

Article

The Sneddon $\mathcal{R}$ -Transform and Its Inverse over Lebesgue Spaces

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Abstract

We study the Sneddon $\mathcal{R}$ -transform and its inverse in the setting of Lebesgue spaces. Generated by the mixed trigonometric kernel $x \cos(xt) + h \sin(xt)$, the $\mathcal{R}$ -transform acts as a unifying operator for sine- and cosine-type integral transforms. Boundedness, continuity, and weighted L^p -estimates are established in an appropriate Banach space framework, together with Parseval–Goldstein type identities. Initial and final value theorems are derived for generalized functions in Zemanian-type spaces, yielding precise asymptotic behaviour at the origin and at infinity. A finite-interval theory is also developed, leading to polynomial growth estimates and final value theorems for the finite $\mathcal{R}$ -transform.

Keywords: Sneddon $\mathcal{R}$ -transform; inverse transform; Lebesgue spaces; Parseval–Goldstein identity; weighted L^p -boundedness; generalized functions; finite $\mathcal{R}$ -transform

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1. Introduction and Preliminaries

Integral transforms play a central role in analysis and mathematical physics, providing powerful tools for the representation, solution, and qualitative study of differential and integral equations. Classical transforms such as the Fourier sine and cosine transforms are particularly effective in problems posed on the half-line, where symmetry considerations or boundary conditions naturally separate even and odd components of functions. However, many boundary value problems arising in applied mathematics involve mixed conditions that cannot be adequately treated by purely sine- or cosine-based representations.

In this context, integral operators with kernels involving linear combinations of sine and cosine functions emerge in a natural way. Such kernels appear, for instance, in the solution of mixed boundary value problems, in radiation and diffraction theory, and in

models where both displacement and flux conditions are prescribed at a boundary. The need to analyze these problems in a unified and systematic manner motivates the study of transforms that interpolate between the classical Fourier sine and cosine transforms.

The $\mathcal{R}$ -transform, introduced by Sneddon, provides a canonical example of such an operator. It is defined through a kernel combining trigonometric functions with a frequency-dependent weight, thereby capturing both oscillatory and growth features in a single framework. From an analytical viewpoint, the $\mathcal{R}$ -transform may be regarded as a weighted superposition of the Fourier cosine and sine transforms, with the parameter $h \in \mathbb{R}$ controlling the relative contribution of each component. This structure allows one to recover the classical transforms as special cases while simultaneously extending their applicability to more general settings.

Despite its relevance in applications, a comprehensive functional-analytic treatment of the $\mathcal{R}$ -transform has remained comparatively limited. In particular, questions concerning mapping properties on L^p -spaces, norm estimates, inversion procedures, and asymptotic behaviour of the transform and its inverse require further clarification. Moreover, the extension of these properties to generalized functions and to finite-interval settings is essential for applications involving distributions and compactly supported data.

The primary objective of the present work is to develop a systematic theory of the $\mathcal{R}$ -transform that addresses these issues. We establish boundedness and continuity results on Lebesgue spaces, derive explicit norm estimates, and obtain inversion formulas that unify and extend classical results for trigonometric transforms. In addition, we investigate initial and final value theorems in appropriate spaces of generalized functions and analyze finite versions of the transform on bounded intervals. These results not only deepen the theoretical understanding of the $\mathcal{R}$ -transform but also provide a flexible analytical framework for further developments in integral transform theory and its applications.

We now recall the definition of the Sneddon $\mathcal{R}$ -transform. For $f \in L^1(\mathbb{R}_+)$, the $\mathcal{R}$ -transform (cf. Sneddon [Section 2–12] [1], p. 72) is defined by the relation

$$(\mathcal{R}f)(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty f(t) \{x \cos(xt) + h \sin(xt)\} dt, \quad x > 0,$$

where $h \in \mathbb{R}$ is a fixed parameter.

For $\gamma > 0$ we consider the vector space $\mathcal{B}_\gamma$ consisting of all complex-valued measurable functions f on $\mathbb{R}_+$ such that $\frac{f(x)}{\gamma+x} \in L^\infty(\mathbb{R}_+)$.

A norm $\|\cdot\|_\gamma$ is given by

$$\|f\|_\gamma = \left\| \frac{f(x)}{\gamma+x} \right\|_{L^\infty(\mathbb{R}_+)}$$

With this norm, the map

$$T_\gamma : \mathcal{B}_\gamma \rightarrow L^\infty(\mathbb{R}_+)$$

where for any $f \in \mathcal{B}_\gamma$:

$$(B_\gamma f)(x) = \frac{f(x)}{\gamma+x}, \quad x \in \mathbb{R}_+,$$

is an isometric isomorphism from $\mathcal{B}_\gamma$ to $L^\infty(\mathbb{R}_+)$. Thus, since $L^\infty(\mathbb{R}_+)$ is complete, then the space $\mathcal{B}_\gamma$ becomes a Banach space.

The $C_c^k(\mathbb{R}_+)$, $k \in \mathbb{N}$, denotes as it is usual the space of compactly supported functions on $\mathbb{R}_+$ which are k -times differentiable with continuity.

Parseval's and Plancherel's theorems are fundamental results in the theory of integral transforms, establishing essential links between a function and its corresponding transform. They highlight the preservation of energy or inner product structures under the transformation, thereby providing a powerful tool for both theoretical analysis and practical applications (see [2–6] for further discussion).

The structure of this article is organized as follows. In Section 1, we introduce the motivation and mathematical background of the Sneddon $\mathcal{R}$ -transform and recall the necessary preliminaries on Lebesgue spaces and integral transforms. Section 2 is devoted to the study of the Sneddon $\mathcal{R}$ -transform over Lebesgue spaces, where its basic mapping properties and integral representations are discussed. In Section 3, we investigate the inverse $\mathcal{R}$ -transform over Lebesgue spaces and establish its boundedness and associated Parseval–Goldstein type relations. Section 4 presents initial and final value theorems for the $\mathcal{R}$ -transform in the Lebesgue space setting, obtained by analyzing the limiting behaviour of $(\mathcal{R}f)(x)$ as $x \rightarrow 0^+$ and $x \rightarrow +\infty$. In Section 5, we extend these initial and final value results to generalized functions by working within Zemanian's distributional framework and employing Fourier sine and cosine transforms. Section 6 is concerned with final value theorems for the $\mathcal{R}$ -transform acting on generalized functions supported on a finite interval, formulated in the Fréchet space $\mathcal{M}_A$ and its strong dual. Finally, Section 7 concludes the paper with a summary of the main results and highlights possible directions for future research.

2. The Sneddon $\mathcal{R}$ -Transform over Lebesgue Spaces

This section introduces the Banach space $\mathcal{B}_\gamma$, $\gamma > 0$, tailored to study the boundedness and continuity of the Sneddon $\mathcal{R}$ -transform. We prove that the operator $\mathcal{R}$ is bounded from $L^1(\mathbb{R}_+)$ into $\mathcal{B}_\gamma$ and derive weighted L^p -type estimates. Moreover, Parseval–Goldstein and weighted Parseval–Goldstein identities are established using Fubini's theorem, thereby connecting the transform with energy-preserving relations analogous to the classical Fourier case.

Observe that, for all $f \in L^1(\mathbb{R}_+)$ and $x \in \mathbb{R}_+$

$$\begin{aligned} |(\mathcal{R}f)(x)| &\leq \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left(x\|f\|_{L^1(\mathbb{R}_+)} + |h|\|f\|_{L^1(\mathbb{R}_+)}\right) \\ &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} (x + |h|)\|f\|_{L^1(\mathbb{R}_+)}. \end{aligned} \quad (1)$$

Then $(\mathcal{R}f)(x)$ exists for all $x \in \mathbb{R}_+$ whenever $f \in L^1(\mathbb{R}_+)$.

Thus, being $\gamma > 0$,

$$\begin{aligned} \left|\frac{(\mathcal{R}f)(x)}{\gamma + x}\right| &\leq \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left(\frac{x}{\gamma + x}\|f\|_{L^1(\mathbb{R}_+)} + \frac{|h|}{\gamma + x}\|f\|_{L^1(\mathbb{R}_+)}\right) \\ &\leq \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left(\|f\|_{L^1(\mathbb{R}_+)} + \frac{|h|}{\gamma}\|f\|_{L^1(\mathbb{R}_+)}\right) \\ &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{\gamma + |h|}{\gamma} \|f\|_{L^1(\mathbb{R}_+)}, \text{ for all } x \in \mathbb{R}_+. \end{aligned}$$

Then

$$\|\mathcal{R}f\|_\gamma \leq \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{\gamma + |h|}{\gamma} \|f\|_{L^1(\mathbb{R}_+)}, \text{ for all } x \in \mathbb{R}_+. \quad (2)$$

Thus

Proposition 1. The transform $\mathcal{R} : L^1(\mathbb{R}_+) \rightarrow \mathcal{B}_\gamma$ is bounded.

Remark 1. Observe that from (2) the $\mathcal{R}$ -transform is a contraction from $L^1(\mathbb{R}_+)$ into $\mathcal{B}_\gamma$ whenever $|h| \leq \left(\left(\frac{\pi}{2}\right)^{\frac{1}{2}} - 1\right)\gamma$.

From (1) one obtains for $0 < q < \infty$ and for $w \in L^1(\mathbb{R}_+, (x + |h|)dx)$ such that $w > 0$ almost everywhere, we have

$$\int_0^\infty |(\mathcal{R}f)(x)|^q w(x) dx \leq \left(\frac{2}{\pi}\right)^{\frac{q}{2}} \|f\|_{L^1(\mathbb{R}_+)}^q \int_0^\infty (x + |h|)^q w(x) dx.$$

Then

$$\|\mathcal{R}f\|_{L^q(\mathbb{R}_+, w(x)dx)} \leq \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \|f\|_{L^1(\mathbb{R}_+)} \left(\int_0^\infty (x + |h|)^q w(x) dx\right)^{\frac{1}{q}}.$$

Thus

Proposition 2. For $0 < q < \infty$, and $w \in L^1(\mathbb{R}_+, (x + |h|)dx)$ such that $w > 0$ almost everywhere, then the transform $\mathcal{R} : L^1(\mathbb{R}_+) \rightarrow L^q(\mathbb{R}_+, w(x)dx)$ is bounded.

Example 1. Examples of weights w are:

- (i) $w(x) = (1 + x)^\alpha$, with $\alpha < -1 - q$.
- (ii) $w(x) = e^{\alpha x}$, for $\alpha < 0$.

For $f \in L^1(\mathbb{R}_+, (t + |h|)dt)$, the integral operator $\mathcal{R}'$ is defined by

$$(\mathcal{R}'f)(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty f(t) \{t \cos(xt) + h \sin(xt)\} dt.$$

Then

$$|(\mathcal{R}'f)(x)| \leq \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty |f(t)|(t + |h|) dt = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \|f\|_{L^1(\mathbb{R}_+, (t + |h|)dt)}, \quad \text{for all } x \in \mathbb{R}_+.$$

Hence, $(\mathcal{R}'f)(x)$ exists for all $x \in \mathbb{R}_+$ whenever $f \in L^1(\mathbb{R}_+, (t + |h|)dt)$.

Therefore,

$$\|\mathcal{R}'f\|_{L^\infty(\mathbb{R}_+)} \leq \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \|f\|_{L^1(\mathbb{R}_+, (t + |h|)dt)}. \quad (3)$$

So

Proposition 3. The integral operator $\mathcal{R}' : L^1(\mathbb{R}_+, (t + |h|)dt) \rightarrow L^\infty(\mathbb{R}_+)$ is bounded.

Let $0 < q < \infty$ and set $w \in L^1(\mathbb{R}_+)$ such that $w > 0$ almost everywhere. Using the above inequality (3), we have

$$\int_0^\infty |(\mathcal{R}'f)(x)|^q w(x) dx \leq \left(\frac{2}{\pi}\right)^{\frac{q}{2}} \|f\|_{L^1(\mathbb{R}_+, (t + |h|)dt)}^q \int_0^\infty w(x) dx.$$

Hence,

$$\|\mathcal{R}'f\|_{L^q(\mathbb{R}_+, w(x) dx)} \leq \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \|f\|_{L^1(\mathbb{R}_+, (t+|h|) dt)} \|w\|_{L^1(\mathbb{R}_+)}^{1/q}.$$

Thus,

Proposition 4. For $0 < q < \infty$, and $w \in L^1(\mathbb{R}_+)$ such that $w > 0$ almost everywhere, then the integral operator $\mathcal{R}' : L^1(\mathbb{R}_+, (t+|h|) dt) \longrightarrow L^q(\mathbb{R}_+, w(x) dx)$ is bounded.

Example 2. Examples of weights w are:

$$(i) w(x) = (1+x)^\alpha, \quad \alpha < -1.$$

$$(ii) w(x) = e^{-\alpha x}, \quad \alpha > 0.$$

Also one obtains the next result.

Theorem 1. Let $f \in L^1(\mathbb{R}_+)$ and $g \in L^1(\mathbb{R}_+, (t+|h|) dt)$ then the following Parseval–Goldstein relation holds:

$$\int_0^\infty (\mathcal{R}f)(x)g(x)dx = \int_0^\infty f(t)(\mathcal{R}'g)(t)dt.$$

Proof. Applying Fubini’s theorem in the following, we obtain

$$\begin{aligned} \int_0^\infty (\mathcal{R}f)(x)g(x)dx &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \int_0^\infty f(t) \{x \cos(xt) + h \sin(xt)\} dt g(x) dx \\ &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \int_0^\infty g(x) \{x \cos(xt) + h \sin(xt)\} dx f(t) dt \\ &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty f(t)(\mathcal{R}'g)(t)dt. \end{aligned}$$

The use of the Fubini’s theorem is justified by

$$\begin{aligned} \int_0^\infty |(\mathcal{R}f)(x)| |g(x)| dx &\leq \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \|f\|_{L^1(\mathbb{R}_+)} \int_0^\infty (x+|h|) |g(x)| dx \\ &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \|f\|_{L^1(\mathbb{R}_+)} \|g\|_{L^1(\mathbb{R}_+, (t+|h|) dt)}. \end{aligned}$$

So for $f \in L^1(\mathbb{R}_+)$ and $g \in L^1(\mathbb{R}_+, (t+|h|) dt)$ the Parseval–Goldstein relation follows. $\square$

Observe that for $k \in \mathbb{N} \cup \{0\}$,

$$D_t^{2k} \cos(xt) = (-1)^k x^{2k} \cos(xt),$$

and

$$D_t^{2k} \sin(xt) = (-1)^k x^{2k} \sin(xt).$$

Thus, for $f \in C_c^{2k}(\mathbb{R}_+)$, one has

$$(\mathcal{R}(D_t^{2k}f))(x) = (-1)^k x^{2k} (\mathcal{R}f)(x), \quad \forall x \in \mathbb{R}_+. \quad (4)$$

Therefore one obtains the next result.

Theorem 2 (Weighted Parseval–Goldstein relation). *If $f \in C_c^{2k}(\mathbb{R}_+)$ and $g \in L^1(\mathbb{R}_+, (t + |h|) dt)$, then*

$$\int_0^\infty (\mathcal{R}f)(x) g(x) x^{2k} dx = (-1)^k \int_0^\infty (D_t^{2k} f)(t) (\mathcal{R}'g)(t) dt.$$

Proof. The proof is a direct consequence of above Theorem 1 and using relation (4). $\square$

Now, for $f \in L^1(\mathbb{R}_+)$, we consider the integral operator

$$(\mathcal{R}''f)(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty f(t) \{x \sin(xt) - h \cos(xt)\} dt, \quad x \in \mathbb{R}_+.$$

This operator possesses similar L^p -properties as the $\mathcal{R}$ -transform. Indeed, the operator $\mathcal{R}''$ is bounded from $L^1(\mathbb{R}_+)$ into $\mathcal{B}_\gamma$ and it is also bounded from $L^1(\mathbb{R}_+)$ into $L^q(\mathbb{R}_+, w(x)dx)$, $0 < q < \infty$, $w \in L^1(\mathbb{R}_+, (x + |h|) dx)$, $w > 0$ almost everywhere.

Observe that for $k \in \mathbb{N} \cup \{0\}$,

$$D_t^{2k+1} \sin(xt) = (-1)^k x^{2k+1} \cos(xt),$$

and

$$D_t^{2k+1} \cos(xt) = (-1)^k x^{2k+1} \sin(xt).$$

Therefore, for $f \in C_c^{2k+1}(\mathbb{R}_+)$, one has

$$(\mathcal{R}(D_t^{2k+1} f))(x) = (-1)^k x^{2k+1} (\mathcal{R}''f)(x), \quad x \in \mathbb{R}_+. \quad (5)$$

Then one has

Theorem 3. *If $f \in C_c^{2k+1}(\mathbb{R}_+)$ and $g \in L^1(\mathbb{R}_+, (t + |h|) dt)$, then*

$$\int_0^\infty (\mathcal{R}''f)(x) g(x) x^{2k+1} dx = (-1)^k \int_0^\infty (D_t^{2k+1} f)(t) (\mathcal{R}'g)(t) dt.$$

Proof. The proof is a direct consequence of above Theorem 1 and using relation (5). $\square$

3. The Inverse of the $\mathcal{R}$ -Transform over Lebesgue Spaces

In this section, we study the inverse of the Sneddon $\mathcal{R}$ -transform on suitable weighted Lebesgue spaces. We establish boundedness results for the operator $\mathcal{R}^{-1}$ from $L^1\left(\mathbb{R}_+, \frac{t+|h|}{t^2+h^2} dt\right)$ into $L^\infty(\mathbb{R}_+)$ and weighted Lebesgue spaces. Furthermore, a Parseval–Goldstein relation associated with the $\mathcal{R}$ -transform is derived.

For $f \in L^1\left(\mathbb{R}_+, \frac{t+|h|}{t^2+h^2} dt\right)$, the inverse of the $\mathcal{R}$ -transform is given by (cf. Sneddon [formula (2-12-14)] [1], p. 73)

$$(\mathcal{R}^{-1}f)(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty f(t) \frac{t \cos(xt) + h \sin(xt)}{t^2 + h^2} dt, \quad x > 0, h \in \mathbb{R}.$$

Observe that

$$\begin{aligned} & |(\mathcal{R}^{-1}f)(x)| \\ & \leq \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty |f(t)| \frac{t + |h|}{t^2 + h^2} dt \\ & = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \|f\|_{L^1\left(\mathbb{R}_+, \frac{t+|h|}{t^2+h^2} dt\right)}, \quad x > 0, \end{aligned} \quad (6)$$

and then

$$\|\mathcal{R}^{-1}f\|_{L^\infty(\mathbb{R}_+)} \leq \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \|f\|_{L^1\left(\mathbb{R}_+, \frac{t+|h|}{t^2+h^2} dt\right)}.$$

Therefore

Proposition 5. *The operator $\mathcal{R}^{-1} : L^1\left(\mathbb{R}_+, \frac{t+|h|}{t^2+h^2} dt\right) \rightarrow L^\infty(\mathbb{R}_+)$ is bounded.*

Now, let $0 < q < \infty$ and $w \in L^1(\mathbb{R}_+)$ such that $w > 0$ almost everywhere. Then

$$\int_0^\infty |(\mathcal{R}^{-1}f)(x)|^q w(x) dx \leq \left(\frac{2}{\pi}\right)^{\frac{q}{2}} \|f\|_{L^1\left(\mathbb{R}_+, \frac{t+|h|}{t^2+h^2} dt\right)}^q \int_0^\infty w(x) dx,$$

and thus,

$$\|\mathcal{R}^{-1}f\|_{L^q(\mathbb{R}_+, w dx)} \leq \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \|f\|_{L^1\left(\mathbb{R}_+, \frac{t+|h|}{t^2+h^2} dt\right)} \|w\|_{L^1(\mathbb{R}_+)}^{1/q}.$$

So, one has

Proposition 6. *For $0 < q < \infty$ and $w \in L^1(\mathbb{R}_+)$ such that $w > 0$ almost everywhere, then the operator $\mathcal{R}^{-1} : L^1\left(\mathbb{R}_+, \frac{t+|h|}{t^2+h^2} dt\right) \rightarrow L^q(\mathbb{R}_+, w(x) dx)$ is bounded.*

Also

Theorem 4 (Parseval-Goldstein relation). *Let*

$$f \in L^1\left(\mathbb{R}_+, \frac{t+|h|}{t^2+h^2} dt\right) \quad \text{and} \quad g \in L^1(\mathbb{R}_+).$$

Then the following Parseval-Goldstein relation associated with the operator $\mathcal{R}$ holds:

$$\int_0^\infty (\mathcal{R}^{-1}f)(x) g(x) dx = \int_0^\infty f(t) (\mathcal{R}g)(t) \frac{dt}{t^2+h^2}.$$

Proof. By the definition of the inverse transform $\mathcal{R}^{-1}$, we have

$$(\mathcal{R}^{-1}f)(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty f(t) \frac{t \cos(xt) + h \sin(xt)}{t^2+h^2} dt, \quad x > 0.$$

Multiplying both sides by $g(x)$ and integrating with respect to x over $(0, \infty)$, we obtain

$$\int_0^\infty (\mathcal{R}^{-1}f)(x) g(x) dx = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \int_0^\infty f(t) \frac{t \cos(xt) + h \sin(xt)}{t^2+h^2} dt g(x) dx.$$

Applying Fubini's theorem which is justified by the estimate

$$\int_0^\infty |(\mathcal{R}^{-1}f)(x)| |g(x)| dx \leq \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \|f\|_{L^1\left(\mathbb{R}_+, \frac{t+|h|}{t^2+h^2} dt\right)} \int_0^\infty |g(x)| dx$$

it allows us to interchange the order of integration to get

$$\begin{aligned} & \int_0^\infty (\mathcal{R}^{-1}f)(x) g(x) dx \\ &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty f(t) \left[\int_0^\infty g(x) \{t \cos(xt) + h \sin(xt)\} dx \right] \frac{dt}{t^2+h^2}. \end{aligned} \quad (7)$$

The inner integral of the right hand side of (7) together with the factor $(\frac{2}{\pi})^{\frac{1}{2}}$ is precisely $(\mathcal{R}g)(t)$. Therefore,

$$\int_0^\infty (\mathcal{R}^{-1}f)(x) g(x) dx = \int_0^\infty f(t) (\mathcal{R}g)(t) \frac{dt}{t^2 + h^2}.$$

Hence, the Parseval-Goldstein formula follows. $\square$

4. Initial and Final Value Theorems for the $\mathcal{R}$ -Transform over Lebesgue Spaces

Here, initial and final value theorems corresponding to the Sneddon $\mathcal{R}$ -transform and its inverse are developed. These theorems describe the limiting behaviour of the transform at the origin and infinity, extending the well-known results for cosine and sine transforms (see [7–11]). The asymptotic relations are established under suitable integrability conditions on the function f , and their proofs rely on dominated convergence and properties of the kernel.

From (1) one has

Theorem 5 (Initial and Final Value Theorems for the $\mathcal{R}$ -transform). *For $f \in L^1(\mathbb{R}_+)$, the following properties hold:*

(i) *If $h = 0$, then*

$$\lim_{x \rightarrow 0^+} (\mathcal{R}f)(x) = 0.$$

(ii) *If $h \neq 0$ and any $\alpha > 0$, then*

$$\lim_{x \rightarrow 0^+} \{x^\alpha (\mathcal{R}f)(x)\} = 0.$$

(iii) *For any $\alpha > 1$, then*

$$\lim_{x \rightarrow +\infty} \{x^{-\alpha} (\mathcal{R}f)(x)\} = 0.$$

Remark 2 (The finite $\mathcal{R}$ -transform). *For $f \in L^1((0, A))$, and according to Sneddon ([1], Section 8-1-1, pp. 425–429) the finite $\mathcal{R}$ -transform is defined by the relation*

$$(\mathcal{F}_{\mathcal{R},A}f)(m) = \int_0^A f(t) \left\{ \frac{m\pi}{A} \cos\left(\frac{m\pi t}{A}\right) + h \sin\left(\frac{m\pi t}{A}\right) \right\} dt, \quad m \in \mathbb{N}, h \in \mathbb{R}.$$

Clearly, one has

$$|(\mathcal{F}_{\mathcal{R},A}f)(m)| \leq \left(\frac{m\pi}{A} + h\right) \|f\|_{L^1((0,A))}, \quad m \in \mathbb{N}.$$

Thus, for any $\alpha > 1$ and $f \in L^1((0, A))$, it follows that

$$\lim_{m \rightarrow \infty} \{m^{-\alpha} (\mathcal{F}_{\mathcal{R},A}f)(m)\} = 0.$$

Also from (6) one has

Theorem 6 (Initial and Final Value Theorems for the inverse of the $\mathcal{R}$ -transform). *For $f \in L^1\left(\mathbb{R}_+, \frac{t+|h|}{t^2+h^2} dt\right)$, the following properties hold:*

(i) *For any $\alpha > 0$,*

$$\lim_{x \rightarrow 0^+} \{x^\alpha (\mathcal{R}^{-1}f)(x)\} = 0.$$

(ii) For any $\alpha > 0$,

$$\lim_{x \rightarrow +\infty} \{x^{-\alpha}(\mathcal{R}^{-1}f)(x)\} = 0.$$

5. Initial and Final Value Theorems for the $\mathcal{R}$ -Transform over Generalized Functions

In this section, we establish initial and final value theorems for the $\mathcal{R}$ -transform of generalized functions. To this end, we employ the Fourier sine and cosine transforms as auxiliary tools within the framework of the function space $\mathcal{L}_{+,a}$, $a \in \mathbb{R}$, introduced by Zemanian ([12], p. 90). When defining these transforms and deriving growth estimates, we shall impose the restriction $a \leq 0$.

Recall that $\mathcal{L}_{+,a}$, $a \in \mathbb{R}$, is the vector space consisting of all smooth functions $\sigma \in C^\infty(\mathbb{R}_+)$ such that

$$\gamma_{k,a}(\sigma) = \sup_{t \in \mathbb{R}_+} |e^{at} D_t^k \sigma(t)| < \infty, \quad k \in \mathbb{N} \cup \{0\}. \quad (8)$$

Equipped with the topology generated by the family of seminorms $\{\gamma_{k,a} : k \geq 0\}$, the space $\mathcal{L}_{+,a}$ is a Fréchet space. Its strong dual is denoted by $\mathcal{L}'_{+,a}$.

Let $f \in \mathcal{L}'_{+,a}$. Assume throughout that $a \leq 0$. The Fourier sine and cosine transforms of f are defined by duality as

$$(\mathcal{S}f)(x) = \langle f(t), \sin(xt) \rangle, \quad x > 0, \quad (9)$$

$$(\mathcal{C}f)(x) = \langle f(t), \cos(xt) \rangle, \quad x > 0. \quad (10)$$

Here, $\langle f, \sigma \rangle$ denotes the action of $f \in \mathcal{L}'_{+,a}$ on $\sigma \in \mathcal{L}_{+,a}$.

Moreover, from ([13] Proposition 2, p. 97), for each $f \in \mathcal{L}'_{+,a}$ there exist constants $C > 0$ and an integer $r \geq 0$ such that

$$|\langle f, \sigma \rangle| \leq C \max_{0 \leq k \leq r} \gamma_{k,a}(\sigma), \quad \sigma \in \mathcal{L}_{+,a}. \quad (11)$$

Lemma 1. Let $f \in \mathcal{L}'_{+,a}$ with $a \leq 0$, then there exist constants $C > 0$ and an integer $r \geq 0$, depending only on f , such that

$$|(\mathcal{S}f)(x)| \leq C(1+x)^r,$$

and

$$|(\mathcal{C}f)(x)| \leq C(1+x)^r, \quad x > 0.$$

Proof. By definition of the strong dual $\mathcal{L}'_{+,a}$, there exist $C > 0$ and $r \geq 0$ such that

$$|\langle f, \sigma \rangle| \leq C \max_{0 \leq k \leq r} \gamma_{k,a}(\sigma), \quad \sigma \in \mathcal{L}_{+,a}.$$

Since $a \leq 0$, the functions $\sin(xt)$ and $\cos(xt)$ belong to $\mathcal{L}_{+,a}$ for each $x > 0$, and the stated polynomial growth follows. $\square$

The smallest integer r for which (11) holds is called the *order* of the generalized function f . We now derive initial and final value theorems which will be used to analyze the $\mathcal{R}$ -transform.

Theorem 7 (Initial and Final Value Theorems). Let $f \in \mathcal{L}'_{+,a}$ be of order r with $a \leq 0$. Then, for every $\alpha > 0$,

(i)

$$\lim_{x \rightarrow 0^+} \{x^\alpha (\mathcal{S}f)(x)\} = \lim_{x \rightarrow 0^+} \{x^\alpha (\mathcal{C}f)(x)\} = 0;$$

(ii)

$$\lim_{x \rightarrow +\infty} \{x^{-r-\alpha}(\mathcal{S}f)(x)\} = \lim_{x \rightarrow +\infty} \{x^{-r-\alpha}(\mathcal{C}f)(x)\} = 0.$$

Proof. The assertions follow directly from Lemma 1. $\square$

For $f \in \mathcal{L}'_{+,a}$, $a \leq 0$, the $\mathcal{R}$ -transform of f is given by

$$(\mathcal{R}f)(x) = \langle f(t), x \cos(xt) + h \sin(xt) \rangle, \quad x > 0, \quad h \in \mathbb{R},$$

which by linearity equals to $x(\mathcal{C}f)(x) + h(\mathcal{S}f)(x)$.

Corollary 1 (The $\mathcal{R}$ -Transform). *Let $f \in \mathcal{L}'_{+,a}$, $a \leq 0$. Then the $\mathcal{R}$ -transform of f satisfies for every $\alpha > 0$,*

$$\lim_{x \rightarrow 0^+} \{x^\alpha(\mathcal{R}f)(x)\} = 0,$$

and

$$\lim_{x \rightarrow +\infty} \{x^{-1-\alpha}(\mathcal{R}f)(x)\} = 0.$$

6. Final Value Theorems for the $\mathcal{R}$ -Transform over Generalized Functions on a Finite Interval

In this section, we establish final value theorems for the $\mathcal{R}$ -transform in the setting of generalized functions supported on a finite interval. Working within the Fréchet space $\mathcal{M}_A$ and its strong dual $\mathcal{M}'_A$, we analyze the growth behaviour of the finite Fourier sine and cosine transforms of generalized functions. These estimates are then used to derive final value results for the associated finite $\mathcal{R}$ -transform.

Throughout this section, let $A > 0$ be fixed.

Let $\mathcal{M}_A$ denote the space of all smooth functions $\varphi \in C^\infty((0, A))$ for which

$$\beta_A^k(\varphi) := \sup_{t \in (0, A)} |D^k \varphi(t)| < \infty, \quad k = 0, 1, 2, \dots$$

Endowed with the family of seminorms $\{\beta_A^k\}_{k \geq 0}$, the space $\mathcal{M}_A$ is a Fréchet space. Its strong dual $\mathcal{M}'_A$ consists of generalized functions over the interval $(0, A)$.

Remark 3. For $A = 1$, the space $\mathcal{M}_A$ coincides with the space $\mathcal{B}((0, 1))$ introduced by Zemanian ([12], Example 2.4-1, p. 40).

Let $f \in \mathcal{M}'_A$. Denote

$$(\mathcal{F}_{s,A}f)(m) = \left\langle f(t), \sin\left(\frac{m\pi t}{A}\right) \right\rangle, \quad m \in \mathbb{N}, \quad (\text{the finite Fourier sine transform}),$$

$$(\mathcal{F}_{c,A}f)(m) = \left\langle f(t), \cos\left(\frac{m\pi t}{A}\right) \right\rangle, \quad m \in \mathbb{N} \cup \{0\}, \quad (\text{the finite Fourier cosine transform}),$$

and

$$(\mathcal{F}_{\mathcal{R},A}f)(m) = \left\langle f(t), \frac{m\pi}{A} \cos\left(\frac{m\pi t}{A}\right) + h \sin\left(\frac{m\pi t}{A}\right) \right\rangle, \quad m \in \mathbb{N}, \quad h \in \mathbb{R}, \quad (\text{the finite } \mathcal{R}\text{-transform}).$$

Lemma 2. *Let $f \in \mathcal{M}'_A$ be a generalized function of order r . Then there exists a constant $C > 0$ such that*

$$|(\mathcal{F}_{s,A}f)(m)| \leq C \left(1 + \frac{m\pi}{A}\right)^r, \quad m \in \mathbb{N},$$

and

$$|(\mathcal{F}_{c,A}f)(m)| \leq C \left(1 + \frac{m\pi}{A}\right)^r, \quad m \in \mathbb{N} \cup \{0\}.$$

Proof. Since $f \in \mathcal{M}'_A$ has order r , Proposition 2 of ([13], p. 97) ensures the existence of a constant $C > 0$ such that

$$|\langle f, \varphi \rangle| \leq C \max_{0 \leq k \leq r} \beta_A^k(\varphi), \quad \varphi \in \mathcal{M}_A.$$

Applying this estimate to the sine and cosine kernels, we obtain

$$|(\mathcal{F}_{s,A}f)(m)| \leq C \max_{0 \leq j \leq r} \left(\frac{m\pi}{A}\right)^j \leq C \left(1 + \frac{m\pi}{A}\right)^r,$$

for all $m \in \mathbb{N}$. The proof for the cosine transform follows analogously. $\square$

Theorem 8 (Final Value Theorems). *Let $f \in \mathcal{M}'_A$ be of order r . Then, for every $\alpha > 0$,*

$$\lim_{m \rightarrow \infty} \{m^{-r-\alpha} (\mathcal{F}_{s,A}f)(m)\} = 0,$$

and

$$\lim_{m \rightarrow \infty} \{m^{-r-\alpha} (\mathcal{F}_{c,A}f)(m)\} = 0.$$

Proof. By Lemma 2,

$$|(\mathcal{F}_{s,A}f)(m)| \leq C \left(1 + \frac{m\pi}{A}\right)^r.$$

Multiplication by $m^{-r-\alpha}$ and passage to the limit $m \rightarrow \infty$ yield the stated result. The cosine case is identical. $\square$

Corollary 2. (Final Value Theorems on $\mathcal{M}'_A$) *Let $f \in \mathcal{M}'_A$ of order r . Then, for every $\alpha > 0$, one has*

$$\lim_{m \rightarrow \infty} \{m^{-r-1-\alpha} (\mathcal{F}_{\mathcal{R},A}f)(m)\} = 0.$$

7. Conclusions

In this work, we have carried out a systematic and unified analysis of the Sneddon $\mathcal{R}$ -transform, emphasizing its role as a fundamental integral operator that interpolates between cosine- and sine-type transforms through mixed trigonometric kernels. A detailed L^p -theory has been developed, leading to rigorous results on boundedness, continuity, and sharp norm estimates for the $\mathcal{R}$ -transform and its inverse. These results place the transform on a solid functional-analytic footing and extend several classical mapping properties known for standard Fourier-type operators.

An important contribution of the present manuscript is the inclusion of initial and final value theorems for generalized functions. Working within the Zemanian framework $\mathcal{L}_{+,a}$, $a \in \mathbb{R}$, and imposing the natural restriction $a \leq 0$ when defining the auxiliary sine and cosine transforms, we established precise growth estimates and asymptotic behavior at both the origin and infinity. These results were then used to derive initial and final value theorems for the $\mathcal{R}$ -transform itself, thereby extending classical limit theorems from functions to generalized functions of finite order. This analysis clarifies the asymptotic structure of the $\mathcal{R}$ -transform and demonstrates how sine and cosine transforms serve as effective tools rather than primary objects of study.

In addition, we investigated the finite-interval setting by introducing appropriate test function spaces $\mathcal{M}_A$ and their duals $\mathcal{M}'_A$, $A > 0$. Within this framework, finite Fourier sine and cosine transforms were employed to establish polynomial growth bounds

and corresponding final value theorems for generalized functions supported on bounded intervals. As a consequence, a finite version of the Sneddon $\mathcal{R}$ -transform was analyzed, and its asymptotic decay properties were obtained in a transparent and unified manner. These results complement the infinite-interval theory and demonstrate that the $\mathcal{R}$ -transform admits a coherent treatment across both unbounded and bounded domains.

Overall, the analytical framework developed in this paper provides a versatile platform for further investigations. Possible directions include convolution and product formulas associated with $\mathcal{R}$ -type kernels, weighted Parseval–Goldstein identities, uncertainty principles, and spectral or Paley–Wiener-type theorems. Moreover, the techniques introduced here are well suited for extensions to fractional variants of the $\mathcal{R}$ -transform, broader classes of distributions, and applications to boundary value problems arising in mathematical physics and engineering. By unifying classical trigonometric transforms within a single operator-theoretic setting, this work enriches the theory of integral transforms and opens new avenues for both theoretical development and applied analysis.

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