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Exact Solution for a Class of First-Order Reflected Equations Involving a Polynomial-Type Forcing Term of Arbitrary Finite Degree

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Abstract

Differential equations involving reflection of the argument arise in numerous mathematical models where the present state is coupled with its symmetric counterpart. In this paper, we investigate a challenging reflected equation, expressed as $y'(t) = a\,y(t) + b\,y(-t) + \sum_{k=0}^{m} c_k t^k$, where the forcing function is an arbitrary polynomial of finite degree. A constructive transformation is developed that converts the nonhomogeneous reflected equation into a homogeneous reflection equation. Explicit closed-form expressions for the transformation coefficients are derived and proved by backward induction. Consequently, the original problem is reduced to a homogeneous reflected model whose exact solution is obtained in power-series form and subsequently expressed through hyperbolic or trigonometric functions according to the sign of $a^2 - b^2$. To complete the analysis, a reduction-of-order technique is employed to establish an equivalent second-order ordinary differential equation without reflection. This approach enables the derivation of exact solutions in the singular cases $a = b$ and $a = -b$, where the transformation method is no longer applicable. The obtained results provide a unified analytical methodology for a broad class of first-order reflection equations with polynomial forcing and furnish exact closed-form solutions for both regular and resonant parameter regimes.

Keywords: scalar differential equation; reflection of argument; exact solution

MSC: 34B15; 47H11

1. Introduction

Reflected ordinary differential equations (RODEs) constitute an important branch of differential equations in which the unknown function depends simultaneously on the current state and its reflected value. Such equations exhibit temporal/spatial symmetry and involution structures involving mirrored states. The mathematical foundations of the RODEs were established through the pioneering works of Wiener and Aftabizadeh [1], Gupta [2,3], Hai [4], and O'Regan [5,6], while comprehensive treatments can be found in the monograph of Cabada and Tojo [7]. Over the past decades, the RODEs have attracted sustained attention due to their connections with boundary value problems [8–10], periodic phenomena [11–13], integral equations [14], nonlocal equations [15], control theory [16], and equations with involutions [17–19].

Considerable efforts have been devoted to investigate the existence and uniqueness [20], Green's functions [21], solvability [22], and qualitative properties of the RODEs [23]. Although the functional differential equations (FDEs) have been well treated in [24–26],

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renewed interest has emerged recently in analytical treatments of delay models [27,28], motivated by applications in physics [29–31], biological sciences and neuroscience [32]. In particular, Ebaid and Al-Jeaid [33] obtained the exact solutions for the homogeneous RODE $y'(t) = a\,y(t) + b\,y(-t)$, revealing elegant representations in terms of hyperbolic and trigonometric functions.

However, many difficulties arise when solving models in the presence of external forcing terms. The objective of this work is to introduce a complete analytical approach for solving the RODE:

$$y'(t) = a\,y(t) + b\,y(-t) + \sum_{k=0}^{m} c_k t^k, \quad t \in \mathbb{R}, \tag{1}$$

subject to

$$y(0) = \lambda, \tag{2}$$

where $a, b, c_k, \lambda \in \mathbb{R}$, $k = 0, 1, 2, \ldots, m$. Rather than treating individual forcing terms separately, a general transformation is to be derived using an auxiliary polynomial whose coefficients are determined recurrently. The proposed transformation eliminates the forcing term and successfully converts the problem (1) and (2) into a homogeneous RODE. A recursive algebraic formula governing the transformation coefficients is established, leading to an exact representation of the solution for arbitrary polynomial degree. The special cases $a = b$ and $a = -b$ are found to be singular with respect to the transformation procedure; this is because the denominators appearing in the coefficient formulas vanish. To overcome this difficulty, an equivalent second-order ODE without reflection is derived, and then a reduction-of-order approach is employed to obtain exact solutions in both singular regimes. So, the analysis remains valid for all admissible parameter configurations. The developed methodology provides a direct and constructive route to exact solutions and broadens the class of RODEs that can be solved analytically.

2. Transformation at Particular Values of m

This section focuses on the derivation of some transformations in some particular cases for m such as $m = 0$ (constant polynomial), $m = 1$ (first-degree polynomial), and $m = 2$ (second-degree polynomial). Then the results are to be generalized for an arbitrary positive integer m in a subsequent section.

2.1. $m = 0$

Consider the initial value problem (IVP)

$$y'(t) = a\,y(t) + b\,y(-t) + c_0, \qquad y(0) = \lambda, \tag{3}$$

Let

$$y(t) = u(t) + d_0, \tag{4}$$

where d_0 is a constant to be determined. Substituting Equation (4) into Equation (3) yields

$$u'(t) = a\big(u(t) + d_0\big) + b\big(u(-t) + d_0\big) + c_0. \tag{5}$$

Setting $(a + b)d_0 + c_0 = 0$ gives

$$d_0 = -\frac{c_0}{a + b}, \tag{6}$$

and consequently the problem (3) reduces to the homogeneous IVP

$$u'(t) = a\,u(t) + b\,u(-t), \quad u(0) = \lambda - d_0 = \lambda + \frac{c_0}{a + b}, \quad a + b \neq 0. \tag{7}$$

2.2. $m = 1$

Consider the IVP

$$y'(t) = a\,y(t) + b\,y(-t) + c_0 + c_1 t, \qquad y(0) = \lambda, \tag{8}$$

and assume that

$$y(t) = u(t) + d_0 + d_1 t, \tag{9}$$

where d_0 and d_1 are constants to be determined. Substituting Equation (9) into Equation (8) leads to

$$u'(t) + d_1 = a\big(u(t) + d_0 + d_1 t\big) + b\big(u(-t) + d_0 - d_1 t\big) + c_0 + c_1 t. \tag{10}$$

Setting

$$\begin{cases} d_1 = (a+b)d_0 + c_0, \\ (a-b)d_1 + c_1 = 0. \end{cases} \tag{11}$$

Solving this system for d_0 and d_1, we obtain

$$\begin{cases} d_1 = -\frac{c_1}{a-b}, \\ d_0 = -\frac{c_1}{a^2-b^2} - \frac{c_0}{a+b}. \end{cases} \tag{12}$$

Accordingly, the problem (8) converts to the homogeneous IVP

$$u'(t) = a\,u(t) + b\,u(-t), \quad u(0) = \lambda - d_0 = \lambda + \frac{c_1}{a^2-b^2} + \frac{c_0}{a+b}, \quad a^2 \neq b^2. \tag{13}$$

2.3. $m = 2$

Consider the IVP

$$y'(t) = a\,y(t) + b\,y(-t) + c_0 + c_1 t + c_2 t^2, \qquad y(0) = \lambda. \tag{14}$$

Suppose that

$$y(t) = u(t) + d_0 + d_1 t + d_2 t^2, \tag{15}$$

where d_0, d_1, and d_2 are unknown constants. Inserting Equation (15) into Equation (14) implies

$$u'(t) + d_1 + 2d_2 t = a\big(u(t) + d_0 + d_1 t + d_2 t^2\big) + b\big(u(-t) + d_0 - d_1 t + d_2 t^2\big) + c_0 + c_1 t + c_2 t^2. \tag{16}$$

Setting

$$\begin{cases} d_1 = (a+b)d_0 + c_0, \\ 2d_2 = (a-b)d_1 + c_1 \\ (a+b)d_2 + c_2 = 0. \end{cases} \tag{17}$$

Solving this system yields

$$\begin{cases} d_2 = -\frac{c_2}{a+b}, \\ d_1 = -\frac{2c_2}{a^2-b^2} - \frac{c_1}{a-b}, \\ d_0 = -\frac{2c_2}{(a^2-b^2)(a+b)} - \frac{c_1}{a^2-b^2} - \frac{c_0}{a+b}. \end{cases} \tag{18}$$

Hence, the problem (14) becomes

$$u'(t) = a\,u(t) + b\,u(-t), \quad u(0) = \lambda - d_0 = \lambda + \frac{2c_2}{(a^2-b^2)(a+b)} + \frac{c_1}{a^2-b^2} + \frac{c_0}{a+b}, \quad a^2 \neq b^2. \tag{19}$$

3. Transformation at Arbitrary Positive Finite Integer m

Consider the IVP (1) with a polynomial of degree m and assume an auxiliary polynomial $P(t)$ of degree at most $m+1$ in the form

$$P(t) = \sum_{k=0}^{m+1} d_k t^k, \tag{20}$$

whose coefficients $d_k, k = 0, 1, 2, \ldots, m+1$ are to be determined. The objective is to construct a transformation that eliminates the polynomial forcing term and converts Equation (1) into a homogeneous equation with reflection. Define

$$y(t) = u(t) + P(t). \tag{21}$$

Substituting Equation (21) into Equation (1) yields

$$u'(t) + P'(t) = a\big(u(t) + P(t)\big) + b\big(u(-t) + P(-t)\big) + \sum_{k=0}^{m} c_k t^k, \tag{22}$$

or

$$u'(t) = a\,u(t) + b\,u(-t) + \Big[aP(t) + bP(-t) - P'(t) + \sum_{k=0}^{m} c_k t^k\Big]. \tag{23}$$

If $P(t)$ satisfies

$$P'(t) - aP(t) - bP(-t) = \sum_{k=0}^{m} c_k t^k, \tag{24}$$

then the bracketed term in Equation (23) vanishes identically and the transformed function $u(t)$ satisfies the following homogeneous IVP with reflection

$$u'(t) = a\,u(t) + b\,u(-t), \quad u(0) = \mu = \lambda - d_0. \tag{25}$$

The next steps show the determination of the polynomial coefficients d_k. From Equation (20), we have

$$P'(t) = \sum_{k=0}^{m} (k+1) d_{k+1} t^k, \tag{26}$$

and

$$P(-t) = \sum_{k=0}^{m+1} (-1)^k d_k t^k. \tag{27}$$

Substituting these expressions into (24) gives

$$\sum_{k=0}^{m} (k+1) d_{k+1} t^k - \sum_{k=0}^{m+1} \Big(a + b(-1)^k\Big) d_k t^k = \sum_{k=0}^{m} c_k t^k. \tag{28}$$

Equating coefficients of equal powers of t yields the triangular algebraic system

$$-\Big(a + b(-1)^{m+1}\Big) d_{m+1} = 0, \tag{29}$$

and

$$(k+1) d_{k+1} - \Big(a + b(-1)^k\Big) d_k = c_k, \quad k = 0, 1, \ldots, m. \tag{30}$$

Assuming $a + b(-1)^{m+1} \neq 0$, then Equation (28) implies $d_{m+1} = 0$. Thus, the coefficients are obtained recursively from

$$\begin{cases} d_k = \frac{(k+1)d_{k+1} - c_k}{a+b(-1)^k}, & k = m, m-1, \ldots, 0, \\ d_{m+1} = 0, \end{cases} \tag{31}$$

provided that $a + b(-1)^k \neq 0$, $k = 0, 1, \ldots, m$. Consequently, a unique polynomial $P(t)$ satisfying Equation (24) is obtained. From (31), one can write

$$\begin{cases} d_m & = \frac{(m+1)d_{m+1} - c_m}{a+b(-1)^m} = -\frac{c_m}{a+b(-1)^m}, \\ d_{m-1} & = \frac{m\, d_m - c_{m-1}}{a+b(-1)^{m-1}} = -\frac{m\, c_m}{a^2-b^2} - \frac{c_{m-1}}{a-b(-1)^m}, \\ d_{m-2} & = \frac{(m-1)\, d_{m-1} - c_{m-2}}{a+b(-1)^{m-2}} = \frac{(m-1)\, d_{m-1}}{a+b(-1)^m} - \frac{c_{m-2}}{a+b(-1)^m}, \\ & = -\frac{m(m-1)\, c_m}{(a^2-b^2)(a+b(-1)^m)} - \frac{(m-1)c_{m-1}}{a^2-b^2} - \frac{c_{m-2}}{a+b(-1)^m}, \\ \vdots & \\ \vdots & \\ d_1 & = \frac{2d_2 - c_1}{a-b}, \\ d_0 & = \frac{d_1 - c_0}{a+b}. \end{cases} \tag{32}$$

The coefficients d_k were obtained above in descending order; however, an explicit formula can be derived in terms of the parameters' model as follows.

3.1. Explicit Formula for d_k

Assume that $h_k = a + b(-1)^k \neq 0$, $k = 0, 1, \ldots, m$. The coefficients d_k are determined recursively from

$$\begin{cases} d_k = \frac{(k+1)d_{k+1} - c_k}{h_k}, & k = m, m-1, \ldots, 0, \\ d_{m+1} = 0. \end{cases} \tag{33}$$

The first few coefficients are

$$\begin{cases} d_m = -\frac{c_m}{h_m}, \\ d_{m-1} = -\frac{c_{m-1}}{h_{m-1}} - \frac{m\, c_m}{h_{m-1}A_m}, \\ d_{m-2} = -\frac{c_{m-2}}{h_{m-2}} - \frac{(m-1)c_{m-1}}{h_{m-2}h_{m-1}} - \frac{m(m-1)c_m}{h_{m-2}h_{m-1}h_m}, \\ \vdots \end{cases} \tag{34}$$

These expressions suggest the general pattern established below.

Theorem 1. *Let $h_k = a + b(-1)^k \neq 0, \quad k = 0, 1, \ldots, m$. Then the unique solution of the recurrence relation (34) is given explicitly by*

$$d_k = -\sum_{j=k}^{m} \frac{j!}{k!} \frac{c_j}{\prod_{r=k}^{j} h_r}, \quad k = 0, 1, \ldots, m. \tag{35}$$

Equivalently,

$$d_k = -\sum_{j=k}^{m} \frac{j!}{k!} \frac{c_j}{\prod_{r=k}^{j}(a + b(-1)^r)}, \quad k = 0, 1, \ldots, m. \tag{36}$$

Proof. The proof proceeds by backward induction. For $k = m$, since $d_{m+1} = 0$, relation (33) yields $d_m = -\frac{c_m}{h_m}$, which agrees with (35). From Equation (35), one can write d_k in the form:

$$d_k = -\frac{c_k}{h_k} - \frac{k+1}{h_k} \sum_{j=k+1}^{m} \frac{j!}{(k+1)!} \frac{c_j}{\prod_{r=k+1}^{j} h_r}. \tag{37}$$

Hence

$$\frac{h_k d_k + c_k}{k+1} = -\sum_{j=k+1}^{m} \frac{j!}{(k+1)!} \frac{c_j}{\prod_{r=k+1}^{j} h_r}. \tag{38}$$

From Equation (30) or Equation (31), we have

$$d_{k+1} = \frac{h_k d_k + c_k}{k+1}. \tag{39}$$

Combining (38) and (39) leads to

$$d_{k+1} = -\sum_{j=k+1}^{m} \frac{j!}{(k+1)!} \frac{c_j}{\prod_{r=k+1}^{j} h_r}. \tag{40}$$

which verifies the Formula (35) for d_{k+1}. □

Remark 1. *In view of the above theorem, the IC of the homogeneous IVP (25) is now given in terms of a, b, λ, c_j, $j = 0, 1, 2, \ldots, m$, as*

$$u(0) = \mu = \lambda - d_0 = \lambda + \sum_{j=0}^{m} \frac{j!\, c_j}{\prod_{r=0}^{j}(a + b(-1)^r)}, \tag{41}$$

where

$$d_0 = -\sum_{j=0}^{m} \frac{j!\, c_j}{\prod_{r=0}^{j}(a + b(-1)^r)}, \tag{42}$$

is deduced from (36) at $k = 0$. The next theorem addresses the power series solution (PSS) of the homogeneous IVP

$$u'(t) = a\, u(t) + b\, u(-t), \quad u(0) = \mu, \tag{43}$$

where μ is already obtained in Equation (41).

3.2. Solution of the Homogeneous IVP

Theorem 2. *The PSS of the homogeneous IVP (43) is given by*

$$u(t) = \sum_{n=0}^{\infty} w_n t^n, \tag{44}$$

with

$$w_n = \frac{\mu}{n!}\prod_{j=0}^{n-1} h_j, \quad h_j = a + b(-1)^j, \quad n \geq 0, \tag{45}$$

i.e.,

$$u(t) = \mu \sum_{n=0}^{\infty} \frac{t^n}{n!}\prod_{j=0}^{n-1} h_j. \tag{46}$$

Proof. Substituting (44) into (43) and following the analysis introduced in Ref. [32], the proof follows immediately. □

4. The Exact Solution

In view of the obtained results in the previous sections, one can write the solution of the IVP (1-2) in the form

$$y(t) = \sum_{n=0}^{\infty} w_n t^n + \sum_{k=0}^{m+1} d_k t^k. \tag{47}$$

Since the general formula of d_k, given by Theorem 1 (Equation (35)), was obtained based on the fact that $d_{m+1} = 0$, then

$$y(t) = \sum_{n=0}^{\infty} w_n t^n + \sum_{k=0}^{m} d_k t^k, \tag{48}$$

where

$$\begin{cases} w_n = \frac{\mu}{n!}\prod_{r=0}^{n-1} h_r, & n \geq 0, \\ d_k = -\sum_{j=k}^{m} \frac{j!}{k!} \frac{c_j}{\prod_{r=k} h_r}, & k = 0, 1, \ldots, m, \\ h_r = a + b(-1)^r, & a + b(-1)^{m+1} \neq 0. \end{cases} \tag{49}$$

It has been shown in [33] that the solution $u(t)$ of the homogeneous IVP (43) can be given in hyperbolic and trigonometric forms under certain relationships of the parameters a and b. For declaration, if we define $\Delta = a^2 - b^2$, then

$$u(t) = \sum_{n=0}^{\infty} w_n t^n = \begin{cases} \mu\left[\cosh(\sqrt{\Delta}\, t) + \frac{a+b}{\sqrt{\Delta}} \sinh(\sqrt{\Delta}\, t)\right], & \Delta > 0, \\ \mu\left[\cos(\sqrt{-\Delta}\, t) + \frac{a+b}{\sqrt{-\Delta}} \sin(\sqrt{-\Delta}\, t)\right], & \Delta < 0. \end{cases} \tag{50}$$

Thus, the exact solution of the original nonhomogeneous reflection problem (1) and (2) can be written as

$$y(t) = \begin{cases} \mu\left[\cosh(\sqrt{\Delta}\, t) + \frac{a+b}{\sqrt{\Delta}} \sinh(\sqrt{\Delta}\, t)\right] + \sum_{k=0}^{m} d_k t^k, & \Delta > 0, \\ \mu\left[\cos(\sqrt{-\Delta}\, t) + \frac{a+b}{\sqrt{-\Delta}} \sin(\sqrt{-\Delta}\, t)\right] + \sum_{k=0}^{m} d_k t^k, & \Delta < 0, \end{cases} \tag{51}$$

where μ is defined by Equation (41) and d_k was obtained by Theorem 1 (Equation (35) or Equation (36)). Graphical representations of the hyperbolic-polynomial solution and the trigonometric-polynomial solution given by Equations (51) are depicted in Figures 1 and 2 for different values m under the cases $\Delta > 0$ and $\Delta < 0$, respectively.

Although the above analysis gave the exact solutions under the cases $\Delta > 0$ and $\Delta < 0$, the case $\Delta = 0$ requires a separate treatment. This is because the constant μ violates $m \geq 1$. For $m = 0$, we have $\mu = \lambda + \frac{c_0}{a+b}$ which requires $b \neq -a$. These challenges are addressed in the next section using another approach.

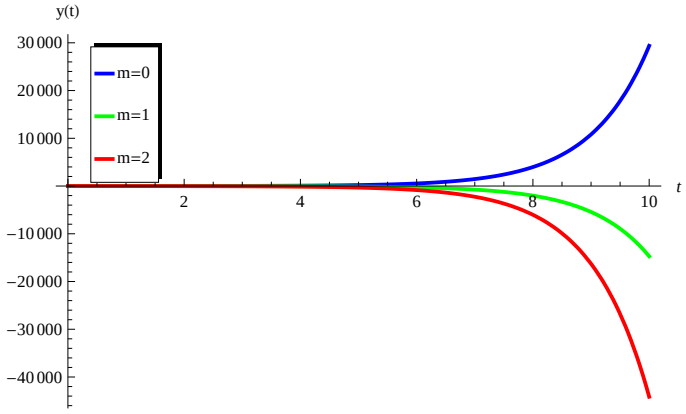


Figure 1. Graphical representation of the hyperbolic-polynomial solution (51) at $\lambda = 1$, $a = \frac{5}{3}$, and $b = \frac{4}{3}$ ($\Delta > 0$) when $c_k = 1$ for $m = 0$, $m = 1$, and $m = 2$.

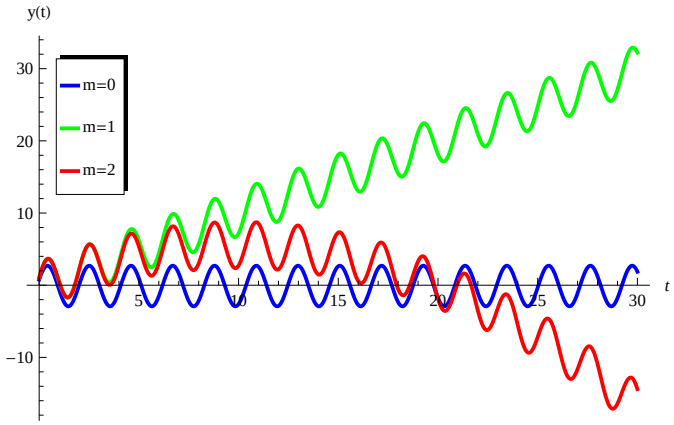


Figure 2. Graphical representation of the trigonometric-polynomial solution (51) at $\lambda = 1$, $a = 4$, and $b = 5$ ($\Delta < 0$) when $c_k = \frac{1}{k!}$ for $m = 0$, $m = 1$, and $m = 2$.

5. Reduction-of-Order Method

Theorem 3. *Consider the IVP*

$$y'(t) = a\,y(t) + b\,y(-t) + Q_m(t), \qquad y(0) = \lambda, \tag{52}$$

where

$$Q_m(t) = \sum_{k=0}^{m} c_k t^k, \tag{53}$$

then the IVP (52) is equivalent to the following second-order ODE without reflection:

$$y''(t) = (a^2 - b^2)y(t) + aQ_m(t) - bQ_m(-t) + Q'_m(t), \tag{54}$$

subject to the ICs

$$y(0) = \lambda, \quad y'(0) = (a+b)\lambda + c_0. \tag{55}$$

Proof. Differentiating Equation (52) once with respect to t yields

$$y''(t) = a\,y'(t) - b\,y'(-t) + Q'_m(t). \tag{56}$$

Replacing t by $-t$ in (52) gives

$$y'(-t) = a\,y(-t) + b\,y(t) + Q_m(-t). \tag{57}$$

Substituting (52) and (57) into (56) yields

$$y''(t) = (a^2 - b^2)y(t) + aQ_m(t) - bQ_m(-t) + Q'_m(t). \tag{58}$$

Therefore

$$y''(t) = (a^2 - b^2)y(t) + a\sum_{k=0}^{m} c_k t^k - b\sum_{k=0}^{m} c_k(-t)^k + \sum_{k=1}^{m} kc_k t^{k-1}. \tag{59}$$

Finally, evaluating (52) at $t = 0$ gives

$$y'(0) = a\,y(0) + b\,y(0) + Q_m(0) = (a+b)\lambda + c_0.$$

This completes the proof. □

5.1. Special Cases and Exact Solutions

One can rewrite Equation (54) in the form

$$y''(t) - \Delta^2 y(t) = F(t), \qquad \Delta^2 = a^2 - b^2, \tag{60}$$

where

$$F(t) = aQ_m(t) - bQ_m(-t) + Q'_m(t), \qquad Q_m(t) = \sum_{k=0}^{m} c_k t^k, \tag{61}$$

subject to

$$y(0) = \lambda, \qquad y'(0) = (a+b)\lambda + c_0. \tag{62}$$

In this section, we are interested only in obtaining the exact solutions in the cases $b = a$ and $b = -a$ for which $\Delta^2 = 0$. These cases fulfill the obtained results of the previous sections and therefore complete the present work.

5.1.1. Case I: $b = a$

In this case, Equation (60) reduces to

$$y''(t) = F(t), \tag{63}$$

where

$$F(t) = a\big(Q_m(t) - Q_m(-t)\big) + Q'_m(t), \tag{64}$$

while the ICs (62) become

$$y(0) = \lambda, \qquad y'(0) = 2a\lambda + c_0. \tag{65}$$

Integrating (63) twice gives

$$y(t) = \lambda + \big(2a\lambda + c_0\big)t + \int_0^t (t-s)F(s)\,ds, \tag{66}$$

or

$$y(t) = \lambda + \big(2a\lambda + c_0\big)t + t\int_0^t F(s)\,ds + \int_0^t sF(s)\,ds. \tag{67}$$

Employing $Q_m(t)$ in Equation (61), one can rewrite $F(t)$ in the form

$$F(t) = a\sum_{k=0}^{m}\left(1-(-1)^k\right)c_k t^k + \sum_{k=1}^{m} k c_k t^{k-1}. \tag{68}$$

Since

$$\sum_{k=0}^{m}\left(1-(-1)^k\right)c_k t^k = 2\left(c_1 t + c_3 t^3 + c_5 t^5 + \cdots + c_m t^m\right), \tag{69}$$

then the equality (69) depends on whether m is odd or even. For odd m, i.e., $m = 2l+1$, $l \geq 0$, we have

$$\sum_{k=0}^{m}\left(1-(-1)^k\right)c_k t^k = 2\sum_{l=0}^{\frac{m-1}{2}} c_{2l+1} t^{2l+1}, \tag{70}$$

while for even m, i.e., $m = 2l$, $l \geq 0$, we have

$$\sum_{k=0}^{m}\left(1-(-1)^k\right)c_k t^k = 2\sum_{l=0}^{\frac{m}{2}-1} c_{2l+1} t^{2l+1}. \tag{71}$$

Hence

$$\sum_{k=0}^{m}\left(1-(-1)^k\right)c_k t^k = \begin{cases} 2\sum\limits_{l=0}^{\frac{m-1}{2}} c_{2l+1} t^{2l+1}, & m \text{ odd}, \\ 2\sum\limits_{l=0}^{\frac{m}{2}-1} c_{2l+1} t^{2l+1}, & m \text{ even}. \end{cases} \tag{72}$$

Thus

$$F(t) = \begin{cases} 2a\sum\limits_{l=0}^{\frac{m-1}{2}} c_{2l+1} t^{2l+1} + \sum\limits_{k=1}^{m} k c_k t^{k-1}, & m \text{ odd}, \\ 2a\sum\limits_{l=0}^{\frac{m}{2}-1} c_{2l+1} t^{2l+1} + \sum\limits_{k=1}^{m} k c_k t^{k-1}, & m \text{ even}. \end{cases} \tag{73}$$

Accordingly, we have the integrals

$$\int_0^t F(s)\,ds = \begin{cases} a\sum\limits_{l=0}^{\frac{m-1}{2}} c_{2l+1}\dfrac{t^{2l+2}}{l+1} + \sum\limits_{k=1}^{m} c_k t^k, & m \text{ odd}, \\ a\sum\limits_{l=0}^{\frac{m}{2}-1} c_{2l+1}\dfrac{t^{2l+2}}{l+1} + \sum\limits_{k=1}^{m} c_k t^k, & m \text{ even}, \end{cases} \tag{74}$$

and

$$\int_0^t sF(s)\,ds = \begin{cases} 2a\sum\limits_{l=0}^{\frac{m-1}{2}} c_{2l+1}\dfrac{t^{2l+3}}{2l+3} + \sum\limits_{k=1}^{m} c_k\dfrac{kt^{k+1}}{k+1}, & m \text{ odd}, \\ 2a\sum\limits_{l=0}^{\frac{m}{2}-1} c_{2l+1}\dfrac{t^{2l+3}}{2l+3} + \sum\limits_{k=1}^{m} c_k\dfrac{kt^{k+1}}{k+1}, & m \text{ even}. \end{cases} \tag{75}$$

Inserting (74) and (75) into (67) gives the exact solution $y(t)$ for the reflected IVP:

$$y'(t) = a(\,y(t) + y(-t)) + \sum_{k=0}^{m} c_k t^k, \quad y(0) = \lambda. \tag{76}$$

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5.1.2. Case II: $b = -a$

If $b = -a$, then

$$F(t) = a\big(Q_m(t) + Q_m(-t)\big) + Q'_m(t), \tag{77}$$

and the exact solution can be obtained by integrating (63) twice under the ICs $y(0) = \lambda$ and $y'(0) = c_0$, given as

$$y(t) = \lambda + c_0 t + t\int_0^t F(s)\,ds + \int_0^t sF(s)\,ds, \tag{78}$$

where $F(t)$ takes the form

$$F(t) = a\sum_{k=0}^{m}\big(1 + (-1)^k\big)c_k t^k + \sum_{k=1}^{m} kc_k t^{k-1}. \tag{79}$$

The expansion

$$\sum_{k=0}^{m}\big(1 + (-1)^k\big)c_k t^k = 2\Big(c_0 + c_2 t^2 + c_4 t^4 + \cdots + c_m t^m\Big), \tag{80}$$

also depends on whether m is even or odd. If m is even, then

$$\sum_{k=0}^{m}\big(1 + (-1)^k\big)c_k t^k = 2\sum_{l=0}^{\frac{m}{2}} c_{2l} t^{2l}, \tag{81}$$

while for odd m, we have

$$\sum_{k=0}^{m}\big(1 + (-1)^k\big)c_k t^k = 2\sum_{l=0}^{\frac{m-1}{2}} c_{2l} t^{2l}. \tag{82}$$

Thus

$$\sum_{k=0}^{m}\big(1 + (-1)^k\big)c_k t^k = \begin{cases} 2\sum\limits_{l=0}^{\frac{m-1}{2}} c_{2l} t^{2l}, & m \text{ odd}, \\ 2\sum\limits_{l=0}^{\frac{m}{2}} c_{2l} t^{2l}, & m \text{ even}. \end{cases} \tag{83}$$

Therefore

$$F(t) = \begin{cases} 2a\sum\limits_{l=0}^{\frac{m-1}{2}} c_{2l} t^{2l} + \sum\limits_{k=1}^{m} kc_k t^{k-1}, & m \text{ odd}, \\ 2a\sum\limits_{l=0}^{\frac{m}{2}} c_{2l} t^{2l} + \sum\limits_{k=1}^{m} kc_k t^{k-1}, & m \text{ even}. \end{cases} \tag{84}$$

So, we have the integrals

$$\int_0^t F(s)\,ds = \begin{cases} 2a\sum\limits_{l=0}^{\frac{m-1}{2}} c_{2l}\dfrac{t^{2l+1}}{2l+1} + \sum\limits_{k=1}^{m} c_k t^k, & m \text{ odd}, \\ 2a\sum\limits_{l=0}^{\frac{m}{2}} c_{2l}\dfrac{t^{2l+1}}{2l+1} + \sum\limits_{k=1}^{m} c_k t^k, & m \text{ even}, \end{cases} \tag{85}$$

and

$$\int_0^t sF(s)\,ds = \begin{cases} a\sum_{l=0}^{\frac{m-1}{2}} c_{2l}\dfrac{t^{2l+2}}{l+1} + \sum_{k=1}^{m} c_k \dfrac{kt^{k+1}}{k+1}, & m \text{ odd}, \\ a\sum_{l=0}^{\frac{m}{2}} c_{2l}\dfrac{t^{2l+2}}{l+1} + \sum_{k=1}^{m} c_k \dfrac{kt^{k+1}}{k+1}, & m \text{ even}. \end{cases} \tag{86}$$

Substituting (85) and (86) into (78) leads to the exact solution $y(t)$ for the reflected IVP:

$$y'(t) = a(\,y(t) - y(-t)) + \sum_{k=0}^{m} c_k t^k, \quad y(0) = \lambda. \tag{87}$$

Remark 2. *This remark highlights the adaptive Fourier principle by Song and Sun [34] which is useful in decomposing $y(t)$ and $Q(t)$ (for any forcing function, not just polynomials) into symmetric (even) and anti-symmetric (odd) components. One can apply this core Fourier principle to bypass the recursive polynomial transformation entirely and arrive directly at the second-order ODE. Let us decompose the solution and the forcing term into their even and odd parts, i.e., $y(t) = y_E(t) + y_O(t)$ and $Q_m(t) = Q_E(t) + Q_O(t)$. Substituting these into the main equation $y'(t) = ay(t) + by(-t) + Q_m(t)$ gives*

$$y'_E(t) + y'_O(t) = a(y_E(t) + y_O(t)) + b(y_E(t) - y_O(t)) + Q_E(t) + Q_O(t).$$

Since the derivative of an even function is odd and vice versa, we can separate the terms by parity: $y'_E(t) = (a-b)y_O(t) + Q_O(t)$ (odd terms) and $y'_O(t) = (a+b)y_E(t) + Q_E(t)$ (even terms). To decouple this system, differentiating the second equation implies

$$y''_O(t) = (a+b)y'_E(t) + Q'_E(t).$$

Substituting the first equation into this new derivative yields

$$y''_O(t) = (a+b)[(a-b)y_O(t) + Q_O(t)] + Q'_E(t),$$

or

$$y''_O(t) = (a^2 - b^2)y_O(t) + (a+b)Q_O(t) + Q'_E(t).$$

By applying the exact same differentiation to $y_E(t)$ and summing the results, we immediately recover the equivalent second-order ODE (54) without reflection. This decomposition proof is highly generalized, mathematically compact, and confirms the results for the resonant parameter regimes while extending naturally to infinite series.

We may summarize the main contributions of the present work as follows:

1. Deriving a unified transformation that removes an arbitrary polynomial forcing term from a first-order RODE.
2. Obtaining explicit expressions for the transformation coefficients associated with any polynomial of degree m.
3. Deducing exact analytical solutions of the resulting homogeneous RODE in both series and explicit forms.
4. Introducing a reduction-of-order formulation yielding an equivalent second-order ODE without reflection.
5. Determining the exact solutions for the resonant cases $a = b$ and $a = -b$, thereby completing the analytical characterization of the considered model.
6. To the author's knowledge, the present model may be analytical/exactly solved in this work for the first time. Moreover, the obtained exact solutions can serve

as benchmark solutions under different parameter regimes, particularly according to the cases $\Delta > 0$, $\Delta < 0$, and $\Delta = 0$ which were effectively discussed in a manner that highlights the transition between hyperbolic, trigonometric, and singular/resonant behaviors.

6. Conclusions

A comprehensive analytical investigation was carried out for a class of first-order RODEs with arbitrary polynomial forcing terms. The central idea was the establishment of an auxiliary polynomial transformation that absorbs the nonhomogeneous term and accordingly converts the original RODEs into a homogeneous version. The developed transformation was first illustrated for low-degree polynomials and then generalized to polynomials of arbitrary finite degree. A recursive formula governing the transformation coefficients was derived and an explicit expression was obtained and rigorously justified. The results yielded a complete characterization of the transformed IC and enabled the exact solution of the homogeneous version through the PSS. By exploiting previously known properties of the homogeneous model, the resulting solutions were successfully determined in hyperbolic/trigonometric forms based on the sign of $a^2 - b^2$. To address the singular parameter cases $b = a$ and $b = -a$ for which the transformation approach loses validity, an equivalent second-order ODE was formulated and then exactly solved using a reduction-of-order approach. The presented work offers a unified methodology for treating polynomial-forced RODEs and demonstrates how nonhomogeneous reflected models can be reduced to analytically tractable forms. The obtained formulas may serve as benchmark solutions for validating numerical algorithms such as the MP-FVM (Message Passing–Finite Volume Method) [35] and may also be extended to broader classes of RODEs involving variable coefficients or those of higher order including other forms of forcing functions. For variable coefficients, this finite-dimensional polynomial structure generally does not persist. In particular, products involving variable coefficients and reflected terms may generate additional functional structures that cannot necessarily be represented by a polynomial of the same finite degree. Consequently, the simple finite recurrence obtained in the present work may no longer close. Similarly, for more general forcing functions, such as arbitrary analytic, exponential, or non-polynomial functions, an appropriate auxiliary transformation may require an infinite series or a different functional basis rather than a finite polynomial. Additional convergence and solvability issues would then have to be addressed.

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