and researchers have extended the theory and applied it to a variety of domains [2,3]. In particular, concepts such as closure operators and related topological ideas have played an important role in these advancements. This study highlights the

importance of a topological viewpoint in the analysis of systems, especially when dealing with uncertainty or incomplete data.

Graph theory, introduced in [4], together with general topology, form two fundamental and strongly interconnected areas of mathematics. Their interaction commonly arises through the construction of topological structures derived from the vertex and edge sets of graphs. Numerous investigations have utilized both directed and undirected graphs to define different types of topologies (see [5–7]), with a primary emphasis placed on simple undirected graphs, particularly their vertex sets. Relations on graphs provide a natural link between graph theory and topology, facilitating the development of new topological frameworks. As stated in [8], there exists a bijective relationship between labeled topologies on n elements and labeled transitive digraphs with n vertices. In 2010, Ref. [9] explored the relationship between directed graphs and finite topological spaces, while in 2013, the work in [10] introduced a topology defined on the vertex set of an undirected graph. More recently, increasing attention has been given to the integration of rough set theory with graph theory, motivated by applications in fields such as physics and medicine. Furthermore, in 2018, the study in [11] proposed an incidence topology on the vertex set of simple graphs without isolated vertices, contributing further to the connection between graph-theoretic and topological approaches.

The concept of fuzzy relations introduced by Zadeh [12] has played a fundamental role in extending classical relational structures to environments characterized by uncertainty. In particular, fuzzy relations provide a flexible mathematical tool for describing the degree of association between elements of a set rather than restricting the relation to binary values. Based on this idea, the notion of a fuzzy graph [13] was formally introduced by employing Zadeh's framework of fuzzy relations, where the strength of connections between vertices is represented by membership values. Later, a more systematic and comprehensive study of fuzzy graph theory was carried out in 1975 [8], in which fuzzy relations defined on fuzzy sets were investigated in detail, and several basic theoretical results were established. This work laid the foundation for the development of fuzzy graph theory by analyzing important structural characteristics of graphs such as trees, general graphs, and path graphs within the fuzzy environment. On the other hand, the integration of rough set theory with fuzzy set theory led to the development of fuzzy rough sets [14], which provide an effective framework for dealing with imprecise and incomplete information. This hybrid model combines the approximation mechanisms of rough sets with the graded membership concept of fuzzy sets, allowing for a more accurate representation of uncertainty in complex systems. In recent years, fuzzy rough set theory has attracted considerable attention due to its ability to handle real-world data containing vagueness and ambiguity. Consequently, it has been successfully applied to a variety of applications, including feature selection [15], data classification, and outlier detection in mixed and high-dimensional datasets [16].

In fuzzy rough set theory, the classical notions of lower and upper approximations from rough set theory are extended within the framework of fuzzy logic, resulting in fuzzy lower and fuzzy upper approximations. This integration offers several important advantages:

1. It provides a flexible and expressive framework for modeling uncertainty and imprecision by describing the boundaries of fuzzy rough membership functions. As a result, elements may belong to a set with varying degrees of membership, which more accurately reflects real-world situations.
2. It enables the direct handling of diverse data types, such as numerical, categorical, and symbolic data, without requiring prior data transformation. This preserves the inherent characteristics and diversity of the data for further analysis.

3. It strengthens reasoning and decision-making under ambiguity by allowing the classification of objects even when their attribute values do not exactly satisfy the criteria of a given class, thus offering a robust mechanism for managing uncertain and imprecise information.

Consequently, fuzzy rough set theory demonstrates strong potential for effectively identifying outliers in heterogeneous datasets characterized by uncertainty and imprecision. Additionally, several results concerning fuzzy rough graphs and their applications to decision-making problems have been investigated in [17–19].

To fully position the proposed fuzzy rough graph framework within the broader landscape of modern uncertainty modeling, it is essential to recognize recent advancements that generalize graph structures and aggregation operators to accommodate highly inconsistent, indeterminate, or vague information. For instance, complex vague graph structures have been developed to model multidimensional uncertainty in networks, illustrating how structural generalizations can effectively manage vagueness. Similarly, single-valued neutrosophic graph models provide powerful mechanisms for capturing indeterminacy that classical fuzzy sets cannot express, demonstrating significant practical utility in sectors such as food and agricultural organization [20]. Parallel to these graph-theoretic generalizations, the aggregation of uncertain information has seen rapid evolution to support complex decision-making. In multi-attribute decision-making scenarios, bipolar neutrosophic environments utilize advanced aggregation mechanisms, such as Dombi operators, to process bipolarity and indeterminacy simultaneously [21]. Furthermore, to capture independent parameter oscillations and complex attribute interactions, recent frameworks have introduced Fermatean fuzzy aggregation operators [22,23], alongside Fermatean hesitant fuzzy distance-based approaches [24]. These methodologies offer robust theoretical contexts for decision-support systems dealing with severe ambiguity. Additionally, the integration of soft set theory has yielded sophisticated structures, such as belief and possibility belief interval-valued N-soft sets [25], as well as m-polar Diophantine fuzzy N-soft sets [26], which excel in handling multi-polar and highly parameterized information.

Considering the complexity of the systems under analysis and the interdisciplinary nature of the associated phenomena, any decision-support protocol must take into account potential parameter oscillations, limited or incomplete data, and the use of inadequate models [27,28]. Despite these advances, uncertainty continues to pose a major challenge in safety management, arising from factors such as differences in expert opinions, insufficient or incomplete data, and semantic ambiguity in the available information. In response to these issues, the authors of [29,30] proposed approaches for large-scale group decision-making that incorporate cloud models within multi-granularity linguistic frameworks and bidirectional trust mechanisms in social networks. These methods are particularly effective for decision problems that demand consensus building and in-depth analysis within complex and dynamic environments.

The connectivity index in graph theory [31] serves as a fundamental measure of how well-connected the vertices of a graph are, traditionally quantifying the robustness and flow efficiency within network structures. The foundational work in [32,33] introduced the connectivity index, average connectivity index, neighborhood connectivity index, and related properties within the framework of directed rough fuzzy graphs, where approximations are derived from binary (crisp) relations. This approach suffers from two critical theoretical and practical limitations. First, the resulting fuzzy lower and upper approximations do not consistently satisfy the fundamental inclusion property of rough set theory: $\underline{A} \subseteq A \subseteq \overline{A}$. This violation leads to an inflated boundary region and consequently a reduced accuracy measure, undermining the reliability of uncertainty quantification in graph-theoretic contexts. Second, the restriction to equivalence relations severely constrains the modeling

capacity, as real-world systems often exhibit graded, non-equivalence relationships that cannot be adequately captured by crisp indiscernibility. To overcome these shortcomings, we propose a significant generalization by extending the connectivity index to directed fuzzy rough graphs, where both vertex and edge approximations are constructed using arbitrary fuzzy relations. This extension preserves the essential axioms of classical rough approximation spaces, including the crucial inclusion property, thereby yielding a more precise representation with a minimized boundary region and enhanced accuracy. Moreover, by liberating the model from the confines of equivalence relations, our framework substantially broadens the scope of applicability to complex, uncertain networks in domains such as intelligent transportation, social dynamics, and decision-support systems under vagueness.

This paper addresses this critical limitation by introducing a novel and significantly more general framework for fuzzy rough graphs based on fuzzy minimal neighborhoods derived from arbitrary fuzzy relations. Our fundamental breakthrough lies in the construction of approximation operators that make no assumptions about the properties of the underlying fuzzy relations on vertex and edge sets. Unlike previous approaches that require predefined relation characteristics, our framework operates on any fuzzy relation R on the vertex set $N(G)$ of a graph G and any fuzzy relation W on the edge set $D(G)$, thereby eliminating the restrictive assumptions that have plagued existing fuzzy rough graph models. The core of our approach centers on the innovative concept of fuzzy minimal neighborhoods, defined through intersection operations over non-zero relation values. Specifically, for any vertex $x \in N(G)$, we construct the fuzzy minimal neighborhood $W < x > W$, which represents the intersection of fuzzy right neighborhoods $< x > W$ and fuzzy left neighborhoods $W < x >$. This construction naturally handles the uncertainty, directionality, and asymmetry inherent in real-world networks without requiring any additional relation properties. Building upon this generalized foundation, we develop a comprehensive connectivity index for fuzzy rough graphs that quantifies network strength by combining connectivity measures from both lower and upper approximations. This index provides a robust quantitative tool for analyzing network robustness under uncertainty, enabling systematic evaluation of structural integrity in complex systems characterized by incomplete or imprecise information. The proposed connectivity index exhibits monotonicity under subgraph operations, with the connectivity index of any partial fuzzy rough subgraph being bounded above by that of the original graph. The removal of fuzzy rough directed bridge arcs whose deletion reduces connectivity in both approximations results in a strict decrease in the overall connectivity index, providing a clear mathematical criterion for identifying critical network components. Our framework encompasses all existing fuzzy rough graph models as special cases while significantly extending their applicability to arbitrary relation structures. The practical effectiveness of our approach is compellingly demonstrated through a case study application in urban traffic flow analysis. In this application, our method successfully identifies the busiest intersections in a city network by quantifying the disruption caused by junction removal through connectivity index calculations.

2. Preliminaries

In this work, we consider simple directed graphs that may either include loops or be loop-free. The notation G is used to denote a graph. For brevity, the phrase "simple directed graph with or without loops" will be referred to simply as a "graph." Let G denote the underlying graph (domain), $\mathbf{I}$ represent the unit interval $[0, 1]$, and $\mathbf{I}^G$ denote the set of all fuzzy graphs defined on G. For a fuzzy subgraph Ξ_G, its complement is denoted by $(\Xi_G)^c$; whenever there is no ambiguity, we use the notation $\overline{\Xi_G}$. A fuzzy graph approximation

space (abbreviated as fuzzy graph AS) is represented by $(\mathrm{G}, \mathrm{R}, \mathrm{W})$. Any terms or notions not explicitly defined in this paper can be found in [8,31,32,34].

A graph G is represented as the pair $(\mathrm{N}(\mathrm{G}), \mathrm{D}(\mathrm{G}))$, where $\mathrm{N}(\mathrm{G})$ is a nonempty finite or infinite set and $\mathrm{D}(\mathrm{G})$ is a set of ordered pairs of members of $\mathrm{N}(\mathrm{G})$. The sets $\mathrm{N}(\mathrm{G}) = \mathrm{N}$ and $\mathrm{D}(\mathrm{G}) = \mathrm{D}$ are the vertex and edge sets of G, respectively. A graph is called an empty graph if it contains a set of vertices but has no edges connecting them. In other words, none of the vertices are adjacent to each other, and the edge set is completely empty. A simple graph, on the other hand, is a graph in which each edge connects two distinct vertices, with no loops (edges connecting a vertex to itself) and no multiple edges between the same pair of vertices. Simple graphs represent the most basic form of a graph, where adjacency between vertices is unambiguous, making them fundamental in both theoretical studies and practical applications of graph theory. A fuzzy graph structured over an underlying graph G is defined by the pair $\mu_{\mathrm{G}} = (\mathrm{N}(\mu), \mathrm{D}(\mu)) = (\mu_{\mathrm{N(G)}}, \mu_{\mathrm{D(G)}}) \in \mathbf{I}^{\mathrm{G}}$. Within this fuzzy graph $(\mu_{\mathrm{N(G)}}, \mu_{\mathrm{D(G)}})$, the functions $\mu_{\mathrm{N(G)}}$ and $\mu_{\mathrm{D(G)}}$ designate the fuzzy vertex and fuzzy edge membership functions, respectively.

For the vertex mapping $\mu_{\mathrm{N(G)}} : \mathrm{N}(\mathrm{G}) \longrightarrow [0,1]$, each element $\mathrm{x} \in \mathrm{N}(\mathrm{G})$ is assigned a membership degree given by $\mu_{\mathrm{N(G)}}(\mathrm{x})$ within $\mu_{\mathrm{N(G)}}$. Similarly, under the edge mapping $\mu_{\mathrm{D(G)}} : \mathrm{D}(\mathrm{G}) \longrightarrow [0,1]$, every edge $\mathrm{y} \in \mathrm{D}(\mathrm{G})$ possesses a membership degree represented by $\mu_{\mathrm{D(G)}}(\mathrm{y})$ in $\mu_{\mathrm{D(G)}}$, where $\mathrm{D}(\mathrm{G}) = \Theta \subseteq \mathrm{N}(\mathrm{G}) \times \mathrm{N}(\mathrm{G})$. A fuzzy graph $(\mu_{\mathrm{N(G)}}, \mu_{\mathrm{D(G)}}) \in \mathbf{I}^{\mathrm{G}}$ over the graph G is called a strong fuzzy graph if $\mu_{\mathrm{D(G)}}(\mathrm{y}) = \mu_{\mathrm{N(G)}}(\mathrm{x}_1) \wedge \mu_{\mathrm{N(G)}}(\mathrm{x}_2)$, where y connects x_1 to x_2, for all $\mathrm{y} \in \mathrm{D}(\mathrm{G})$. Let $\mu^1_{\mathrm{G}} = (\mu^1_{\mathrm{N(G)}}, \mu^1_{\mathrm{D(G)}}), \mu^2_{\mathrm{G}} = (\mu^2_{\mathrm{N(G)}}, \mu^2_{\mathrm{D(G)}}), \in \mathbf{I}^{\mathrm{G}}$ be two fuzzy graphs over the graph G. Then, $(\mu^1_{\mathrm{N(G)}}, \mu^2_{\mathrm{D(G)}})$ is called a fuzzy subgraph of the fuzzy graph $(\mu^2_{\mathrm{N(G)}}, \mu^2_{\mathrm{D(G)}})$ if $\mu^1_{\mathrm{N(G)}}(\mathrm{x}) \leq \mu^2_{\mathrm{N(G)}}(\mathrm{x})$ and $\mu^1_{\mathrm{D(G)}}(\mathrm{y}) \leq \mu^2_{\mathrm{D(G)}}(\mathrm{y})$. The union and intersection of fuzzy graphs extend the classical set operations to graphs with fuzzy membership values. In the union of two fuzzy graphs, each vertex or edge is assigned a membership value equal to the maximum of its membership values in the individual graphs, effectively combining their structures while retaining the highest degree of membership. Conversely, in the intersection of two fuzzy graphs, each vertex or edge is assigned a membership value equal to the minimum of its membership values in the original graphs, capturing only the elements that are strongly present in both graphs. These operations provide a systematic way to combine or compare fuzzy graph structures while preserving the inherent uncertainty represented by fuzzy memberships.

Definition 1 ([14]). *Let (G, R, W) be a fuzzy graph approximation space, where R is an equivalence relation on the vertices of G and let Ξ be a fuzzy subgraph of G. For each vertex $x \in N(G)$, define the lower and upper approximations of the fuzzy vertex set $N(\Xi) \in \mathbf{I}^{N(G)}$ as the fuzzy sets $(N(\Xi))_*, (N(\Xi))^* \in \mathbf{I}^{N(G)}$ given by*

$$(N(\Xi))_*(x) = \bigwedge_{x' \in N(G)} \Big(1 - R(x,x') \vee \mu_{N(\Xi)}(x')\Big),$$

$$(N(\Xi))^*(x) = \bigvee_{x' \in N(G)} \Big(R(x,x') \wedge \mu_{N(\Xi)}(x')\Big).$$

The pair $(N(\Xi)_*, N(\Xi)^*)$ is called fuzzy rough set if $N(\Xi))^* - N(\Xi))_* \neq \emptyset$.

Definition 2 ([17,18]). *Consider a fuzzy graph approximation space denoted by (G, R, W), where Ξ represents a fuzzy subgraph of G. Let $([N(\Xi)]^R, [N(\Xi)]_R)$ be a fuzzy rough vertex set defined on G. Furthermore, assume W is an equivalence fuzzy relation acting on the edge set $D(G)$ of the graph G, satisfying the condition $W(x_1x_2, x_1'x_2') \leq R(x_1, x_1') \wedge R(x_2, x_2')$ for all $x_1x_2, x_1'x_2' \in D(G)$. Additionally, suppose γ is a fuzzy subgraph of G such that $\gamma_{D(G)}(xx') \leq [N(\Xi)]_R(x) \wedge [N(\Xi)]_R(x')$*

holds for every $xx' \in D(G)$. *Then, for every edge point* $y \in D(G)$, *define* $(D(\Xi))_*, (D(\Xi))^* \in \mathbf{I}^{D(G)}$ *of the fuzzy edge set* $D(\Xi) \in \mathbf{I}^{D(G)}$ *by*

$$(D(\Xi))_*(y) = \bigwedge_{y' \in D(G)} \Big(1 - W(y,y') \vee \mu_{D(\gamma)}\Big),$$

$$(D(\Xi))^*(y) = \bigvee_{y' \in D(G)} W(y,y') \wedge \mu_{D(\gamma)}).$$

The pair $(D(\Xi))_*, D(\Xi))^*$ *is called fuzzy rough relation.*

3. Fuzzy Rough Graphs Based on Fuzzy Minimal Neighborhoods

This section presents the formulation of fuzzy graph approximation spaces and investigates their key properties. It introduces the concept of fuzzy minimal neighborhoods in the setting of fuzzy graph theory and explains their role in defining new forms of fuzzy approximations. In particular, fuzzy lower and upper approximation operators are constructed using fuzzy relations together with fuzzy minimal neighborhoods. In addition, the section studies the connections between these approximation operators and compares them with existing methods in the literature, highlighting the enhanced generality and adaptability of the proposed approach. Moreover, the notions of fuzzy interior and fuzzy closure graphs associated with fuzzy rough graphs are examined, demonstrating their effectiveness in providing a deeper analysis of the structural features and properties of fuzzy graphs.

Definition 3. *Let* $G = (N(G), D(G))$ *be a crisp graph. The notation* $\underline{0}_G = (\underline{0}_{N(G)}, \underline{0}_{D(G)}) \in \mathbf{I}^G$ *denotes the empty fuzzy graph on G, where all vertices and edges are assigned membership value* 0. *Similarly,* $\underline{1}_G = (\underline{1}_{N(G)}, \underline{1}_{D(G)}) \in \mathbf{I}^G$ *represents the full fuzzy graph on G, in which every vertex and edge has membership value* 1. *It is worth noting that the crisp graph G coincides with the full fuzzy graph.*

Definition 4. *Let* $\mu_G = (\mu_{N(G)}, \mu_{D(G)}) \in \mathbf{I}^G$ *be a fuzzy graph defined on G. The complement of* μ_G *is defined as* $\mu^c G = (\overline{\mu}_{N(G)}, \overline{\mu}_{D(G)}) \in \mathbf{I}^G$, *where* $\overline{\mu}_{N(G)}(x) = 1 - \mu_{N(G)}(x)$ *for each vertex* $x \in N(G)$, *and* $\overline{\mu}_{D(G)}(y) = \min \overline{\mu}_{N(G)}(x_1), \overline{\mu}_{N(G)}(x_2)$ *for every edge* $y \in D(G)$ *connecting the vertices* x_1 *and* x_2. *It follows that the complement of any fuzzy graph is a strong fuzzy graph.*

Let us define the fuzzy difference between two fuzzy graphs as follows:

$$(\Xi_G \overline{\wedge} \complement_G) = \begin{cases} \underline{0}_G, & \text{if } \Xi_G \leq \complement_G; \\ \Xi_G \wedge \complement_G^c, & \text{otherwise.} \end{cases}$$

Definition 5. *Let* $N(G)$ *denote the vertex set of a finite graph G, and let R be a fuzzy relation on* $N(G)$. *For each* $x \in N(G)$, *define the fuzzy sets* $xR, Rx \in \mathbf{I}^{N(G)}$ *by*

$$xR(x') = R(x,x'), \quad \text{and} \quad Rx(x') = R(x',x), \ \forall x' \in N(G).$$

For any $x \in N(G)$, *define the fuzzy sets* $< x > R, R < x > \in \mathbf{I}^{N(G)}$ *as*

$$< x > R = \bigwedge_{x' \in N(G): R(x',x) > 0} x'R, \quad R < x > = \bigwedge_{x' \in N(G): R(x,x') > 0} Rx'.$$

Then, for each $x \in N(G)$, the fuzzy minimal neighborhood $R < x > R : N(G) \to \mathbf{I}$ is defined by

$$R < x > R =< x > R \wedge R < x > .$$

Definition 6. *Let $D(G)$ be the edge set of a finite graph G, and let W be a fuzzy relation on $D(G)$. For each $y \in D(G)$, define the fuzzy sets $yW, Wy \in \mathbf{I}^{D(G)}$ by*

$$yW(y') = W(y, y'), \quad \text{and} \quad Wy(y') = W(y', y), \ \forall y' \in D(G).$$

For any $y \in D(G)$, define the fuzzy sets $< y > W, W < y >\in \mathbf{I}^{D(G)}$ as

$$< y > W = \bigwedge_{y' \in D(G): W(y', y) > 0} y'W, \quad W < y >= \bigwedge_{y' \in D(G): W(y, y') > 0} Wy'.$$

Accordingly, for each $y \in D(G)$, the fuzzy minimal neighborhood $W < y > W : D(G) \to \mathbf{I}$ is given by

$$W < y > W =< y > W \wedge W < y > .$$

Definition 7. *Let (G, R, W) be a fuzzy graph approximation space and Ξ be a fuzzy subgraph of the graph G. For every vertex point $x \in N(G)$, define $(N(\Xi))_*, (N(\Xi))^* \in \mathbf{I}^{N(G)}$ of a fuzzy vertex set $N(\Xi) \in \mathbf{I}^{N(G)}$ by*

$$(N(\Xi))_*(x) = \bigwedge_{x' \in N(G)} \Big(1 - \big(R < x' > R(x)\big) \vee \mu_{N(\Xi)}(x')\Big),$$

$$(N(\Xi))^*(x) = \bigvee_{x' \in N(G)} \Big(R < x' > R(x)) \wedge \mu_{N(\Xi)}(x')\Big).$$

The roughness of a fuzzy vertex set $N(\Xi) \in \mathbf{I}^{N(G)}$ is defined by

$$[N(\Xi)]_R = N(\Xi) \wedge (N(\Xi))_* \text{ and } [N(\Xi)]^R = N(\Xi) \vee (N(\Xi))^*$$

$[N(\Xi)]_R$ is the lower fuzzy vertex set of $N(\Xi)$ and $N(\Xi)^R$ is the upper fuzzy vertex set of $N(\Xi)$. The boundary fuzzy region of $N(\Xi)$ is $(N(\Xi))^B$ given by $(N(\Xi))^B = [N(\Xi)]^R \overline{\wedge} [N(\Xi)]_R$. The pair $([N(\Xi)]^R, [N(\Xi)]_R)$ will be called a fuzzy rough vertex set if $[N(\Xi)]^R \overline{\wedge} [N(\Xi)]_R \neq \underline{0}_{N(G)}$.

Definition 8. *Let (G, R, W) be a fuzzy graph approximation space, and let Ξ be a fuzzy subgraph of G. Assume that $([N(\Xi)]^R, [N(\Xi)]_R)$ forms a fuzzy rough vertex set on G. Let W be a fuzzy relation on the edge set $D(G)$ satisfying*

$$W(x_1x_2, x_1'x_2') \leq R(x_1, x_1') \wedge R(x_2, x_2')$$

for all $x_1x_2, x_1'x_2' \in D(G)$. Moreover, let γ be a fuzzy subgraph of G such that

$$\gamma_{D(G)}(xx') \leq [N(\Xi)]_R(x) \wedge [N(\Xi)]_R(x')$$

for every $xx' \in D(G)$.

Then, for each $y \in D(G)$, define the fuzzy sets $(D(\Xi))_*, (D(\Xi))^* \in \mathbf{I}^{D(G)}$ associated with $D(\Xi)$ by

$$(D(\Xi))_*(y) = \bigwedge_{y' \in D(G)} \Big(1 - \big(W < y' > W(y)\big) \vee \mu_{D(\gamma)}\Big),$$

$$(D(\Xi))^*(y) = \bigvee_{y' \in D(G)} \Big(W < y' > W(y) \wedge \mu_{D(\gamma)}\Big).$$

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The rough approximations of the fuzzy edge set $D(\Xi) \in \mathbf{I}^{D(G)}$ *are defined as*

$$[D(\Xi)]_W = D(\Xi) \wedge (D(\Xi))_*, \quad [D(\Xi)]^W = D(\Xi) \vee (D(\Xi))^*.$$

Here, $[D(\Xi)]_W$ *and* $[D(\Xi)]^W$ *are called the lower and upper fuzzy edge approximations of* $D(\Xi)$*, respectively. The boundary region of* $D(\Xi)$*, denoted by* $(D(\Xi))^B$*, is given by*

$$(D(\Xi))^B = [D(\Xi)]^W \overline{\wedge} [D(\Xi)]_W.$$

The pair $([D(\Xi)]^W, [D(\Xi)]_W)$ *is said to be a fuzzy rough edge set (or fuzzy rough relation) whenever*

$$[D(\Xi)]^W \overline{\wedge} [D(\Xi)]_W \neq \underline{0}_{D(G)}.$$

Example 1. *Assume that* G *is the graph* $(N(G), D(G))$*, where* $N(G) = \{x_1, x_2, x_3, x_4, x_5\}$ *and* $D(G) = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_7, y_{10}\}$.

Now, we define a fuzzy graph $\Xi_G = (\Xi_{N(G)}, \Xi_{D(G)})$ *over the crisp graph* G*. We assume that the fuzzy vertex set is*

$$\Xi_{N(G)} = \{(x_1, 0.9), (x_2, 1.0), (x_3, 0.8), (x_4, 0.9), (x_5, 1.0)\}$$

and the fuzzy edge set is

$$\Xi_{D(G)} = \{(y_1, 0.5), (y_2, 0.7), (y_3, 0.6), (y_4, 0.7), (y_5, 0.5), (y_6, 0.5), (y_7, 0.7), (y_8, 0.4), (y_9, 0.6), (y_{10}, 0.7)\}.$$

The fuzzy subgraph Ξ_G *is represented in Figure 1.*

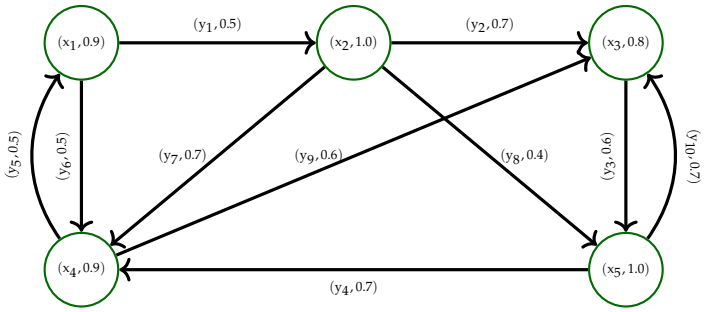


Figure 1. The fuzzy subgraph Ξ_G given in Example 1.

Define a fuzzy binary relation R *on* $N(G)$ *as given in Table 1.*

Table 1. Fuzzy relation R in Example 1.

R	x_1	x_2	x_3	x_4	x_5
x_1	0.5	0.6	0.4	0.4	0.4
x_2	0.7	0.3	0.4	0.5	0.3
x_3	0.4	0.4	0.2	0.6	0.5
x_4	0.4	0.4	0.4	0.7	0.6
x_5	0.8	0.8	0.3	0.8	0.7

By computing, $< x_1 > R =< x_2 > R =< x_3 > R =< x_4 > R =< x_5 > R = \{0.4, 0.3, 0.2, 0.4, 0.3\}$ *and* $R < x_1 >= R < x_2 >= R < x_3 >= R < x_4 >= R < x_5 >= \{0.4, 0.3, 0.2, 0.4, 0.3\}$*. Therefore,* $R < x_1 > R = R < x_2 > R = R < x_3 > R = R < x_4 > R = R < x_5 > R = R < x_6 > R = \{0.4, 0.3, 0.2, 0.4, 0.3\}$*. By computing,*

$$\begin{aligned}
(N(\Xi))_*(x_1) =& (N(\Xi))_*(x_2) = (N(\Xi))_*(x_3) = (0.6 \vee 0.9) \wedge (0.7 \vee 1.0) \wedge (0.8 \vee 0.8) \wedge (0.6 \vee 0.9) \wedge (0.7 \vee 1.0) \\
&= 0.9 \wedge 1.0 \wedge 0.8 \wedge 0.9 \wedge 1.0 = 0.8, \\
(N(\Xi))_*(x_4) =& (N(\Xi))_*(x_5) = (N(\Xi))_*(x_6) = (0.6 \vee 0.9) \wedge (0.7 \vee 1.0) \wedge (0.8 \vee 0.8) \wedge (0.6 \vee 0.9) \wedge (0.7 \vee 1.0) \\
&= 0.9 \wedge 1.0 \wedge 0.8 \wedge 0.9 \wedge 1.0 = 0.8, \\
(N(\Xi))^*(x_1) &= (N(\Xi))^*(x_2) = (N(\Xi))^*(x_3) = (0.4 \wedge 0.9) \vee (0.3 \wedge 1.0) \vee (0.2 \wedge 0.8) \vee (0.4 \wedge 0.9) \vee (0.3 \wedge 1.0) \\
&= 0.4 \vee 0.3 \vee 0.2 \vee 0.4 \vee 0.3 = 0.4, \\
(N(\Xi))^*(x_4) &= (N(\Xi))^*(x_6) = (N(\Xi))^*(x_6) = (0.4 \wedge 0.9) \vee (0.3 \wedge 1.0) \vee (0.2 \wedge 0.8) \vee (0.4 \wedge 0.9) \vee (0.3 \wedge 1.0) \\
&= 0.4 \vee 0.3 \vee 0.2 \vee 0.4 \vee 0.3 = 0.4.
\end{aligned}$$

This implies that the lower and upper approximations of $\Xi_{N(G)}$ with respect to R are given by

$$[N(\Xi)]_R = \{(x_1, 0.8), (x_2, 0.8), (x_3, 0.8), (x_4, 0.8), (x_5, 0.8)\},$$

$$[N(\Xi)]^R = \{(x_1, 0.9), (x_2, 1.0), (x_3, 0.8), (x_4, 0.9), (x_5, 1.0)\}.$$

Let $\gamma_{D(G)} = \{(x_1x_2, 0.5), (x_2x_3, 0.6), (x_3x_5, 0.6), (x_5x_4, 0.6), (x_4x_1, 0.7), (x_1x_4, 0.6), (x_2x_4, 0.5), (x_2x_5, 0.7), (x_4x_3, 0.5), (x_5x_3, 0.7)\}$ be a fuzzy edge set defined on $D(G)$ and W be a fuzzy relation defined on $D(G)$ given as in Table 2.

Table 2. Fuzzy relation W in Example 1.

W	x_1x_2	x_2x_3	x_3x_5	x_5x_4	x_4x_1	x_1x_4	x_2x_4	x_2x_5	x_4x_3	x_5x_3
x_1x_2	0.2	0.2	0.1	0.2	0.3	0.1	0.2	0.1	0.2	0.1
x_2x_3	0.2	0.1	0.1	0.2	0.3	0.4	0.3	0.1	0.2	0.1
x_3x_5	0.1	0.1	0.1	0.3	0.4	0.2	0.3	0.3	0.2	0.1
x_5x_4	0.2	0.2	0.1	0.1	0.1	0.1	0.3	0.2	0.2	0.1
x_4x_1	0.3	0.3	0.3	0.1	0.1	0.1	0.1	0.1	0.2	0.1
x_1x_4	0.1	0.4	0.2	0.1	0.1	0.1	0.3	0.3	0.2	0.1
x_2x_4	0.2	0.3	0.3	0.2	0.1	0.3	0.1	0.2	0.2	0.1
x_2x_5	0.1	0.1	0.3	0.2	0.1	0.3	0.2	0.1	0.2	0.1
x_4x_3	0.2	0.2	0.2	0.4	0.1	0.3	0.1	0.3	0.2	0.1
x_5x_3	0.1	0.1	0.3	0.2	0.1	0.3	0.2	0.1	0.2	0.1

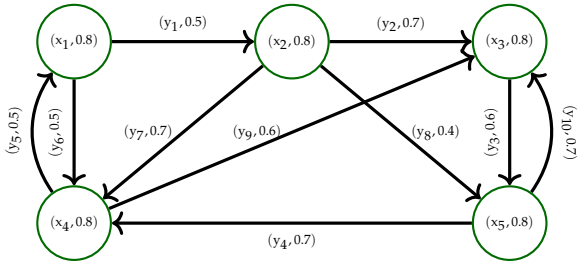
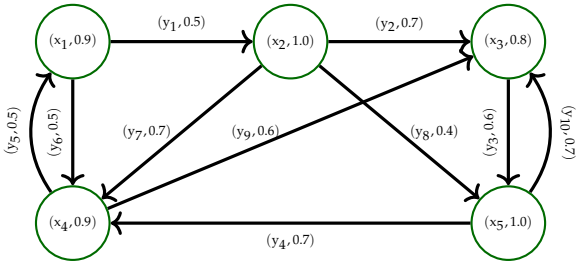
By computing, $W < x_1x_2 > W = W < x_2x_3 > W = W < x_3x_5 > W = W < x_5x_4 > W = W < x_4x_1 > W = W < x_1x_4 > W = W < x_2x_4 > W = W < x_2x_5 > W = W < x_4x_3 > W = W < x_5x_3 > W = \{0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1\}$. Therefore, the lower and upper approximations of $\Xi_{D(G)}$ with respect to W are given by

$$[D(\Xi)]_W = \{(x_1x_2, 0.5), (x_2x_3, 0.7), (x_3x_5, 0.6), (x_5x_4, 0.7), (x_4x_1, 0.5), (x_1x_4, 0.5), (x_2x_4, 0.7), (x_2x_5, 0.4), (x_4x_3, 0.6), (x_5x_3, 0.7)\},$$

$$[D(\Xi)]^W = \{(x_1x_2, 0.5), (x_2x_3, 0.7), (x_3x_5, 0.6), (x_5x_4, 0.7), (x_4x_1, 0.5), (x_1x_4, 0.5), (x_2x_4, 0.7), (x_2x_5, 0.4), (x_4x_3, 0.6), (x_5x_3, 0.7)\}.$$

As a result, the lower and upper approximations $\underline{\Xi_G} = ([N(\Xi)]_R, [D(\Xi)]_W)$ and $\overline{\Xi_G} = ([N(\Xi)]^R, [D(\Xi)]^W)$ of Ξ_G are shown in Table 3.

Table 3. The lower and upper approximations $\underline{\Xi_G}$ and $\overline{\Xi_G}$ in Example 1.

The Fuzzy Lower Approximation $\underline{\Xi_G}$	**The Fuzzy Upper Approximation $\overline{\Xi_G}$**
	

Definition 9. *Let (G, R, W) be a fuzzy graph approximation space. For every rough fuzzy vertex set $\Xi_{N(G)} \in \mathbf{I}^{N(G)}$, define the accuracy fuzzy set $\mu(\Xi_{N(G)}) \in \mathbf{I}^{N(G)}$, for all $x \in \Xi_{N(G)}$, by the following*

$$\mu(\Xi_{N(G)})(x) = \begin{cases} 0 & \text{if } [N(\Xi)]^R = \underline{1}_{N(G)} \text{ and } [N(\Xi)]_R = \underline{0}_{N(G)}, \\ \Big([N(\Xi)]^R(x) - \Xi_{N(G)}(x)\Big)^c \wedge \Big(\Xi_{N(G)}(x) - [N(\Xi)]_R(x)\Big)^c & \text{if } [N(\Xi)]^R \not\leq [N(\Xi)]_R, \\ 1 & \text{otherwise,} \end{cases}$$

Moreover, the accuracy value of the rough fuzzy vertex set $\Xi_{N(G)}$ is given by $\mathrm{Inf}(\mu(\Xi_{N(G)}))$.

Whenever $[N(\Xi)]^R$ is such that $[N(\Xi)]^R \leq [N(\Xi)]_R$, we obtain $\Xi_{N(G)} = [N(\Xi)]_R = [N(\Xi)]^R$ and then $(\Xi_{N(G)})^B = \underline{0}_{N(G)}$ and $\mathrm{Inf}(\mu(\Xi_{N(G)})) = 1$. If $[N(\Xi)]_R = \underline{0}_{N(G)}$ and $[N(\Xi)]^R = \underline{1}_{N(G)}$, then $(\Xi_{N(G)})^B = \underline{1}_{N(G)}$ and $\mathrm{Inf}(\mu([N(\Xi)])) = 0$. Otherwise, $(\Xi_{N(G)})^B = [N(\Xi)]^R \wedge ([N(\Xi)]_R)^c$ and $0 < \mathrm{Inf}(\mu(\Xi_{N(G)})) < 1$. That is, the largest boundary fuzzy vertex set is associated with the lowest accuracy value and the converse is true. If $\mathrm{Inf}(\mu(\Xi_{N(G)})) = 1$, then $\Xi_{N(G)}$ is crisp with respect to $R\big([N(\Xi)]_R = [N(\Xi)]^R$ and $\Xi_{N(G)}$ is precise with respect to R). If $\mathrm{Inf}(\mu(\Xi_{N(G)})) = 0$, then $\Xi_{N(G)}$ is totally rough with respect to R. Moreover, if $0 < \mathrm{Inf}(\mu(\Xi_{N(G)})) < 1$, then $\Xi_{N(G)}$ is rough with respect to μ.

It is worth noting that a similar formulation applies to rough fuzzy edge sets. Consequently, the accuracy degree of the rough fuzzy subgraph Ξ_G is determined by $(\mathrm{Inf}(\mu(\Xi_{N(G)})), \mathrm{Inf}(\mu(\Xi_{D(G)})))$.

Example 2. *Consider Example 1, where G is the graph $(N(G), D(G))$, where $N(G) = \{x_1, x_2, x_3, x_4, x_5\}$ and $D(G) = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_7, y_{10}\}$. The computations are presented in Table 4. Table 4 compares the fuzzy boundary regions and the accuracy measures obtained by our minimal neighborhood method with those produced by previous approaches in Definition 1 as used by [14,17,18,32,33].*

Table 4 presents a comprehensive comparative analysis between the existing approximation framework in Definition 1 [14,17,18,32,33] and our proposed technique in Definition 7. The evaluation is based on two critical rough set metrics: the fuzzy boundary region $(\Xi_{N(G)})^B$ and the accuracy measure $\mathrm{Inf}(\mu(\Xi_{N(G)}))$, computed across a variety of fuzzy sets $\Xi_{N(G)}$. The empirical results explicitly illustrate the superiority and theoretical significance of our proposed definitions. Under the previous technique, evaluating heterogeneous fuzzy sets reveals substantial limitations. For instance, given the input set $\{0.2, 0.4, 0.6, 0.4, 0.5, 0.8\}$, the previous method generates a prominent boundary region of $\{0.5, 0.6, 0.6, 0.6, 0.6, 0.5\}$. This significant non-zero boundary reflects a high degree of uncertainty in the approximation space, which directly translates to a degraded accuracy measure of 0.7. A similar limitation is observed for the set $\{0.2, 0.3, 0.3, 0.3, 0.3, 0.3\}$, where the previous method yields a boundary of $\{0.3, 0.3, 0.3, 0.3, 0.3, 0.3\}$ and an accuracy of only 0.9. Conversely,

our proposed technique significantly eliminates this uncertainty. For the identical heterogeneous set $\{0.2, 0.4, 0.6, 0.4, 0.5, 0.8\}$*, our method aggressively reduces the boundary region to an absolute null state,* $\{0.0, 0.0, 0.0, 0.0, 0.0, 0.0\}$*, thereby achieving a perfect accuracy measure of* 1.0*. This perfect accuracy is consistently maintained across nearly all evaluated sets, demonstrating that our proposed technique mathematically tightens the gap between the upper and lower approximations. By doing so, it effectively neutralizes the vague boundary area that plagues traditional definitions.*

Table 4. Comparison of fuzzy boundary and fuzzy accuracy measures.

$\Xi_{N(G)} = \{x_1, \ldots, x_6\}$	Previous Technique in Definition 1 Used by [14,17,18,32,33]		Our Proposed Technique in Definition 7	
	$(\Xi_{N(G)})^B$	$\mathrm{Inf}(\mu(\Xi_{N(G)}))$	$(\Xi_{N(G)})^B$	$\mathrm{Inf}(\mu(\Xi_{N(G)}))$
$\{0.2, 0.4, 0.6, 0.4, 0.5, 0.8\}$	$\{0.5, 0.6, 0.6, 0.6, 0.6, 0.5\}$	0.7	$\{0.0, 0.0, 0.0, 0.0, 0.0, 0.0\}$	1.0
$\{0.1, 0.0, 0.0, 0.0, 0.0, 0.0\}$	$\{0.1, 0.1, 0.1, 0.1, 0.1, 0.1\}$	0.9	$\{0.1, 0.1, 0.1, 0.1, 0.1, 0.1\}$	0.9
$\{0.1, 0.1, 0.1, 0.1, 0.1, 0.1\}$	$\{0.0, 0.0, 0.0, 0.0, 0.0, 0.0\}$	1.0	$\{0.0, 0.0, 0.0, 0.0, 0.0, 0.0\}$	1.0
$\{0.2, 0.2, 0.2, 0.2, 0.2, 0.2\}$	$\{0.0, 0.0, 0.0, 0.0, 0.0, 0.0\}$	1.0	$\{0.0, 0.0, 0.0, 0.0, 0.0, 0.0\}$	1.0
$\{0.2, 0.4, 0.6, 0.4, 0.5, 0.8\}$	$\{0.5, 0.6, 0.6, 0.6, 0.6, 0.5\}$	0.7	$\{0.0, 0.0, 0.0, 0.0, 0.0, 0.0\}$	1.0
$\{0.2, 0.3, 0.3, 0.3, 0.3, 0.3\}$	$\{0.3, 0.3, 0.3, 0.3, 0.3, 0.3\}$	0.9	$\{0.0, 0.0, 0.0, 0.0, 0.0, 0.0\}$	1.0
$\{0.4, 0.4, 0.4, 0.4, 0.4, 0.4\}$	$\{0.0, 0.0, 0.0, 0.0, 0.0, 0.0\}$	1.0	$\{0.0, 0.0, 0.0, 0.0, 0.0, 0.0\}$	1.0
$\{0.5, 0.5, 0.5, 0.5, 0.5, 0.5\}$	$\{0.0, 0.0, 0.0, 0.0, 0.0, 0.0\}$	1.0	$\{0.0, 0.0, 0.0, 0.0, 0.0, 0.0\}$	1.0
$\{0.6, 0.6, 0.6, 0.6, 0.6, 0.6\}$	$\{0.0, 0.0, 0.0, 0.0, 0.0, 0.0\}$	1.0	$\{0.0, 0.0, 0.0, 0.0, 0.0, 0.0\}$	1.0
$\{0.7, 0.7, 0.7, 0.7, 0.7, 0.7\}$	$\{0.0, 0.0, 0.0, 0.0, 0.0, 0.0\}$	1.0	$\{0.0, 0.0, 0.0, 0.0, 0.0, 0.0\}$	1.0
$\{0.8, 0.8, 0.8, 0.8, 0.8, 0.8\}$	$\{0.0, 0.0, 0.0, 0.0, 0.0, 0.0\}$	1.0	$\{0.0, 0.0, 0.0, 0.0, 0.0, 0.0\}$	1.0
$\{0.9, 0.9, 0.9, 0.9, 0.9, 0.9\}$	$\{0.0, 0.0, 0.0, 0.0, 0.0, 0.0\}$	1.0	$\{0.0, 0.0, 0.0, 0.0, 0.0, 0.0\}$	1.0

Furthermore, for uniform fuzzy sets ranging from $\{0.2, \ldots, 0.2\}$ *to* $\{0.9, \ldots, 0.9\}$*, both techniques successfully yield a zero boundary and perfect accuracy. However, the proposed technique's ability to maintain a perfect accuracy of* 1.0 *even for highly fluctuating input sets highlights its robustness and superior generalization. The only edge case,* $\{0.1, 0.0, 0.0, 0.0, 0.0, 0.0\}$*, yields identical results (accuracy of* 0.9*) for both methods, proving that our technique performs at least as well as the previous method in extreme low-value scenarios, while vastly outperforming it in complex, heterogeneous scenarios. In conclusion, the proposed definitions articulated in Definition 7 provide a mathematically rigorous and precise approximation space. By effectively shrinking the boundary region to zero in complex cases, the proposed model drastically enhances the exactness of fuzzy rough set approximations, establishing the model as a highly reliable and superior framework for managing uncertainty in fuzzy systems.*

A parallel comparative analysis of the fuzzy edge sets confirms identical trends, demonstrating that the proposed edge approximation operators in Definition 8 consistently yield tighter boundary regions and superior accuracy measures compared to the traditional framework of Definition 2 in [17,18,32,33].

Theorem 1. *Let* (G, R, W) *be a fuzzy graph approximation space, where* R *is an equivalence fuzzy relation on* $N(G)$*, and let* Ξ *be a fuzzy subgraph of* G*. Suppose* $(N(\Xi))^1_*$ *and* $(N(\Xi))^{*1}$ *denote the lower and upper approximations of* $N(\Xi)$ *defined in Definition 1, while* $(N(\Xi))^7_*$ *and* $(N(\Xi))^{*7}$ *denote the minimal neighborhood-based lower and upper approximations defined in Definition 7. Then, the following relations hold:*

(i) $(N(\Xi))^1_* \leq (N(\Xi))^7_*$;
(ii) $(N(\Xi))^{*7} \leq (N(\Xi))^{*1}$;
(iii) $[N(\Xi)]^1_R \leq [N(\Xi)]^7_R \leq N(\Xi) \leq [N(\Xi)]^R_7 \leq [N(\Xi)]^R_1$;
(iv) $(N(\Xi))^B_7 \leq (N(\Xi))^B_1$;
(v) $\mathrm{Inf}(\mu(\Xi_{N(G)})^7) \geq \mathrm{Inf}(\mu(\Xi_{N(G)})^1)$.

Proof. Let $x, x' \in N(G)$ be arbitrary vertices. We first establish that for any reflexive fuzzy relation R on $N(G)$, the inequality $R < x' > R(x) \leq R(x, x')$ holds. By the definition of the fuzzy minimal neighborhood, we have $R < x' > R(x) = < x' > R(x) \wedge R < x' > (x)$. Recall that $< x' > R(x) = \bigwedge_{\substack{x'' \in N(G) \\ R(x'', x') > 0}} R(x'', x)$.

Since R is reflexive, $R(x', x') = 1 > 0$. Taking $x'' = x'$ in the infimum over x'', we obtain

$$< x' > R(x) \leq R(x', x).$$

Similarly, we have

$$R < x' > (x) = \bigwedge_{\substack{x'' \in N(G) \\ R(x', x'') > 0}} R(x, x'') \leq R(x, x').$$

When R is symmetric (as required for an equivalence relation in Definition 1), $R(x', x) = R(x, x')$. Therefore,

$$R < x' > R(x) \leq R(x, x') \quad \forall x, x' \in N(G).$$

Using this fundamental inequality, we prove each part as follows:

(i) Since $R < x' > R(x) \leq R(x, x')$, taking complements yields

$$1 - R < x' > R(x) \geq 1 - R(x, x').$$

Thus, for any membership degree $\mu_{N(\Xi)}(x')$, it follows that

$$\left(1 - R < x' > R(x)\right) \vee \mu_{N(\Xi)}(x') \geq \left(1 - R(x, x')\right) \vee \mu_{N(\Xi)}(x').$$

Taking the infimum over all $x' \in N(G)$ gives $(N(\Xi))^1_*(x) \leq (N(\Xi))^7_*(x)$ for every $x \in N(G)$. Hence, $(N(\Xi))^1_* \leq (N(\Xi))^7_*$.

(ii) From $R < x' > R(x) \leq R(x, x')$, we immediately have

$$R < x' > R(x) \wedge \mu_{N(\Xi)}(x') \leq R(x, x') \wedge \mu_{N(\Xi)}(x').$$

Taking the supremum over all $x' \in N(G)$ yields $(N(\Xi))^{*7}(x) \leq (N(\Xi))^{*1}(x)$ for all $x \in N(G)$. Hence, $(N(\Xi))^{*7} \leq (N(\Xi))^{*1}$.

(iii) By Definition 7, the rough vertex approximations are defined by $[N(\Xi)]_R = N(\Xi) \wedge (N(\Xi))_*$ and $[N(\Xi)]^R = N(\Xi) \vee (N(\Xi))^*$. Applying parts (i) and (ii), we obtain

$$[N(\Xi)]^1_R = N(\Xi) \wedge (N(\Xi))^1_* \leq N(\Xi) \wedge (N(\Xi))^7_* = [N(\Xi)]^7_R,$$

and

$$[N(\Xi)]^R_7 = N(\Xi) \vee (N(\Xi))^{*7} \leq N(\Xi) \vee (N(\Xi))^{*1} = [N(\Xi)]^R_1.$$

Combining this with Lemma 1(1), which states $[N(\Xi)]_R \leq N(\Xi) \leq [N(\Xi)]^R$, establishes the chain of inequalities.

(iv) By definition, the boundary region is given by $(N(\Xi))^B = [N(\Xi)]^R \bar{\wedge} [N(\Xi)]_R$. Since $[N(\Xi)]^R_7 \leq [N(\Xi)]^R_1$ and $[N(\Xi)]^7_R \geq [N(\Xi)]^1_R$, the difference between the upper and lower approximations in Definition 7 is pointwise smaller than or equal to that in Definition 1. Consequently, $(N(\Xi))^B_7 \leq (N(\Xi))^B_1$.

(v) Since the boundary region $(N(\Xi))^B_7$ is smaller than or equal to $(N(\Xi))^B_1$, the accuracy membership values satisfy $\mu(\Xi_{N(G)})^7 \geq \mu(\Xi_{N(G)})^1$ pointwise. Taking the infimum over $N(G)$ proves that $\text{Inf}(\mu(\Xi_{N(G)})^7) \geq \text{Inf}(\mu(\Xi_{N(G)})^1)$.

□

Theorem 2. *Let (G, R, W) be a fuzzy graph approximation space, where W is an equivalence fuzzy relation on $\mathfrak{E}(G)$, and let Ξ be a fuzzy subgraph of G. Suppose $(\mathfrak{E}(\Xi))^2_*$ and $(\mathfrak{E}(\Xi))^{*2}$ denote the lower and upper approximations of $\mathfrak{E}(\Xi)$ defined in Definition 2, while $(\mathfrak{E}(\Xi))^8_*$ and $(\mathfrak{E}(\Xi))^{*8}$ denote the minimal neighborhood-based lower and upper approximations defined in Definition 8. Then, the following relations hold:*

(i) $(\mathfrak{E}(\Xi))^2_* \leq (\mathfrak{E}(\Xi))^8_*$;
(ii) $(\mathfrak{E}(\Xi))^{*8} \leq (\mathfrak{E}(\Xi))^{*2}$;
(iii) $[\mathfrak{E}(\Xi)]^2_W \leq [\mathfrak{E}(\Xi)]^8_W \leq \mathfrak{E}(\Xi) \leq [\mathfrak{E}(\Xi)]^W_8 \leq [\mathfrak{E}(\Xi)]^W_2$;
(iv) $(\mathfrak{E}(\Xi))^B_8 \leq (\mathfrak{E}(\Xi))^B_2$;
(v) $\mathrm{Inf}(\mu(\Xi_{\mathfrak{E}(G)})^8) \geq \mathrm{Inf}(\mu(\Xi_{\mathfrak{E}(G)})^2)$.

Proof. The proof is similar to that of Theorem 1. □

Remark 1. *The mathematical relations established in Theorems 1 and 2 demonstrate that the minimal neighborhood-based rough approximations introduced in Definition 7 (for vertices) and Definition 8 (for edges) represent a strict theoretical improvement over the standard equivalence-based approximations of Definitions 1 and 2. Specifically, the importance of these definitions can be understood through three core advantages:*

1. *By utilizing the minimal neighborhood relations $R\langle x'\rangle R(x)$ and $W\langle R'\rangle W(R)$, the lower approximations satisfy $(N(\Xi))^1_* \leq (N(\Xi))^7_*$ and $(\mathfrak{E}(\Xi))^2_* \leq (\mathfrak{E}(\Xi))^8_*$. This guarantees that a greater number of vertices and edges are positively classified as belonging to the target fuzzy subgraph Ξ with absolute certainty.*
2. *The upper approximations satisfy $(N(\Xi))^{*7} \leq (N(\Xi))^{*1}$ and $(\mathfrak{E}(\Xi))^{*8} \leq (\mathfrak{E}(\Xi))^{*2}$. This tighter containment filters out false positives and extraneous elements that would otherwise be incorrectly included in the upper approximation under standard equivalence relations.*
3. *As a direct consequence of expanding the lower bounds and shrinking the upper bounds, the boundary regions of both vertices and edges are strictly reduced: $(N(\Xi))^B_7 \leq (N(\Xi))^B_1$ and $(\mathfrak{E}(\Xi))^B_8 \leq (\mathfrak{E}(\Xi))^B_2$.*
   Consequently, the accuracy degrees are guaranteed to be higher; that is, $\mathrm{Inf}(\mu(\Xi_{N(G)})^7) \geq \mathrm{Inf}(\mu(\Xi_{N(G)})^1)$ and $\mathrm{Inf}(\mu(\Xi_{\mathfrak{E}(G)})^8) \geq \mathrm{Inf}(\mu(\Xi_{\mathfrak{E}(G)})^2)$. This provides a mathematical justification for adopting minimal neighborhoods when modeling uncertain or granular network structures where precision is critical.

Definition 10. *Let (G, R, W) be a fuzzy graph AS, and let Ξ be a fuzzy subgraph of G. Then:*

(i) *R denotes an arbitrary fuzzy binary relation defined on the vertex set $N(G)$.*
(ii) *W denotes an arbitrary fuzzy binary relation defined on the edge set $D(G)$.*
(iii) *$\Xi^R_{N(G)} = ([N(\Xi)]_R, [N(\Xi)]^R)$ represents a rough set on $N(G)$.*
(iv) *$\Xi^W_{D(G)} = ([D(\Xi)]_W, [D(\Xi)]^W)$ represents a rough relation on $D(G)$.*
(v) *Define $\underline{\Xi_G} = ([N(\Xi)]_R, [D(\Xi)]_W)$ and $\overline{\Xi_G} = ([N(\Xi)]^R, [D(\Xi)]^W)$ as fuzzy graphs, corresponding to the lower and upper approximations of Ξ_G, respectively, such that*

$$[D(\Xi)]_W(xx') \leq \min\{[N(\Xi)]_R(x), [N(\Xi)]_R(x')\},$$

$$[D(\Xi)]^W(xx') \leq \min\{[N(\Xi)]^R(x), [N(\Xi)]^R(x')\}, \quad \text{for all } xx' \in D(G).$$

The pair

$$\Xi_G = (\underline{\Xi_G}, \overline{\Xi_G}) = \left(\Xi^R_{N(G)}, \Xi^W_{D(G)}\right)$$

is called a fuzzy rough graph.

Lemma 1. *Let (G, R, W) be a fuzzy graph AS and $\Xi_G, \complement_G \in \mathbf{I}^G$. Then,*

(1) $[N(\Xi)]_R \leq [N(\Xi)] \leq [N(\Xi)]^R$ *and* $[D(\Xi)]_W \leq [D(\Xi)] \leq [D(\Xi)]^W$;
(2) $[\underline{0}_{N(G)}]_R = [\underline{0}_{N(G)}]^R = [\underline{0}_{N(G)}]$ *and* $[\underline{0}_{D(G)}]_W = [\underline{0}_{D(G)}]^W = [\underline{0}_{D(G)}]$;
(3) $[\underline{1}_{N(G)}]_R = [\underline{1}_{N(G)}]^R = [\underline{1}_{N(G)}]$ *and* $[\underline{1}_{D(G)}]_W = [\underline{1}_{D(G)}]^W = [\underline{1}_{D(G)}]$;
(4) $([N(\Xi)] \vee [N(\complement)])_R \geq [N(\Xi)]_R \vee [N(\complement)]_R$ *and* $([D(\Xi)] \vee [D(\complement)])_W \geq [D(\Xi)]_W \vee [D(\complement)]_W$;
(5) $([N(\Xi)] \wedge [N(\complement)])^R \leq [N(\Xi)]^R \wedge [N(\complement)]^R$ *and* $([D(\Xi)] \wedge [D(\complement)])^W \leq [D(\Xi)]^W \wedge [D(\complement)]^W$;
(6) $\Xi_G \leq \complement_G$ *implies that* $[N(\Xi)]_R \leq [N(\complement)]_R$ *and* $[D(\Xi)]_W \leq [D(\complement)]_W$;
(7) $\Xi_G \leq \complement_G$ *implies that* $[N(\Xi)]^R \leq [N(\complement)]^R$ *and* $[D(\Xi)]^W \leq [D(\complement)]^W$;
(8) $([N(\Xi)] \vee [N(\complement)])^R = [N(\Xi)]^R \vee [N(\complement)]^R$ *and* $([D(\Xi)] \vee [D(\complement)])^W = [D(\Xi)]^W \vee [D(\complement)]^W$;
(9) $([N(\Xi)] \wedge [N(\complement)])_R = [N(\Xi)]_R \wedge [N(\complement)]_R$ *and* $([D(\Xi)] \wedge [D(\complement)])_W = [D(\Xi)]_W \wedge [D(\complement)]_W$;
(10) $\left([N(\Xi)]^R\right)^c = ([N(\Xi)]^c)_R$ *and* $\left([D(\Xi)]^W\right)^c = ([D(\Xi)]^c)_W$;
(11) $([N(\Xi)]_R)^c = ([N(\Xi)]^c)^R$ *and* $([D(\Xi)]_W)^c = ([D(\Xi)]^c)^W$;
(12) $([N(\Xi)]_R)^R = [N(\Xi)]_R = ([N(\Xi)]_R)_R$ *and* $([D(\Xi)]_W)^W = [D(\Xi)]_W = ([D(\Xi)]_W)_W$;
(13) $\left([N(\Xi)]^R\right)_R = [N(\Xi)]^R = \left([N(\Xi)]^R\right)^R$ *and* $\left([D(\Xi)]^W\right)_W = [D(\Xi)]^W = \left([D(\Xi)]^W\right)^W$.

Proof. Straightforward from Definitions 7 and 8. □

Lemma 2. *Let* (G, R, W) *be a fuzzy graph AS and* $\Xi_G, \complement_G \in \mathbf{I}^G$. *Then,*

(1) $\underline{\Xi_G} \leq \Xi_G \leq \overline{\Xi_G}$;
(2) $\underline{0_G} = \overline{0_G} = 0_G$ *and* $\underline{1_G} = \overline{1_G} = 1_G$;
(3) $\underline{(\Xi_G \vee \complement_G)} \geq \underline{\Xi_G} \vee \underline{\complement_G}$;
(4) $\overline{(\Xi_G \wedge \complement_G)} \leq \overline{\Xi_G} \wedge \overline{\complement_G}$;
(5) $\Xi_G \leq \complement_G$ *implies that* $\underline{\Xi_G} \leq \underline{\complement_G}$ *and* $\overline{\Xi_G} \leq \overline{\complement_G}$;
(6) $\overline{(\Xi_G \vee \complement_G)} = \overline{\Xi_G} \vee \overline{\complement_G}$;
(7) $\underline{(\Xi_G \wedge \complement_G)} = \underline{\Xi_G} \wedge \underline{\complement_G}$,
(8) $\left(\overline{\Xi_G}\right)^c = \underline{\left(\Xi_G^c\right)}$ *and* $\left(\underline{\Xi_G}\right)^c = \overline{\left(\Xi_G^c\right)}$;
(9) $\overline{\left(\underline{\Xi_G}\right)} = \underline{\Xi_G} = \underline{\left(\underline{\Xi_G}\right)}$;
(10) $\underline{\left(\overline{\Xi_G}\right)} = \overline{\Xi_G} = \overline{\left(\overline{\Xi_G}\right)}$.

Proof. Straightforward from Definition 10 and Lemma 1. □

Remark 2. *Let* (G, R, W) *represent a fuzzy graph AS. It is observed that Definitions 7 and 8 effectively reduce the upper approximation while expanding the lower approximation. This novel strategy offers greater generality compared to several existing frameworks. It can be observed that, under specific conditions, the following cases are noted:*

(1) *In the presence of equivalence fuzzy relations R and W, and assuming the graph G contains no edges, Definition 7 serves as the fuzzification of the fundamental definition proposed by Pawlak [1].*
(2) *Under the conditions of equivalence fuzzy relations R and W for a graph G without edges, Definition 7 is consistent with the formulation provided by Pawlak [14].*
(3) *For symmetric fuzzy relations R and W in an edgeless graph G, Definition 7 represents the fuzzification of the model introduced by Allam [35].*
(4) *The fuzzification of the definition established by El-Atik [36] is represented by Definition 7.*
(5) *When R and W are equivalence fuzzy relations, Definitions 7 and 8 align with the results described by Akram [17].*

(6) *If R and W are considered tolerance fuzzy relations (implying both reflexivity and symmetry), then Definitions 7 and 8 coincide with the definition presented by Akram [18].*

The Implication of Remark 2 is represented in Figure 2.

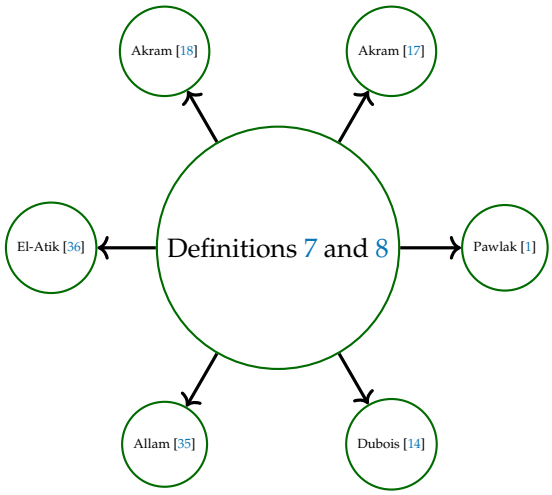


Figure 2. Implication of Remark 2.

4. Connectivity Index of Fuzzy Rough Graphs

Definition 11. *Let $\Xi_G = (\underline{\Xi_G}, \overline{\Xi_G})$ be a fuzzy rough graph on a nonempty graph G. A sequence of vertices*

$$\mathfrak{P} : x_1 \to x_2 \to \cdots \to x_n$$

is called a directed path of length n in Ξ_G if it forms a directed path of the same length from x_1 to x_n in both $\underline{\Xi_G}$ and $\overline{\Xi_G}$.

Definition 12. *Let $\Xi_G = (\underline{\Xi_G}, \overline{\Xi_G})$ be a fuzzy rough graph on a nonempty graph G. For a directed path $\mathfrak{P}$ in Ξ_G, an arc $x_{i-1}x_i \in Y^*$ is called the* weakest arc *if it has the minimum membership degree along $\mathfrak{P}$ in both $\underline{\Xi_G}$ and $\overline{\Xi_G}$. In other words, among all arcs of $\mathfrak{P}$, it has the smallest value in both fuzzy approximations.*

Definition 13. *Let $\Xi_G = (\underline{\Xi_G}, \overline{\Xi_G})$ be a fuzzy rough graph on a nonempty graph G. The* strength *of a directed path $\mathfrak{P}$ is defined by combining the membership values of the weakest arcs in both lower and upper fuzzy approximations:*

$$S(\mathfrak{P}) = \bigwedge_{i=1}^{n} [D(\Xi)]_W(x_{i-1}x_i) + \bigwedge_{i=1}^{n} [D(\Xi)]^W(x_{i-1}x_i).$$

This quantity measures the robustness of $\mathfrak{P}$ within the fuzzy rough graph.

Definition 14. *Let $\Xi_G = (\underline{\Xi_G}, \overline{\Xi_G})$ be a fuzzy rough graph on a nonempty graph G. The* strength of connectedness *from a vertex x_0 to x_1 in Ξ_G, denoted by $\mathrm{Connev}_{\Xi_G}(x_0, x_1)$, is described in terms of the connectedness in both approximations:*

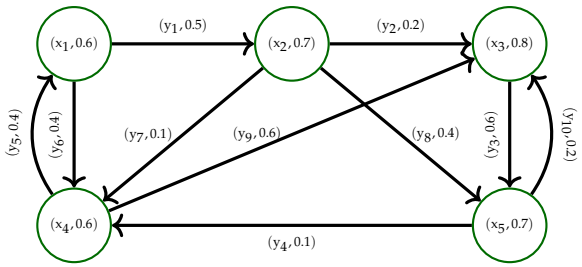
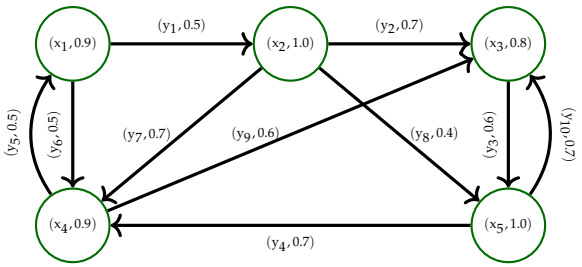
$$\mathrm{Connev}_{\Xi_G}(x_0, x_1) = \left(\mathrm{Connev}_{\underline{\Xi_G}}(x_0, x_1), \mathrm{Connev}_{\overline{\Xi_G}}(x_0, x_1)\right).$$

Here, $\mathrm{Connev}_{\underline{\Xi_G}}(x_0, x_1)$ is the maximum strength among all directed paths from x_0 to x_1 in $\underline{\Xi_G}$, and $\mathrm{Connev}_{\overline{\Xi_G}}(x_0, x_1)$ is the maximum strength among all directed paths in $\overline{\Xi_G}$.

Definition 15. *Let $\Xi_G = (\underline{\Xi_G}, \overline{\Xi_G})$ be a fuzzy rough graph on a nonempty graph G. A directed path $\mathfrak{P}$ connecting vertices x_1 and x_2 is called a* strongest directed path *in Ξ_G if it has the maximum strength among all paths between x_1 and x_2 in both $\underline{\Xi_G}$ and $\overline{\Xi_G}$. In other words, $\mathfrak{P}$ is the strongest if no other path has greater strength in either fuzzy approximation.*

Example 3. *Assume that G is the graph $(N(G), D(G))$, where $N(G) = \{x_1, x_2, x_3, x_4, x_5\}$ and $D(G) = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}\}$. Let $\Xi_G = (\underline{\Xi_G}, \overline{\Xi_G})$ be a fuzzy rough graph on G, as shown in Table 5.*

Table 5. The fuzzy rough graph Ξ_G in Example 3.

The Fuzzy Lower Approximation $\underline{\Xi_G}$	**The Fuzzy Upper Approximation $\overline{\Xi_G}$**
	

In both $\underline{\Xi_G}$ and $\overline{\Xi_G}$, the sequence $x_1 \to x_2 \to x_3$ forms a directed path of length 3. Hence, this sequence is also considered a directed path of length 3 in the fuzzy rough graph Ξ_G. The strengths of this path differ between the two approximations: it has a value of 0.2 in $\underline{\Xi_G}$ and 0.5 in $\overline{\Xi_G}$, giving a combined path strength of $S(\mathfrak{P}) = 0.7$.

Similarly, the path $x_1 \to x_2 \to x_5$ is a directed path of length 3 in Ξ_G, with the arc x_2x_5 being identified as the weakest arc along this path. From routine calculations, we have the strengths of connectedness between all pairs of vertices in both $\underline{\Xi_G}$ and $\overline{\Xi_G}$:

$$\begin{array}{llll}
\text{Connev}_{\underline{\Xi_G}}(x_1,x_2) = 0.5, & \text{Connev}_{\underline{\Xi_G}}(x_2,x_1) = 0.1, & \text{Connev}_{\overline{\Xi_G}}(x_1,x_2) = 0.5, & \text{Connev}_{\overline{\Xi_G}}(x_2,x_1) = 0.5,\\
\text{Connev}_{\underline{\Xi_G}}(x_1,x_3) = 0.4, & \text{Connev}_{\underline{\Xi_G}}(x_3,x_1) = 0.1, & \text{Connev}_{\overline{\Xi_G}}(x_1,x_3) = 0.5, & \text{Connev}_{\overline{\Xi_G}}(x_3,x_1) = 0.5,\\
\text{Connev}_{\underline{\Xi_G}}(x_1,x_4) = 0.4, & \text{Connev}_{\underline{\Xi_G}}(x_4,x_1) = 0.4, & \text{Connev}_{\overline{\Xi_G}}(x_1,x_4) = 0.5, & \text{Connev}_{\overline{\Xi_G}}(x_4,x_1) = 0.5,\\
\text{Connev}_{\underline{\Xi_G}}(x_1,x_5) = 0.4, & \text{Connev}_{\underline{\Xi_G}}(x_5,x_1) = 0.1, & \text{Connev}_{\overline{\Xi_G}}(x_1,x_5) = 0.5, & \text{Connev}_{\overline{\Xi_G}}(x_5,x_1) = 0.5,\\
\text{Connev}_{\underline{\Xi_G}}(x_2,x_3) = 0.2, & \text{Connev}_{\underline{\Xi_G}}(x_3,x_2) = 0.1, & \text{Connev}_{\overline{\Xi_G}}(x_2,x_3) = 0.7, & \text{Connev}_{\overline{\Xi_G}}(x_3,x_2) = 0.3,\\
\text{Connev}_{\underline{\Xi_G}}(x_2,x_4) = 0.1, & \text{Connev}_{\underline{\Xi_G}}(x_4,x_2) = 0.4, & \text{Connev}_{\overline{\Xi_G}}(x_2,x_4) = 0.7, & \text{Connev}_{\overline{\Xi_G}}(x_4,x_2) = 0.5,\\
\text{Connev}_{\underline{\Xi_G}}(x_2,x_5) = 0.4, & \text{Connev}_{\underline{\Xi_G}}(x_5,x_2) = 0.1, & \text{Connev}_{\overline{\Xi_G}}(x_2,x_5) = 0.6, & \text{Connev}_{\overline{\Xi_G}}(x_5,x_2) = 0.5,\\
\text{Connev}_{\underline{\Xi_G}}(x_3,x_4) = 0.1, & \text{Connev}_{\underline{\Xi_G}}(x_4,x_3) = 0.6, & \text{Connev}_{\overline{\Xi_G}}(x_3,x_4) = 0.6, & \text{Connev}_{\overline{\Xi_G}}(x_4,x_3) = 0.6,\\
\text{Connev}_{\underline{\Xi_G}}(x_3,x_5) = 0.6, & \text{Connev}_{\underline{\Xi_G}}(x_5,x_3) = 0.2, & \text{Connev}_{\overline{\Xi_G}}(x_3,x_5) = 0.6, & \text{Connev}_{\overline{\Xi_G}}(x_5,x_3) = 0.7,\\
\text{Connev}_{\underline{\Xi_G}}(x_4,x_5) = 0.6, & \text{Connev}_{\underline{\Xi_G}}(x_5,x_4) = 0.1, & \text{Connev}_{\overline{\Xi_G}}(x_4,x_5) = 0.6, & \text{Connev}_{\overline{\Xi_G}}(x_5,x_4) = 0.7.
\end{array}$$

It can be observed that there are four directed paths connecting the vertex x_1 to x_5 in both $\underline{\Xi_G}$ and $\overline{\Xi_G}$. Consequently, the complete set of directed paths from x_1 to x_5 in the fuzzy rough graph Ξ_G is:

(i) $x_1 \to x_2 \to x_5$ *with strength 0.8 ;*
(ii) $x_1 \to x_2 \to x_3 \to x_5$ *with strength 0.7;*
(iii) $x_1 \to x_4 \to x_3 \to x_5$ *with strength 0.9;*
(iv) $x_1 \to x_2 \to x_4 \to x_3 \to x_5$ *with strength 0.6.*

Thus, $x_1 \to x_4 \to x_3 \to x_5$ is the strongest path in Ξ_G.

5. Types of Arcs and Vertices in Fuzzy Rough Graphs

Definition 16. *Consider a fuzzy rough graph $\Xi_G = (\underline{\Xi_G}, \overline{\Xi_G})$ on a nonempty graph G. Then*

- *An* arc x_1x_2 *in* Ξ_G *is called y-strong if it is y-strong in both* $\underline{\Xi_G}$ *and* $\overline{\Xi_G}$*; thus,* $[D(\Xi)]_W(x_1x_2) > \text{Connev}_{\underline{\Xi_G}-x_1x_2}(x_1, x_2)$ *and* $[D(\Xi)]^W(x_1x_2) > \text{Connev}_{\overline{\Xi_G}-x_1x_2}(x_1, x_2)$*, respectively.*
- *An arc* x_1x_2 *in* Ξ_G *is called β-strong if it is β-strong in both* $\underline{\Xi_G}$ *and* $\overline{\Xi_G}$*; thus,* $[D(\Xi)]_W(x_1x_2) = \text{Connev}_{\underline{\Xi_G}-x_1x_2}(x_1, x_2)$ *and* $[D(\Xi)]^W(x_1x_2) = \text{Connev}_{\overline{\Xi_G}-x_1x_2}(x_1, x_2)$*, respectively.*
- *An* arc x_1x_2 *in* Ξ_G *is called* $\delta -$ arc *if it is* $\delta -$ arc *in both* $\underline{\Xi_G}$ *and* $\overline{\Xi_G}$*; thus,* $[D(\Xi)]_W(x_1x_2) < \text{Connev}_{\underline{\Xi_G}-x_1x_2}(x_1, x_2)$ *and* $[D(\Xi)]^W(x_1x_2) < \text{Connev}_{\overline{\Xi_G}-x_1x_2}(x_1, x_2)$*, respectively.*
- *An arc* x_1x_2 *is strong if it is either y-strong or β-strong.*
- *A path* $\mathfrak{P}$ *in* Ξ_G *is termed a* strong directed path *if it qualifies as a strong directed path in both* $\underline{\Xi_G}$ *and* $\overline{\Xi_G}$*. In other words, every arc in* $\mathfrak{P}$ *exhibits strength in both approximations.*

Definition 17. *Consider a fuzzy rough graph* $\Xi_G = (\underline{\Xi_G}, \overline{\Xi_G})$ *on a nonempty graph G. An arc* x_1x_2 *is called a strong arc in* Ξ_G *if it is strong in both* $\underline{\Xi_G}$ *and* $\overline{\Xi_G}$*. In this case, the vertex* x_2 *is referred to as a strong neighbor of* x_1 *in* Ξ_G*.*

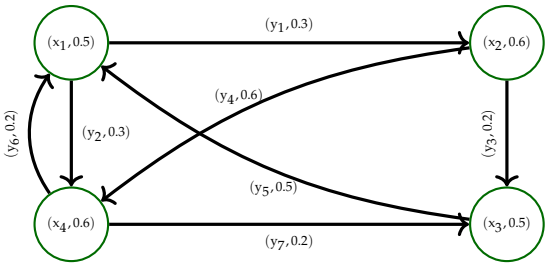
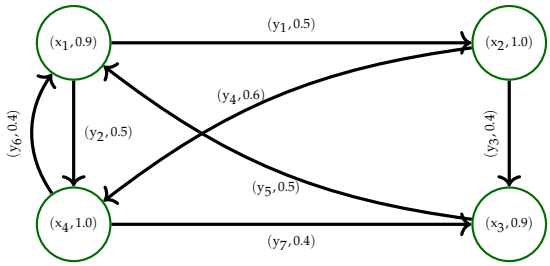
Definition 18. *Consider a fuzzy rough graph* $\Xi_G = (\underline{\Xi_G}, \overline{\Xi_G})$ *on a nonempty graph G. A vertex* $x_1 \in \Xi_G$ *is called a fuzzy rough end vertex if it is an end vertex in both* $\underline{\Xi_G}$ *and* $\overline{\Xi_G}$*. That is,* x_1 *must have at least one strong neighbor in* Ξ_G*.*

Definition 19. *A fuzzy rough graph* Ξ_G *is a strongly connected fuzzy rough graph if both* $\underline{\Xi_G}$ *and* $\overline{\Xi_G}$ *are strongly connected fuzzy digraphs. That is,* $\forall x_1, x_2 \in X^*$*,*

$$\text{Connev}_{\underline{\Xi_G}}(x_1, x_2) > 0 \text{ and } \text{Connev}_{\underline{\Xi_G}}(x_2, x_1) > 0, \quad \text{Connev}_{\overline{\Xi_G}}(x_1, x_2) > 0 \text{ and } \text{Connev}_{\overline{\Xi_G}}(x_2, x_1) > 0.$$

Example 4. *Assume that G is the graph* $(N(G), D(G))$*, where* $N(G) = \{x_1, x_2, x_3, x_4\}$ *and* $D(G) = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7\}$*. Let* $\Xi_G = (\underline{\Xi_G}, \overline{\Xi_G})$ *be a fuzzy rough graph on G, as shown in Table 6.*

Table 6. The fuzzy rough graph Ξ_G in Example 4.

The Fuzzy Lower Approximation $\underline{\Xi_G}$	**The Fuzzy Upper Approximation** $\overline{\Xi_G}$
	

As indicated by Table 6, every vertex is reachable from every other vertex via a directed path in both $\underline{\Xi_G}$ *and* $\overline{\Xi_G}$*. This implies that both* $\underline{\Xi_G}$ *and* $\overline{\Xi_G}$ *constitute strongly connected fuzzy digraphs; hence,* Ξ_G *is a strongly connected fuzzy rough graph.*

Definition 20. *A fuzzy rough graph* Ξ_G *is classified as unilaterally connected if both* $\underline{\Xi_G}$ *and* $\overline{\Xi_G}$ *function as unilaterally connected fuzzy digraphs. In other words, for every* $x_1, x_2 \in X^*$*,*

$$\text{Connev}_{\underline{\Xi_G}}(x_1, x_2) > 0 \text{ or } \text{Connev}_{\underline{\Xi_G}}(x_2, x_1) > 0, \quad \text{Connev}_{\overline{\Xi_G}}(x_1, x_2) > 0 \text{ or } \text{Connev}_{\overline{\Xi_G}}(x_2, x_1) > 0.$$

Example 5. *Assume that G is the graph* $(N(G), D(G))$*, where* $N(G) = \{x_1, x_2, x_3, x_4, x_5\}$ *and* $D(G) = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7\}$*. Let* $\Xi_G = (\underline{\Xi_G}, \overline{\Xi_G})$ *be a fuzzy rough graph on G, as shown in Table 7.*

Table 7. The fuzzy rough graph Ξ_G in Example 5.

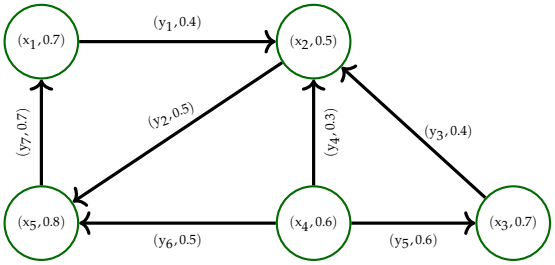
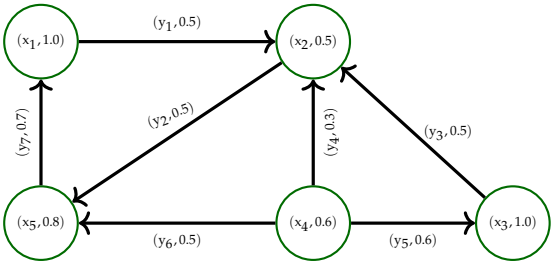
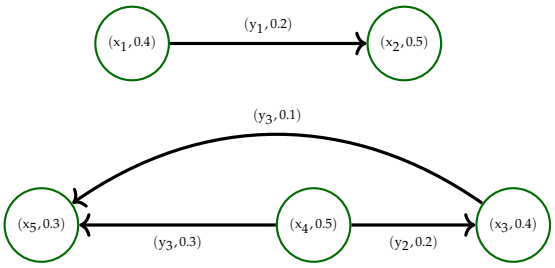
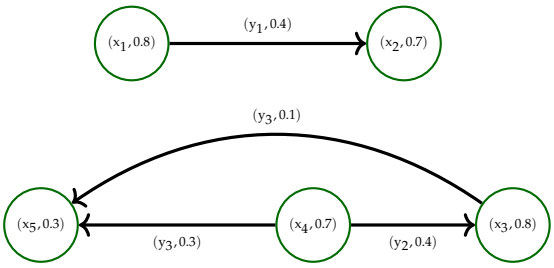
The Fuzzy Lower Approximation $\underline{\Xi_G}$	**The Fuzzy Upper Approximation $\overline{\Xi_G}$**
	

Table 7 illustrates that while a directed path exists from x_3 to x_1, no directed path can be found from x_1 to x_3 in either $\underline{\Xi_G}$ or $\overline{\Xi_G}$. Accordingly, both $\underline{\Xi_G}$ and $\overline{\Xi_G}$ are considered unilaterally connected fuzzy digraphs, confirming that Ξ_G is a unilaterally connected fuzzy rough graph.

Definition 21. *A fuzzy rough graph Ξ_G is characterized as a disconnected fuzzy rough graph if both $\underline{\Xi_G}$ and $\overline{\Xi_G}$ represent disconnected fuzzy digraphs. This implies that the vertex set is partitioned into two disjoint subsets such that no vertex in the first subset is connected to any vertex in the second subset within both $\underline{\Xi_G}$ and $\overline{\Xi_G}$.*

Example 6. *Assume that G is the graph $(N(G), D(G))$, where $N(G) = \{x_1, x_2, x_3, x_4, x_5\}$ and $D(G) = \{y_1, y_2, y_3, y_4\}$. Let $\Xi_G = (\underline{\Xi_G}, \overline{\Xi_G})$ be a fuzzy rough graph on G, as shown in Table 8.*

Table 8. The fuzzy rough graph Ξ_G in Example 6.

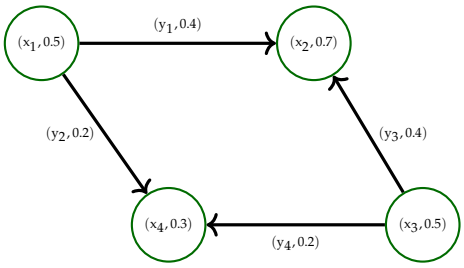
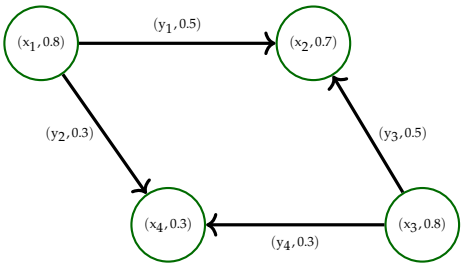
The Fuzzy Lower Approximation $\underline{\Xi_G}$	**The Fuzzy Upper Approximation $\overline{\Xi_G}$**
	

From Table 8, it is clear that both $\underline{\Xi_G}$ and $\overline{\Xi_G}$ are disconnected fuzzy digraphs. Thus, Ξ_G is a disconnected fuzzy rough graph.

Definition 22. *A fuzzy rough graph Ξ_G is said to be a weakly connected fuzzy rough graph if both $\underline{\Xi_G}$ and $\overline{\Xi_G}$ are weakly connected fuzzy digraphs. That is, both $\underline{\Xi_G}$ and $\overline{\Xi_G}$ are not disconnected fuzzy digraphs.*

Example 7. *Assume that G is the graph $(N(G), D(G))$, where $N(G) = \{x_1, x_2, x_3, x_4\}$ and $D(G) = \{y_1, y_2, y_3, y_4\}$. Let $\Xi_G = (\underline{\Xi_G}, \overline{\Xi_G})$ be a fuzzy rough graph on G, as shown in Table 9.*

Table 9. The fuzzy rough graph Ξ_G in Example 7.

The Fuzzy Lower Approximation $\underline{\Xi_G}$	**The Fuzzy Upper Approximation $\overline{\Xi_G}$**
	

As shown in Table 9, no directed paths exist between x_1 and x_3 in either $\underline{\Xi_G}$ or $\overline{\Xi_G}$. Furthermore, because these vertices cannot be isolated into two disjoint sets, Ξ_G does not meet the criteria for a disconnected fuzzy rough graph. Consequently, Ξ_G is classified as a weakly connected fuzzy rough graph.

Definition 23. *Consider a fuzzy rough graph $\Xi_G = (\underline{\Xi_G}, \overline{\Xi_G})$ on a nonempty graph G. That is, for some $w_0, w_1 \in X^*$,*

$$\text{Connev}_{\underline{\Xi_G}-x_1x_2}(w_0, w_1) < \text{Connev}_{\underline{\Xi_G}}(w_0, w_1),$$
$$\text{Connev}_{\overline{\Xi_G}-x_1x_2}(w_0, w_1) < \text{Connev}_{\overline{\Xi_G}}(w_0, w_1).$$

Then, "x_1x_2" is called a fuzzy directed bridge of $\underline{\Xi_G}$ and $\overline{\Xi_G}$ and also called a fuzzy rough directed bridge of Ξ_G.

Definition 24. *Let $\Xi_G = (\underline{\Xi_G}, \overline{\Xi_G})$ and $\Gamma_G = (\underline{\Gamma_G}, \overline{\Gamma_G})$ be two fuzzy rough graphs, where $\underline{\Xi_G} = ([N(\Xi)]_R, [D(\Xi)]_W)$ and $\overline{\Xi_G} = ([N(\Xi)]^R, [D(\Xi)]^W)$, $\underline{\Gamma_G} = ([N(\Gamma)]_R, [D(\Gamma)]_W)$, $\overline{\Gamma_G} = ([N(\Gamma)]^R, [D(\Gamma)]^W)$ are fuzzy graphs and $N(\Gamma) \subseteq N(\Xi), D(\Gamma) \subseteq D(\Xi)$. Γ_G is called a partial fuzzy rough subgraph of Ξ_G if $\underline{\Gamma_G}$ and $\overline{\Gamma_G}$ are partial fuzzy subgraphs of $\underline{\Xi_G}$ and $\overline{\Xi_G}$, respectively. That is, for all $x_1 \in N(\Gamma)$ and $x_1x_2 \in D(\Gamma)$, the following inequalities are satisfied:*

1. $[N(\Gamma)]_R(x_1) \leq [N(\Xi)]_R(x_1)$;
2. $[D(\Gamma)]_W(x_1x_2) \leq [D(\Xi)]_W(x_1x_2)$;
3. $[N(\Gamma)]^R(x_1) \leq [N(\Xi)]^R(x_1)$;
4. $[D(\Gamma)]^W(x_1x_2) \leq [D(\Xi)]^W(x_1x_2)$.

Definition 25. *Consider a fuzzy rough graph $\Xi_G = (\underline{\Xi_G}, \overline{\Xi_G})$ and $\Gamma_G = (\underline{\Gamma_G}, \overline{\Gamma_G})$ is a partial fuzzy rough subgraph of Ξ_G, where $\underline{\Gamma_G} = ([N(\Gamma)]_R, [D(\Gamma)]_W)$ and $\overline{\Gamma_G} = ([N(\Gamma)]^R, [D(\Gamma)]^W)$ are partial fuzzy subgraphs of $\underline{\Xi_G} = ([N(\Xi)]_R, [D(\Xi)]_W)$ and $\overline{\Xi_G} = ([N(\Xi)]^R, [D(\Xi)]^W)$. Γ_G is called a fuzzy rough subgraph of Ξ_G if $\underline{\Gamma_G}$ and $\overline{\Gamma_G}$ are fuzzy subgraphs of $\underline{\Xi_G}$ and $\overline{\Xi_G}$, respectively. That is, for all $x_1 \in N(\Gamma)$ and $x_1x_2 \in D(\Gamma)$, the following equations are satisfied:*

1. $[N(\Gamma)]_R(x_1) = [N(\Xi)]_R(x_1)$;
2. $[D(\Gamma)]_W(x_1x_2) = [D(\Xi)]_W(x_1x_2)$;
3. $[N(\Gamma)]^R(x_1) = [N(\Xi)]^R(x_1)$;
4. $[D(\Gamma)]^W(x_1x_2) = [D(\Xi)]^W(x_1x_2)$.

Proposition 1. *If $\Gamma_G = (\underline{\Gamma_G}, \overline{\Gamma_G})$ is a partial fuzzy rough subgraph of $\Xi_G = (\underline{\Xi_G}, \overline{\Xi_G})$, then*

$$\text{Connev}_{\underline{\Gamma_G}}(x_1, x_2) \leq \text{Connev}_{\underline{\Xi_G}}(x_1, x_2), \forall x_1, x_2 \in \Gamma,$$
$$\text{Connev}_{\overline{\Gamma_G}}(x_1, x_2) \leq \text{Connev}_{\overline{\Xi_G}}(x_1, x_2), \forall x_1, x_2 \in \Gamma.$$

Definition 26. *Consider a fuzzy rough graph* $\Xi_G = (\underline{\Xi_G}, \overline{\Xi_G})$. *The connectivity index of* Ξ_G *denoted by* $CI(\Xi_G)$ *and represented by*

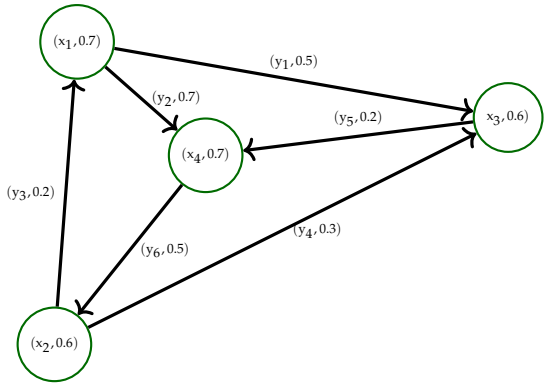
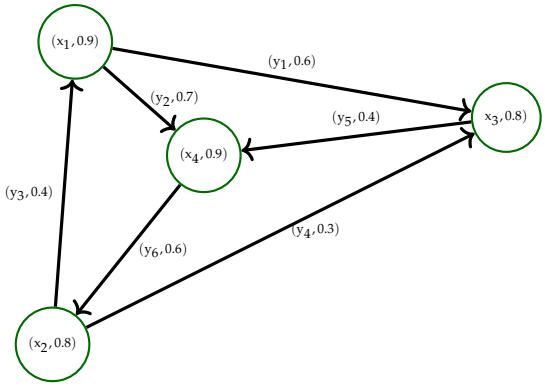
$$CI(\Xi_G) = CI(\underline{\Xi_G}) + CI(\overline{\Xi_G}),$$

where

$$CI(\underline{\Xi_G}) = \sum_{x_1,x_2 \in G} [N(\Xi)]_R(x_1)[N(\Xi)]_R(x_2)\,\mathrm{Connev}_{\underline{\Xi_G}}(x_1,x_2),$$
$$CI(\overline{\Xi_G}) = \sum_{x_1,x_2 \in G} [N(\Xi)]^R(x_1)[N(\Xi)]^R(x_2)\,\mathrm{Connev}_{\overline{\Xi_G}}(x_1,x_2).$$

Example 8. *Assume that* G *is the graph* $(N(G), D(G))$, *where* $N(G) = \{x_1, x_2, x_3, x_4\}$ *and* $D(G) = \{y_1, y_2, y_3, y_4, y_5, y_6\}$. *Let* $\Xi_G = (\underline{\Xi_G}, \overline{\Xi_G})$ *be a fuzzy rough graph on* G, *as shown in Table 10.*

Table 10. The lower and upper approximations $\underline{\Xi_G}$ and $\overline{\Xi_G}$ in Example 8.

Fuzzy Lower $\underline{\Xi_G} = ([N(\Xi)]_R, [D(\Xi)]_W)$	**Fuzzy Upper** $\overline{\Xi_G} = ([N(\Xi)]^R, [D(\Xi)]^W)$
	

From Table 10 and calculations, we have

$$CI(\underline{\Xi_G}) = 1.713, \quad CI(\overline{\Xi_G}) = 4.139, \text{ and } CI(\Xi_G) = 5.852.$$

Proposition 2. *If* $\Gamma_G = (\underline{\Gamma_G}, \overline{\Gamma_G})$ *is a partial fuzzy rough subgraph of* $\Xi_G = (\underline{\Xi_G}, \overline{\Xi_G})$, *then* $CI(\Gamma_G) \leq CI(\Xi_G)$.

Proof. Let $\Gamma_G = (\underline{\Gamma_G}, \overline{\Gamma_G})$ be a partial fuzzy rough subgraph of Ξ_G, then by definition,

$$[N(\Gamma)]_R(x_1) \leq [N(\Xi)]_R(x_1), \text{ for any } x_1 \in \Gamma,$$
$$[N(\Gamma)]^R(x_1) \leq [N(\Xi)]^R(x_1), \text{ for any } x_1 \in \Gamma.$$

By Proposition 1.8 and the inequality (1.1), we obtain

$$\sum_{x_1,x_2 \in \Gamma} [N(\Gamma)]_R(x_1)[N(\Gamma)]_R(x_2)\,\mathrm{Connev}_{\underline{\Gamma_G}}(x_1,x_2) \leq \sum_{x_1,x_2 \in G} [N(\Xi)]_R(x_1)[N(\Xi)]_R(x_2)\,\mathrm{Connev}_{\underline{\Xi_G}}(x_1,x_2)$$

Also, by Proposition 1.8 and the inequality (1.2), we obtain

$$\sum_{x_1,x_2 \in \Gamma} [N(\Gamma)]^R(x_1)[N(\Gamma)]^R(x_2)\,\mathrm{Connev}_{\overline{\Gamma_G}}(x_1,x_2) \leq \sum_{x_1,x_2 \in G} [N(\Xi)]^R(x_1)[N(\Xi)]^R(x_2)\,\mathrm{Connev}_{\overline{\Xi_G}}(x_1,x_2).$$

Adding the above inequalities, we obtain

$$\left\{\sum_{x_1,x_2\in\Gamma}[N(\Gamma)]_R(x_1)[N(\Gamma)]_R(x_2)\,Connev_{\underline{\Gamma_G}}(x_1,x_2)+\sum_{x_1,x_2\in\Gamma}[N(\Gamma)]^R(x_1)[N(\Gamma)]^R(x_2)\,Connev_{\overline{\Gamma_G}}(x_1,x_2)\right\}\leq$$
$$\left\{\sum_{x_1,x_2\in G}[N(\Xi)]_R(x_1)[N(\Xi)]_R(x_2)\,Connev_{\underline{\Xi_G}}(x_1,x_2)+\sum_{x_1,x_2\in G}[N(\Xi)]^R(x_1)[N(\Xi)]^R(x_2)\,Connev_{\overline{\Xi_G}}(x_1,x_2)\right\}.$$

Hence, $CI(\Gamma_G)\leq CI(\Xi_G)$. □

Proposition 3. *If $\Gamma_G=(\underline{\Gamma_G},\overline{\Gamma_G})$ is a fuzzy rough subgraph of $\Xi_G=(\underline{\Xi_G},\overline{\Xi_G})$, then $CI(\Gamma_G)\leq CI(\Xi_G)$.*

Theorem 3. *Consider a fuzzy rough graph $\Xi_G=(\underline{\Xi_G},\overline{\Xi_G})$ and $\Gamma_G=(\underline{\Xi_G}-x_1x_2\,,\overline{\Xi_G}-x_1x_2)$ a fuzzy rough subgraph of Ξ_G. Then, $CI(\Gamma_G)<CI(\Xi_G)$ if and only if for some $x_1x_2\in\Theta$,*

$$Connev_{\underline{\Xi_G}-x_1x_2}(x_1,x_2)<Connev_{\underline{\Xi_G}}(x_1,x_2),$$
$$Connev_{\overline{\Xi_G}-x_1x_2}(x_1,x_2)<Connev_{\overline{\Xi_G}}(x_1,x_2).$$

Proof. Let x_1x_2 be a fuzzy rough directed bridge in Ξ_G. Assume that

$$Connev_{\underline{\Xi_G}-x_1x_2}(x_1,x_2)<Connev_{\underline{\Xi_G}}(x_1,x_2),$$
$$Connev_{\overline{\Xi_G}-x_1x_2}(x_1,x_2)<Connev_{\overline{\Xi_G}}(x_1,x_2).$$

This implies that $CI(\underline{\Gamma_G})<CI(\underline{\Xi_G})$ and $CI(\overline{\Gamma_G})<CI(\overline{\Xi_G})$. Adding these inequalities, we have

$$CI(\underline{\Gamma_G})+CI(\overline{\Gamma_G})<CI(\underline{\Xi_G})+CI(\overline{\Xi_G}).$$

Hence, $CI(\Gamma_G)<CI(\Xi_G)$. Conversely, suppose that $CI(\underline{\Gamma_G})<CI(\underline{\Xi_G})$. Then, we consider the following cases:

- Case I. If x_1x_2 is a δ - arc, then $Connev_{\underline{\Gamma_G}}(x_1,x_2)=Connev_{\underline{\Xi_G}}(x_1x_2)$ and $Connev_{\overline{\Gamma_G}}(x_1,x_2)=Connev_{\overline{\Xi_G}}(x_1x_2)$. Hence, $CI(\underline{\Gamma_G})=CI(\underline{\Xi_G})$.
- Case II. If x_1x_2 is a β-strong arc, then $[D(\Xi)]_W(x_1x_2)=Connev_{\underline{\Gamma_G}}(x_1,x_2)$ and $[D(\Xi)]^W(x_1x_2)=Connev_{\overline{\Gamma_G}}(x_1,x_2)$; that is, there exists another strongest directed path from x_1 to x_2 other than the arc " x_1x_2 " in both $\underline{\Xi_G}$ and $\overline{\Xi_G}$. Thus, $Connev_{\underline{\Gamma_G}}(x_1,x_2)=Connev_{\underline{\Xi_G}}(x_1,x_2)$ and $Connev_{\overline{\Gamma_G}}(x_1,x_2)=Connev_{\overline{\Xi_G}}(x_1,x_2)$. Hence, $CI(\underline{\Gamma_G})=CI(\underline{\Xi_G})$.
- Case III. If x_1x_2 is a y-strong arc, then $Connev_{\underline{\Gamma_G}}(x_1,x_2)<[D(\Xi)]_W(x_1x_2)$ and $Connev_{\underline{\Gamma_G}}(x_1,x_2)<(\underline{SY})(x_1x_2)$. Thus, the arc x_1x_2 is a unique strongest directed path having strength equal to $[D(\Xi)]_W(x_1x_2)$ and $[D(\Xi)]^W(x_1x_2)$, respectively. Clearly $CI(\underline{\Gamma_G})<CI(\underline{\Xi_G})$.

The three cases imply that $Connev_{\underline{\Xi_G}-x_1x_2}(x_1,x_2)<Connev_{\underline{\Xi_G}}(x_1,x_2)$ and $Connev_{\overline{\Xi_G}-x_1x_2}(x_1,x_2)<Connev_{\overline{\Xi_G}}(x_1,x_2)$, which completes the proof. □

Corollary 1. *Consider a fuzzy rough graph $\Xi_G=(\underline{\Xi_G},\overline{\Xi_G})$. Then, $CI(\Xi_G-x_1x_2)=CI(\Xi_G)$ if and only if x_1x_2 is either a β-strong or δ-arc of Ξ_G.*

6. Applications

The connectivity index of directed fuzzy rough graphs plays an important role in analyzing the strength and reliability of relationships within systems characterized by uncertainty and incomplete information. It is particularly useful in applications such as decision-making, network reliability assessment, pattern recognition, and information flow analysis, where relationships are not strictly binary but exist with different degrees

of certainty. By quantifying how strongly components of a system are connected under vagueness and roughness, the connectivity index helps identify critical nodes, evaluate system stability, and support optimal choices in complex real-world problems such as communication networks, social interaction modeling, and uncertain knowledge systems. This section presents a method for making decisions in uncertain systems: the connectivity index of directed fuzzy rough graphs via fuzzy minimal neighborhoods. This technique provides an in-depth analysis of the problem by using lower and upper approximations for unknown data.

Example 9. ***Case Study: Application in Traffic Flow Network:*** *Traffic flow theory focuses on the movement of vehicles along roadways and their interactions with other vehicles, pedestrians, and traffic control systems. Several factors influence traffic flow, including road design speed, the proportion of heavy vehicles, and the number of lanes and intersections. Traffic flow is typically defined as the number of vehicles passing a specific point on a roadway within a given period of time, usually measured through traffic counts collected at fixed locations. These counts vary throughout the day, with peak-hour traffic playing a crucial role in traffic analysis. Measuring traffic flow is essential for identifying congestion points, determining the need for traffic signals at intersections, evaluating roadway capacity, and supporting effective transportation planning. Additionally, traffic flow is closely related to vehicle speed and traffic density on the road. In this case study, a directed fuzzy rough graph is employed to identify the busiest intersection when traveling between locations within a city. The application determines the most congested intersection by utilizing the concept of the minimal neighborhoods and the connectivity index within the traffic flow network. For this purpose, we consider a collection of intersections or junctions denoted by* $N(G)$*, consisting of* $N(G) = \{x_1, x_2, x_3, x_4, x_5\}$*.* $D(G) = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}, y_{12}\}$*.*

Now, we define a fuzzy graph $\Xi_G = (\Xi_{N(G)}, \Xi_{D(G)})$ *over the crisp graph* G *as* $\Xi_{N(G)} = \{(x_1, 0.5), (x_2, 0.7), (x_3, 0.6), (x_4, 0.6), (x_5, 0.6)\}$*, which indicates the traffic density and* $\Xi_{D(G)} = \{(y_1, 0.5), (y_2, 0.5), (y_3, 0.5), (y_4, 0.5), (y_5, 0.5), (y_6, 0.5), (y_7, 0.4), (y_8, 0.5), (y_9, 0.6), (y_{10}, 0.5), (y_{11}, 0.5), (y_{12}, 0.5)\}$*, which indicates the degree of traffic density among different junctions under consideration. For example,* $(y_7 = x_3x_4, 0.4)$ *shows that the traffic density of junction* x_3 *is* 40% *more than that of junction* x_4*. The fuzzy subgraph* Ξ_G *is represented in Figure 3. We construct the fuzzy binary relation R, as presented in Table 11, to indicate the comparison of characteristics of the roads under consideration, where* $R(x_i, x_j)$*,* $(i, j \in \{1, 2, \ldots, 5\})$*.*

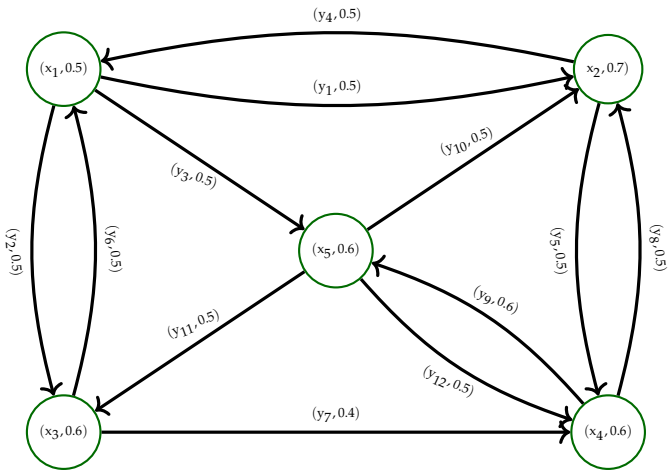


Figure 3. The fuzzy subgraph Ξ_G given in Example 9.

Table 11. Fuzzy relation R in Example 9.

R	x_1	x_2	x_3	x_4	x_5
x_1	1	0.4	0.5	0.5	0.3
x_2	0.4	1	0.4	0.5	0.6
x_3	0.5	0.4	1	0.7	0.6
x_4	0.5	0.5	0.7	1	0.4
x_5	0.3	0.6	0.6	0.4	1
x_6	0.5	0.6	0.6	0.6	0.9
x_7	0.6	0.6	0.5	0.6	0.2

By computing, $R < x_1 > R = R < x_2 > R = R < x_3 > R = R < x_4 > R = R < x_5 > R = \{0.3, 0.4, 0.4, 0.4, 0.2\}$. *The lower and upper approximations of* $\Xi_{N(G)}$ *with respect to R are given by*

$$[N(\Xi)]_R = \{(x_1, 0.5), (x_2, 0.6), (x_3, 0.6), (x_4, 0.6), (x_5, 0.6)\},$$

$$[N(\Xi)]^R = \{(x_1, 0.5), (x_2, 0.7), (x_3, 0.6), (x_4, 0.6), (x_5, 0.6)\}.$$

Also, let $\gamma_{D(G)} = \{(x_1x_2, 0.5), (x_1x_3, 0.5), (x_1x_5, 0.5), (x_2x_1, 0.5), (x_2x_4, 0.5), (x_3x_1, 0.5), (x_3x_4, 0.4), (x_4x_2, 0.5), (x_4x_5, 0.6), (x_5x_2, 0.5), (x_5x_3, 0.5), (x_5x_4, 0.5)\}$ *be a fuzzy edge set defined on* $D(G)$ *and* $\gamma_{D(G)}(x_ix_j)$ *be a fuzzy set on* $D(G)$. *Let W be a fuzzy relation defined on* $D(G)$, *given as in Table 12.*

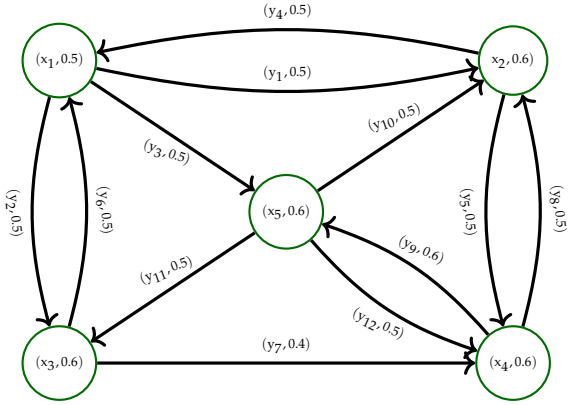
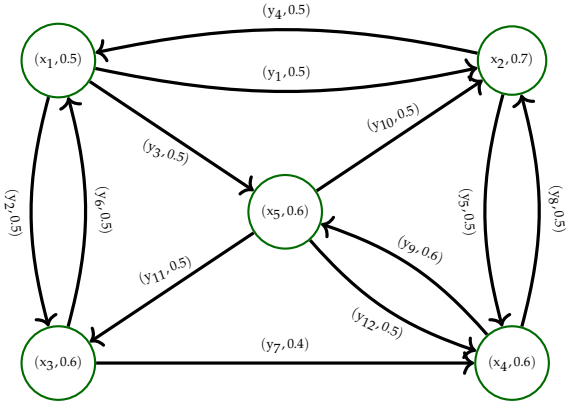
Table 12. Fuzzy relation W in Example 9.

W	x_1x_2	x_1x_3	x_1x_5	x_2x_1	x_2x_4	x_3x_1	x_3x_4	x_4x_2	x_4x_5	x_5x_2	x_5x_3	x_5x_4
x_1x_2	1	0.3	0.3	0.4	0.4	0.5	0.5	0.4	0.2	0.2	0.5	0.6
x_1x_3	0.3	1	0.3	0.4	0.5	0.3	0.4	0.5	0.6	0.3	0.6	0.7
x_1x_5	0.3	0.3	1	0.5	0.5	0.6	0.5	0.5	0.6	0.2	0.7	0.7
x_2x_1	0.4	0.4	0.5	1	0.5	0.5	0.4	0.5	0.6	0.3	0.8	0.7
x_2x_4	0.4	0.5	0.5	0.5	1	0.5	0.6	0.5	0.2	0.2	0.8	0.2
x_3x_1	0.5	0.3	0.6	0.5	0.5	1	0.3	0.3	0.2	0.2	0.2	0.8
x_3x_4	0.5	0.4	0.5	0.4	0.6	0.3	1	0.5	0.4	0.4	0.3	0.3
x_4x_2	0.4	0.5	0.5	0.5	0.5	0.3	0.5	1	0.6	0.2	0.8	0.8
x_4x_5	0.2	0.6	0.6	0.6	0.2	0.2	0.4	0.6	1	0.4	0.2	0.2
x_5x_2	0.2	0.3	0.2	0.3	0.2	0.2	0.4	0.2	0.4	1	0.2	0.9
x_5x_3	0.2	0.3	0.2	0.3	0.2	0.2	0.4	0.2	0.4	1	0.7	0.2
x_5x_4	0.2	0.3	0.2	0.3	0.2	0.2	0.4	0.2	0.4	1	0.6	0.5

By computing, $W < x_1x_2 > W = W < x_1x_3 > W = W < x_1x_5 > W = W < x_2x_1 > W = W < x_2x_4 > W = W < x_3x_1 > W = W < x_3x_4 > W = W < x_4x_2 > W = W < x_4x_5 > W = W < x_5x_2 > W = W < x_5x_3 > W = W < x_5x_4 > W = \{0.2, 0.3, 0.2, 0.3, 0.2, 0.2, 0.3, 0.2, 0.2, 0.2, 0.2, 0.2\}$. *The lower and upper approximations of* $\Xi_{D(G)}$ *with respect to W are given by* $[D(\Xi)]_W = \{(x_1x_2, 0.5), (x_1x_3, 0.5), (x_1x_5, 0.5), (x_2x_1, 0.5), (x_2x_4, 0.5), (x_3x_1, 0.5), (x_3x_4, 0.4), (x_4x_2, 0.5), (x_4x_5, 0.6), (x_5x_2, 0.5), (x_5x_3, 0.5), (x_5x_4, 0.5)\}$, $[D(\Xi)]^W = \{(x_1x_2, 0.5), (x_1x_3, 0.5), (x_1x_5, 0.5), (x_2x_1, 0.5), (x_2x_4, 0.5), (x_3x_1, 0.5), (x_3x_4, 0.4), (x_4x_2, 0.5), (x_4x_5, 0.6), (x_5x_2, 0.5), (x_5x_3, 0.5), (x_5x_4, 0.5)\}$.

The problem can be represented by fuzzy rough graphs as shown in Table 13, which is a traffic flux network.

Table 13. The lower and upper approximations $\underline{\Xi_G}$ and $\overline{\Xi_G}$ representing traffic flow among different junctions in Example 9.

Fuzzy Lower $\underline{\Xi_G} = ([\mathbf{N}(\Xi)]_{\mathbf{R}}, [\mathbf{D}(\Xi)]_{\mathbf{W}})$	**Fuzzy Upper** $\overline{\Xi_G} = ([\mathbf{N}(\Xi)]^{\mathbf{R}}, [\mathbf{D}(\Xi)]^{\mathbf{W}})$
	

Consider the vertex or junction x_1 and delete this junction from the given network. Now $CI(\underline{\Xi_G} - x_1)$, $CI(\overline{\Xi_G} - x_1)$ and $CI(\Xi_G - x_1)$ can be calculated by

$$CI(\underline{\Xi_G} - x_1) = 0.54 + 0.54 + 0.6 \times 0.6 \times 1.2 + 0.6 \times 0.6 \times 1.0 + 0.6 \times 0.6 \times 0.6 = 1.08 + 0.36 + 0.432 + 0.216 = 2.088,$$
$$CI(\overline{\Xi_G} - x_1) = 0.63 + 0.57 + 0.6 \times 0.6 \times 0.8 + .6 \times 0.7 \times 0.4 + 0.6 \times 0.7 \times 0.5 + 0.6 \times 0.6 \times 0.5 + 0.6 \times 0.6 \times 0.6$$
$$= 1.2 + 0.288 + 0.168 + 0.21 + 0.18 + 0.216 = 2.262,$$
$$CI(\Xi_G - x_1) = CI(\underline{\Xi_G} - x_1) + CI(\overline{\Xi_G} - x_1) = 2.088 + 2.262 = 4.35,$$

Similarly, we have the following calculations for the fuzzy subgraphs $\Xi_G - x_2$,$\Xi_G - x_2$,$\Xi_G - x_4$ and $\Xi_G - x_5$.

$$CI(\underline{\Xi_G} - x_2) = 0.45 + 0.51 + 0.474 + 0.516 = 1.95,$$
$$CI(\overline{\Xi_G} - x_2) = 0.45 + 0.51 + 0.474 + 0.516 = 1.95,$$
$$CI(\Xi_G - x_2) = CI(\underline{\Xi_G} - x_2) + CI(\overline{\Xi_G} - x_2) = 3.90,$$
$$CI(\underline{\Xi_G} - x_3) = 0.45 + 0.51 + 0.51 + 0.546 = 2.016,$$
$$CI(\overline{\Xi_G} - x_3) = 0.475 + 0.595 + 0.576 + 0.54 = 2.186,$$
$$CI(\Xi_G - x_3) = CI(\underline{\Xi_G} - x_3) + CI(\overline{\Xi_G} - x_3) = 4.202,$$

$$CI(\underline{\Xi_G} - x_4) = 0.45 + 0.51 + 0.51 + 0.51 = 1.98,$$
$$CI(\overline{\Xi_G} - x_4) = 0.475 + 0.595 + 0.505 + 0.54 = 2.115,$$
$$CI(\Xi_G - x_4) = CI(\underline{\Xi_G} - x_4) + CI(\overline{\Xi_G} - x_4) = 4.095,$$
$$CI(\underline{\Xi_G} - x_5) = 0.45 + 0.51 + 0.51 + 0.51 = 1.98,$$
$$CI(\overline{\Xi_G} - x_5) = 0.475 + 0.57 + 0.51 + 0.54 = 2.095,$$
$$CI(\Xi_G - x_5) = CI(\underline{\Xi_G} - x_5) + CI(\overline{\Xi_G} - x_5) = 4.075.$$

The intersections (or junctions) can be ordered based on their connectivity index values in the given traffic flow model, ranking them from the least busy junction to the most busy junction in the traffic network as follows: $CI(\Xi_G - x_1) > CI(\Xi_G - x_3) > CI(\Xi_G - x_4) > CI(\Xi_G - x_5) > CI(\Xi_G - x_2)$. The connectivity index of the deleted directed fuzzy rough subgraph $\Xi_G - x_2$ is the lowest, whereas that of $\Xi_G - x_1$ is the highest. This indicates that junction x_2 is the busiest intersection in the traffic network, while junction x_1 is the least busy according to the connectivity index. Consequently, the traffic load is greatest at x_2 and lowest at x_1 within this network. The above method for identifying the busiest junction of a city in the traffic model is illustrated in Algorithm 1.

Algorithm 1 The algorithm for identifying the busiest junction of a city in the traffic model

Input: A finite fuzzy graph $\Xi_G = (N_G, D_G)$ and two finite fuzzy relations R, W.
Output: The busiest junction x_i.

Input the vertex or junction set $N(G) = \{x_1, x_2, \ldots, x_n\}$.
Input the fuzzy vertex set $\Xi_{N(G}$ on $N(G)$.
Input the fuzzy association set R on $N(G)$.
Input the fuzzy edges set $D(G) = \{y_1, y_2, \ldots, y_r\}$, where $y_i = x_j x_k$, for some $1 \leq j, k \leq n$.
Input the fuzzy association set W on $D(G)$.
for $i = 1 : n$ **do**
  $[N(\Xi)]_R(x_i) = 1$ and $[N(\Xi)]^R(x_i) = 0$
  for $j = 1 : n$ **do**
    $(N(\Xi))_*(x_j) = \bigwedge_{x' \in N(G)} \Big(1 - (R < x' > R(x)) \vee \mu_{N(\Xi)}(x')\Big)$
    $(N(\Xi))^*(x_j) = \bigvee_{x' \in N(G)} \Big(R < x' > R(x)) \wedge \mu_{N(\Xi)}(x')\Big)$
    $[N(\Xi)]_R(x_i) = \min\{[N(\Xi)]_*(x_i), (N(\Xi))(x_j)\}$
    $[N(\Xi)]^R(x_i) = \max\{[N(\Xi)]^*(x_i), (N(\Xi))(x_j)\}$
  end for
end for
for $i = 1 : r$ **do**
  $[D(\Xi)]_W(y_i) = 1$
  $[D(\Xi)]^W(y_i) = 0$
  for $j = 1 : n$ **do**
    $(D(\Xi))_*(y_j) = \bigwedge_{y' \in D(G)} \Big(1 - (W < y' > W(y)) \vee \mu_{D(\gamma)}\Big)$
    $(D(\Xi))^*(y_j) = \bigvee_{y' \in D(G)} \Big(W < y' > W(y) \wedge \mu_{D(\gamma)}\Big)$
    $[D(\Xi)]_W(x_i) = \min\{[D(\Xi)]_*(y_i), (N(\Xi))(y_j)\}$
    $[D(\Xi)]^W(y_i) = \max\{[D(\Xi)]^*(y_i), (D(\Xi))(y_j)\}$
  end for
end for
for $i = 1 : n$ **do**
  if $i \neq k$ **then**
    for $j = 1 : n$ **do**
      $CI(\underline{\Xi_G} - x_k) = \sum_{x_1, x_2 \in G} [N(\Xi)]_R(x_1)[N(\Xi)]_R(x_2)\, Connev_{\underline{\Xi_G}}(x_1, x_2)$
      $CI(\overline{\Xi_G} - x_k) = \sum_{x_1, x_2 \in G} [N(\Xi)]^R(x_1)[N(\Xi)]^R(x_2)\, Connev_{\overline{\Xi_G}}(x_1, x_2)$
      $CI(\Xi_G - x_k) = CI(\underline{\Xi_G} - x_k) + CI(\overline{\Xi_G} - x_k)$
    end for
  end if
end for
if $CI(\Xi_G - x_s) = \min_i CI(\Xi_G - x_k), k = 1, 2, \ldots, |G|$ **then**
  Print: The decision is x_s is the busiest junction.
  if s has more than one value **then**
    Print: Any one of x_s.
  end if
end if

Discussion

Following the computation of the fuzzy minimal neighborhoods and the subsequent lower and upper approximations for both the vertex and edge sets, we obtain the fully constructed fuzzy rough graph $\Xi_G = (\underline{\Xi_G}, \overline{\Xi_G})$. By calculating the connectivity index for these lower and upper bounding graphs, we can successfully pinpoint the exact intersections acting as the primary bottlenecks in the traffic flow network. The significance of this traffic network application extends beyond simple bottleneck identification; it effectively illustrates the fundamental necessity of our proposed framework over previous methodologies. Traditional fuzzy rough graph models [17,18,32,33] heavily depend on equivalence

relations or at least tolerance and symmetric relations to construct their approximation spaces. Applying such classical models to the traffic flow scenario presented in this example exposes severe theoretical and practical limitations:

- Handling Asymmetry and Directionality: Real-world traffic flow is inherently asymmetric. The congestion or flow capacity from junction x_1 to x_2 rarely equals the flow from x_2 to x_1. Utilizing an equivalence relation would forcefully impose symmetry and transitivity upon the network, distorting the actual traffic data and leading to highly inaccurate connectivity indices. Our framework, which operates on arbitrary fuzzy relations R and W, naturally accommodates the directional complexities of urban transportation.
- Boundary Region Minimization and Mathematical Precision: In prior models, attempting to force non-binary, directed network data into rigid equivalence classes frequently results in the violation of the fundamental rough set inclusion property ($\underline{A} \subseteq A \subseteq \overline{A}$). This violation artificially inflates the boundary region and degrades the accuracy measure of the entire system. By refining the mathematical calculations to utilize fuzzy minimal neighborhoods, our proposed method consistently preserves this crucial inclusion property. This approach guarantees minimized boundary regions and highly reliable accuracy measures, ensuring absolute mathematical precision before the final connectivity index is derived.
- Graded Relationships over Crisp Indiscernibility: Previous connectivity indices derived from crisp binary indiscernibility fail to capture the continuous severity of traffic dynamics. An intersection is rarely just "congested" or "clear"; it exists on a probabilistic spectrum. Because our lower and upper approximation operators process arbitrary fuzzy relations without demanding predefined structural constraints, they successfully capture these nuanced, real-world uncertainties.

Consequently, the connectivity index evaluated under this new fuzzy rough framework yields a highly realistic, mathematically sound evaluation of the network's robustness. Removing a critical fuzzy directed bridge within this model demonstrably degrades the overall connectivity index in both approximations, providing transportation planners with a rigorously verified, precise metric for decision-making under uncertainty that far surpasses the capabilities of equivalence-based models. To validate the performance of the newly introduced model, a comparative evaluation against several established techniques is detailed in Table 14, based on the scenario outlined in Example 9 in Section 6. Existing mathematical structures such as rough fuzzy graphs [17,19,32], fuzzy rough graphs [18,33], and soft rough fuzzy graphs [37] lack built-in mechanisms for parameterization. Because of this absence of parametric features, the algorithms discussed in [17–19,32,33,37] are fundamentally incapable of resolving the presented example. The methodology introduced herein efficiently bypasses this representational bottleneck by formulating a robust framework capable of utilizing arbitrary fuzzy binary relations to generate granular minimal fuzzy neighborhoods. By decoupling the approximation spaces from the stringent requirements of traditional equivalence relations [38–40], the current model offers exceptional adaptability and extensive mathematical generalization. Consequently, this comparative analysis demonstrates that the advanced technique outperforms prior models, granting decision-makers substantially greater autonomy and flexibility when designating membership and non-membership degrees relative to the existing literature.

Table 14. The rankings based on different methods.

Method	The Ranking of Alternatives
The rough fuzzy graphs method [17,19,32]	Failed to handle
The fuzzy rough graphs method [18,33]	Failed to handle
The soft rough fuzzy graphs method [37]	Failed to handle
The intuitionistic fuzzy rough graphs method [38,39]	Failed to handle
The Pythagorean rough fuzzy graphs method [40]	Failed to handle
Our method based on fuzzy minimal neighborhoods and fuzzy rough graphs $\Xi_G = (\underline{\Xi_G}, \overline{\Xi_G})$	$x_1 > x_3 > x_4 > x_5 > x_2$

While this demonstration confirms the mathematical viability of our theory, its applicability is not confined to traffic analysis. The same underlying principles harnessing minimal fuzzy neighborhoods to compute precise approximation operators can be seamlessly adapted for a variety of complex decision paradigms. By appropriately contextualizing the vertex and edge relationships, this fuzzy rough graph framework can be deployed to optimize supply chains, evaluate financial risks, assess healthcare diagnostics, or dissect social networks, utilizing Algorithm 1 to minimize boundaries and boost choice accuracy. Ultimately, this example establishes a flexible blueprint capable of resolving multi-layered uncertainty across numerous disciplines.

7. Conclusions

This study developed a novel theoretical framework for fuzzy rough graphs based on fuzzy minimal neighborhoods, introducing a comprehensive connectivity index that effectively captures the structural strength and reliability of uncertain networks. By integrating fuzzy set theory with rough set approximation through vertex- and edge-based minimal neighborhoods, we established rigorous mathematical foundations for lower and upper approximations in fuzzy graph contexts. The proposed connectivity index, defined as the sum of connectivity measures from both lower and upper approximations, provides a robust quantitative tool for analyzing network robustness under uncertainty. Our key theoretical contributions include: (i) the formal construction of fuzzy rough graphs using fuzzy minimal neighborhoods derived from any fuzzy relation on both vertex and edge sets; (ii) the systematic development of connectivity concepts, including strongest paths, various arc classifications, and different connectivity types (strongly, unilaterally, weakly, and disconnected); (iii) the introduction and mathematical characterization of the connectivity index with proven monotonicity properties under subgraph operations; and (iv) the establishment of critical relationships between connectivity indices and network structural properties, particularly through the identification of fuzzy rough directed bridges.

The practical utility of our framework was demonstrated through a case study application in urban traffic flow analysis, where the connectivity index effectively identified the busiest intersections in a city network by quantifying the disruption caused by junction removal. This application showcases the method's ability to handle incomplete and uncertain data while providing actionable insights for decision-making in complex systems. The integration of fuzzy rough sets with graph theory through minimal neighborhoods offers several advantages over existing approaches: it provides greater flexibility in handling uncertainty than rough fuzzy graphs, as we use any fuzzy relations rather than the equivalence relations used in [14,17,18,32,33]; allows for more nuanced approximations through intersection-based neighborhood systems; and enables comprehensive analysis of both local (neighborhood-level) and global (network-level) connectivity properties.

Several promising research directions emerge from this work:

- **Dynamic and Temporal Extensions:** Extending the current static framework to dynamic fuzzy rough graphs where vertex/edge memberships and relations evolve over time, enabling applications in evolving social networks, adaptive transportation systems, and real-time monitoring scenarios.
- **Computational Complexity and Algorithmic Optimization:** Developing efficient algorithms for computing connectivity indices in large-scale networks, including parallel and distributed computing approaches, and establishing complexity bounds for various computational problems within the fuzzy rough graph framework.
- **Theoretical Extensions:** Investigating spectral properties of fuzzy rough graphs, developing centrality measures based on the connectivity index, and exploring relationships with other uncertainty theories such as intuitionistic fuzzy sets, neutrosophic sets, or probabilistic rough sets.
- **Hybrid Decision-Making Frameworks:** Combining the connectivity index with multi-criteria decision-making methods to create comprehensive decision-support systems for complex network optimization problems under uncertainty.

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