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Left-Symmetric Algebras Arising from Modified DNA Insertion Operations

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Abstract

DNA recombination is a fundamental biological process that encodes genetic information for organism development and function. In this study, we construct left-symmetric algebras arising from DNA insertion operations. That is, we define a modified insertion operation by weighting the simplified insertion. It generalizes the left-symmetric algebra constructed from the simplified DNA insertion operation. We prove that the algebra $\mathbb{F}(R)$ (over a field $\mathbb{F}$ of characteristic 0, with R being an infinite free semigroup generated by DNA nucleotides $\{A, G, C, T\}$) forms a left-symmetric algebra if and only if the function f satisfies a certain multiplicative condition for all positive integers m, n, and p. A key example of such a function is $f(m, n) = \exp\{g(m, n)\}$, where $g(m, n) = k \cdot mn$, and k is a fixed positive number, which effectively models length-dependent DNA insertion dynamics. This work contributes an algebraic framework that may be useful for quantitative modeling of DNA recombination processes.

Keywords: left-symmetric algebras; DNA insertion operations; mathematical model

MSC: 17D25

1. Introduction

Genetic information is obtained through DNA recombination. Recombination provides the genetic program for the development and functioning of all living organisms. The algebraic formalization of DNA recombination is represented in the form of a linear space $\mathbb{F}(R)$ over a field $\mathbb{F}$ of characteristic 0, where R is an infinite free semigroup generated by the set of DNA nucleotides $\{A, G, C, T\}$. The schematic model of non-homologous DNA recombination can be represented in the form: $c \cdot (ab) \to acb$, where two chromosomes (ab) and c participate in non-homologous recombination. The algebraic formalization of the above recombination, with respect to all possible insertions of DNA c in DNA (ab), defines the algebra of simplified insertions. Let $X \in R$. Then X can be presented as $X = x_1 x_2 \cdots x_n$, where $x_i \in \{A, G, C, T\}$. The operation, defined in [1], is given by

$$a \to X = \sum_{i=0}^{n} x_1 \cdots x_i a x_{i+1} \cdots x_n, \tag{1}$$

where a is an arbitrary element in $\mathbb{F}(R)$; and $a \to b = \sum \alpha_s (a \to u_s)$, where $b = \sum \alpha_s u_s$, $\alpha_s \in \mathbb{F}$, and u_s are monomials in $\mathbb{F}(R)$. This operation $\to$ is called the operation of left simplified insertion. And it satisfies the left-symmetric identity $(x, y, z) = (y, x, z)$. (Notice that the categories of the left-symmetric and right-symmetric algebras are equivalent [1].)

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The operation of simplified insertion was first introduced by Bremner [2], and it is an algebraic formalization of the operation of normal insertion in the theory of DNA computing. This algebra defined on $\mathbb{F}(R)$ is a very important class of non-associative algebras. In the interdisciplinary field of mathematics and biology, the study of non-associative algebras has become an important branch. They have previously been applied to population genetics in the well-developed theory of genetic algebras, see [3]. In 2016, Tian abstracted evolution algebras from biological processes, demonstrating the direct impetus that biology provides for the development of mathematical theories [4]. Later, Vidal et al. constructed genetic evolution algebras based on Gibbs measures, offering new perspectives for modeling complex genetic dynamic processes [5].

As an important class of non-associative algebras, left-symmetric algebras (also called pre-Lie algebras) have attracted sustained interest due to their rich algebraic structure and wide-ranging applications in geometry [6], mathematical physics [7,8], and combinatorial algebra [9,10]. In 1890, A. Cayley first introduced rooted tree algebras into the study of left-symmetric algebras, see [11]. By the 1960s, left-symmetric algebras began to emerge in geometric and algebraic contexts such as convex homogeneous cones [6] and affinely flat manifolds [12,13] and the deformation theory of associative algebras [14]. In addition, remarkable progress has been made in the theoretical and geometric generalizations of left-symmetric algebras. To extend the theory to more general coefficient rings and incorporate symmetry variants, Bakalov et al. [15] proposed the concept of left-symmetric pseudoalgebras, from which several algebraic forms have been developed, including left-symmetric superalgebras [16,17]. As a geometric generalization, left-symmetric algebroids have been extensively studied [18–20], with the concept of Nijenhuis operators defined on them [20]. Researchers subsequently studied left-symmetric algebras, focusing on issues such as their structures, classification, and important applications in different fields [21–23]. In this paper, we construct new left-symmetric algebras through DNA recombination to enrich the research field.

The left simplified insertion left-symmetric algebra is infinite-dimensional as a vector space. However, one finds that there are only finitely many types of DNA in nature. Therefore, in this paper, we modify the simplified insertion operation by introducing a length-dependent function. This operation does not reduce the dimension of the vector space; rather, it ensures that the insertion probability approaches zero when the fragment length is large enough. Moreover, it is similar to ordered constraint semigroups [24] in the sense that both extend the original algebraic framework by imposing constraints, thereby making the operations more consistent with realistic scenarios.

The paper is organized as follows. In Section 2, we recall left-symmetric algebras and the simplified insertion, and prove that the simplified insertion satisfies the left-symmetric identity. We then examine the intermediate version of the insertion operation proposed by Bremner [2] and demonstrate through a counterexample that it does not satisfy the left-symmetric identity. And we prove that the operation of synchronized insertion given in [25] does not satisfy the left-symmetric identity, either. In Section 3, we specify the conditions under which the simplified insertion operation satisfies the left-symmetric identity. In Section 4, we give the definition of modified simplified insertion and construct a new left-symmetric structure for the modified simplified insertion. Specifically, we define a new operation of insertion as follows:

$$x \Rightarrow y := f(|x|, |y|) \sum_{i=0}^{q} y_1 y_2 \cdots y_i x y_{i+1} \cdots y_q,$$

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where $x = x_1x_2 \cdots x_p$, $y = y_1y_2 \cdots y_q$, and $|x|$, $|y|$ denote the lengths of x and y, respectively. This operation is called modified simplified insertion. Then $\mathbb{F}(R)$ is a left-symmetric algebra if and only if $f(m,n)$ satisfies the following:

$$f(m,n)f(m+n,p) = f(n,p)f(m,n+p) = f(m,p)f(n,m+p),$$

where $m,n,p \in \mathbb{N}$. For example, if the operation is given by

$$f(m,n) := e^k(m \times n),$$

where k is a fixed negative integer, then $\mathbb{F}(R)$ is still a left-symmetric algebra. And the new genetics generated by the left simplified insertion can be suppressed by this coefficient $e^k(m \times n)$ if m or n is large enough. Finally, we illustrate properties of these left-symmetric algebras and certain applications in genetics and molecular. In Section 5, we summarize the main results, limitations, and future directions.

2. Definitions and Notions

In this section, we mainly recall the definitions of left-symmetric algebra and the simplified insertion operation. And we show that the simplified insertion operation satisfies the left-symmetric identity.

2.1. The Left-Symmetric Algebra

Let $\mathbb{F}$ be a field of characteristic zero. Let A be an algebra over $\mathbb{F}$. For elements x and y in A, the bilinear product is denoted by $(x,y) \to x \circ y$. Given three elements x, y and z, we denote the associator of these elements by the following:

$$(x,y,z) = (x \circ y) \circ z - x \circ (y \circ z).$$

Definition 1. *A left-symmetric algebra A is an algebra whose associator satisfies*

$$\forall x,y,z \in A, (x,y,z) = (y,x,z) \text{ or equivalently } (x \circ y) \circ z - x \circ (y \circ z) = (y \circ x) \circ z - y \circ (x \circ z).$$

And the identity $(x,y,z) = (y,x,z)$ is called the left-symmetric identity.

From the perspective of DNA insertion, the left-symmetric identity $(x,y,z) = (y,x,z)$ indicates that swapping the order of the first two inserted fragments does not alter the outcome of the triple-insertion operation. This property ensures that, regardless of the sequence in which multiple DNA fragments are inserted, the algebraic representation of the recombination process remains consistent.

Example 1.

(1) Every associative algebra is a left-symmetric algebra with the left-symmetric structure given by the following:

$$x \circ y := xy.$$

Obviously, $(x \circ y) \circ z - x \circ (y \circ z) = 0$.

(2) Let A be the vector space $C^\infty(\mathbb{R},\mathbb{R})$ of smooth functions. For f and g in A, we set $f \circ g := f\frac{dg}{dx}$. It is easy to check that A is a left-symmetric algebra.

2.2. Languages and Simplified Insertion

Assume that I is a finite set. Let S be a finite non-empty set, denoted by $S = \{a_i \mid i \in I\}$. A word over S is a finite string $w = a_1a_2 \cdots a_p$, where $p \geqslant 0$ and $a_i \in S (1 \leqslant i \leqslant p)$. For

$p = 0$, we have the empty word denoted 1. We denote the length of w by $|w| = p$. We write $M(S)$ for the set of all words, and any subset of $M(S)$ is called a language.

Let $A[S]$ be the free associative algebra over $\mathbb{F}$ generated by S.

Remark 1. *Notice that $M(S)$ is a basis of $A[S]$. Then all elements in $M(S)$ are linearly independent and every element in $A[S]$ can be expressed as a linear combination of the elements in $M(S)$. More precisely,*

$$A[S] = \bigoplus_{w \in M(S)} \mathbb{F} \cdot w.$$

Consider the operation of DNA insertion of one word into another. This provides a method for combining DNA molecules that are well-studied in molecular genetics. Given $x, y \in M(S)$, we consider all insertions of x into y:

$$x \triangleright y = \{y^1 x y^2 \mid y = y^1 y^2, y^1, y^2 \in M(S)\}.$$

Note that we allow both y^1 and y^2 to be empty.

Firstly, we recall the simplified insertion.

Definition 2. *The simplified insertion of x into y is the linearized form of this operation, that is, the sum of all insertions, as*

$$x \to y = \sum_{i=0}^{q} y_1 y_2 \cdots y_i x y_{i+1} \cdots y_q,$$

where $y = y_1 y_2 \cdots y_q$, $y_i \in S$ and $x = x_1 x_2 \cdots x_p$, $x_i \in S$.

We can define a new operation of multiplication on $M(S)$ by

$$x \circ y = x \to y = \sum_{i=0}^{q} y_1 y_2 \cdots y_i x y_{i+1} \cdots y_q, \tag{2}$$

where $y = y_1 y_2 \cdots y_q$, $y_i \in S$. And we extend it to $A[S]$ bilinearly. In detail, if $x = \sum \alpha_i x_i$ and $y = \sum \beta_j y_j$, where $\alpha_i, \beta_j \in \mathbb{F}$ and $x_i, y_j \in M(S)$, then

$$x \circ y = x \to y = \sum_{i,j} \alpha_i \beta_j (x_i \to y_j). \tag{3}$$

The associator for this new operation, that is simplified insertion operation, is defined as usual by the following:

$$(x, y, z)_1 = (x \circ y) \circ z - x \circ (y \circ z) = ((x \to y) \to z) - (x \to (y \to z)). \tag{4}$$

Example 2. *Let $x = a_1 a_2 a_3$, $y = a_4 a_5$ and $z = a_6$. Then $x \to y = a_1 a_2 a_3 a_4 a_5 + a_4 a_1 a_2 a_3 a_5 + a_4 a_5 a_1 a_2 a_3$ and $(x, y, z)_1 = -a_1 a_2 a_3 a_6 a_4 a_5 - a_4 a_5 a_6 a_1 a_2 a_3$.*

Theorem 1. *The algebra $A[S]$ is a left-symmetric algebra whose associator is defined by (4).*

Proof. Let

$$x = x_1 x_2 \cdots x_p, \ \ y = y_1 y_2 \cdots y_q, \ \ z = z_1 z_2 \cdots z_r.$$

Using (2), we obtain

$$x \to y = \sum_{j=0}^{q} y_1 y_2 \cdots y_j x y_{j+1} \cdots y_q,$$

$$y \to z = \sum_{k=0}^{r} z_1 z_2 \cdots z_k y z_{k+1} \cdots z_r.$$

Then

$$x \to (y \to z) = \sum_{k=0}^{r} (\sum_{j=0}^{k-1} z_1 z_2 \cdots z_j x z_{j+1} \cdots z_k y z_{k+1} \cdots z_r$$

$$+ z_1 z_2 \cdots z_k (x \to y) z_{k+1} \cdots z_r + \sum_{j=k+1}^{r} z_1 z_2 \cdots z_k y z_{k+1} \cdots z_j x z_{j+1} \cdots z_r),$$

$$(x \to y) \to z = \sum_{j=0}^{q} \sum_{k=0}^{r} z_1 z_2 \cdots z_k y_1 y_2 \cdots y_j x y_{j+1} \cdots y_q z_{k+1} \cdots z_r.$$

Hence, we obtain

$$\begin{aligned} & ((x \to y) \to z) - (x \to (y \to z)) \\ = \; & -\sum_{k=0}^{r} (\sum_{j=0}^{k-1} z_1 z_2 \cdots z_j x z_{j+1} \cdots z_k y z_{k+1} \cdots z_r \\ & + \sum_{j=k+1}^{r} z_1 z_2 \cdots z_k y z_{k+1} \cdots z_j x z_{j+1} \cdots z_r) \\ := \; & LHS, \end{aligned} \tag{5}$$

$$\begin{aligned} & ((y \to x) \to z) - (y \to (x \to z)) \\ = \; & -\sum_{k=0}^{r} (\sum_{j=0}^{k-1} z_1 z_2 \cdots z_j y z_{j+1} \cdots z_k x z_{k+1} \cdots z_r \\ & + \sum_{j=k+1}^{r} z_1 z_2 \cdots z_k x z_{k+1} \cdots z_j y z_{j+1} \cdots z_r) \\ := \; & RHS. \end{aligned} \tag{6}$$

The monomials in the LHS correspond precisely to insertions of x and y into z at distinct positions separated by at least one letter; the same set of monomials occurs in the RHS. Indeed, there is a natural bijection between the monomials of LHS and RHS that swaps the insertion positions of x and y while maintaining the result. Hence, $LHS = RHS$. We immediately obtain the following:

$$((x \to y) \to z) - (x \to (y \to z)) = ((y \to x) \to z) - (y \to (x \to z)). \tag{7}$$

It still holds when

$$x = \sum \alpha_i x_i,\ y = \sum \beta_j y_j,\ z = \sum \gamma_k z_k,$$

where $\alpha_i, \beta_j, \gamma_k \in \mathbb{F}$ and $x_i, y_j, z_k \in M(S)$. Using (3), we obtain

$$x \to y = \sum_{i,j} \alpha_i \beta_j (x_i \to y_j),$$

$$y \to z = \sum_{j,k} \beta_j \gamma_k (y_j \to z_k).$$

Then

$$(x \to y) \to z = \sum_{i,j,k} \alpha_i \beta_j \gamma_k ((x_i \to y_j) \to z_k),$$

$$x \to (y \to z) = \sum_{i,j,k} \alpha_i \beta_j \gamma_k (x_i \to (y_j \to z_k)).$$

Hence

$$((x \to y) \to z) - (x \to (y \to z)) = \sum_{i,j,k} \alpha_i \beta_j \gamma_k \big[((x_i \to y_j) \to z_k) - (x_i \to (y_j \to z_k))\big],$$

$$((y \to x) \to z) - (y \to (x \to z)) = \sum_{i,j,k} \beta_j \alpha_i \gamma_k \big[((y_j \to x_i) \to z_k) - (y_j \to (x_i \to z_k))\big].$$

Therefore, the left-symmetric identity holds on $A[S]$. □

Remark 2. *We consider the following open problem given in [2] by Bremner.*
Problem: An intermediate version of the insertion operation is obtained when we regard the set-theoretic definition as producing a set rather than a multiset. That is, on words $x = x_1 \cdots x_p$ and $y = y_1 \cdots y_q$ with $x_i, y_j \in S$, we define

$$\delta(x,y) = \begin{cases} 1, & \text{if } x_1 = y_1, \\ 0, & \text{otherwise,} \end{cases}$$

where $\delta : M(S) \times M(S) \to \{0,1\}$. We then consider the following operation:

$$x \circ y = \sum_{j=0}^{q} \delta(x, y_j \cdots y_q) y_1 y_2 \cdots y_j x y_{j+1} \cdots y_q.$$

What are the polynomial identities satisfied by this operation?

One checks that this operation does not satisfy the left-symmetric identity $(x,y,z) = (y,x,z)$. We present the following counterexample. Let $S = \{a,b,c\}$, $x = ab$, $y = abc$ and $z = ac$. Then $x \circ y = ababc$, $(x \circ y) \circ z = ababcac$, $x \circ (y \circ z) = ababcac + abcabac$ and hence $(x \circ y) \circ z - x \circ (y \circ z) = -abcabac$. Similarly, $(y \circ x) \circ z - y \circ (x \circ z) = -ababcac$. That is $(x,y,z) \neq (y,x,z)$.

Remark 3. *Recall another DNA insertion operation, the linearized version of the synchronized insertion introduced in [25]. We show that it does not satisfy the left-symmetric identity $(x,y,z) = (y,x,z)$, either.*

$$x \rightrightarrows y = \sum_{j=0}^{q} t(x, y_j \cdots y_q) y_1 y_2 \cdots y_j x y_{j+1} \cdots y_q,$$

where $t(x,y)$ is defined by

$$t(x,y) = k \ \text{ when } \ x_i = y_i \ \text{ for } \ 1 \leqslant i \leqslant k, \ \text{but } \ x_{k+1} \neq y_{k+1} (\text{or } k+1 \geqslant min(p,q)).$$

Without loss of generality, we choose the simplest case of the linearized version of synchronized insertion in which S contains one letter: $S = \{a\}$. Then the synchronized insertion becomes

$$a^p \rightrightarrows a^q = c(p,q) a^{p+q},$$

where

$$c(p,q) = \begin{cases} \frac{1}{2} p(2q - p + 1), & p < q, \\ \frac{1}{2} q(q+1), & \text{otherwise.} \end{cases}$$

Let $x = a^p$, $y = a^q$ and $z = a^r$, where $p = 2$, $q = 3$ and $r = 6$. Then $(x \rightrightarrows y) \rightrightarrows z - x \rightrightarrows (y \rightrightarrows z) - ((y \rightrightarrows x) \rightrightarrows z - y \rightrightarrows (x \rightrightarrows z)) = a^{16} \neq 0$. Hence, it does not satisfy the left-symmetric identity.

The linearized form of synchronized insertion also fails to satisfy the left-symmetric identity in the multi-letter case. Let $S = \{a, b, c\}$, $x = a^2$, $y = ab$ *and* $z = ac$. *Then* $(x \rightrightarrows y) \rightrightarrows z - x \rightrightarrows (y \rightrightarrows z) - ((y \rightrightarrows x) \rightrightarrows z - y \rightrightarrows (x \rightrightarrows z)) = -2a^4bc$.

3. Adjacency Constraints and Subspace Properties

In this section, we mainly study whether a subspace of the left-symmetric algebra $A[S]$ (from Section 2) remains a left-symmetric algebra when considered as a linear space.

Set $S = \{a_1, a_2, \cdots, a_p\}$. From a biological perspective, certain DNA insertion operations fail to yield specific DNA fragments. Therefore, we introduce the function $\delta' : S \times S \to \{0, 1\}$ by

$$\delta'(a_i, a_j) = \begin{cases} 1, & \text{if } a_i \text{ and } a_j \text{ can be adjacent in a word,} \\ 0, & \text{otherwise.} \end{cases}$$

for any $a_i, a_j \in S$. That is, we say that a_i and a_j can be adjacent in a word if there are some words like $\cdots a_i a_j \cdots$ and therefore $\delta'(a_i, a_j) = 1$; otherwise, $\delta'(a_i, a_j) = 0$. In particular, we set $\delta'(a_i, a_i) = 1$, for $\forall\, a_i \in S$. Then we consider the operation

$$\begin{aligned} x \dashrightarrow y &= \delta(x_p, y_1) x_1 x_2 \cdots x_p y_1 y_2 \cdots y_q + \delta(y_q, x_1) y_1 y_2 \cdots y_q x_1 x_2 \cdots x_p \\ &+ \sum_{i=1}^{q-1} \delta(y_i, x_1)\delta(y_{i+1}, x_p) y_1 y_2 \cdots y_i x y_{i+1} \cdots y_q. \end{aligned}$$

We give examples to illustrate the role of the function δ'.

Example 3.

(1) Let $S = \{a, b, c\}$ *and* $\delta'(a, b) = \delta'(b, c) = \delta'(a, c) = 1$. *The adjacency graph is shown in Figure 1a. Set* $x = abc$ *and* $y = bc$. *Then* $x \dashrightarrow y = abcbc + babcc + bcabc$.

(2) Let $S = \{a, b, c\}$, $\delta'(a, b) = \delta'(b, c) = 1$ *and* $\delta'(a, c) = 0$. *The adjacency graph is shown in Figure 1b. Set* $x = abc$ *and* $y = bc$. *Then* $x \dashrightarrow y = abcbc + babcc$.



$(a)\ \delta'(a, b) = \delta'(b, c) = \delta'(a, c) = 1$ $(b)\ \delta'(a, b) = \delta'(b, c) = 1,\ \delta'(a, c) = 0$

Figure 1. Adjacency graphs for Example 3.

Notice that $A[S]$ is the associative algebra over $\mathbb{F}$ generated by S. Consider the subspace of $A[S]$ for which there exist $a_i, a_j \in S$ such that $\delta'(a_i, a_j) = 0$.

Theorem 2. *Let* $S = \{a_1, \ldots, a_p\}$, $|S| = p$.

(1) If $|S| = 1$, *then the operation* $\dashrightarrow$ *satisfies the left-symmetric identity.*

(2) If $|S| = 2$ *and* $\delta'(a_1, a_2) = 1$, *then the operation* $\dashrightarrow$ *satisfies the left-symmetric identity. If* $|S| = 2$ *and* $\delta'(a_1, a_2) = 0$, *then the operation* $\dashrightarrow$ *is associative. It obviously satisfies the left-symmetric identity.*

(3) If $|S| \geqslant 3$ *and* $\delta'(a_i, a_j) = 1$ *for all* $a_i, a_j \in S$, *then the operation* $\dashrightarrow$ *is non-associative. However, it satisfies the left-symmetric identity. If there exists* $a_i, a_j \in S$ *such that* $\delta'(a_i, a_j) = 0$, *and its adjacency graph can be decomposed into a family of adjacency subgraphs that satisfy the*

left-symmetric identity with $|S| \leqslant p-1$, then the operator $\dashrightarrow$ satisfies the left-symmetry identity; in other cases, the operator $\dashrightarrow$ does not satisfy the left-symmetry identity.

Proof.

(1) Since $|S| = 1$ and $\delta'(a_i, a_i) = 1$ for $a_i \in S$, the operation obviously satisfies the left-symmetric identity.

(2) If $|S| = 2$ and $\delta'(a_i, a_j) = 1$ for all $a_i, a_j \in S$, then the operation satisfies the left-symmetric identity by Theorem 1.
If $|S| = 2$ and $\delta'(a_1, a_2) = 0$, then the operation obviously satisfies $(a_i \dashrightarrow a_j) \dashrightarrow a_k = a_i \dashrightarrow (a_j \dashrightarrow a_k) = 0$ for $a_i, a_j, a_k \in \{a_1, a_2\}$.

(3) We discuss the validity of the left-symmetric identity for the operation on S by classifying the cardinality of S and the value of $\delta'(a_i, a_j)$.

Case 1. $|S| = 3$, i.e., $S = \{a_1, a_2, a_3\}$. We divide this case into four subcases according to the number of pairs (a_i, a_j) satisfying $\delta'(a_i, a_j) = 1$.

Subcase 1.1. $\delta'(a_i, a_j) = 1$ for any $a_i, a_j, a_k \in \{a_1, a_2, a_3\}$. Then the operation satisfies the left-symmetric identity by Theorem 1.

Subcase 1.2. $\delta'(a_i, a_j) = 0$ for any $a_i, a_j, a_k \in \{a_1, a_2, a_3\}$. Then the operation obviously satisfies the left-symmetric identity.

Suncase 1.3. There exist two pairs of elements in S such that $\delta'(a_i, a_j) = 0$ for $a_i, a_j, a_k \in \{a_1, a_2, a_3\}$. Without loss of generality, assume $\delta'(a_i, a_k) = \delta'(a_j, a_k) = 0$ and $\delta'(a_i, a_j) = 1$. Then $(a_i, a_j, a_k) = (a_j, a_i, a_k) = 0$. Hence, the operation satisfies the left-symmetric identity.

Subcase 1.4. There exist two pairs of elements in S such that $\delta'(a_i, a_j) = 1$ for $a_i, a_j, a_k \in \{a_1, a_2, a_3\}$. Without loss of generality, assume $\delta'(a_i, a_j) = \delta'(a_j, a_k) = 1$ and $\delta'(a_i, a_k) = 0$. Then $(a_i \dashrightarrow a_j) \dashrightarrow a_k - a_i \dashrightarrow (a_j \dashrightarrow a_k) = 0$ and $(a_j \dashrightarrow a_i) \dashrightarrow a_k - a_j \dashrightarrow (a_i \dashrightarrow a_k) = a_k a_j a_i + a_i a_j a_k$. Hence, it does not satisfy the left-symmetric identity.

Case 2. $|S| > 3$. We prove this case by mathematical induction on the cardinality of S. When $|S| = 3$, the conclusion has been verified. Assume the conclusion holds when $|S| = p - 1$. We now consider the case $|S| = p$.

Subcase 2.1. $\delta'(a_i, a_j) = 1$ for all $a_i, a_j \in S$. Then the operation satisfies the left-symmetric identity by Theorem 1.

Subcase 2.2. There exist $a_i, a_j \in S$ such that $\delta'(a_i, a_j) = 0$ and its adjacency graph can be decomposed into a family of adjacency subgraphs that satisfy the left-symmetric identity. For any $a_i, a_j, a_k \in S$, two situations arise. If a_i, a_j, a_k do not come from the same subgraph, then the associator $(a_i, a_j, a_k) = (a_j, a_i, a_k) = 0$, which satisfies the left-symmetric identity. If a_i, a_j, a_k come from the same subgraph, then the left-symmetric identity is obviously satisfied by the inductive hypothesis.

Subcase 2.3. If the adjacency graph cannot be decomposed into such subgraphs, then it does not satisfy the left-symmetric identity by the inductive hypothesis. □

Example 4.

(1) Let $S = \{a, b, c\}$, $\delta'(a, b) = \delta'(b, c) = 1$ and $\delta'(a, c) = 0$. Set $x = a$, $y = b$ and $z = c$, then $x \dashrightarrow y = ab + ba$ and $y \dashrightarrow z = bc + cb$.
Then

$$
\begin{aligned}
&(x \dashrightarrow y) \dashrightarrow z = abc + cba,\ (y \dashrightarrow x) \dashrightarrow z = abc + cba,\\
&x \dashrightarrow (y \dashrightarrow z) = abc + cba,\ y \dashrightarrow (x \dashrightarrow z) = 0,\\
&LHS := (x \dashrightarrow y) \dashrightarrow z - x \dashrightarrow (y \dashrightarrow z) = 0,\\
&RHS := (y \dashrightarrow x) \dashrightarrow z - y \dashrightarrow (x \dashrightarrow z) = abc + cba.
\end{aligned}
$$

Hence, it does not satisfy the left-symmetric identity.

(2) Let $S = \{a, b, c\}$, $\delta'(a, b) = \delta'(b, c) = 0$ *and* $\delta'(a, c) = 1$. *Set* $x = a$, $y = b$ *and* $z = c$, *then* $x \dashrightarrow z = ac + ca$.
Then

$$\begin{aligned}
&(x \dashrightarrow y) \dashrightarrow z = 0,\ (y \dashrightarrow x) \dashrightarrow z = 0,\\
&x \dashrightarrow (y \dashrightarrow z) = 0,\ y \dashrightarrow (x \dashrightarrow z) = 0,\\
&LHS := (x \dashrightarrow y) \dashrightarrow z - x \dashrightarrow (y \dashrightarrow z) = 0,\\
&RHS := (y \dashrightarrow x) \dashrightarrow z - y \dashrightarrow (x \dashrightarrow z) = 0.
\end{aligned}$$

Hence, it satisfies the left-symmetric identity.

Corollary 1. *Let* $G = (V, E)$ *be the adjacency graph associated with* δ', *where the vertex set* $V = S$ *and the edge set* $E = \{(a_i, a_j) \mid \delta'(a_i, a_j) = 1\}$. *Then*

(1) If G *is a complete graph (i.e.,* $\delta'(a_i, a_j) = 1$ *for all* $a_i, a_j \in S$*), the operation* $\dashrightarrow$ *satisfies the left-symmetric identity.*

(2) If G *is not a complete graph but can be decomposed into a family of adjacency subgraphs that satisfy the left-symmetry identity with* $|S| \leqslant p - 1$, *the operation* $\dashrightarrow$ *satisfies the left-symmetric identity.*

4. The Modified Simplified Insertion

In this section, we mainly introduce the modified simplified insertion and establish the core Theorem 3.

4.1. The Modified Simplified Insertion

Definition 3. *We define the modified simplified insertion as follows:*

$$x \Rightarrow y := f(|x|, |y|) \sum_{i=0}^{q} y_1 y_2 \cdots y_i x y_{i+1} \cdots y_q,$$

where the function $f : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{F}$ *is given by* $(m, n) \mapsto f(m, n)$. *Then, we have the following:*

$$(x, y, z)_2 = (x \circ y) \circ z - x \circ (y \circ z) = ((x \Rightarrow y) \Rightarrow z) - (x \Rightarrow (y \Rightarrow z)). \tag{8}$$

Example 5. *Let* $x = a_1 a_2 a_3$ *and* $y = a_4 a_5$, *then* $x \Rightarrow y = f(3, 2)(a_1 a_2 a_3 a_4 a_5 + a_4 a_1 a_2 a_3 a_5 + a_4 a_5 a_1 a_2 a_3)$.

4.2. The Left-Symmetric Algebra for Modified Simplified Insertion

Theorem 3. $A[S]$ *is the left-symmetric algebra where* (x, y, z) *is defined as* (8) *if and only if* f *satisfies the following conditions:*

$$f(m, n) f(m + n, p) = f(n, p) f(m, n + p) = f(m, p) f(n, m + p), \tag{9}$$

for any $m, n, p \in \mathbb{N}$.

Proof. Let x, y, z be the basis vectors of $A[S]$. Let

$$x = x_1 x_2 \cdots x_p,\ \ y = y_1 y_2 \cdots y_q,\ \ z = z_1 z_2 \cdots z_r.$$

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Then $|x| = p$, $|y| = q$, $|z| = r$. We obtain the following:

$$x \Rightarrow y = f(|x|,|y|)\sum_{j=0}^{q} y_1y_2\cdots y_jxy_{j+1}\cdots y_q = f(p,q)\sum_{j=0}^{q} y_1y_2\cdots y_jxy_{j+1}\cdots y_q,$$

$$y \Rightarrow z = f(|y|,|z|)\sum_{k=0}^{r} z_1z_2\cdots z_kyz_{k+1}\cdots z_r = f(q,r)\sum_{k=0}^{r} z_1z_2\cdots z_kyz_{k+1}\cdots z_r.$$

Then

$$\begin{aligned}
& x \Rightarrow (y \Rightarrow z) \\
= \; & f(|y|,|z|)f(|x|,|y|+|z|)\sum_{k=0}^{r}(\sum_{j=0}^{k-1} z_1z_2\cdots z_jxz_{j+1}\cdots z_kyz_{k+1}\cdots z_r \\
& +z_1z_2\cdots z_k(x\to y)z_{k+1}\cdots z_r + \sum_{j=k+1}^{r} z_1z_2\cdots z_kyz_{k+1}\cdots z_jxz_{j+1}\cdots z_r) \\
= \; & f(q,r)f(p,q+r)\sum_{k=0}^{r}(\sum_{j=0}^{k-1} z_1z_2\cdots z_jxz_{j+1}\cdots z_kyz_{k+1}\cdots z_r \\
& +z_1z_2\cdots z_k(x\to y)z_{k+1}\cdots z_r + \sum_{j=k+1}^{r} z_1z_2\cdots z_kyz_{k+1}\cdots z_jxz_{j+1}\cdots z_r), \\
& (x \Rightarrow y) \Rightarrow z \\
= \; & f(|x|,|y|)f(|x|+|y|,|z|)\sum_{j=0}^{q}\sum_{k=0}^{r} z_1\cdots z_ky_1\cdots y_jxy_{j+1}\cdots y_qz_{k+1}\cdots z_r \\
= \; & f(p,q)f(p+q,r)\sum_{j=0}^{q}\sum_{k=0}^{r} z_1\cdots z_ky_1\cdots y_jxy_{j+1}\cdots y_qz_{k+1}\cdots z_r.
\end{aligned}$$

Denote $H(p,q,r)$ and $H_2(p,q,r)$ as

$$f(p,q)f(p+q,r) - f(q,r)f(p,q+r) \text{ and } f(q,r)f(p,q+r),$$

respectively. Hence, we obtain the following:

$$\begin{aligned}
& ((x \Rightarrow y) \Rightarrow z) - (x \Rightarrow (y \Rightarrow z)) \\
= \; & H(p,q,r)\sum_{k=0}^{r}\sum_{j=0}^{q} z_1\cdots z_ky_1\cdots y_jxy_{j+1}\cdots y_qz_{k+1}\cdots z_r \\
& -H_2(p,q,r)\sum_{k=0}^{r}\sum_{j=0}^{k-1} z_1\cdots z_jxz_{j+1}\cdots z_kyz_{k+1}\cdots z_r \\
& -H_2(p,q,r)\sum_{k=0}^{r}\sum_{j=k+1}^{r} z_1\cdots z_kyz_{k+1}\cdots z_jxz_{j+1}\cdots z_r \\
:= \; & LHS,
\end{aligned} \tag{10}$$

$$\begin{aligned}
& ((y \Rightarrow x) \Rightarrow z) - (y \Rightarrow (x \Rightarrow z)) \\
= \; & H(q,p,r) \sum_{k=0}^{r} \sum_{j=0}^{p} z_1 \cdots z_k x_1 \cdots x_j y x_{j+1} \cdots x_p z_{k+1} \cdots z_r \\
& - H_2(q,p,r) \sum_{k=0}^{r} \sum_{j=0}^{k-1} z_1 \cdots z_j y z_{j+1} \cdots z_k x z_{k+1} \cdots z_r \\
& - H_2(q,p,r) \sum_{k=0}^{r} \sum_{j=k+1}^{r} z_1 \cdots z_k x z_{k+1} \cdots z_j y z_{j+1} \cdots z_r \\
:= \; & RHS. \qquad (11)
\end{aligned}$$

From (10), we observe that the beginning term of LHS is yxz with the coefficient $H(p,q,r)$ and there are no other terms containing yxz. The remaining terms are those in which both x and y are inserted into z and separated by at least one letter of z. Therefore, by Remark 1, for $\forall\, x, y, z$, $LHS = RHS$ if and only if $H(p,q,r) = H(q,p,r) = 0$ and $H_2(p,q,r) = H_2(q,p,r)$. This directly implies the following:

$$f(p,q)f(p+q,r) = f(q,r)f(p,q+r) = f(p,r)f(q,p+r).$$

Since x, y, z are the basis vectors and any element in $A[S]$ is a linear combination of such monomials, the result extends to all elements in $A[S]$. Hence,

$$f(m,n)f(m+n,p) = f(n,p)f(m,n+p) = f(m,p)f(n,m+p),$$

for any $m, n, p \in \mathbb{N}$. □

Remark 4. *If we denote $H_1(|x|,|y|,|z|)$ as $f(|x|,|y|)f(|x|+|y|,|z|)$, then we have*

$$H_1(|x|,|y|,|z|) = H_2(|x|,|y|,|z|), \;\; H_2(|x|,|y|,|z|) = H_2(|y|,|x|,|z|).$$

That is

$$f(|x|,|y|)f(|x|+|y|,|z|) = f(|y|,|z|)f(|x|,|y|+|z|)$$

and

$$f(|y|,|z|)f(|x|,|y|+|z|) = f(|x|,|z|)f(|y|,|x|+|z|).$$

Example 6. *Let $S = \{A, C, G, T\}$. Set $X = AC$, $Y = G$, $Z = T$ and $f(m,n) = e^{mn}$. We obtain the following:*

$$X \Rightarrow Y = e^2(ACG + GAC),$$
$$Y \Rightarrow Z = e(GT + TG).$$

Then

$$\begin{aligned}
(X \Rightarrow Y) \Rightarrow Z &= e^5(ACGT + GACT + TACG + TGAC), \\
X \Rightarrow (Y \Rightarrow Z) &= e^5(ACGT + ACTG + GACT + TACG + GTAC + TGAC).
\end{aligned}$$

Hence,

$$\begin{aligned}
((X \Rightarrow Y) \Rightarrow Z) - (X \Rightarrow (Y \Rightarrow Z)) &= -e^5(ACTG + GTAC), \qquad (12) \\
((Y \Rightarrow X) \Rightarrow Z) - (Y \Rightarrow (X \Rightarrow Z)) &= -e^5(ACTG + GTAC).
\end{aligned}$$

It satisfies the left-symmetric identity.

We compare the properties of the four types of DNA insertion operations in Table 1.

Table 1. Algebraic properties of different DNA insertion operations.

Operation Type	Assoc.	Comm.	Left Sym.
Simplified Insertion ($\rightarrow$)	No	No	Yes
Intermediate Insertion ($\circ$)	No	No	No
Synchronized Insertion ($\rightrightarrows$)	No	No	No
Modified Insertion ($\Rightarrow$)	No	No	Yes *

Note: * This holds when f satisfies conditions (12) in Theorem 3.

4.3. The Existence of f

Recall that f satisfies the following equations:

$$f(m,n)f(m+n,p) = f(n,p)f(m,n+p), \tag{13}$$

$$f(n,p)f(m,n+p) = f(m,p)f(n,m+p), \tag{14}$$

where $\forall\, m,n,p \in \mathbb{N}$.

Equations (13) and (14) admit a broad family of solutions, the general form of which is characterized by the following proposition.

Proposition 1. *Let $\varphi : \mathbb{N} \to \mathbb{F}$ be an arbitrary non-zero function. Define the following:*

$$f(m,n) = \frac{\varphi(m)\varphi(n)}{\varphi(m+n)}, \quad \textit{for } m,n \in \mathbb{N}.$$

Then f satisfies (13) and (14).

Proof. For any $m,n,p \in \mathbb{N}$,

$$f(m,n)f(m+n,p) = \frac{\varphi(m)\varphi(n)}{\varphi(m+n)} \cdot \frac{\varphi(m+n)\varphi(p)}{\varphi(m+n+p)} = \frac{\varphi(m)\varphi(n)\varphi(p)}{\varphi(m+n+p)},$$

$$f(n,p)f(m,n+p) = \frac{\varphi(n)\varphi(p)}{\varphi(n+p)} \cdot \frac{\varphi(m)\varphi(n+p)}{\varphi(m+n+p)} = \frac{\varphi(m)\varphi(n)\varphi(p)}{\varphi(m+n+p)}.$$

Hence $f(m,n)f(m+n,p) = f(n,p)f(m,n+p)$. Similarly,

$$f(n,p)f(m,n+p) = f(m,p)f(n,m+p),$$

Hence f satisfies (13) and (14). □

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Example 7. *We choose a symmetric bilinear function $g : \mathbb{N} \times \mathbb{N} \to \mathbb{F}$ given by $(m, n) \mapsto g(m, n)$. For example, $g(m, n) = k \cdot mn$, where k is a fixed positive number. Let $f(m, n) = \exp\{g(m, n)\}$. Then*

$$\begin{aligned} & f(m,n)f(m+n,p) \\ = \ & \exp\{g(m,n)\}\exp\{g(m+n,p)\} \\ = \ & \exp\{g(m,n)+g(m+n,p)\} \\ = \ & \exp\{g(m,n)+g(m,p)+g(n,p)\} \\ = \ & \exp\{g(m,n+p)+g(n,p)\} \\ = \ & \exp\{g(m,n+p)\}\exp\{g(n,p)\} \\ = \ & f(m,n+p)f(n,p), \end{aligned}$$

$$\begin{aligned} & f(n,p)f(m,n+p) \\ = \ & \exp\{g(n,p)\}\exp\{g(m,n+p)\} \\ = \ & \exp\{g(n,p)+g(m,n+p)\} \\ = \ & \exp\{g(m,n)+g(m,p)+g(n,p)\} \\ = \ & \exp\{g(n,m+p)+g(m,p)\} \\ = \ & \exp\{g(n,m+p)\}\exp\{g(m,p)\} \\ = \ & f(n,m+p)f(m,p). \end{aligned}$$

Hence, f satisfies (13) and (14).

Example 8. *Assume that the function $f : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{F}$ is defined by $f(m, n) := \frac{m!\,n!}{(m+n)!}$, where $m, n \in \mathbb{N}$. Then*

$$\begin{aligned} & f(m,n)f(m+n,p) \\ = \ & \frac{m!\,n!}{(m+n)!}\cdot\frac{(m+n)!\,p!}{(m+n+p)!} \\ = \ & \frac{m!\,n!\,p!}{(m+n+p)!} \\ = \ & \frac{n!\,p!}{(n+p)!}\cdot\frac{m!\,(n+p)!}{(m+n+p)!} \\ = \ & f(n,p)f(m,n+p) \end{aligned}$$

$$\begin{aligned} & f(n,p)f(m,n+p) \\ = \ & \frac{n!\,p!}{(n+p)!}\cdot\frac{m!\,(n+p)!}{(m+n+p)!} \\ = \ & \frac{m!\,n!\,p!}{(m+n+p)!} \\ = \ & \frac{m!\,p!}{(m+p)!}\cdot\frac{n!\,(m+p)!}{(m+n+p)!} \\ = \ & f(m,p)f(n,m+p). \end{aligned}$$

Hence, f satisfies (13) and (14).

Example 9.

(1) Let $f(m, n) \equiv C_0$, where C_0 is a constant. It is the same as the simplified insertion.

(2) We can easily check that f satisfies the (13) and (14), where f is defined as follows

$$f(m,n) = \begin{cases} 1, & \text{if } m,n \text{ are both odd,} \\ 0, & \text{otherwise.} \end{cases}$$

Corollary 2. *Let $n = p$ in (13) and (14). Then we get $f(m+p,p) = f(p,m+p)$, for $m,\ p \in \mathbb{N}$. Hence, f is symmetric, i.e.,*

$$f(m,n) = f(n,m), \quad \text{for } m,n \in \mathbb{N}.$$

Based on Proposition 1, we compare the algebraic properties of three characteristic functions that satisfy condition (12) in Table 2.

Table 2. Comparison of algebraic properties of f.

Functions	$f(m,n) = 1$	$f(m,n) = e^{kmn}$	$f(m,n) = \frac{m!n!}{(m+n)!}$
Symmetry	Yes	Yes	Yes
Satisfying (12)	Yes	Yes	Yes
Left-symmetric algebra	Yes	Yes	Yes
General solution form	$\varphi \equiv 1$	$\varphi(n) = e^{\frac{k}{2}n^2}$	$\varphi(n) = n!$

4.4. Application in Genetics

The algebraic framework of left-symmetric algebras constructed from DNA insertion operations provides insights into DNA recombination mechanisms. The operation of the modified insertion with functions $f(m,n)$ establishes an algebraic foundation for the quantitative study of length-dependent insertion, such as in mathematical descriptions of transposition processes. And the constraint condition

$$f(m,n)f(m+n,p) = f(n,p)f(m,n+p) = f(m,p)f(n,m+p)$$

provides a mathematical model for transposon expansion dynamics and selection pressure. This can be interpreted as a model assumption. Then we consider a simple example: $f(m,n) := e^{k\cdot mn}$, where $k = -0.01$. From Figure 2, we can clearly observe that $f(m,n)$ decays exponentially with the increase in fragment length. In other words, when the fragment length is sufficiently large, the new genetic features induced by the left simplified insertion can be suppressed by coefficient adjustment. This finding provides a mathematical basis for relevant studies in biological genetics and genomics.

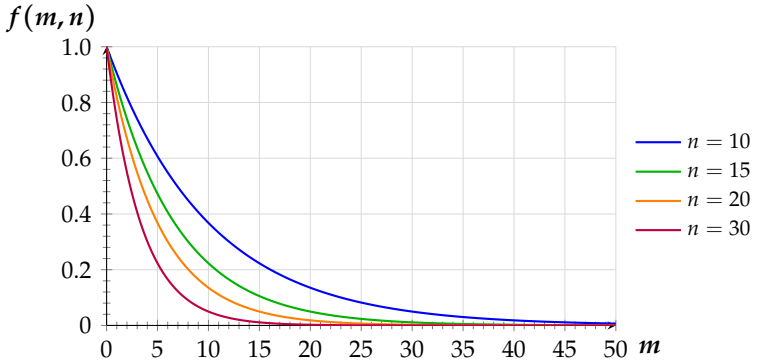


Figure 2. The trend of the coefficient $f(m,n) = e^{-0.01mn}$.

5. Conclusions

In this paper, we construct a new class of left-symmetric algebra structures from DNA insertion operations. The main result is Theorem 3, which states that the algebra is left-symmetric if and only if f satisfies

$$f(m,n)f(m+n,p) = f(n,p)f(m,n+p) = f(m,p)f(n,m+p),$$

for any $m, n, p \in \mathbb{N}$. While this formula is structurally similar to associativity-type cocycle conditions [14], it is not imported from abstract theory; rather, it is derived as a necessary and sufficient condition for the DNA insertion operation to satisfy the left-symmetric identity. This derivation also provides a novel interpretation of this cocycle condition in genetics.

However, the proposed algebraic structure remains a theoretical construct and has not yet been validated against real-world DNA recombination data. Future research may seek to deduce the specific form of the function $f(m,n)$ and verify its predictive performance. Furthermore, the functional equations for f are formally analogous to studies on the stability of bi-derivations and bihomomorphisms in Banach algebras and approximation problems of homomorphisms and derivations in Lie C^*-algebras [26,27]. This, in turn, paves the way for further exploration of left-symmetric algebras arising from DNA insertions and their related stability problems.

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