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Multidimensional Widder–Arendt Theorem in Locally Convex Spaces

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Abstract

In this research article, we formulate and prove the multidimensional Widder–Arendt theorem and the integrated form of the multidimensional Widder–Arendt theorem for functions with values in sequentially complete locally convex spaces. Established results seem to be new even for scalar-valued functions.

Keywords: multidimensional vector-valued Laplace transform; multidimensional Widder–Arendt theorem; sequentially complete locally convex spaces

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1. Introduction and Preliminaries

In 1785, Pierre-Simon, Marquis de Laplace, introduced a new type of integral transform for applications to probability. Speaking matter-of-factly, he sometimes introduced what is nowadays called the transform, and he was the predecessor of what is today called Mellin's transform. The common opinion is that P.-S. Laplace was not the first to invent his transform. The generating functions, used by Laplace, have already been known from the work of A. de Moivre (1718), who can be called the father of modern harmonic analysis. Furthermore, many historians of mathematics highlight that the modern form of the Laplace transform was probably used for the first time by L. Euler in 1744. The Laplace transform has been systematically investigated by the British physicist Oliver Heaviside (1850–1925).

The Widder representation theorem for the Laplace transform was proved in 1934 [1,2]. Following the ideas of I. Miyadera ([3]), S. Zaidman proved in 1960 that the Widder representation theorem holds for functions with values in a reflexive Banach space X ([4]). In 1987, W. Arendt proved that the Widder representation theorem holds for functions with values in a Banach space X if and only if X has the Radon–Nikodym property, which was essentially important for a further expansion of the theory of abstract ill-posed Cauchy problems ([5]). In 1991, M. Hieber proved an integrated version of the Widder theorem in an arbitrary Banach space ([6]). After that, the Widder–Arendt theorem has been seriously reconsidered by J. Kisyński [7], Y.-C. Li, S.-Y. Shaw [8] and T.-J. Xiao, J. Liang [9]; cf. also [10] (Theorem 2.2.1, Theorem 2.4.1, Theorem 2.4.2). The main aim of this paper is to reconsider the Widder–Arendt theorem for multidimensional vector-valued Laplace transform ([11]).

The structure and main ideas of the paper can be briefly summarized as follows. We first explain the basic notation used throughout the paper and provide some useful observations about the Widder–Arendt theorem in the one-dimensional setting; cf. Theorems 1 and 2 in Section 1.1. We recall the basic definitions and results about the multidimensional vector-valued Laplace transform in Section 1.2. The multidimensional

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Widder–Arendt theorem in locally convex spaces with Lipschitz–Radon–Nikodym property, which seems to be introduced here for the first time, is formulated and proved in Section 2; cf. Theorem 3. Motivated by results established in [6,8,9], in Section 3 we prove integrated form of the multidimensional Widder–Arendt theorem in an arbitrary sequentially complete locally convex space; cf. Theorem 4. We can freely say that the results established in this paper, like the famous Hille-Yosida theorem, are of pure theoretical interest in functional analysis.

Notation and preliminaries. By X we denote a Hausdorff sequentially complete locally convex space over the field of complex numbers (SCLCS). The abbreviation $\circledast$ stands for the fundamental system of seminorms which defines the topology of X; by X^* we denote the dual space of X equipped with the strong topology. For further information concerning the integration of functions with values in SCLCSs, we refer the reader to [12]. If $p \in \circledast$, then we define $U_p := \{x \in X : p(x) \leq 1\}$ and $U_p^\circ := \{x^* \in X^* : |\langle x^*, x\rangle| \leq 1 \text{ for all } x \in U_p\}$.

The Gamma function is denoted by $\Gamma(\cdot)$ and the principal branch is always used to take the powers. Define $g_\zeta(t) := t^{\zeta-1}/\Gamma(\zeta)$ and $0^\zeta := 0$ ($\zeta > 0$, $t > 0$); $g_0(t) := \delta(t)$ is the Dirac distribution. Generally, the finite convolution product $*_0$ is defined by $(a *_0 b)(t) := \int_0^t a(t-s)b(s)\,ds$, $t \geq 0$; $\delta *_0 a \equiv a$. If $0 < T < +\infty$, then we say that a function $f : [0,T] \to X$ is Lipschitz continuous if for each seminorm $p \in \circledast$ there exists a real constant $M_p > 0$ such that $p(f(t) - f(t')) \leq M_p|t - t'|$ for all t, $t' \in [0,T]$; a function $f : [0,+\infty) \to X$ is said to be Lipschitz continuous if for each $T \in (0,+\infty)$ the function $f_{|[0,T]} : [0,T] \to X$ is Lipschitz continuous.

The Radon–Nikodym property in locally convex spaces has been investigated by many authors; see, e.g., [13–17] and the doctoral dissertation of Y. J. Yaseen [18]. We say that X has the Lipschitz–Radon–Nikodym property if any Lipschitz continuous function $f : [0,+\infty) \to X$ is differentiable almost everywhere, and its first derivative is locally integrable (in the sense of [12] (Definition 1.1.2(i))).

In the setting of complex Banach spaces, the Lipschitz–Radon–Nikodym property is equivalent to the Radon–Nikodym property and the Gel'fand property; it is completely without scope of this paper to further relate these concepts in the setting of locally convex spaces (cf. [10] (Section 1.2), [18] (Definition 3.1, p. 36), [19] and [20] (p. 371) for more details). We will only emphasize here that a Fréchet space X has the Lipschitz–Radon–Nikodym property if any Lipschitz continuous function $f : [0,+\infty) \to X$ is differentiable almost everywhere because its first derivative is automatically locally integrable (the assumption $p(f(t) - f(t')) \leq M_p|t - t'|$ for all t, $t' \in [0,T]$ implies $p(f'(t)) \leq M_p$ for a.e. $t \in [0,T]$ and this is sufficient to ensure the local integrability of $f'(\cdot)$); to the best knowledge of the author, this cannot be so easily done in general SCLCSs. Since any Lipschitz continuous function $f : [0,+\infty) \to X$ is locally of bounded variation, we consequently obtain that any Fréchet space with the Gel'fand property has the Lipschitz–Radon–Nikodym property ([20]).

1.1. Widder–Arendt Theorem in Locally Convex Spaces

In this subsection, we recall and provide some new results about the Widder–Arendt theorem in locally convex spaces. First of all, we will slightly reformulate the integrated version of Widder–Arendt theorem in locally convex spaces (cf. [9] (Theorem 2.3); part (iii)' is new and not considered elsewhere):

Theorem 1. *Suppose that $\omega \in \mathbb{R}$, X is an* SCLCS, *$F : (\omega, +\infty) \to X$ and $(M_p)_{p\in\circledast}$ is a family of positive real numbers. Then the following statements are equivalent:*

(i) *$F(\cdot)$ is infinitely differentiable, and for each seminorm $p \in \circledast$ we have*

$$p\Big(F^{(v)}(\lambda)\Big) \leq M_p \frac{v!}{(\lambda-\omega)^{v+1}}, \quad \lambda > \omega,\ v \in \mathbb{N}_0. \tag{1}$$

(i)′ *There exists $a \geq \max(\omega, 0)$ such that $F_{|(a,+\infty)}(\cdot)$ is infinitely differentiable and for each seminorm $p \in \circledast$ the estimate (1) holds for any $\lambda > a$ and $v \in \mathbb{N}_0$.*

(ii) *$F(\cdot)$ admits an analytical extension to the right half-plane $\{z \in \mathbb{C} : \Re z > \omega\}$, denoted by the same symbol, and for each seminorm $p \in \circledast$ we have*

$$p\Big(F^{(v)}(\lambda)\Big) \leq M_p \frac{v!}{(\Re\lambda-\omega)^{v+1}}, \quad \Re\lambda > \omega,\ v \in \mathbb{N}_0.$$

(iii) *For every $r \in (0, 1]$, there exists a continuous function $f_r : [0, +\infty) \to X$ such that $f_r(0) = 0$,*

$$F(\lambda) = \lambda^r \int_0^{+\infty} e^{-\lambda t} f_r(t)\, dt, \quad \lambda > \max(\omega, 0),$$

for each seminorm $p \in \circledast$ we have

$$p\Big(f_r(t+h) - f_r(t)\Big) \leq \frac{2M_p}{r\Gamma(r)} \max\Big(e^{\omega(t+h)}, 1\Big) h^r, \quad t \geq 0,\ h \geq 0,$$

and the function $H_r(\cdot) := (g_{1-r} *_0 f_r)(\cdot)$ satisfies that for each seminorm $p \in \circledast$ we have

$$p\Big(H_r(t+h) - H_r(t)\Big) \leq M_p e^{\omega t} \max\Big(e^{\omega h}, 1\Big) h, \quad t \geq 0,\ h \geq 0.$$

(iii)′ *For every $r \in (0, 1]$, there exists a continuous function $h_r : [0, +\infty) \to X$ such that $h_r(0) = 0$,*

$$F(\lambda) = (\lambda-\omega)^r \int_0^{+\infty} e^{-(\lambda-\omega)t} h_r(t)\, dt, \quad \lambda > \omega,$$

for each seminorm $p \in \circledast$ we have

$$p\Big(h_r(t+h) - h_r(t)\Big) \leq \frac{2M_p}{r\Gamma(r)} h^r, \quad t \geq 0,\ h \geq 0,$$

and the function $G_r(\cdot) := (g_{1-r} *_0 h_r)(\cdot)$ satisfies that for each seminorm $p \in \circledast$ we have

$$p\Big(G_r(t+h) - G_r(t)\Big) \leq M_p h, \quad t \geq 0,\ h \geq 0. \tag{2}$$

Proof. The implication (iii) $\Rightarrow$ (ii) follows from the proof of implication (ii) $\Rightarrow$ (i) in the above-mentioned theorem, with $a = \max(\omega, 0)$, the implication (ii) $\Rightarrow$ (i)′ is trivial, and the implication (i) $\Rightarrow$ (iii) follows from the proof of implication (i)′ $\Rightarrow$ (ii) in the above-mentioned theorem, with $a = \max(\omega, 0)$. Therefore, the statements (i)′, (ii) and (iii) are equivalent. The equivalence of these conditions with (i) follows from the trivial implications (ii) $\Rightarrow$ (i) $\Rightarrow$ (i)′. If (iii) holds, then we define the function $G : (0, +\infty) \to X$ by $G(\lambda) := F(\lambda + \omega)$, $\lambda > 0$. Then, due to (i), it follows that for each seminorm $p \in \circledast$ we have $p(G^{(v)}(\lambda)) \leq M_p v!/\lambda^{v+1}$, $\lambda > 0$, $v \in \mathbb{N}_0$. Applying (i) $\Rightarrow$ (iii) to function $G(\cdot)$, with $\omega = 0$, we get (iii)′. Finally, if (iii)′ holds, then (iii) holds for function $G(\cdot)$ defined above, with $\omega = 0$. Hence, (i) holds for function

$G(\cdot)$, with $\omega = 0$, so that for each seminorm $p \in \circledast$ we have $p(G^{(v)}(\lambda)) \leq M_p v!/\lambda^{v+1}$, $\lambda > 0$, $v \in \mathbb{N}_0$. This simply implies (i) and completes the proof. □

The following result is not well formulated in the setting of locally convex spaces by now (cf. [10] (Theorem 2.4.2) for the Banach space setting):

Theorem 2. *Suppose that $\omega \in \mathbb{R}$, X is an* SCLCS*, X has the Lipschitz–Radon–Nikodym property, $F : (\omega, +\infty) \to X$ and $(M_p)_{p \in \circledast}$ is a family of positive real numbers. Then the following statements are equivalent:*

(i) *$F(\cdot)$ is infinitely differentiable and (1) holds for all $p \in \circledast$, $\lambda > \omega$ and $v \in \mathbb{N}_0$.*

(ii) *There exists $f \in L^1_{loc}([0, +\infty) : X)$ such that*

$$F(\lambda) = \int_0^{+\infty} e^{-\lambda t} f(t)\, dt, \quad \lambda > \omega,$$

and for each seminorm $p \in \circledast$ we have $p(f(t)) \leq M_p e^{\omega t}$, $t \geq 0$.

Proof. Let us assume first (ii). Then $F(\cdot)$ is analytic in the right half-plane $\{\lambda \in \mathbb{C} : \Re\lambda > \omega\}$ and

$$\frac{d^v}{d\lambda^v} F(\lambda) = \int_0^{+\infty} e^{-\lambda t} (-t)^v f(t)\, dt, \quad \lambda > \omega;$$

cf. the equation [12] ((34), p. 58). Then a simple calculation yields that (i) holds. To prove the converse statement, let us consider the function $h_1(\cdot)$ given in part (iii)′ of Theorem 1. Then we know that (2) holds with $r = 1$ as well as that for each seminorm $p \in \circledast$ the estimate (2) holds with the function $G_r(\cdot)$ replaced by the function $f_1(\cdot)$ therein. Therefore, for each seminorm $p \in \circledast$ we have $p(h_1(t + h) - h_1(t)) \leq M_p h$, $t \geq 0$, $h \geq 0$ and the function $h_1(\cdot)$ is Lipschitz continuous. Since X has the Lipschitz–Radon–Nikodym property, its first derivative $h_1'(t)$ exists for a.e. $t \geq 0$ and it is locally integrable on the non-negative real axis. It is clear that for each seminorm $p \in \circledast$ we have $p(h_1'(t)) \leq M_p$ for a.e. $t \geq 0$. Applying the formula for partial integration established in [12] (Theorem 1.1.4(iii)) and keeping in mind the equality $h_1(0) = 0$, it readily follows that

$$F(\lambda) = \int_0^{+\infty} e^{-(\lambda-\omega)t} h_1'(t)\, dt, \quad \lambda > \omega.$$

This yields

$$F^{(v)}(\lambda) = \int_0^{+\infty} e^{-(\lambda-\omega)t} (-t)^v h_1'(t)\, dt, \quad \lambda > \omega,\ v \in \mathbb{N}_0,$$

which immediately implies (i). □

1.2. Multidimensional Vector-Valued Laplace Transform

The double Laplace transform of scalar-valued functions was initially analyzed by D. L. Bernstein [21] (1939), J. C. Jaeger [22] (1939–1941) and L. Amerio [23] (1940). In our recent paper [11], we have analyzed the multidimensional Laplace transform of functions with values in SCLCSs. We recall the following notion:

Definition 1. *Suppose that X is an SCLCS, $f : [0,+\infty)^n \to X$ is a locally integrable function and $(\lambda_1,\ldots,\lambda_n) \in \mathbb{C}^n$. If*

$$F(\lambda_1,\ldots,\lambda_n) := \lim_{t_1\to+\infty;\ldots;t_n\to+\infty} \int_0^{t_1}\cdots\int_0^{t_n} e^{-\lambda_1 s_1-\cdots-\lambda_n s_n} f(s_1,\ldots,s_n)\, ds_1\ldots ds_n$$
$$:= \int_0^{+\infty}\cdots\int_0^{+\infty} e^{-\lambda_1 t_1-\cdots-\lambda_n t_n} f(t_1,\ldots,t_n)\, dt_1\ldots dt_n,$$

exists in topology of X, then we say that the Laplace integral $(\mathcal{L}f)(\lambda_1,\ldots,\lambda_n) := \hat{f}(\lambda_1,\ldots,\lambda_n) := F(\lambda_1,\ldots,\lambda_n)$ exists. We define the region of convergence of the Laplace integral $\Omega(f)$ by $\Omega(f) := \{(\lambda_1,\ldots,\lambda_n) \in \mathbb{C}^n : F(\lambda_1,\ldots,\lambda_n)$ exists$\}$. Furthermore, if for each seminorm $p \in \circledast$ we have

$$\int_0^{+\infty}\cdots\int_0^{+\infty} p\Big(e^{-\lambda_1 t_1-\cdots-\lambda_n t_n} f(t_1,\ldots,t_n)\Big)\, dt_1\ldots dt_n < +\infty,$$

then we say that the Laplace integral $F(\lambda_1,\ldots,\lambda_n)$ converges absolutely. We define the region of absolute convergence of the Laplace integral $\Omega_{abs}(f)$ by $\Omega_{abs}(f) := \{(\lambda_1,\ldots,\lambda_n) \in \mathbb{C}^n :$ $F(\lambda_1,\ldots,\lambda_n)$ converges absolutely$\}$. Finally, we define $\Omega_b(f)$ as the set of all tuples $(\lambda_1,\ldots\lambda_n) \in \mathbb{C}^n$ such that the set

$$\left\{\int_0^{t_1}\int_0^{t_2}\cdots\int_0^{t_n} e^{-\lambda_1 s_1-\lambda_2 s_2-\cdots-\lambda_n s_n} f(s_1,s_2,\ldots,s_n)\, ds_1\, ds_2\ldots ds_n : t_1 \geq 0,\ldots,t_n \geq 0\right\}$$

is bounded in X.

If $f : [0,+\infty)^n \to X$ is locally integrable, $\emptyset \neq \Omega \subseteq \Omega(f) \cap \Omega_b(f)$ is open and $(\lambda_1,\ldots,\lambda_n) \in \Omega$, then the mapping $F : \Omega \to X$ is holomorphic and we have

$$F^{(v_1,\ldots,v_n)}(\lambda_1,\ldots,\lambda_n) = (-1)^{v_1+\cdots+v_n}$$
$$\times\left[\int_0^{+\infty}\cdots\int_0^{+\infty} e^{-\lambda_1 t_1-\cdots-\lambda_n t_n} t_1^{v_1}\cdot\ldots\cdot t_n^{v_n} f(t_1,\ldots,t_n)\, dt_1\ldots dt_n\right], \tag{3}$$

for all $\lambda = (\lambda_1,\ldots,\lambda_n) \in \Omega$ and $(v_1,\ldots,v_n) \in \mathbb{N}_0^n$; for the basic information concerning holomorphic vector-valued functions of several variables, we refer the reader to the research article [24] by K. Kruse and a list of references quoted in [11].

We will employ the following result from [11]:

Lemma 1. *Suppose that $f : [0,+\infty)^n \to X$ is a locally integrable function, $(\lambda_1^0,\ldots,\lambda_n^0) \in \Omega_b(f)$ and*

$$G(t_1,\ldots,t_n) := \int_0^{t_1}\cdots\int_0^{t_n} f(s_1,\ldots,s_n)\, ds_1\ldots ds_n, \quad (t_1,\ldots,t_n) \in [0,+\infty)^n.$$

Then, the following Post–Widder inversion formula holds as:

$$G(t_1,\ldots,t_n) = \lim_{k_1\to+\infty}\ldots\lim_{k_n\to+\infty} \frac{(-1)^{k_1+\cdots+k_n}}{k_1!\cdot\ldots\cdot k_n!}\left(\frac{k_1}{t_1}\right)^{k_1+1}\cdot\ldots\cdot\left(\frac{k_n}{t_n}\right)^{k_n+1}$$
$$\times\frac{\partial^{k_1+\cdots+k_n}(\mathcal{L}G)}{\partial x_1^{k_1}\cdot\ldots\cdot\partial x_n^{k_n}}\left(\frac{k_1}{t_1},\ldots,\frac{k_n}{t_n}\right), \quad t_1 > 0,\ \ldots,\ t_n > 0.$$

Suppose, finally, that $f : [0,+\infty)^n \to X$ is a locally integrable function, $f_{|(0,+\infty)^n}(\cdot)$ is continuous, and there exist real constants $\omega_1 \in \mathbb{R}$, $\ldots$, $\omega_n \in \mathbb{R}$, $\eta_1 \in (-1,+\infty)$, $\ldots$, $\eta_n \in (-1,+\infty)$ and $\zeta_1 \in (-1,+\infty)$, $\ldots$, $\zeta_n \in (-1,+\infty)$ such that, for every seminorm $p \in \circledast$, there exists a finite real constant $M_p > 0$ such that

$$p\Big(f(t_1,\ldots,t_n)\Big) \leq M_p\Big(t_1^{\eta_1}+t_1^{\zeta_1}\Big)\cdot\ldots\cdot\Big(t_n^{\eta_n}+t_n^{\zeta_n}\Big)\exp(\omega_1 t_1+\cdots+\omega_n t_n),$$

for a.e. $t_1 \geq 0,\ldots,\ t_n \geq 0$. Then, the following multidimensional Post–Widder inversion formula holds:

$$\begin{aligned} f(t_1,\ldots,t_n) = \lim_{k_1\to+\infty}\cdots\lim_{k_n\to+\infty}\frac{(-1)^{k_1+\cdots+k_n}}{k_1!\cdot\ldots\cdot k_n!}\Big(\frac{k_1}{t_1}\Big)^{k_1+1}\cdot\ldots\cdot\Big(\frac{k_n}{t_n}\Big)^{k_n+1} \\ \times\frac{\partial^{k_1+\cdots+k_n}F}{\partial x_1^{k_1}\cdot\ldots\cdot\partial x_n^{k_n}}\Big(\frac{k_1}{t_1},\ldots,\frac{k_n}{t_n}\Big), \quad t_1>0,\ \ldots,\ t_n>0. \end{aligned}$$

2. Multidimensional Widder–Arendt Theorem in Locally Convex Spaces with Lipschitz–Radon–Nikodym Property

The main result of this section reads as follows:

Theorem 3. *Let $\omega_1 \in \mathbb{R}$, $\ldots$, $\omega_n \in \mathbb{R}$, X is an* SCLCS*, X has the Lipschitz–Radon–Nikodym property, $(M_p)_{p\in\circledast}$ is a family of positive real numbers, and $F : (\omega_1,+\infty)\times(\omega_2,+\infty)\times\cdots\times(\omega_n,+\infty)\to X$. Let us consider the following statements:*

(i) *$F(\cdot,\ldots,\cdot)$ is infinitely differentiable, and for each seminorm $p \in \circledast$ we have*

$$p\Big(F^{(v_1,\ldots,v_n)}(\lambda_1,\ldots,\lambda_n)\Big) \leq M_p\frac{v_1!\cdots v_n!}{(\lambda_1-\omega_1)^{v_1+1}\cdots(\lambda_n-\omega_n)^{v_n+1}}, \tag{4}$$

for any $\lambda_1>\omega_1,\ldots,\lambda_n>\omega_n$ and $(v_1,\ldots,v_n)\in\mathbb{N}_0^n$.

(ii) *There exists $f \in L^1_{loc}([0,+\infty)^n : X)$ such that*

$$F(\lambda) = \int_0^{+\infty}\cdots\int_0^{+\infty} e^{-\lambda_1 t_1-\cdots-\lambda_n t_n}f(t_1,\ldots,t_n)\,dt_1\ldots dt_n, \quad \lambda_i>\omega_i\ \ (1\leq i\leq n),$$

and for each seminorm $p \in \circledast$ we have $p(f(t_1,\ldots,t_n)) \leq M_p e^{\omega_1 t_1+\cdots+\omega_n t_n}$, $t=(t_1,\ldots,t_n)\in[0,+\infty)^n$.

Then we have (ii) $\Rightarrow$ (i)*. Moreover, if X is a Fréchet space, then* (i) $\Leftrightarrow$ (ii)*.*

Proof. If (ii) holds, then the mapping $(\lambda_1,\ldots,\lambda_n)\mapsto F(\lambda_1,\ldots,\lambda_n)$, $\Re\lambda_1>\omega_1,\ldots,\Re\lambda_n>\omega_n$ is analytic and (3) holds for $(v_1,\ldots,v_n)\in\mathbb{N}_0$ and $\Re\lambda_1>\omega_1,\ldots,\ \Re\lambda_n>\omega_n$. Since for each seminorm $p\in\circledast$ we have $p(f(t_1,\ldots,t_n))\leq M_p e^{\omega_1 t_1+\cdots+\omega_n t_n}$, $t=(t_1,\ldots,t_n)\in[0,+\infty)^n$, (i) follows from a simple computation involving the Fubini theorem.

Assume now that (i) holds and X is a Fréchet space. We will prove that (ii) holds by induction on n. By Theorem 2, (ii) is true in dimension $n=1$; suppose that (ii) is true in any dimension $<n$, with the meaning clear. Let us fix $\lambda_2>\omega_2,\ldots,\ \lambda_n>\omega_n$ and consider the function $F_{\lambda_2,\ldots,\lambda_n} : (\omega_1,+\infty)\to X$ defined by $F_{\lambda_2,\ldots,\lambda_n}(\lambda_1) := F(\lambda_1,\lambda_2,\ldots,\lambda_n)$, $\lambda_1>\omega_1$. Since for each seminorm $p\in\circledast$ we have

$$\begin{aligned} &p\Big(F^{(v_1)}(\lambda_1)\Big) = p\Big(F^{(v_1,0,\ldots,0)}(\lambda_1,\lambda_2,\ldots,\lambda_n)\Big) \\ &\leq M_p\frac{v_1!}{(\lambda_1-\omega_1)^{v_1+1}(\lambda_2-\omega_2)\cdots(\lambda_n-\omega_n)}, \quad v_1\in\mathbb{N}_0,\ \lambda_1>\omega_1, \end{aligned}$$

an application of Theorem 2 yields the existence of a locally integrable function $f_{\lambda_2,\ldots,\lambda_n} : [0,+\infty) \to X$ such that

$$F(\lambda_1,\lambda_2,\ldots,\lambda_n) = \int_0^{+\infty} e^{-\lambda_1 t_1} f_{\lambda_2,\ldots,\lambda_n}(t_1)\,dt_1, \quad \lambda_1 > \omega_1, \tag{5}$$

and for each seminorm $p \in \circledast$ we have

$$p\Big(f_{\lambda_2,\ldots,\lambda_n}(t_1)\Big) \leq M_p e^{\omega_1 t_1}(\lambda_2-\omega_2)^{-1}\cdot\ldots\cdot(\lambda_n-\omega_n)^{-1}, \quad t_1 \geq 0. \tag{6}$$

Since X is a Fréchet space, the Post–Widder inversion formula and (5) and (6) together imply that there exists a Lebesgue measurable set $N \subseteq (0,\infty)$ whose Lebesgue measure is equal to zero, such that

$$f_{\lambda_2,\ldots,\lambda_n}(t_1) = \lim_{k\to+\infty} \frac{(-1)^k}{k!}\left(\frac{k}{t_1}\right)^{k+1} F^{(k,0,\ldots,0)}\left(\frac{k}{t_1},\lambda_2,\ldots,\lambda_n\right),\ t_1 \in (0,+\infty)\setminus N.$$

Now we will prove that, for every $t_1 \in (0,+\infty)\setminus N$, the mapping $(\lambda_2,\ldots,\lambda_n) \to f_{\lambda_2,\ldots,\lambda_n}(t_1)$, $\lambda_2 > \omega_2,\ldots,\ \lambda_n > \omega_n$ is infinitely differentiable as well as that

$$\frac{\partial^{v_2+\cdots+v_n}}{\partial\lambda_2^{v_2}\cdot\ldots\cdot\partial\lambda_n^{v_n}} f_{\lambda_2,\ldots,\lambda_n}(t_1) = \lim_{k\to+\infty} \frac{(-1)^k}{k!}\left(\frac{k}{t_1}\right)^{k+1} F^{(k,v_2,\ldots,v_n)}\left(\frac{k}{t_1},\lambda_2,\ldots,\lambda_n\right), \tag{7}$$

for any $t_1 \in (0,+\infty)\setminus N$. Suppose first that $v_2 = 1$ and $v_3 = \cdots = v_n = 0$. Let $\delta > 0$ be such that $\lambda_2 - \delta > \omega_2$ and let $h \in (-\delta,\delta)$. If $p \in \circledast$, then we have

$$\begin{aligned}
&p\left(\frac{f_{\lambda_2+h,\ldots,\lambda_n}(t_1)-f_{\lambda_2,\ldots,\lambda_n}(t_1)}{h} - \lim_{k\to+\infty}\frac{(-1)^k}{k!}\Big(\frac{k}{t_1}\Big)^{k+1} F^{(k,1,0,\ldots,0)}(k/t_1,\lambda_2,\ldots,\lambda_n)\right)\\
&= \sup_{x^*\in U_p^\circ}\left|\left\langle x^*, \frac{f_{\lambda_2+h,\ldots,\lambda_n}(t_1)-f_{\lambda_2,\ldots,\lambda_n}(t_1)}{h} - \lim_{k\to+\infty}\frac{(-1)^k}{k!}\Big(\frac{k}{t_1}\Big)^{k+1} F^{(k,1,0,\ldots,0)}\big(\frac{k}{t_1},\lambda_2,\ldots,\lambda_n\big)\right\rangle\right|\\
&\leq \sup_{x^*\in U_p^\circ}\limsup_{k\to+\infty}\frac{1}{k!}\Big(\frac{k}{t_1}\Big)^{k+1}\\
&\times\left|\left\langle x^*, \frac{F^{(k,0,0,\ldots,0)}\big(\frac{k}{t_1},\lambda_2+h,\ldots,\lambda_n\big)-F^{(k,0,0,\ldots,0)}\big(\frac{k}{t_1},\lambda_2,\ldots,\lambda_n\big)-hF^{(k,1,0,\ldots,0)}\big(\frac{k}{t_1},\lambda_2,\ldots,\lambda_n\big)}{h}\right\rangle\right|\\
&\leq h\sup_{x^*\in U_p^\circ}\limsup_{k\to+\infty}\left[\frac{1}{k!}\Big(\frac{k}{t_1}\Big)^{k+1}\sup_{\lambda'\in[\lambda_2-\delta,\lambda_2+\delta]}\left|\left\langle x^*, F^{(k,2,0,\ldots,0)}(k/t_1,\lambda_2,\ldots,\lambda_n)\right\rangle\right|\right]\\
&\leq h\limsup_{k\to+\infty}\left[\frac{1}{k!}\Big(\frac{k}{t_1}\Big)^{k+1}\sup_{\lambda'\in[\lambda_2-\delta,\lambda_2+\delta]} p\left(F^{(k,2,0,\ldots,0)}(k/t_1,\lambda_2,\ldots,\lambda_n)\right)\right]\\
&\leq h(\lambda_2-\delta-\omega_2)^{-3}(\lambda_3-\omega_3)^{-1}\cdot\ldots\cdot(\lambda_n-\omega_n)^{-1}\limsup_{k\to+\infty}\left[\frac{1}{k!}\Big(\frac{k}{t_1}\Big)^{k+1}\frac{2M_p k!}{((k/t_1)-\omega_1)^{k+1}}\right]\\
&= 2M_p h(\lambda_2-\delta-\omega_2)^{-3}(\lambda_3-\omega_3)^{-1}\cdot\ldots\cdot(\lambda_n-\omega_n)^{-1}e^{\omega_1 t_1} \to 0,
\end{aligned}$$

as $h \to 0$. Repeating this argumentation, we obtain that (7) holds for $\lambda_2 > \omega_2,\ldots,\ \lambda_n > \omega_n$ and $(v_2,\ldots,v_n) \in \mathbb{N}_0^{n-1}$.

Further on, by the induction hypothesis, for every $t_1 \in (0,+\infty)\setminus N$, there exists a locally integrable function $f_{t_1} : [0,+\infty)^{n-1} \to X$ such that

$$f_{\lambda_2,\ldots,\lambda_n}(t_1) = \int_0^{+\infty}\cdots\int_0^{+\infty} e^{-\lambda_2 t_2-\cdots-\lambda_n t_n} f_{t_1}(t_2,\ldots,t_n)\,dt_2\ldots dt_n,\ \lambda_i > \omega_i\ \ (2 \leq i \leq n), \tag{8}$$

and for each seminorm $p \in \circledast$ we have

$$p\Big(f_{t_1}(t_2,\dots,t_n)\Big) \leq M_p e^{\omega_1 t_1+\cdots+\omega_n t_n}, \quad t_1 \in (0,+\infty)\setminus N,\ (t_2,\dots,t_n) \in [0,+\infty)^{n-1}. \tag{9}$$

Define now $f(t_1,\dots,t_n) := f_{t_1}(t_2,\dots,t_n)$ for a.e. $(t_1,\dots,t_n) \in [0,+\infty)^n$; then we have $f \in L^1_{loc}([0,+\infty)^n : X)$. Using the Fubini theorem, (5) and (9), we get

$$\begin{aligned} F(\lambda_1,\lambda_2,\dots,\lambda_n) &= \int_0^{+\infty} e^{-\lambda_1 t_1} f_{\lambda_2,\dots,\lambda_n}(t_1)\,dt_1 \\ &= \int_0^{+\infty} e^{-\lambda_1 t_1} \int_0^{+\infty}\cdots\int_0^{+\infty} e^{-\lambda_2 t_2-\cdots-\lambda_n t_n} f_{t_1}(t_2,\dots,t_n)\,dt_2\ \dots\ dt_n\,dt_1 \\ &= \int_0^{+\infty}\cdots\int_0^{+\infty} e^{-\lambda_1 t_1-\lambda_2 t_2-\cdots-\lambda_n t_n} f(t_1,t_2,\dots,t_n)\,dt_1\ \dots\ dt_n, \end{aligned}$$

for $\lambda_1 > \omega_1,\dots,\ \lambda_n > \omega_n$. Keeping in mind the estimate (9), this implies that (ii) is true in dimension n, which completes the proof of the theorem. □

3. Integrated Form of Multidimensional Widder–Arendt Theorem in Arbitrary Locally Convex Spaces

In this section, we will prove the following integrated form of Widder–Arendt theorem in arbitrary locally convex spaces:

Theorem 4. *Suppose that $\omega_1 \in \mathbb{R},\dots,\omega_n \in \mathbb{R}$, $F : (\omega_1,+\infty)\times\cdots\times(\omega_n,+\infty) \to X$ and $(M_p)_{p\in\circledast}$ is a family of positive real numbers. Then the following statements are equivalent:*

(i) *$F(\cdot)$ is infinitely differentiable and the estimate (4) holds for any $p \in \circledast$, $\lambda_1 > \omega_1,\dots,\lambda_n > \omega_n$ and $(v_1,\dots,v_n) \in \mathbb{N}_0^n$.*

(i)′ *There exist real numbers $a_1 \geq \max(\omega_1,0),\dots,a_n \geq \max(\omega_n,0)$ such that the function $F_{|(a_1,+\infty)\times\cdots\times(a_n,+\infty)}(\cdot)$ is infinitely differentiable and for each seminorm $p \in \circledast$ the estimate (4) holds for any $\lambda_1 > a_1,\dots,\lambda_n > a_n$ and $(v_1,\dots,v_n) \in \mathbb{N}_0^n$.*

(ii) *$F(\cdot)$ admits an analytical extension to the region $\{z_1 \in \mathbb{C} : \Re z_1 > \omega_1\}\times\cdots\times\{z_n \in \mathbb{C} : \Re z_n > \omega_n\}$, denoted by the same symbol, and for each seminorm $p \in \circledast$ we have*

$$p\Big(F^{(v_1,\dots,v_n)}(\lambda_1,\dots,\lambda_n)\Big) \leq M_p \frac{v_1!\cdot\ldots\cdot v_n!}{(\Re\lambda_1-\omega_1)^{v_1+1}\cdot\ldots\cdot(\Re\lambda_n-\omega_n)^{v_n+1}},$$

for any $\Re\lambda_1 > \omega_1,\dots,\Re\lambda_n > \omega_n$ and $(v_1,\dots,v_n) \in \mathbb{N}_0^n$.

(iii) *We have* (iii.1) and (iii.2) *, where:*

(iii.1) For every $r_1 \in (0,1],\dots,\ r_n \in (0,1]$, there exists a continuous function $f_{r_1,\dots,r_n} : [0,+\infty) \to X$ such that $f_{r_1,\dots,r_n}(0) = 0$, for each seminorm $p \in \circledast$ we have

$$\begin{aligned} &p\Big(f_{r_1,\dots,r_n}(t_1,\dots,t_n)\Big) \\ &\leq M_p \int_0^{t_1}\cdots\int_0^{t_n} g_{r_1}(t_1-s_1)\cdot\ldots\cdot g_{r_n}(t_n-s_n)e^{\omega_1 s_1+\cdots+\omega_n s_n}\,ds_1\dots\,ds_n, \end{aligned} \tag{10}$$

for any $(t_1,\dots,t_n) \in [0,+\infty)^n$, and

$$F(\lambda_1,\dots,\lambda_n) = \lambda_1^{r_1}\cdot\ldots\cdot\lambda_n^{r_n}\int_0^{+\infty}\cdots\int_0^{+\infty} e^{-\lambda_1 t_1-\cdots-\lambda_n t_n} f_{r_1,\dots,r_n}(t_1,\dots,t_n)\,dt_1\ \dots\ dt_n, \tag{11}$$

for $\lambda_1 > \max(\omega_1,0),\dots,\lambda_n > \max(\omega_n,0)$.

(iii.2) For each seminorm $p \in \circledast$ and for each functional $x^* \in U_p^\circ$, there exists a locally integrable function $f_{x^*} : [0,+\infty)^n \to X$ such that $p(f_{x^*}(t_1,\dots,t_n)) \leq M_p \exp(\omega_1 t_1+\cdots+\omega_n t_n)$, $(t_1,\dots,t_n) \in [0,+\infty)^n$ and

$$\left\langle x^*, F(\lambda_1,\dots,\lambda_n)\right\rangle = \int_0^{+\infty}\cdots\int_0^{+\infty} e^{-\lambda_1 t_1-\cdots-\lambda_n t_n}\left\langle x^*, f_{x^*}(t_1,\dots,t_n)\right\rangle dt_1\ \dots\ dt_n, \tag{12}$$

for any $\lambda_1 > \max(\omega_1, 0), \dots, \lambda_n > \max(\omega_n, 0)$.

If this is the case, then we have

$$\begin{aligned}&\left\langle x^*, f_{r_1,\dots,r_n}(t_1,\dots,t_n)\right\rangle\\ &= \int_0^{t_1}\cdots\int_0^{t_n} g_{r_1}(t_1-s_1)\cdot\ldots\cdot g_{r_n}(t_n-s_n)\left\langle x^*, f_{x^*}(s_1,\dots,s_n)\right\rangle ds_1\dots ds_n,\end{aligned} \tag{13}$$

for any $x^* \in U_p^\circ$ *and* $(t_1,\dots,t_n) \in [0,+\infty)^n$.

(iii)′ We have (iii.1)′ and (iii.2)′ *, where:*

(iii.1)′ For every $r_1 \in (0,1], \dots, r_n \in (0,1]$, *there exists a continuous function* $h_{r_1,\dots,r_n} : [0,+\infty)^n \to X$ *such that* $h_{r_1,\dots,r_n}(0) = 0$, *for each seminorm* $p \in \circledast$ *we have*

$$p\left(h_{r_1,\dots,r_n}(t_1,\dots,t_n)\right) \leq M_p g_{r_1+1}(t_1)\cdot\ldots\cdot g_{r_n+1}(t_n), \quad (t_1,\dots,t_n) \in [0,+\infty)^n, \tag{14}$$

and

$$\begin{aligned}&F(\lambda_1,\dots,\lambda_n) = (\lambda_1-\omega_1)^{r_1}\cdot\ldots\cdot(\lambda_n-\omega_n)^{r_n}\\ &\times\int_0^{+\infty}\cdots\int_0^{+\infty} e^{-(\lambda_1-\omega_1)t_1-\cdots-(\lambda_n-\omega_n)t_n} h_{r_1,\dots,r_n}(t_1,\dots,t_n)\,dt_1\ \dots\ dt_n,\end{aligned} \tag{15}$$

for any $\lambda_1 > \omega_1, \dots, \lambda_n > \omega_n$.

(iii.2)′ For each seminorm $p \in \circledast$ *and for each functional* $x^* \in U_p^\circ$, *there exists a locally integrable function* $h_{x^*} : [0,+\infty)^n \to X$ *such that* $p(h_{x^*}(t_1,\dots,t_n)) \leq M_p$, $(t_1,\dots,t_n) \in [0,+\infty)^n$ *and*

$$\left\langle x^*, F(\lambda_1,\dots,\lambda_n)\right\rangle = \int_0^{+\infty}\cdots\int_0^{+\infty} e^{-(\lambda_1-\omega_1)t_1-\cdots-(\lambda_n-\omega_n)t_n}\left\langle x^*, h_{x^*}(t_1,\dots,t_n)\right\rangle dt_1\ \dots\ dt_n,$$

for any $\lambda_1 > \omega_1, \dots, \lambda_n > \omega_n$.

Proof. It is clear that (ii) implies (i) and (i)′. If (i)′ holds, then we can use the usual series expansion to prove (ii); see also the proof of [8] (Theorem 2.1, p. 78). Therefore, the assertions (i), (i)′ and (ii) are equivalent.

If (iii) holds, then we can use the convolution theorem for Laplace transform and the uniqueness theorem for Laplace transform in order to see that (13) holds for any $x^* \in U_p^\circ$ and $(t_1,\dots,t_n) \in [0,+\infty)^n$; cf. [11] for more details. Now we will prove that (iii) implies (ii). Let $r_1 = \cdots = r_n = 1$ and $p \in \circledast$. Using (10) and (11), it readily follows that $F(\cdot)$ admits an analytical extension to the region $\{z_1 \in \mathbb{C} : \Re z_1 > \omega_1\} \times \cdots \times \{z_n \in \mathbb{C} : \Re z_n > \omega_n\}$. Keeping in mind this fact and the uniqueness theorem for analytic functions, it follows that (12) holds for any $\Re\lambda_1 > \omega_1, \dots, \Re\lambda_n > \omega_n$. Now, by the generalization of the norm formula given in [25] (Proposition 22.14), we have

$$p\Big(F^{(v_1,\ldots,v_n)}(\lambda_1,\ldots,\lambda_n)\Big) = \sup_{x^*\in U_p^\circ}\left|\frac{\partial^{v_1+\cdots+v_n}}{\partial\lambda_1^{v_1}\cdot\ldots\cdot\partial\lambda_n^{v_n}}\Big\langle x^*, F(\lambda_1,\ldots,\lambda_n)\Big\rangle\right|$$

$$= \sup_{x^*\in U_p^\circ}\left|\int_0^{+\infty}\cdots\int_0^{+\infty} e^{-\lambda_1 t_1-\cdots-\lambda_n t_n}(-t_1)^{v_1}\cdot\ldots\cdot(-t_n)^{v_n}\Big\langle x^*, f_{x^*}(t_1,\ldots,t_n)\Big\rangle\, dt_1\ldots dt_n\right|$$

$$\leq \int_0^{+\infty}\cdots\int_0^{+\infty} e^{-\Re\lambda_1 t_1-\cdots-\Re\lambda_n t_n} t_1^{v_1}\cdot\ldots\cdot t_n^{v_n}\, p\Big(f_{x^*}(t_1,\ldots,t_n)\Big)\, dt_1\ldots dt_n$$

$$\leq M_p\int_0^{+\infty}\cdots\int_0^{+\infty} e^{-\Re\lambda_1 t_1-\cdots-\Re\lambda_n t_n} t_1^{v_1}\cdot\ldots\cdot t_n^{v_n} e^{\omega_1 t_1+\cdots+\omega_n t_n}\, dt_1\ldots dt_n$$

$$= M_p\frac{v_1!\cdot\ldots\cdot v_n!}{(\Re\lambda_1-\omega_1)^{v_1+1}\cdot\ldots\cdot(\Re\lambda_n-\omega_n)^{v_n+1}},$$

for any $\Re\lambda_1>\omega_1,\ldots,\Re\lambda_n>\omega_n$ and $(v_1,\ldots,v_n)\in\mathbb{N}_0^n$. This yields (ii).

The equivalence between (iii) and (iii)′ can be shown using the same arguments as in the corresponding part of proof of Theorem 1. Therefore, it remains to be proved that (ii) implies (iii). Since for each seminorm $p\in\circledast$ we have $|\langle x^*,x\rangle|\leq p(x)$, $x\in X$, $x^*\in U_p^\circ$, an application of Theorem 3 implies that (iii.2) holds.

The part (iii.1) can be deduced similarly to the one-dimensional case, and we will only provide the main details of the proof here. Let us consider the space $\tilde{X}\subseteq X^{**}$ consisting of all linear functionals $\tilde{x}$ on X^* such that $P_G(\tilde{x}):=\sup_{x^*\in G}|\langle\tilde{x},x^*\rangle|<+\infty$ for each equicontinuous family $G\subseteq X^*$; the fundamental system of seminorms which defines a sequentially complete locally convex topology on $\tilde{X}$ is given by $(P_G)_{G\in\mathcal{T}}$, where $\mathcal{T}$ denotes the collection of all equicontinuous families $G\subseteq X^*$. We know that X and $\tilde{X}$ are linearly and topologically isomorphic ([9]).

Suppose now that the numbers $r_1\in(0,1],\ldots,r_n\in(0,1]$ are given. Using Theorem 3, it follows that for each functional $x^*\in X^*$ there exists a function $f_{x^*}\in L^1_{loc}([0,+\infty)^n)$ such that

$$\Big\langle x^*, F(\lambda)\Big\rangle = \int_0^{+\infty}\cdots\int_0^{+\infty} e^{-\lambda_1 t_1-\cdots-\lambda_n t_n} f_{x^*}(t_1,\ldots,t_n)\, dt_1\ldots dt_n,\ \lambda_i>\omega_i\ \ (1\leq i\leq n).$$

Define

$$f_{r_1,\ldots,r_n;x^*}(t_1,\ldots,t_n):=\int_0^{t_1}\cdots\int_0^{t_n} g_{r_1}(t_1-s_1)\cdot\ldots\cdot g_{r_n}(t_n-s_n) f_{x^*}(s_1,\ldots,s_n)\, ds_1\ldots ds_n,$$

for any $x^*\in X^*$ and $(t_1,\ldots,t_n)\in[0,+\infty)^n$. Using the convolution theorem for multidimensional Laplace transform [11] (Proposition 2.17), it follows that

$$\Big\langle x^*, F(\lambda)\Big\rangle = \lambda_1^{r_1}\cdot\ldots\cdot\lambda_n^{r_n}\int_0^{+\infty}\cdots\int_0^{+\infty} e^{-\lambda_1 t_1-\cdots-\lambda_n t_n} f_{r_1,\ldots,r_n;x^*}(t_1,\ldots,t_n)\, dt_1\ldots dt_n,$$

for $\lambda_1>\max(\omega_1,0),\ldots,\lambda_n>\max(\omega_n,0)$. For every $t=(t_1,\ldots,t_n)\in[0,+\infty)^n$, we define $F_t: X^*\to\mathbb{C}$ by $F_t(x^*):=\langle x^*, f_{r_1,\ldots,r_n;x^*}(t)\rangle$, $x^*\in X^*$. Arguing as in the proof of [9] (Theorem 2.3), we can show that $F_t\in\tilde{X}$ for all $t\in[0,+\infty)^n$ as well as that the mapping $t\mapsto F_t$, $t\in[0,+\infty)^n$ is continuous with respect to the topology of $\tilde{X}$; let us only mention here that the linearity of mapping $F_t(\cdot)$ for fixed $t\in[0,+\infty)^n$ follows from the continuity of function $f_{r_1,\ldots,r_n;x^*}(\cdot)$ for each $x^*\in X^*$ and the uniqueness theorem for multidimensional Laplace transform [11] (Theorem 2.23). The existence of a continuous function $f_{r_1,\ldots,r_n}:[0,+\infty)\to X$ such that $f_{r_1,\ldots,r_n}(0)=0$ and (11) holds for $\lambda_1>\max(\omega_1,0),\ldots,\lambda_n>\max(\omega_n,0)$ now follows by identification of X and $\tilde{X}$ through evaluation and the multidimensional Post–Widder formula given in Lemma 1; cf. also the paragraph following Lemma 1 and [9] (l.-1–l.-3, p. 457). Using the generalized norm formula, we can simply prove that for each seminorm $p\in\circledast$ the estimate (10) holds for any $(t_1,\ldots,t_n)\in[0,+\infty)^n$. □

Remark 1. *Alternatively, the proof of implication (ii) $\Rightarrow$ (iii.1) can be given following the argumentation proposed by Y.-C. Li and S.-Y. Shaw in the proof of [8] (Theorem 2.1, pp. 79–80). In actual fact, we can use the complex characterization theorem for multidimensional vector-valued Laplace transform [11] (Theorem 2.13) in order to see that, for every real numbers $r_1 \in (0,1], \ldots, r_n \in (0,1]$, there exists a continuous function $w_{r_1,\ldots,r_n} : [0,+\infty)^n \to X$ such that $w_{r_1,\ldots,r_n}(0) = 0$ and*

$$F(\lambda_1,\ldots,\lambda_n) = \lambda_1^{r_1+1}\cdot\ldots\cdot\lambda_n^{r_n+1}\int_0^{+\infty}\cdots\int_0^{+\infty} e^{-\lambda_1 t_1-\cdots-\lambda_n t_n} w_{r_1,\ldots,r_n}(t_1,\ldots,t_n)\,dt_1\ldots dt_n,$$

for $\Re\lambda_1 > \max(\omega_1,0),\ldots,\ \Re\lambda_n > \max(\omega_n,0)$. The function $f_{r_1,\ldots,r_n}(\cdot)$ constructed in (iii.1) and the function $w_{r_1,\ldots,r_n}(\cdot)$ satisfy the following equality

$$w_{r_1,\ldots,r_n}(t_1,\ldots,t_n) = \int_0^{t_1}\cdots\int_0^{t_n} f_{r_1,\ldots,r_n}(s_1,\ldots,s_n)\,ds_1\ldots ds_n, \quad t_1 \geq 0,\ \ldots,\ t_n \geq 0,$$

so that the only problem in this approach is to prove that the partial derivative $(\partial^n/\partial t_1\ldots\partial t_n)w_{r_1,\ldots,r_n}(\cdot)$ continuously exists on $[0,+\infty)^n$. It is our strong belief that this problem can be solved with the help of estimate (4) and the bipolar theorem.

In the proof of [8] (Theorem 2.1), the authors do not use the identification of pivot space X and a closed subspace of its bidual X^{**}, which has been originally used in the Banach space setting by M. Hieber in the proof of [6] (Theorem 3.2). The proof of [9] (Theorem 2.3) is also meaningful because T.-J. Xiao and J. Liang have explicitly constructed the space $\tilde{X} \subseteq X^{**}$ which is linearly and topologically isomorphic to the initial space X in the locally convex space setting.

In [26], we have recently introduced the class of multidimensional (a,k)-regularized C-resolvent solution operator families subgenerated by multivalued linear operators. The class of exponentially equicontinuous (a,k)-regularized C-resolvent solution operator families can be precisely characterized with the help of a multidimensional vector-valued Laplace transform. The multidimensional Widder–Arendt type theorems established in this paper can serve to provide the necessary and sufficient conditions for the generation of multidimensional (a,k)-regularized C-resolvent solution operator families. These conditions can be clarified in a rather technical way; cf. also [12] (Theorem 3.2.12) for the one-dimensional setting.

4. Conclusions and Final Remarks

In this paper, we have proved a multidimensional Widder–Arendt theorem and an integrated form of the multidimensional Widder–Arendt theorem for functions with values in sequentially complete locally convex spaces. We have specifically considered the class of locally convex spaces with Lipschitz–Radon–Nikodym property. The results obtained in this paper seem to be new even for scalar-valued functions.

Finally, let us observe that it would also be interesting to extend the statement of the Widder–Arendt–Kisýnski theorem ([7]) and the remarkable results established by A. Bobrowski [27,28] and W. Chojnacki [29,30] to the multidimensional setting, for functions with values in locally convex spaces.

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